Automorphic Representations and "Golden" Quantum Logic Gates

Rahul Dalal (Joint w/Shai Evra and Ori Parzanchevski)

University of Vienna

November 5, 2024

Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange we will only explain intuitively and imprecisely due to time constraints.

Outline

- Quantum Computing Motivation
- Results/Summary of Argument
- Argument step details

Draft available at: `https://www.mat.univie.ac.at/~rdalal/GoldenGatesDraft.pdf`

The Problem

Classical computers use classical circuits:

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

What about Quantum computers? Quantum Circuits

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

What about Quantum computers? Quantum Circuits

- Input: quantum superposition of all possible strings of n bits: unit-norm vector in $\mathbb{C}^{\{0,1\}^n} \cong \mathbb{C}^{2^n}$.

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

What about Quantum computers? Quantum Circuits

- Input: quantum superposition of all possible strings of n bits: unit-norm vector in $\mathbb{C}^{\{0,1\}^n} \cong \mathbb{C}^{2^n}$.
- Circuit: Projective Unitary map $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$ + measurements

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

What about Quantum computers? Quantum Circuits

- Input: quantum superposition of all possible strings of n bits: unit-norm vector in $\mathbb{C}^{\{0,1\}^n} \cong \mathbb{C}^{2^n}$.
- Circuit: Projective Unitary map $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$ + measurements

Problem: Find a finite set S of "universal gates" in $PU(2^n)$ that can be multiplied to realize any unitary matrix $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$.

The Problem

Classical computers use classical circuits:

- Input: String of n bits in $\{0,1\}^n$: 01100....
- Circuit: some function $\{0,1\}^n \to \{0,1\}^m$.
- Universal Gates: e.g. NAND and NOR can be used to build any such function—need a good set to build computers

What about Quantum computers? Quantum Circuits

- Input: quantum superposition of all possible strings of n bits: unit-norm vector in $\mathbb{C}^{\{0,1\}^n} \cong \mathbb{C}^{2^n}$.
- Circuit: Projective Unitary map $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$ + measurements

Problem: Find a finite set S of "universal gates" in $PU(2^n)$ that can be multiplied to ~~realize~~ approximate any unitary matrix $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$.

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.
- Def: invar. distance on $PU(2^n)$ e.g. $d(x, y) = 1 - \mathrm{tr}(x^* y)/n$.

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.
- Def: invar. distance on $PU(2^n)$ e.g. $d(x, y) = 1 - \text{tr}(x^* y)/n$.
- $B(x, \delta)$: ball of *volume* δ around x w/res to $d(\cdot, \cdot)$.
    - Normalization: $\text{Vol}\, PU(2^n) = 1$

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.
- Def: invar. distance on $PU(2^n)$ e.g. $d(x,y) = 1 - \mathrm{tr}(x^* y)/n$.
- $B(x,\delta)$: ball of *volume* δ around x w/res to $d(\cdot,\cdot)$.
    - Normalization: $\mathrm{Vol}\, PU(2^n) = 1$
- For each $\delta > 0$, there should be a "small" ℓ such that

$$PU(2^n) \subseteq \bigcup_{s \in S^{(\ell)}} B(s,\delta)$$

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.
- Def: invar. distance on $PU(2^n)$ e.g. $d(x,y) = 1 - \mathrm{tr}(x^* y)/n$.
- $B(x,\delta)$: ball of *volume* δ around x w/res to $d(\cdot,\cdot)$.
    - Normalization: $\mathrm{Vol}\, PU(2^n) = 1$
- For each $\delta > 0$, there should be a "small" ℓ such that

$$PU(2^n) \subseteq \bigcup_{s \in S^{(\ell)}} B(s,\delta)$$

- Absolute best possible:

$$|S^{(\ell)}| = 1/\delta, \qquad |S^{(\ell)}| = |S|^{\ell} \implies \ell \propto \log(1/\delta)$$

Mathematical Formulation

What does it mean for a universal gate set S to approximate well?

- $S^{(\ell)}$: set of words of minimum length exactly ℓ in S.
- Def: invar. distance on $PU(2^n)$ e.g. $d(x,y) = 1 - \mathrm{tr}(x^* y)/n$.
- $B(x,\delta)$: ball of *volume* δ around x w/res to $d(\cdot,\cdot)$.
  - Normalization: $\mathrm{Vol}\, PU(2^n) = 1$
- For each $\delta > 0$, there should be a "small" ℓ such that

$$PU(2^n) \subseteq \bigcup_{s \in S^{(\ell)}} B(s,\delta)$$

- Absolute best possible:

$$|S^{(\ell)}| = 1/\delta, \qquad |S^{(\ell)}| = |S|^{\ell} \implies \ell \propto \log(1/\delta)$$

- In addition: approximation should be efficiently computable.

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

1. Covering: There is $c > 1$ s.t.

$$\delta_\ell = \frac{(\log |S^{(\ell)}|)^c}{|S^{(\ell)}|} \implies \mathrm{Vol}\left(PU(2^n) - \bigcup_{s \in S^{(\ell)}} B(s, \delta) \right) \xrightarrow{\ell} 0$$

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

1. Covering: There is $c > 1$ s.t.

$$\delta_\ell = \frac{(\log|S^{(\ell)}|)^c}{|S^{(\ell)}|} \implies \mathrm{Vol}\left(PU(2^n) - \bigcup_{s \in S^{(\ell)}} B(s,\delta)\right) \xrightarrow{\ell} 0$$

2. Growth: $|S^{(\ell)}|$ grows exponentially in ℓ.

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

1. Covering: There is $c > 1$ s.t.

$$\delta_\ell = \frac{(\log|S^{(\ell)}|)^c}{|S^{(\ell)}|} \implies \mathrm{Vol}\left(PU(2^n) - \bigcup_{s \in S^{(\ell)}} B(s,\delta) \right) \xrightarrow{\ell} 0$$

2. Growth: $|S^{(\ell)}|$ grows exponentially in ℓ.
3. Navigation: given $s \in \langle S \rangle$, there is an efficient algorithm that writes it as a word in S of the shortest possible length .

