Applications of the Endoscopic Classification to Statistics of Cohomological Automorphic Representations on Unitary Groups

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange I will only explain intuitively and imprecisely due to time constraints.

Outline

- Motivation: Understanding $\mathcal{AR}_{\mathrm{disc}}$.
- Statement of Results
- Background: Arthur's Classification
- Background: Taïbi's Inductive Analysis
- Tricks for computation

See ArXiv for details.

WARNING: This work depends on Arthur's classification for non-quasisplit unitary groups! This uses unpublished/unwritten references

What is an Automorphic Representation?

Modular Forms:

- Functions on upper-half plane symmetric space $\mathrm{GL}_2\mathbb{R}/O_2\mathbb{R}$
- w/ symmetries translation by "arithmetic" lattice in $\mathrm{GL}_2\mathbb{R}$

Automorphic Representations: generalize beyond GL_2

- Exact generalization very non-obvious: black box for this talk
- Representations: notion of newform doesn't generalize, analog of space generated by newform

Why do we care?

Just like modular forms:

- They have a lot of handles to grab onto when studying
  - representation theory of reductive groups
  - harmonic analysis
- They mysteriously encode information about so much else:
  - Number Theory: Galois representations (Langlands conjectures)
  - Computer Science: expander graphs/higher-dimensional expanders
  - Differential Geometry: spectra of Laplacians on locally symmetric spaces
  - Combinatorics: identities for the partition function
  - Finite Groups: representation theory of large sporadic simple groups (moonshine)
  - Mathematical Physics: representations of infinite-dimensional Lie algebras, certain scattering amplitudes in string theory

Black-Box Defintion

Definition

Let G be a reductive group over a number field F. A discrete automorphic representation for G is an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$.

- Reductive group: algeberaic group with nice representation theory (root and weight theory works).
  - ex. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{U}_n, \mathrm{SO}_n, \mathrm{Sp}_n$.
  - Non ex. Upper triangular matrices.
- L^2: square-integrable functions as a unitary representation of $G(\mathbb{A}_F)$ under right-translation.
- $$\mathbb{A}_F = \prod_{\text{places } v}' F_v \qquad \left(\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_{\text{primes } p}' \mathbb{Q}_v\right)$$
- Intuition: $\mathbb{Z}$ is to $\mathbb{R}$ as F is to $\mathbb{A}_F$.
- subrepresentation: analysis issue—infinite-dimensional

Perspective on Automorphic Representations

- What does G do?
  - G_∞: determines symmetric space G_∞/K_∞
  - G^∞: determines possible lattices Γ: "Levels"
- Factor into local components:

$$\pi = {\bigotimes_v}' \pi_v, \qquad \pi_v \text{ rep. of } G(F_v)$$

  - π_∞: "qualitative type" of the representation: modular vs. Maass, holomorphic, algebraic, cohomological.
  - π^∞: information analogous to level and Hecke eigenvalues

Perspective cont.

Key Problem: Which combinations of π_v actually produce an automorphic representation?

- e.g. which combinations of Hecke eigenvalues do the modular forms of weight k and level N have?

Most Basic Version: counts/statistics w/ local restrictions

- e.g. what fraction modular forms of weight k have Hecke eigenvalue at p with norm bigger than something as level $N \to \infty$?

Complexity Ranking

Informal ranking of complexity based on qualitative type π_∞:

- Discrete-at-∞: π_∞ discrete inside $L^2(G(F_\infty))$.
- Cohomological: π_∞ regular, integral infinitesimal character
- Algebraic: π_∞ integral infinitesimal character
- General: all π_∞

Different application need different generality:

- Cohomology of locally symmetric spaces
- Galois Representations

Example: Modular Forms

Fix $G = \mathrm{GL}_2/\mathbb{Q}$

- Automorphic Representations on $G \approx$ classical modular and Maass forms
- Discrete-at-∞: modular forms of weight ≥ 2
- Cohomological: add in the trivial rep, (there is more to add on other groups)
- Algebraic: add in weight 1 modular and Maass forms
- General: add in other Maass forms

