Statistics of automorphic representations through simplified trace formulas

Rahul Dalal

UC Berkeley

December 3, 2020

Note on technical details

This subject has a lot of details that are distractions at the level of a 45 minute talk. Therefore,

- Feel free to ignore anything in gray if you aren't familiar with the subject.
- Anything in orange will be explained only intuitively and imprecisely.

Outline

- Background: Automorphic Representations
- Background: Trace Formulas
- Background: Simple Trace Formula
- Results of Shin-Templier
- New Work

Unmotivated Defintion

Definition

Let G be a reductive group over a number field F. A discrete automorphic representation for G is an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$.

- Reductive group: algeberaic group with nice representation theory (root and weight theory works).
  - ex. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{U}_n, \mathrm{SO}_n, \mathrm{Sp}_n$.
  - Non ex. Upper triangular matrices.
- L^2: square-integrable functions as a unitary representation of $G(\mathbb{A}_F)$ under right-translation.
- subrepresentation: analysis issue—infinite-dimensional representations can be direct integrals instead of direct sums
- discrete: There is a definition for non-discrete

Motivation

Why do we care about such bizarre objects?

- They have a lot of handles to grab onto when studying
  - representation theory of reductive groups
  - Fourier analysis
- They mysteriously encode information about so much else:
  - Number Theory: Galois representations (Langlands conjectures)
  - Computer Science: expander graphs/higher-dimensional expanders
  - Differential Geometry: spectra of Laplacians on locally symmetric spaces
  - Combinatorics: identities for the partition function
  - Finite Groups: representation theory of large sporadic simple groups (moonshine)
  - Mathematical Physics: Scattering amplitudes in string theory, black hole partition functions

Example

If $G = \mathrm{GL}_2/\mathbb{Q}$

$$\{\text{aut. reps. for } G\} \approx \{\text{new, eigen modular/Maass forms}\}$$

- This is NOT obvious
- Key step: If K^∞ is a maximal compact subgroup at the finite places,

$$\mathrm{GL}_2(\mathbb{Q})\mathbb{R}^\times\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{SO}_2(\mathbb{R})K^\infty = \Gamma_{K^\infty}\backslash\mathcal{H}$$

where Γ_{K^∞} is some arithmetic subgroup of $\mathrm{SL}_2\mathbb{R}$ and $\mathcal{H}$ is upper-half plane

Flath Decomposition

Theorem

Let π be an automorphic representation for group G/F. Then π factors over places v of F:

$$\pi = \widehat{\bigotimes}' \pi_v$$

where each π_v is an admissible, unitary representation of $G(F_v)$.

For $G = \mathrm{GL}_2/\mathbb{Q}$:

- π_∞ is the qualitative "type" of π: modular vs. Maass, weight
- π_p relates to the p^nth Fourier coefficients of π.

Key Question: Which combinations of π_v actually appear in L^2?

Motivation

First trick to try for decomposing a representation: look at traces.

- Assume for a moment

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi) = \bigoplus_{\pi \text{ d.a.}} \pi$$

- Then if R is an operator on L^2

$$\operatorname{tr}_{L^2} R = \sum_{\pi \text{ d.a.}} \operatorname{tr}_\pi R$$

- Choose R cleverly $\implies$ information towards key question: distribution of fixed component π_v "in families"

Test Functions Example

Idea: $f = \prod_v f_v$ so $\mathrm{tr}_\pi(f) = \prod_v \mathrm{tr}_{\pi_v}(f_v)$.

- Choose one test place v. Everything else is condition places.
- For lots of reasonable conditions on π_w, can find

$$\mathrm{tr}_{\pi_w}(f_w) = \begin{cases} 1 & \text{condition at } w \text{ satisfied} \\ 0 & \text{else} \end{cases}$$

  (In general: more complicated weights $\mathrm{tr}_{\pi_w}(f_w) = a_w(\pi_w)$)
- Set family weight: $a_{\mathcal{F}}(\pi) = \prod_{w \neq v} a_w(\pi_w)$
- Choose probe function f_v
- $$\sum \mathrm{tr}_\pi(f) = \sum a_{\mathcal{F}}(\pi) \mathrm{tr}_{\pi_v}(f_v)$$

  average over harmonic family of local statistic

Fantasy

How do we compute these traces?