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

1. Covering: There is $c > 1$ s.t.

$$\delta_\ell = \frac{(\log |S^{(\ell)}|)^c}{|S^{(\ell)}|} \implies \mathrm{Vol}\left(PU(2^n) - \bigcup_{s \in S^{(\ell)}} B(s, \delta) \right) \xrightarrow{\ell} 0$$

2. Growth: $|S^{(\ell)}|$ grows exponentially in ℓ.
3. Navigation: given $s \in \langle S \rangle$, there is an efficient algorithm that writes it as a word in S of the shortest possible length .
4. Approximation: There is constant N such that there is a (randomized, heuristic) efficient algorithm inputting ℓ, δ, x such that there is $s \in S^{(\ell)}$ with $x \in B(s, \delta)$ and outputting $s' \in |S^{(\ell N)}|$ with $x \in B(s', \delta)$.

Golden Gates

Definition

A finite subset $S \subseteq PU(2^n)$ is a set of golden gates if:

1. Covering: There is $c > 1$ s.t. (slightly weaker!)

$$\delta_\ell = \frac{(\log |S^{(\ell)}|)^c}{|S^{(\ell)}|} \implies \mathrm{Vol}\left(PU(2^n) - \bigcup_{s \in S^{(\ell)}} B(s,\delta)\right) \xrightarrow{\ell} 0$$

2. Growth: $|S^{(\ell)}|$ grows exponentially in ℓ.
3. Navigation: given $s \in \langle S \rangle$, there is an efficient algorithm that writes it as a word in S of the shortest possible length .
4. Approximation: There is constant N such that there is a (randomized, heuristic) efficient algorithm inputting ℓ, δ, x such that there is $s \in S^{(\ell)}$ with $x \in B(s,\delta)$ and outputting $s' \in |S^{(\ell N)}|$ with $x \in B(s',\delta)$.

Main Result

Theorem ([DEP24])

There are sets of golden gates on $PU(2^n)$ *for* $n = 2, 3$.

Main Result

Theorem ([DEP24])

There are sets of golden gates on $PU(2^n)$ *for* $n = 2, 3$.

- $U(2^n)$ can be written as a product of smaller unitary groups $\implies$ efficient gate sets for small n give less efficient gate sets for larger n

Main Result

Theorem ([DEP24])

There are sets of golden gates on $PU(2^n)$ for $n = 2, 3$.

- $U(2^n)$ can be written as a product of smaller unitary groups $\implies$ efficient gate sets for small n give less efficient gate sets for larger n
- Previous work: only $n = 1$

Main Result

Theorem ([DEP24])

There are sets of golden gates on $PU(2^n)$ for $n = 2, 3$.

- $U(2^n)$ can be written as a product of smaller unitary groups $\implies$ efficient gate sets for small n give less efficient gate sets for larger n
- Previous work: only $n = 1$
- $n = 2$: explicit matrices computed

Main Result

Theorem ([DEP24])

There are sets of golden gates on $PU(2^n)$ for $n = 2, 3$.

- $U(2^n)$ can be written as a product of smaller unitary groups $\implies$ efficient gate sets for small n give less efficient gate sets for larger n
- Previous work: only $n = 1$
- $n = 2$: explicit matrices computed
- $n = 3$: explicit matrices can be computed from [MSG12]

Summary of Construction

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.
- Step 2: golden $\implies$ $\langle S \rangle$ has a set of generators S such that the corresponding Cayley graph is the type-0 or hyperspecial vertices in a Bruhat-Tits building.
  - basic props. of Bruhat-Tits buildings $\implies$ growth, navigation

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.
- Step 2: golden $\implies$ $\langle S \rangle$ has a set of generators S such that the corresponding Cayley graph is the type-0 or hyperspecial vertices in a Bruhat-Tits building.
  - basic props. of Bruhat-Tits buildings $\implies$ growth, navigation
- Step 3: p-Arithmetic $\implies$ covering rewritten as Sarnak-Xue type bound on counts of automorphic representations

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.
- Step 2: golden $\implies$ $\langle S \rangle$ has a set of generators S such that the corresponding Cayley graph is the type-0 or hyperspecial vertices in a Bruhat-Tits building.
  - basic props. of Bruhat-Tits buildings $\implies$ growth, navigation
- Step 3: p-Arithmetic $\implies$ covering rewritten as Sarnak-Xue type bound on counts of automorphic representations
- Step 4: Prove bound w/ endoscopic classification [KMSW14]

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.
- Step 2: golden $\implies$ $\langle S \rangle$ has a set of generators S such that the corresponding Cayley graph is the type-0 or hyperspecial vertices in a Bruhat-Tits building.
  - basic props. of Bruhat-Tits buildings $\implies$ growth, navigation
- Step 3: p-Arithmetic $\implies$ covering rewritten as Sarnak-Xue type bound on counts of automorphic representations
- Step 4: Prove bound w/ endoscopic classification [KMSW14]
- Step 5: approximation from orthogonal CS result [RS15]

Summary of Construction

- Step 1: Pick $\langle S \rangle$: "golden" p-arithmetic lattice in $PU(2^n)$
  - These only exist when $n \leq 3$.
- Step 2: golden $\implies$ $\langle S \rangle$ has a set of generators S such that the corresponding Cayley graph is the type-0 or hyperspecial vertices in a Bruhat-Tits building.
  - basic props. of Bruhat-Tits buildings $\implies$ growth, navigation
- Step 3: p-Arithmetic $\implies$ covering rewritten as Sarnak-Xue type bound on counts of automorphic representations
- Step 4: Prove bound w/ endoscopic classification [KMSW14]
- Step 5: approximation from orthogonal CS result [RS15]

Rest of the talk: steps 1-4 in more detail

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

- e.g. polynomial condition invertible determinant $\to \mathrm{GL}_n/\mathbb{Z}$.

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

- e.g. polynomial condition invertible determinant $\to \mathrm{GL}_n/\mathbb{Z}$.
- Given an R-algbera S, $G(S)$ is the group of S-entried matrices satisfying the polynomial conditions.

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

- e.g. polynomial condition invertible determinant $\rightarrow \mathrm{GL}_n/\mathbb{Z}$.
- Given an R-algbera S, $G(S)$ is the group of S-entried matrices satisfying the polynomial conditions.