Answering Key Question

How far can we go? Basic Version: use Arthur's trace formula

- Discrete-at-∞: coarse info. [Art89], fine info. [Fer07].
  - Need: orbital integrals, endoscopic transfers
  - Exact counts: many, many results for low level on small rank
  - Statistics: most powerful/general [ST16] coarse, [Dal22] fine
- Cohomological: inductive arg. w/ endoscopic class. [Taï17]
  - Need: orbital integrals, endoscopic transfers, stable transfers
  - Exact counts: [Taï17] +Chenevier, Renard, Taïbi at level-1
  - Statistics: [MS19] + Marshall, Gerbelli-Gauthier upper bounds, this work many exact asymptotics and more upper bounds
- Beyond: very hard—asymptotic counts not known even for weight-1 modular forms :'(

Classical Version

Consider:

- Symmetric space $X = U(p,q)/(U(p) \times U(q))$
- A specific type of tower of arithmetic lattices $\cdots \subseteq \Gamma_2 \subseteq \Gamma_1$
- $h_n^i := H^i(\Gamma_n \backslash X, V_\lambda) = H^i(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma_n \backslash G(\mathbb{R})) \otimes V_\lambda)$ as reps of $U(p,q)$.

Problem: Given π_0 unirrep of $G(R)$, understand asymptotics of count of $\pi_0 \in h_n^i$ weighted by arbitrary moment of Satake parameters.

- Analogue: weight-2 modular forms in $H^1(\Gamma(N))$ weighted by power of Hecke eigenvalue
- Matsushima's formula: translate to counting $\pi \in \mathcal{AR}_{\mathrm{disc}}(G)$ with $\pi_\infty = \pi_0$.

Main Result

Theorem

Let E/F be an unram. CM-extension and G an unram. inner form of $U_{E/F}(N)$. Fix π_0 cohom. on G_∞. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ only divisible by primes split in E/F and f_S an unram. test function at some set of places S not dividing $\mathfrak{n}$. Then for good π_0

$$|\mathfrak{n}|^{-R(\pi_0)} L_{\pi_0}(\mathfrak{n})^{-1} \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim((\pi^\infty)^{K(\mathfrak{n})}) \mathrm{tr}_{\pi_S} f_S$$

$$= M(\pi_0)\mu_S^{\mathrm{pl}(\pi_0)}(f_S) + O(|\mathfrak{n}|^{-C} q_S^{A+B\kappa(f_S)}).$$

- There are some strong conditions: E/F, level, and π_0
- Good π_0: Explicit: combinatorial data classifying π_0.

Main Result Cont.

$$|\mathfrak{n}|^{-R(\pi_0)} L_{\pi_0}(\mathfrak{n})^{-1} \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim((\pi^\infty)^{K(\mathfrak{n})}) \operatorname{tr}_{\pi_S} f_S$$
$$= M(\pi_0)\mu_S^{\mathrm{pl}(\pi_0)}(f_S) + O(|\mathfrak{n}|^{-C} {q_S}^{A+B\kappa(f_S)}).$$

- Asymptotic in $\mathfrak{n}$, S, f_S
- $\mathfrak{n}$: Counting fixed vectors in aut. reps with component $\pi_\infty = \pi_0$ (i.e. aut. forms of level $\mathfrak{n}$)
- f_S: averaging a Satake parameter over these forms (e.g. moment of Hecke eigenvalue)
- Constants: combo. param. of π_0 , Plancherel equidistribution
- Constants: Inexplcit

Example: parallel $U(N-1,1)$

Assume $\deg F/\mathbb{Q} = d$, $G_\infty \cong U(N-1,1)^d$ (if possible) $\pi_0 \cong \pi^d$

- Cohomological Reps of $U(N-1,1)$ at inf. char of trivial:
  - ordered partitions $(a_1, \dots, a_k)$ of N
  - one marked index $1 \le m \le k$, $a_i = 1$ for $i \neq m$.
  - Discrete series: all $a_i = 1$.
- "good" class: a_m is odd
- If π_0 d.s. $R(\pi_0) = N^2$, $M(\pi_0) = 1$. Otherwise:

$$R(\pi_0) = \frac{1}{2}(N^2 + (N-a_m)^2 - a_m^2) + 1$$

$$M(\pi_0) = \begin{cases} N^{-d} \dim(\pi_{a_m \lambda_{m-1}}) \tau'(G) & d \text{ even or } m \text{ correct parity} \\ 0 & d \text{ odd and } m \text{ wrong parity} \end{cases}$$