- Convolution operators: f compactly supported smooth on $G(\mathbb{A}_F)$:
$$R_f : v \mapsto \int_{G(\mathbb{A}_F)} f(g) g v \, dg$$
- If $G(F) \backslash G(\mathbb{A}_F)$ is compact,
$$\operatorname{tr}_{L^2} R_f = \sum_{[\gamma] \in [G(F)]} \mathrm{Vol}(G(F)_\gamma \backslash G(\mathbb{A}_F)_\gamma) \int_{G(\mathbb{A}_F)_\gamma \backslash G(\mathbb{A}_F)} f(g^{-1} \gamma g) \, dg$$
- conjugacy classes, volume term, orbital integral $O_\gamma(f)$

Reality

Without compactness, nothing converges. There are various truncations to use instead. Each complicates the spectral expansion and the three pieces of the geometric expansion.

coarse expansion	↑	explicit
		elementary
fine expansion $J_{\mathrm{geom}} = J_{\mathrm{spec}}$		bad abstract properties
invariant $I_{\mathrm{geom}} = I_{\mathrm{spec}}$		
		inexplicit
stable $S_{\mathrm{geom}} = S_{\mathrm{spec}}$		complicated
		good abstract properties
beyond endoscopic??	↓	simple in nice cases

Discrete Series

- Restrict attention to the nicest "qualitative type" of automorphic representations$\leftrightarrow$the nicest real representations
- Discrete series: appear discretely in $L^2(G(F_\infty))$.
- Classified into L-packets Π_λ
- $G = \mathrm{GL}_2/\mathbb{Q}$
  - L-packets singletons parameterized by $k \geq 2$.
  - $\pi_\infty \in \Pi_k$ means π a holomorphic modular form of weight k.
- The invariant trace formula dramatically simplifies when restricted to representations with discrete series at infinity.

"Simple" trace formula

Theorem ([Art89])

Let G/F be a cuspidal reductive group and let Π_λ be a regular discrete series L-packet. Let $\mathcal{A}_\lambda$ be the set of automorphic representations π of G with $\pi_\infty \in \Pi_\lambda$. Then for any compactly supported smooth test function f on $G(\mathbb{A}^\infty)$

$$\sum_{\pi \in \mathcal{A}_\lambda} \mathrm{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{ell}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- "Conjugacy classes" counted with principle of inclusion-exclusion
- "Volume term"
- "Orbital integral" factored into infinite and finite places

Some Ideas in Proof that May Come Up Later

- Discrete Series π come with pseudocoefficients φ_π. For ρ a basic representation, $\mathrm{tr}_\rho(\varphi_\pi) = \mathbf{1}_{\pi=\rho}$
- η_λ Euler-Poincaré function

$$\eta_\lambda = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|} \sum_{\pi \in \Pi_\lambda} \varphi_\pi$$

- When λ regular, for ρ any unitary representation: $\mathrm{tr}_\rho(\eta_\lambda) = |\Pi_{\mathrm{disc}}(\lambda)|^{-1} \mathbf{1}_{\pi \in \Pi_\lambda}$
- Use Euler-Poincaré's as infinite component of test function: $\eta_\lambda f^\infty$, the above computes spectral side
- Much more!

Prototypical Result

This allows us to get good enough error bounds on statistics over these families for applications. The shape of the result is:

Theorem (prototype)

Fix G. Let $\mathcal{F}_k$ be a sequence of increasing-size families of aut. reps. of G with regular discrete series at infinity. Then for any unramified test function φ_v at large enough place v:

$$\frac{1}{|\mathcal{F}_k|} \sum_{\pi \in \mathcal{F}_k} a_{\mathcal{F}_k} \operatorname{tr}_{\pi_v}(\varphi_v) = \mu^{\mathrm{pl}}(\varphi_v) + O(\|\varphi_v\|_\infty q_v^{A+B\kappa} |\mathcal{F}_k|^{-C}).$$

- Families: set of aut. reps weighted by $a_{\mathcal{F}}$, total weight is $|\mathcal{F}|$.
- μ^{pl}: average over space of representations
- κ: measure of size of support of unramified φ.

Which Families?

Shin-Templier '16:

- $\mathcal{F}_k$: Level condition on π^∞ and π_∞ in a fixed discrete series L-packet

$$a_{\mathcal{F}_k}(\pi) = \mathbf{1}_{\pi_\infty \in \Pi_{\mathrm{disc}}(\lambda)} \dim((\pi^{v,\infty})^{U^{v,\infty}})$$

- $k \to \infty$: level $\to \infty$ or if G has trivial center, weight of L-packet $\to \infty$
- Applications:
  - Automorphic Sato-Tate—equidstribution of unramified π_v over all v.
  - Distributions of low-lying zeros of L-functions in families
- Error bound essential for applications!
- Proof-of-concept that detailed information can be computed

Computing terms

Reasonably general reductive groups, but terms somewhat explicit!

$$\sum_{\pi \in \mathcal{A}_\lambda} \mathrm{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{\mathrm{ell}}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- Sums: reductive group theory, Steinberg-Hitchen base?
- Φ: Weyl character formula + more root combinatorics
- O_γ: Counting points moved some amount by automorphisms of Bruhat-Tits buildings
- f_M: The p-adic integrals are easier OR branching laws + Kato-Lusztig formula
- a_γ: L-functions of Gross motives for red. groups

Goal

We want to compute detailed information about families that distinguish representations with π_∞ in the same L-packet:

$$a_{\mathcal{F}_{\pi_0}}(\pi) = \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^{v,\infty})^{U^{v,\infty}})$$

[Dal19]: Develop the necessary techniques and generalize Shin-Templier's results as a proof-of-concept.