Important example: $U(N) = G(\mathbb{R})$ for G a matrix group over $\mathbb{R}$

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

- e.g. polynomial condition invertible determinant $\to \mathrm{GL}_n/\mathbb{Z}$.
- Given an R-algbera S, $G(S)$ is the group of S-entried matrices satisfying the polynomial conditions.

Important example: $U(N) = G(\mathbb{R})$ for G a matrix group over $\mathbb{R}$

- $2N \times 2N$-matrices: each 2×2-block is a complex number

Algebraic Matrix Groups

We need a more general perspective on matrix groups:

Definition

A matrix group G over R is a set of R-coefficient polynomial conditions on the entries of matrices that is closed under matrix multiplication

- e.g. polynomial condition invertible determinant $\to \mathrm{GL}_n/\mathbb{Z}$.
- Given an R-algbera S, $G(S)$ is the group of S-entried matrices satisfying the polynomial conditions.

Important example: $U(N) = G(\mathbb{R})$ for G a matrix group over $\mathbb{R}$

- $2N \times 2N$-matrices: each 2×2-block is a complex number
- preserving a positive-definite Hermitian form.

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.
- Choose: positive-definite $2^n \times 2^n$-Hermitian matrix H with entries in integers $\mathcal{O}_E$.
  - Same def. w/res to complex conjugation!
  - other possibility: involution of second kind on division algebra

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.
- Choose: positive-definite $2^n \times 2^n$-Hermitian matrix H with entries in integers $\mathcal{O}_E$. $2^{n+1} \times 2^{n+1}$ integer matrix
    - Same def. w/res to complex conjugation!
    - other possibility: involution of second kind on division algebra

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.
- Choose: positive-definite $2^n \times 2^n$-Hermitian matrix H with entries in integers $\mathcal{O}_E$. $2^{n+1} \times 2^{n+1}$ integer matrix
  - Same def. w/res to complex conjugation!
  - other possibility: involution of second kind on division algebra
- Define: $U^{E,H}$ as matrices g such that $gH\bar{g}^T = H$.
  - AG def: $U^{E,H}(R)$ is such matrices g with entries in $R \otimes_{\mathbb{Z}} \mathcal{O}_E$

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.
- Choose: positive-definite $2^n \times 2^n$-Hermitian matrix H with entries in integers $\mathcal{O}_E$. $2^{n+1} \times 2^{n+1}$ integer matrix
  - Same def. w/res to complex conjugation!
  - other possibility: involution of second kind on division algebra
- Define: $U^{E,H}$ as matrices g such that $gH\bar{g}^T = H$.
  - AG def: $U^{E,H}(R)$ is such matrices g with entries in $R \otimes_{\mathbb{Z}} \mathcal{O}_E$
- Depends on choice of H and d

p-Arithmetic Lattices

Key Motivating Example: $\mathrm{SL}_2(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_2\mathbb{R}$
Goal: Generalize to $U(2^n)$.

- Idea: find matrix group G over $\mathbb{Z}[1/p]$ s.t. $G(\mathbb{R}) = U(2^n)$.
- Choose: imaginary quadratic extension $E = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$.
- Choose: positive-definite $2^n \times 2^n$-Hermitian matrix H with entries in integers $\mathcal{O}_E$. $2^{n+1} \times 2^{n+1}$ integer matrix
  - Same def. w/res to complex conjugation!
  - other possibility: involution of second kind on division algebra
- Define: $U^{E,H}$ as matrices g such that $gH\bar{g}^T = H$.
  - AG def: $U^{E,H}(R)$ is such matrices g with entries in $R \otimes_{\mathbb{Z}} \mathcal{O}_E$
- Depends on choice of H and d

Upshot: $\mathrm{SL}_2(\mathbb{Z}[1/p])$ generalizes to matrices with entries in $\mathbb{Z}[\sqrt{-d}, 1/p]$ preserving H for p inert in $\mathcal{O}_E$.

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact
- Define: $U^{E,H}(\mathbb{A})$, $U^{E,H}(\mathbb{Q})$, $U^{E,H}(\mathbb{Q}_p)$, $U^{E,H}(\mathbb{Z}_p)$.

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact
- Define: $U^{E,H}(\mathbb{A})$, $U^{E,H}(\mathbb{Q})$, $U^{E,H}(\mathbb{Q}_p)$, $U^{E,H}(\mathbb{Z}_p)$.
- Fact: $U^{E,H}(\mathbb{Q}) \hookrightarrow U^{E,H}(\mathbb{A})$ discrete and cocompact

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact
- Define: $U^{E,H}(\mathbb{A})$, $U^{E,H}(\mathbb{Q})$, $U^{E,H}(\mathbb{Q}_p)$, $U^{E,H}(\mathbb{Z}_p)$.
- Fact: $U^{E,H}(\mathbb{Q}) \hookrightarrow U^{E,H}(\mathbb{A})$ discrete and cocompact

Arithmetic lattices $\leftrightarrow$ open compact subgroups $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact
- Define: $U^{E,H}(\mathbb{A})$, $U^{E,H}(\mathbb{Q})$, $U^{E,H}(\mathbb{Q}_p)$, $U^{E,H}(\mathbb{Z}_p)$.
- Fact: $U^{E,H}(\mathbb{Q}) \hookrightarrow U^{E,H}(\mathbb{A})$ discrete and cocompact

Arithmetic lattices $\leftrightarrow$ open compact subgroups $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$

- For each p: define p-arithmetic lattice $\Lambda_p := K^{\infty,p} \cap U^{E,H}(\mathbb{Q})$.

Adelic Perspective

Arithmetic lattices are simpler from an adelic perspective:

- Recall: $\mathbb{Q}$ has completions $\mathbb{R}$ and $\mathbb{Q}_p$. $\mathbb{A}$ is the restricted direct product, diagonal $\mathbb{Q} \hookrightarrow \mathbb{A}$ discrete and cocompact
- Define: $U^{E,H}(\mathbb{A})$, $U^{E,H}(\mathbb{Q})$, $U^{E,H}(\mathbb{Q}_p)$, $U^{E,H}(\mathbb{Z}_p)$.
- Fact: $U^{E,H}(\mathbb{Q}) \hookrightarrow U^{E,H}(\mathbb{A})$ discrete and cocompact

Arithmetic lattices $\leftrightarrow$ open compact subgroups $K^{\infty} \subseteq U^{E,H}(\mathbb{A}^{\infty})$

- For each p: define p-arithmetic lattice $\Lambda_p := K^{\infty,p} \cap U^{E,H}(\mathbb{Q})$.
- Example: If $K_p = U^{E,H}(\mathbb{Z}_p)$, then $\Lambda_p = U^{E,H}(\mathbb{Z}[1/p])$.