($\pi_{a_m \lambda_{m-1}}$: f.d. rep. of GL_{N-a_m}, λ_i: ith fundamental weight)

- Vary m: different masses, growth rates

Main Result: other π_0

Remove conditions $\Longrightarrow$ upper bound instead of exact asymptotic:

Theorem

Recall the setup for the main result except E/F can be ramified. Let S_0 be a set of places containing all the ramified ones and disjoint from S and $\mathfrak{n}$. Let φ_{S_0} be a test function on G_{S_0}. Then for all π_0:

$$\sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim((\pi^\infty)^{K(\mathfrak{n}_i)}) \operatorname{tr}_{\pi_S} f_S \operatorname{tr}_{\pi_{S_0}} \varphi_{S_0} = O(|\mathfrak{n}_i|^{R(\pi_0)} q_{S_1}^{A+B\kappa(f_S)}).$$

Corollaries

This gives us many corollaries:

- Sato-Tate equidistribution in families
  - GL_2 version: Hecke eigenvalues over all primes over all of $S_k(N)$ follow semicircle rule
  - Prove: expectation from interpreting π with $\pi_\infty = \pi_0$ as *non-endoscopic* functorial transfers from smaller group depending on π_0
- Sarnak density
  - $R(\pi_0)$ achieves a certain bound depending on matrix coefficient decay of π_0, useful in analytic number theory applications
  - Prove: for all cohomological π_0 except a single rep. on $U(2, 2)$
- Growth rates of $H^{p,q}$ of towers of locally symmetric spaces
  - Exact asymptotics: e.g. every other degree for $U(N, 1)$ with certain towers of lattices

Overview

Goal: Parametrize discrete automorphic representations for G in terms of all automorphic representations on GL_n.

$\implies$ Known info on GL_n gives info on G

- Moeglin-Waldspurger classification in terms of cuspidals
- Local Langlands

Stated in terms of two key concepts:

- Parameters: ψ: reps on GL_n encoded in a way to emphasize known info
- Packets: $\psi \mapsto \Pi_\psi$: subsets of $\mathcal{AR}_{\mathrm{disc}}(G)$ with determined structure of local components

G can be: SO_n or Sp_{2n} (Arthur), q-split $U_{E/F}(N)$ (Mok), General unitary groups [KMSW14].

Parameters

Some details:

Definition

An elliptic A-parameter for $U_{E/F,+}(N)$ is a formal sum

$$\psi = \bigoplus_i \tau_i[d_i]$$

where each τ_i is a conjugate self-dual cuspidal automorphic representation of GL_{t_i}/E and $\sum_i t_i d_i = N$ and each τ_i has the appropriate parity.

- ψ determines *local paramters* ψ_v by LL + lots of work

$$\psi_v : L_{F_v} \times \mathrm{SL}_2 \to {}^L U_{E/F}(N) : \bigoplus_i LL(\tau_{i,v}) \boxtimes [d_i]$$

Packets

Some details:

Theorem (KMSW classification)

Let G be an extended pure inner form of $G^* = U_{E/F}(N)$. To each elliptic parameter ψ of $U_{E/F}(N)$, there is an associated packet $\Pi_\psi^G \subseteq \mathcal{AR}_{\mathrm{disc}}(G)$ such that for any test function f on $G(\mathbb{A})$:

$$\mathrm{tr}_{\mathcal{AR}_{\mathrm{disc}}(G)}(f) = \sum_{\psi \in \Psi_{\mathrm{ell}}(G^*)} I_\psi(f) := \sum_{\psi \in \Psi_{\mathrm{ell}}(G^*)} \sum_{\pi \in \Pi_\psi^G} \mathrm{tr}_\pi(f)$$

- Π_ψ is a subset of a restricted product of local packets Π_{ψ_v} determined by a multiplicity formula

Stable Multiplicity

I_ψ: summands of Arthur's $I_{\text{disc}} \to S_\psi$: summands of S_{disc}

- Stabilization: $I_\psi^G = \sum_{H,\psi^H} S_{\psi^H}^H$, H smaller endoscopic groups