Why?

Different members of the L-packet are the same for Galois Representation applications so why do we care about distinguishing them?

- Ex. $G = \mathrm{SL}_2$: one member holomorphic, one antiholomorphic
- Ex. $G = \mathrm{Sp}_{2n}$: one member holomorphic
- Ex. G exceptional: one member quaternionic
- When λ non-regular, some members may be "entangled" with non-tempered reps at infinity when trying to pick them out with the trace formula

Spectral Side

Strategy: plug $\varphi_{\pi_0} f^\infty$ into the trace formula

Lemma

If π_0 is a regular discrete series representation of G_∞, then for all unitary representations of G_∞, $\mathrm{tr}_\rho \varphi_{\pi_0} = \mathbf{1}_{\rho=\pi_0}$.

Proof.

(Idea) Expand ρ as a sum of basic representations in the Grothendieck group. All of π_0's L-packet has the same sign. □

Corollary

$$I_{\mathrm{spec}}(\varphi_{\pi_0} f^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} m_{\mathrm{disc}}(\pi) a_{\mathcal{F}_{\pi_0}}(\pi) \, \mathrm{tr}_{\pi_S}(f_S)$$

Geometric Side: Endoscopy and Stabilization

Goal:

- Rational conjugacy is too complicated, work with stable conjugacy instead
- $\implies$ want a trace formula with stably-invariant terms: SO's

How?

- G has elliptic endoscopic groups $H \in \mathcal{E}_{\mathrm{ell}}(G)$ if G^{der} simply connected
  - (H, s, η): $\widehat{H} = Z_{\widehat{G}}(s)$, $\eta : {}^L H \hookrightarrow {}^L G$
- f on G has a transfer f^H on H
  - κ-orbital integral identity locally: $O^{\kappa_H}_{\gamma_G}(f) = SO_{\gamma_H}(f^H)$
- For $S_\star$ stably-invariant:

$$I^G_\star(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, H) S^H_\star(f^H)$$

How to compute I_{geom}?

- $I_{\mathrm{geom}}(f_\infty f^\infty)$ simplifies if f_∞ linear combination of η_λ's.
- Try: write $I_{\mathrm{geom}}(\varphi_{\pi_0} f^\infty)$ in terms of $I_{\mathrm{geom}}(\eta_\lambda f^\infty)$'s

Lemma

If $\pi_0 \in \Pi_{\mathrm{disc}}(\lambda)$, φ_{π_0} has the same stable orbital integrals as η_λ. Furthermore, all endoscopic transfers $(\varphi_{\pi_0})^H$'s can be taken to be linear combinations of η_λ's.

- Therefore stabilization will help!

Hyperendscopy Outline

Trick from [Fer07]:

- Rearrange the stabilization of the spectral side

$$S^{G^*}_{\mathrm{disc}}(f^{G^*}) = I^G_{\mathrm{disc}}(f) + \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} (-\iota(G,H)) S^H_{\mathrm{disc}}(f^H)$$

- Inductively continue expanding each of the $S^H_{\mathrm{disc}} \to$ a linear combination of I^H_{disc}'s that is stable
- Forward substitution terminates at tori: $I^T_\star = S^T_\star$
- Set this equal for f, f' with the same stable orbital integrals:

$$I^G_{\mathrm{disc}}(f) = I^G_{\mathrm{disc}}(f') + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{\mathcal{H}}_{\mathrm{disc}}((f'-f)^{\mathcal{H}})$$

Hyperendoscopy Application Outline

We use this with $f = \varphi_{\pi_0} f^\infty$ and $f' = \eta_\lambda f^\infty$:

$$I^G_{\mathrm{disc}}(f) = I^G_{\mathrm{disc}}(f') + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G, \mathcal{H}) I^{\mathcal{H}}_{\mathrm{disc}}((f' - f)^{\mathcal{H}})$$

- These are linear combinations of η_λ's
- I_{disc} should therefore be computable by just applying Shin-Templier

Issues:

- Sum: which terms appear depend on f^∞
- This: needs to be bounded
- $\mathcal{HE}_{\mathrm{ell}}$: Major technical issue coming from precise definition, extend Arthur/Shin-Templier to arbitrary center

Papers Mentioned

- James Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290. MR 1001841
- Rahul Dalal, *Sato-tate equidistribution for families of automorphic representations through the stable trace formula*, arXiv preprint arXiv:1910.10800 (2019).
- Axel Ferrari, *Théorème de l'indice et formule des traces*, Manuscripta Math. **124** (2007), no. 3, 363–390. MR 2350551
- Sug Woo Shin and Nicolas Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, Invent. Math. **203** (2016), no. 1, 1–177, Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok. MR 3437869

Contact info: dalal@berkeley.edu