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty$$

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q}))$$

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \overset{2}{=} U(2^n)$$

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \overset{2}{=} U(2^n)$$

Variant: For all p: Λ_p acts on $U^{E,H}(\mathbb{Q}_p)/K_p$

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \overset{2}{=} U(2^n)$$

Variant: For all p: Λ_p acts on $U^{E,H}(\mathbb{Q}_p)/K_p$ simply

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \overset{2}{=} U(2^n)$$

Variant: For all p: Λ_p acts on $U^{E,H}(\mathbb{Q}_p)/K_p$ simply transitively

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \stackrel{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \stackrel{2}{=} U(2^n)$$

Variant: For all p: Λ_p acts on $U^{E,H}(\mathbb{Q}_p)/K_p$ simply transitively

Key Limitation:

Golden Arithmetic Subroups

Definition

A compact open $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden* if

1. $K^\infty U^{E,H}(\mathbb{Q}) = U^{E,H}(\mathbb{A}^\infty)$,
2. $K^\infty \cap U^{E,H}(\mathbb{Q}) = 1$.

Key Property: if K^∞ is golden, then;

$$U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})/K^\infty \overset{1}{=} U^{E,H}(\mathbb{R})/(K^\infty \cap U^{E,H}(\mathbb{Q})) \overset{2}{=} U(2^n)$$

Variant: For all p: Λ_p acts on $U^{E,H}(\mathbb{Q}_p)/K_p$ simply transitively

Key Limitation: 1 is rarely satisfied (finitely many examples with rank > 4, none with rank > 8)

Bruhat-Tits Building

Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

Bruhat-Tits Building

Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

- Ex: if $G = \mathrm{GL}_2/\mathbb{Q}_p$, then $\mathcal{B}$ is an infinite $(p+1)$-regular tree.

Bruhat-Tits Building

Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

- Ex: if $G = \mathrm{GL}_2/\mathbb{Q}_p$, then $\mathcal{B}$ is an infinite $(p+1)$-regular tree.
- Higher-dimensional generalization for higher-rank groups

Bruhat-Tits Building

Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

- Ex: if $G = \mathrm{GL}_2/\mathbb{Q}_p$, then $\mathcal{B}$ is an infinite $(p+1)$-regular tree.
- Higher-dimensional generalization for higher-rank groups

Properties:

Bruhat-Tits Building

Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

- Ex: if $G = \mathrm{GL}_2/\mathbb{Q}_p$, then $\mathcal{B}$ is an infinite $(p+1)$-regular tree.
- Higher-dimensional generalization for higher-rank groups

Properties:

- $\mathcal{B}$: union of equidimensional Euclidean subsets, apartments
  - Any two simplices share a common apartment
  - $\mathrm{GL}_2/\mathbb{Q}_p$: apartments are infinite two-sided paths, each $\cong \mathbb{R}$.

Bruhat-Tits Building

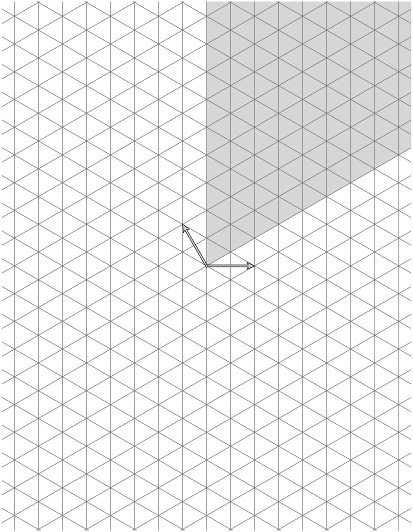
Given reductive matrix group $G/\mathbb{Q}_p$, there is an associated contractible simplicial complex $\mathcal{B}$ with a $G(\mathbb{Q}_p)$-action.

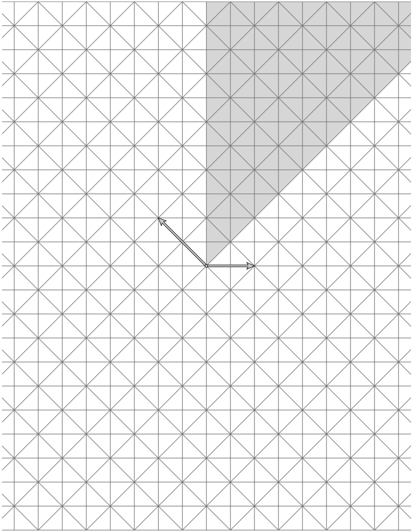
- Ex: if $G = \mathrm{GL}_2/\mathbb{Q}_p$, then $\mathcal{B}$ is an infinite $(p+1)$-regular tree.
- Higher-dimensional generalization for higher-rank groups

Properties:

- $\mathcal{B}$: union of equidimensional Euclidean subsets, apartments
  - Any two simplices share a common apartment
  - $\mathrm{GL}_2/\mathbb{Q}_p$: apartments are infinite two-sided paths, each $\cong \mathbb{R}$.
- If K is a maximal compact special subgroup, $G(\mathbb{Q}_p)/K$ embeds as a subset of the vertices of $\mathcal{B}$.
  - Consistent with $G(\mathbb{Q}_p)$-action
  - K is the stabilizer of fixed vertex v_0.
  - $\mathrm{GL}_2/\mathbb{Q}_p$: $G(\mathbb{Q}_p)/K$ is the vertices of the tree

Example Apartments





Golden Lattice to Gate Set

Definition

A golden subgroup $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden at* p if K_p is a hyperspecial maximal compact.

Golden Lattice to Gate Set

Definition

A golden subgroup $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden at p* if K_p is a hyperspecial maximal compact.

For Simplicity: Assume $U^{E,H}(\mathbb{Q}_p)/K_p$ is all vertices of $\mathcal{B}$. Then:

Golden Lattice to Gate Set

Definition

A golden subgroup $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden at p* if K_p is a hyperspecial maximal compact.