Formula:

$$S_\psi^H(f) = \epsilon_\psi C_\psi \operatorname{tr}_\psi(f)$$

- very difficult sign attached to ψ
- easy constant attached to ψ
- Stable trace $\sum_{\pi \in \Pi_\psi} \pm \operatorname{tr}_\pi(f)$.
  - related to trace of a rep π_ψ on some twisted GL_n
  - π_ψ explicitly described as Langlands quotient of π_{τ_i} with very complicated twist

AJ-packets

We care about a special kind of packet at ∞:

- Parameters ψ_∞ at ∞ have associated infinitesimal characters
- If the infinitesimal character is regular integral, then Π_{π_∞} is an Adams-Johnson packet $\Longrightarrow$ explicit combinatorial description of elements
- Exactly that packets that contain cohomological representations
- Key property: for cohom. π_0, there exists pseudocoefficient φ such that among the π that share an A-packet with π_0:

$$\mathrm{tr}_\pi \varphi = \mathbf{1}_{\pi=\pi_0}$$

Shapes

The inductive analysis depends on a key definition:

Definition

The refined shape Δ of A-parameter

$$\psi = \bigoplus_i \tau_i[d_i]$$

is $\Delta = (T_i, d_i, \lambda_i, \eta_i)_i$ where

- T_i is the dimension of τ_i
- λ_i is the infinitesimal character of $\tau_{i,\infty}$.

Key Property: Δ determines ψ_∞ among AJ-params if λ_i regular integral

Step 1: Induction Setup

Let $\psi_{i,\infty}$ be list of AJ-parameters such that $\pi_0 \in \Pi_{\psi_{i,\infty}}$. Let $\Delta(\pi_0)$ be the set of Δ that determine ψ_∞ to be one of the $\psi_{i,\infty}$:

$$\sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \mathrm{tr}_{\pi^\infty}(f^\infty) = \sum_{\Delta \in \Delta(\pi_0)} I_\Delta(\varphi f^\infty)$$

where

$$I_\Delta(f) := \sum_{\psi \in \Delta} I_\psi(f) = \sum_{\psi \in \Delta} \sum_{\pi \in \Pi_\psi} \mathrm{tr}_\pi(f)$$

- Stabilization + hyperendoscopy: Can switch freely between $I_\Delta(\varphi f^\infty), S_\Delta(EP_\lambda f^\infty)$ by adding lower order terms in $\mathfrak{n}_i$
- Goal: Understand $S_\Delta(EP_\lambda f^\infty)$ for shapes Δ.

Induction: Base Case

What is the base case at the bottom?

- Arthur's simple trace formula: Euler-Poincaré function EP_λ

$$I^H(\mathrm{EP}_\lambda f^\infty) = \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(H) \\ \text{inf. char. } \pi_\infty = \lambda}} \mathcal{L}(\pi_\infty)\, \mathrm{tr}_{\pi^\infty}(f^\infty)$$

  (similar result holds for pseudocoefficient φ).

- Shin-Templier's analysis: geometric expression for $I^H(\mathrm{EP}_\lambda f^\infty)$ can be bounded very explicitly (error terms as in main theorem)
- $f^\infty = \mathbf{1}_{K(\mathfrak{n}_i)} f_{S_1} \implies \mathrm{tr}_{\pi^\infty}(f^\infty) = \dim((\pi^\infty)^{K(\mathfrak{n}_i)})\, \mathrm{tr}_{\pi_{S_1}} f_{S_1}$.
- Recall: we don't care S^H vs. I^H

The Induction: Heuristic Dream

Trivial Shape: $\Sigma_{\lambda,\eta} = (T, 1, \lambda, \eta)$, cuspidal parameters on GL_n:

$$S^H_{\Sigma_\lambda}(EP_\lambda f^\infty) = S^H(EP_\lambda f^\infty) - \sum_{\substack{\Delta \neq \Sigma_\lambda \\ \text{inf. char. } \Delta = \lambda}} S^H_\Delta(EP_\lambda f^\infty)$$