For Simplicity: Assume $U^{E,H}(\mathbb{Q}_p)/K_p$ is all vertices of $\mathcal{B}$. Then:

- Λ_p acts simply transitively on the vertices of $\mathcal{B}$.

Golden Lattice to Gate Set

Definition

A golden subgroup $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden at p* if K_p is a hyperspecial maximal compact.

For Simplicity: Assume $U^{E,H}(\mathbb{Q}_p)/K_p$ is all vertices of $\mathcal{B}$. Then:

- Λ_p acts simply transitively on the vertices of $\mathcal{B}$.
- The 1-skeleton of $\mathcal{B}$ is a Cayley graph for Λ_p w/res to generators taking v_0 to its neighbors.

Golden Lattice to Gate Set

Definition

A golden subgroup $K^\infty \subseteq U^{E,H}(\mathbb{A}^\infty)$ is *golden at p* if K_p is a hyperspecial maximal compact.

For Simplicity: Assume $U^{E,H}(\mathbb{Q}_p)/K_p$ is all vertices of $\mathcal{B}$. Then:

- Λ_p acts simply transitively on the vertices of $\mathcal{B}$.
- The 1-skeleton of $\mathcal{B}$ is a Cayley graph for Λ_p w/res to generators taking v_0 to its neighbors.
- These generators are our gate set $S_p^{K^\infty}$ corresp. to K^∞ and p
- simplifying condition holds when p is split
- when p is non-split, can use distance-2 vertices instead of neighbors—specific trick to unitary groups!

Growth and Navigation

Immediately get growth and navigation properties.

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.
- Standard properties of buildings $\implies$ exponential size in ℓ.

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.
- Standard properties of buildings $\implies$ exponential size in ℓ.

Navigation:

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.
- Standard properties of buildings $\implies$ exponential size in ℓ.

Navigation:

- Cartan Decomposition of γ: finds relative position of v_0 and γv_0 in shared apartment
  - Can be computed by integer normal form algorithm

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.
- Standard properties of buildings $\implies$ exponential size in ℓ.

Navigation:

- Cartan Decomposition of γ: finds relative position of v_0 and γv_0 in shared apartment
  - Can be computed by integer normal form algorithm
- find shortest path from γv_0 to v_0 in apartment

Growth and Navigation

Immediately get growth and navigation properties.

Growth:

- $S^{(\ell)}$ is all vertices at distance exactly ℓ in the 1-skeleton of $\mathcal{B}$.
- Standard properties of buildings $\implies$ exponential size in ℓ.

Navigation:

- Cartan Decomposition of γ: finds relative position of v_0 and γv_0 in shared apartment
  - Can be computed by integer normal form algorithm
- find shortest path from γv_0 to v_0 in apartment
- Cayley graph structure gives decomposition as a word in S
  - iterative process: try all generators at each step to find which follows path

Intuitive Idea

$\mathbf{1}_{B_\delta}$: indicator function of a ball of volume δ around $1 \subseteq U(2^n)$.

Intuitive Idea

$\mathbf{1}_{B_\delta}$: indicator function of a ball of volume δ around $1 \subseteq U(2^n)$.

- If $S^{(\ell)}$ covers $U(2^n)$ efficiently $\leftrightarrow$ it is evenly distributed

Intuitive Idea

$\mathbf{1}_{B_\delta}$: indicator function of a ball of volume δ around $1 \subseteq U(2^n)$.

- If $S^{(\ell)}$ covers $U(2^n)$ efficiently $\leftrightarrow$ it is evenly distributed
- $\implies$ this should be close to the identity function $\mathbf{1}$:

$$\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta} := |S^{(\ell)}|^{-1} \sum_{s \in S^{(\ell)}} \mathbf{1}_{B_\delta}(s^{-1}(*))$$

Intuitive Idea

$\mathbf{1}_{B_\delta}$: indicator function of a ball of volume δ around $1 \subseteq U(2^n)$.

- If $S^{(\ell)}$ covers $U(2^n)$ efficiently $\leftrightarrow$ it is evenly distributed
- $\implies$ this should be close to the identity function $\mathbf{1}$:

$$\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta} := |S^{(\ell)}|^{-1} \sum_{s \in S^{(\ell)}} \mathbf{1}_{B_\delta}(s^{-1}(*))$$

Quantitative Bound

$$\| \operatorname{Proj}_{\mathbf{1}^\perp}(\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta})\|_2^2 = \|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta} - \delta \mathbf{1}\|_2^2$$

$$\geq \delta^2 \operatorname{Vol}\left(U(2^n) - \bigcup_{s \in S^{(\ell)}} B(s, \delta) \right)$$

Intuitive Idea

$\mathbf{1}_{B_\delta}$: indicator function of a ball of volume δ around $1 \subseteq U(2^n)$.

- If $S^{(\ell)}$ covers $U(2^n)$ efficiently $\leftrightarrow$ it is evenly distributed
- $\implies$ this should be close to the identity function $\mathbf{1}$:

$$\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta} := |S^{(\ell)}|^{-1} \sum_{s \in S^{(\ell)}} \mathbf{1}_{B_\delta}(s^{-1}(*))$$

Quantitative Bound

$$\| \operatorname{Proj}_{\mathbf{1}^\perp}(\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta})\|_2^2 = \|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta} - \delta \mathbf{1}\|_2^2$$

$$\geq \delta^2 \operatorname{Vol}\left(U(2^n) - \bigcup_{s \in S^{(\ell)}} B(s, \delta) \right)$$

$\implies$ Goal: Upper bound $\|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta}\|_2^2 / \|\mathbf{1}_{B_\delta}\|_2^2$

Automorphic Interpretation

Definition

An *automorphic representation* on $U^{E,H}$ is an irreducible subrep. of $L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))$ under right translation by $U^{E,H}(\mathbb{A})$.

Automorphic Interpretation

Definition

An *automorphic representation* on $U^{E,H}$ is an irreducible subrep. of $L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))$ under right translation by $U^{E,H}(\mathbb{A})$.

- Fact: $G(\mathbb{R})$ compact $\implies$ L^2 decomposes as a $\oplus$ of irreps.

$$L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})) = \bigoplus_{\pi \in \mathcal{AR}(U^{E,H})} \pi$$

Automorphic Interpretation

Definition

An *automorphic representation* on $U^{E,H}$ is an irreducible subrep. of $L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))$ under right translation by $U^{E,H}(\mathbb{A})$.