- "Just" need to reduce S^H_Δ to $S^{H_i}_\Sigma$ for smaller H_i.
- Step 1: "Stable transfer" $\epsilon \, \mathrm{tr}_{\bigoplus_i \tau_i[d_i]} f = \prod_i \mathrm{tr}_{\tau_i[d_i]} f_i$
- Step 2: "Speh transfer" $\mathrm{tr}_{\tau_i[d_i]} f_i = \mathrm{tr}_{\tau_i} f_i'$

Total:

$$S^H_{(T_i, d_i, \lambda_i)_i}(EP_\lambda f^\infty) = \prod_i S^{H_i}_{(T_i, 1, \lambda_i)}(EP_{\lambda_i}(f^\infty)_i')$$

The Induction: Reality

Stable transfer and Speh transfer are hard, open problems in general :(

- Main work in analysis: Find an easy special case where you can compute them!
- General idea: use relation to twisted representations on GL_n and Langlands quotients
- $\Delta^{\max}(\pi_0)$: shapes with dominant-in-$|\mathfrak{n}_i|$ contribution, need transfers computed exactly here
- The rest of $\Delta(\pi_0)$: error term, only need upper bounds here.
- Rest of talk: explaining which easy special case we use

The ϵ-sign: $\epsilon_\psi C_\psi \operatorname{tr}_\psi f$

For upper bounds:

- If ψ has one summand, then $\epsilon_\psi = 1$ and the signs in tr_ψ are all $+1$.
- $\implies$ if $\operatorname{tr}_{\pi^\infty}(f^\infty) \geq 0$ always, can take absolute value and get upper bound

$$\operatorname{tr}_{\bigoplus_i \tau_i[d_i]} f = \prod_i \operatorname{tr}_{\tau_i[d_i]} f_i \implies S^H_{\bigoplus_i \tau_i[d_i]}(f) \leq \prod_i S^{H_i}_{\tau_i[d_i]}(f_i)$$

For exact computation:

- If all the d_i are odd, then $\epsilon_\psi = 1$.
- Restriction : $\Delta^{\max}(\pi_0)$ can only have shapes with all d_i odd.

Unramified Places: $\epsilon_\psi C_\psi \mathrm{tr}_\psi f$

At places v where f_v unramfied:

- Π_{ψ_v} has at most one unramified member $\pi^{\mathrm{ur}}_{\psi_v}$. This always has coefficient $+1$ in tr_{ψ_v}.
- $\implies \mathrm{tr}_{\psi_v} f_v = \mathrm{tr}_{\pi^{\mathrm{ur}}_{\psi_v}} f_v$
- Its Satake parameters are determined explicitly by those of the unramified members in $\Pi_{\tau_{i,v}}$.

$\implies$ can compute stable and Speh transfers of f_v dual to transfer of Satake parameters through Satake isomorphism (analogy—full fundamental lemma).

Split Places: $\epsilon_\psi C_\psi \mathrm{tr}_\psi f$

At split v, $G_v \cong \mathrm{GL}_N(F)$

- Check: stable transfer = constant term (=end. trans.)
- Check: Π_{ψ_v} singleton: from π_{ψ_v} from before on $\mathrm{GL}_N(E)$.

Speh transfer upper bounds: If $tr_{\pi_v}(f_v) \geq 0$:

- Can bound trace aganst Langlands quotient $\mathrm{tr}_{\pi_{\tau[d]}} f_v$ by trace against parabolic induction
- $\implies$ constant term integral upper bounds

Speh transfer exact computation

- If $T_i = 1$, then $\pi_{\tau[d]}$ is a character $\implies$ Speh transfer is integration against G^{der}.
- Restriction : $\Delta^{\max}(\pi_0)$ can only have shapes where all summands have either $T_i = 1$ or $d_i = 1$.

Conclusion

These are the only cases we needed with our setup:

- f^∞ is only ramified at split places

The "good" class of π_0 becomes π_0 such that for $\Delta \in \Delta^{\max}(\pi_0)$

- All summands have d_i odd
- All summands have $T_i = 1$ or $d_i = 1$
- There is a relatively simple equivalent combinatorial condition

Last Technicality: Need slightly stronger upper bounds of Marshall-Shin for $d_i = 2, 3$ to get that those terms are truly errors

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