- Fact: $G(\mathbb{R})$ compact $\implies$ L^2 decomposes as a $\oplus$ of irreps.

$$L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})) = \bigoplus_{\pi \in \mathcal{AR}(U^{E,H})} \pi$$

- Recall: if K^∞ is golden

$$L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))^{K^\infty} = L^2(U(2^n))$$

Automorphic Interpretation

Definition

An *automorphic representation* on $U^{E,H}$ is an irreducible subrep. of $L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))$ under right translation by $U^{E,H}(\mathbb{A})$.

- Fact: $G(\mathbb{R})$ compact $\implies$ L^2 decomposes as a $\bigoplus$ of irreps.

$$L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A})) = \bigoplus_{\pi \in \mathcal{AR}(U^{E,H})} \pi$$

- Recall: if K^∞ is golden

$$L^2(U^{E,H}(\mathbb{Q})\backslash U^{E,H}(\mathbb{A}))^{K^\infty} = L^2(U(2^n))$$

- Output: extra $U^{E,H}(\mathbb{A}^\infty)$ action on $L^2(U(2^n))$ understandable through information about the set $\mathcal{AR}(U^{E,H})$.

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S(\ell)}\star$ operator in this extra action

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S(\ell)}\star$ operator in this extra action

- nice function f on $U^{E,H}(\mathbb{Q}_p) \mapsto$ covolution operator on reps:

$$f : \pi_p \to \pi_p : f \star v = \int_{U^{E,H}(\mathbb{Q}_p)} f(g) g \cdot v \, dg$$

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S(\ell)}\star$ operator in this extra action

- nice function f on $U^{E,H}(\mathbb{Q}_p) \mapsto$ covolution operator on reps:

$$f : \pi_p \to \pi_p : f \star v = \int_{U^{E,H}(\mathbb{Q}_p)} f(g) g \cdot v \, dg$$

- K_p compact open $\implies \mathbf{1}_{K_p g K_p}$ acts on $\pi_p^{K_p}$.

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S(\ell)}\star$ operator in this extra action

- nice function f on $U^{E,H}(\mathbb{Q}_p) \mapsto$ covolution operator on reps:

$$f : \pi_p \to \pi_p : f \star v = \int_{U^{E,H}(\mathbb{Q}_p)} f(g) g \cdot v \, dg$$

- K_p compact open $\implies \mathbf{1}_{K_p g K_p}$ acts on $\pi_p^{K_p}$.
- K_p from golden $K^\infty \implies \mathbf{1}_{K_p S^{(\ell)} K_p}$ acts on $L^2(U(2^n))$.

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S^{(\ell)}}\star$ operator in this extra action

- nice function f on $U^{E,H}(\mathbb{Q}_p) \mapsto$ covolution operator on reps:

$$f : \pi_p \to \pi_p : f \star v = \int_{U^{E,H}(\mathbb{Q}_p)} f(g) g \cdot v \, dg$$

- K_p compact open $\implies \mathbf{1}_{K_p g K_p}$ acts on $\pi_p^{K_p}$.
- K_p from golden $K^\infty \implies \mathbf{1}_{K_p S^{(\ell)} K_p}$ acts on $L^2(U(2^n))$.
- Recall: $S^{(\ell)} K_p$ is all vertices at distance ℓ from v_0

Automorphic Interpretation: Hecke Operators

Goal: Realize $\mathbf{1}_{S^{(\ell)}}\star$ operator in this extra action

- nice function f on $U^{E,H}(\mathbb{Q}_p) \mapsto$ covolution operator on reps:

$$f : \pi_p \to \pi_p : f \star v = \int_{U^{E,H}(\mathbb{Q}_p)} f(g) g \cdot v \, dg$$

- K_p compact open $\implies \mathbf{1}_{K_p g K_p}$ acts on $\pi_p^{K_p}$.
- K_p from golden $K^\infty \implies \mathbf{1}_{K_p S^{(\ell)} K_p}$ acts on $L^2(U(2^n))$.
- Recall: $S^{(\ell)} K_p$ is all vertices at distance ℓ from v_0

$$\implies K_p S^{(\ell)} K_p = S^{(\ell)} K_p \implies \mathbf{1}_{K_p S^{(\ell)} K_p} \star f = \mathbf{1}_{S^{(\ell)}} \star f$$

p-Matrix Coefficient Decay

Recall: $\pi \in \mathcal{AR}(U^{E,H}) \implies$ irreps π_∞, π_p of $U^{E,H}(\mathbb{R})$, $U^{E,H}(\mathbb{Q}_p)$:

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p,$$

p-Matrix Coefficient Decay

Recall: $\pi \in \mathcal{AR}(U^{E,H}) \implies$ irreps π_∞, π_p of $U^{E,H}(\mathbb{R})$, $U^{E,H}(\mathbb{Q}_p)$:

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p,$$

Define: $\sigma(\pi, p) := \inf\{q > 2 : \pi_p \text{ has matrix coefficients in } L^q\}$

p-Matrix Coefficient Decay

Recall: $\pi \in \mathcal{AR}(U^{E,H}) \implies$ irreps π_∞, π_p of $U^{E,H}(\mathbb{R})$, $U^{E,H}(\mathbb{Q}_p)$:

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p,$$

Define: $\sigma(\pi, p) := \inf\{q > 2 : \pi_p$ has matrix coefficients in $L^q\}$

- The (false!) naïve Ramanujan Conjecture: $\sigma(\pi, p) = 2$ always.

p-Matrix Coefficient Decay

Recall: $\pi \in \mathcal{AR}(U^{E,H}) \implies$ irreps π_∞, π_p of $U^{E,H}(\mathbb{R})$, $U^{E,H}(\mathbb{Q}_p)$:

$$\pi = \pi_\infty \otimes {\bigotimes_p}' \pi_p,$$

Define: $\sigma(\pi, p) := \inf\{q > 2 : \pi_p$ has matrix coefficients in $L^q\}$

- The (false!) naïve Ramanujan Conjecture: $\sigma(\pi, p) = 2$ always.

Theorem ([Kam16])

Let finite $S \subseteq U^{E,H}(\mathbb{Q}_p)^{\mathrm{der}}$ and K_p Iwahori or maximal compact hyperspecial. Then for $\pi \in \mathcal{AR}(U^{E,H})$ and all $\epsilon > 0$

$$\|\mathbf{1}_{K_pSK_p}|_\pi\|_{op} \ll_\epsilon |K_pSK_p/K_p|^{(1+\epsilon)\left(1-\frac{1}{\sigma(\pi,p)}\right)}$$

A Sarnak-Xue-Type Bound

Final Goal: Control $\|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta}\|_2^2$ by bounding projections of $\mathbf{1}_{B_\delta}$ onto $\pi \in \mathcal{AR}(U^{E,H})$ with large $\sigma(\pi, p)$.

A Sarnak-Xue-Type Bound

Final Goal: Control $\|\mathbf{1}_{S^{(\ell)}} \star \mathbf{1}_{B_\delta}\|_2^2$ by bounding projections of $\mathbf{1}_{B_\delta}$ onto $\pi \in \mathcal{AR}(U^{E,H})$ with large $\sigma(\pi, p)$.

Theorem ([DEP24])

For $\pi \in \mathcal{AR}(U^{E,H})$, define

$$a(\delta, \pi) := \frac{\| \mathrm{Proj}_\pi \mathbf{1}_{\widetilde{B}_\delta} \|_2^2}{\|\mathbf{1}_{\widetilde{B}_\delta}\|_2^2}.$$

Then, for all $\epsilon > 0$,

$$\sum_{\pi:\sigma(\pi,p)\geq\sigma_0} a(\delta, \pi) \ll_\epsilon \delta^{(1-\epsilon)\left(1-\frac{2}{\sigma_0}\right)}.$$

A Sarnak-Xue-Type Bound

Final Goal: Control $\|\mathbf{1}_{S(\ell)} \star \mathbf{1}_{B_\delta}\|_2^2$ by bounding projections of $\mathbf{1}_{B_\delta}$ onto $\pi \in \mathcal{AR}(U^{E,H})$ with large $\sigma(\pi, p)$.

Theorem ([DEP24])

For $\pi \in \mathcal{AR}(U^{E,H})$, define

$$a(\delta, \pi) := \frac{\| \mathrm{Proj}_\pi \mathbf{1}_{\widetilde{B}_\delta} \|_2^2}{\|\mathbf{1}_{\widetilde{B}_\delta}\|_2^2}.$$

Then, for all $\epsilon > 0$,

$$\sum_{\pi : \sigma(\pi, p) \geq \sigma_0} a(\delta, \pi) \ll_\epsilon \delta^{(1-\epsilon)\left(1 - \frac{2}{\sigma_0}\right)}.$$

Interpretation: most of $\mathbf{1}_{\widetilde{B}_\delta}$ avoids violations of Ramanujan

Endoscopic Classification Input

How to prove bound? First,

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p
  - Intuitively: Ramanujan conjecture for $U^{E,H}$

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p
  - Intuitively: Ramanujan conjecture for $U^{E,H}$
  - Requires: +theory of Shimura Varieties and their integral models, Weil conjectures, etc.

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p
  - Intuitively: Ramanujan conjecture for $U^{E,H}$
  - Requires: +theory of Shimura Varieties and their integral models, Weil conjectures, etc.
- +[Mœg09]: Bound on $\sigma(\pi, p)$ in terms of Arthur-SL_2.

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p
  - Intuitively: Ramanujan conjecture for $U^{E,H}$
  - Requires: +theory of Shimura Varieties and their integral models, Weil conjectures, etc.
- +[Mœg09]: Bound on $\sigma(\pi, p)$ in terms of Arthur-SL_2.
  - Requires: classifications of p-adic reps of GL_n/classical groups.

Endoscopic Classification Input

How to prove bound? First, Deep input from Aut. Rep. theory

- [KMSW14]: $\pi \in \mathcal{AR}(U^{E,H}) \mapsto$ Arthur-SL_2, $\mathrm{SL}_2 \to \mathrm{GL}_{2^n}/\mathbb{C}$
  - Invariant determining much useful info about π
  - Requires: Arthur's trace formula, Ngo's proof the fundamental lemma, stabilization of the trace formula, Moeglin and Waldspurger's work on twisted versions, relations between Aubert involutions and intertwining operators, etc.
- +[Shi11, Clo13]: triv. Arthur-$\mathrm{SL}_2 \implies \sigma(\pi, p) = 2$ for all p
  - Intuitively: Ramanujan conjecture for $U^{E,H}$
  - Requires: +theory of Shimura Varieties and their integral models, Weil conjectures, etc.
- +[Mœg09]: Bound on $\sigma(\pi, p)$ in terms of Arthur-SL_2.
  - Requires: classifications of p-adic reps of GL_n/classical groups.

Upshot: rewrite bound in terms of Arthur-SL_2 instead of $\sigma(\pi, p)$.

Computing $a(\delta, \pi)$

Next, understand $a(\delta, \pi)$

Computing $a(\delta, \pi)$

Next, understand $a(\delta, \pi)$

- Compute:

$$a(\delta, \pi) = \| \mathrm{Proj}_{\pi_\infty} \mathbf{1}_{B_\delta} \|\|_2^2 \dim((\pi^\infty)^{K^\infty}) \\ = \mathrm{tr}_{\pi_\infty}(\mathbf{1}_{B_\delta} \star \mathbf{1}_{B_\delta}) \dim((\pi^\infty)^{K^\infty})$$

Computing $a(\delta, \pi)$

Next, understand $a(\delta, \pi)$

- Compute:

$$
\begin{aligned}
a(\delta,\pi) &= \| \mathrm{Proj}_{\pi_\infty} \mathbf{1}_{B_\delta} \|_2^2 \dim((\pi^\infty)^{K^\infty}) \\
&= \mathrm{tr}_{\pi_\infty}(\mathbf{1}_{B_\delta} \star \mathbf{1}_{B_\delta}) \dim((\pi^\infty)^{K^\infty})
\end{aligned}
$$

- $U(2^n)$ compact $\implies$ π_∞ is some finite dimensional π_{λ_∞} with highest weight λ_∞.

Computing $a(\delta, \pi)$

Next, understand $a(\delta, \pi)$

- Compute:

$$a(\delta,\pi) = \| \mathrm{Proj}_{\pi_\infty} \mathbf{1}_{B_\delta} \|_2^2 \dim((\pi^\infty)^{K^\infty}) = \mathrm{tr}_{\pi_\infty}(\mathbf{1}_{B_\delta} \star \mathbf{1}_{B_\delta}) \dim((\pi^\infty)^{K^\infty})$$

- $U(2^n)$ compact $\implies$ π_∞ is some finite dimensional π_{λ_∞} with highest weight λ_∞.
- Kirilov's orbit-method character formula explicitly computes

$$a(\lambda_\infty, \delta) := \mathrm{tr}_{\pi_{\lambda_\infty}}(\mathbf{1}_{\widetilde{B}_\delta} \star \mathbf{1}_{\widetilde{B}_\delta})$$

Computing $a(\delta, \pi)$

Next, understand $a(\delta, \pi)$

- Compute:

$$a(\delta,\pi) = \| \mathrm{Proj}_{\pi_\infty} \mathbf{1}_{B_\delta} \| \|_2^2 \dim((\pi^\infty)^{K^\infty}) \\ = \mathrm{tr}_{\pi_\infty}(\mathbf{1}_{B_\delta} \star \mathbf{1}_{B_\delta}) \dim((\pi^\infty)^{K^\infty})$$

- $U(2^n)$ compact $\implies$ π_∞ is some finite dimensional π_{λ_∞} with highest weight λ_∞.
- Kirilov's orbit-method character formula explicitly computes

$$a(\lambda_\infty, \delta) := \mathrm{tr}_{\pi_{\lambda_\infty}}(\mathbf{1}_{\widetilde{B}_\delta} \star \mathbf{1}_{\widetilde{B}_\delta})$$

- $\mathbf{1}_{\widetilde{B}_\delta}$: slight modification of $\mathbf{1}_{B_\delta}$ for computational simplicity

Putting it Together

Goal: $\square$: subset of $\mathcal{AR}(U^{E,H})$ w/ some fixed Arthur-SL_2. Bound:

$$\sum_{\pi \in \square} a(\pi, \delta) =$$

Putting it Together

Goal: $\square$: subset of $\mathcal{AR}(U^{E,H})$ w/ some fixed Arthur-SL_2. Bound:

$$\sum_{\pi\in\square} a(\pi,\delta) = \sum_{\lambda_\infty} a(\lambda_\infty,\delta) \sum_{\substack{\pi\in\square \\ \pi_\infty=\pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty})$$

Putting it Together

Goal: $\square$: subset of $\mathcal{AR}(U^{E,H})$ w/ some fixed Arthur-SL_2. Bound:

$$\sum_{\pi \in \square} a(\pi, \delta) = \sum_{\lambda_\infty} a(\lambda_\infty, \delta) \sum_{\substack{\pi \in \square \\ \pi_\infty = \pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty})$$

Key Input: [DGG23, DGG24] finds explicit function $d(\lambda_\infty)$ s.t.

$$\sum_{\substack{\pi \in \square \\ \pi_\infty = \pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty}) \ll d(\lambda_\infty)$$

Putting it Together

Goal: $\square$: subset of $\mathcal{AR}(U^{E,H})$ w/ some fixed Arthur-SL_2. Bound:

$$\sum_{\pi \in \square} a(\pi, \delta) = \sum_{\lambda_\infty} a(\lambda_\infty, \delta) \sum_{\substack{\pi \in \square \\ \pi_\infty = \pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty})$$

Key Input: [DGG23, DGG24] finds explicit function $d(\lambda_\infty)$ s.t.

$$\sum_{\substack{\pi \in \square \\ \pi_\infty = \pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty}) \ll d(\lambda_\infty)$$

- Req: End. class. [KMSW14] + inductive method of Taïbi.
- $d(\lambda_\infty)$ is conjecturally optimal

Putting it Together

Goal: $\square$: subset of $\mathcal{AR}(U^{E,H})$ w/ some fixed Arthur-SL_2. Bound:

$$\sum_{\pi\in\square} a(\pi,\delta) = \sum_{\lambda_\infty} a(\lambda_\infty,\delta) \sum_{\substack{\pi\in\square \\ \pi_\infty=\pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty})$$

Key Input: [DGG23, DGG24] finds explicit function $d(\lambda_\infty)$ s.t.

$$\sum_{\substack{\pi\in\square \\ \pi_\infty=\pi_{\lambda_\infty}}} \dim((\pi^\infty)^{K^\infty}) \ll d(\lambda_\infty)$$

- Req: End. class. [KMSW14] + inductive method of Taïbi.
- $d(\lambda_\infty)$ is conjecturally optimal

Final Step: plug in formulas for $d(\lambda_\infty), a(\lambda_\infty,\delta)$ and sum!

Papers Mentioned

L. Clozel, *Purity reigns supreme*, International Mathematics Research Notices **2013** (2013), no. 2, 328–346.

Rahul Dalal, Shai Evra, and Ori Parzanchevski, *Golden gates in PU(N)*, "https://www.mat.univie.ac.at/~rdalal/GoldenGatesDraft.pdf", 2024.

Rahul Dalal and Mathilde Gerbelli-Gauthier, *Statistics of cohomological automorphic representations on unitary groups via the endoscopic classification*, 2023.

———, *Root number equidistribution for self-dual automorphic representations on* gl_n, 2024.

Amitay Kamber, L_p*-expander complexes*, arXiv:1701.00154 (2016).

Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: inner forms of unitary groups*, arXiv preprint arXiv:1409.3731 (2014).

Colette Mœglin, *Comparaison des paramètres de Langlands et des exposants à l'intérieur d'un paquet d'Arthur*, J. Lie Theory **19** (2009), no. 4, 797–840.

A. Mohammadi and A. Salehi Golsefidy, *Discrete subgroups acting transitively on vertices of a Bruhat–Tits building*, Duke Mathematical Journal **161** (2012), no. 3, 483–544.

Neil J Ross and Peter Selinger, *Optimal ancilla-free Clifford+V approximation of z-rotations*, Quantum Information & Computation **15** (2015), no. 11-12, 932–950.

Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741. MR 2800722

Contact info: rahul.dalal@univie.ac.at