Root Number Equidistribution in the Weight Aspect for Cuspidal, Self-Dual, Automorphic Representations on GL_N

Rahul Dalal
(joint w/ Mathilde Gerbelli-Gauthier)

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange I will only explain intuitively and imprecisely due to time constraints.
- blue is miscellaneous highlighting

Outline

- The Question/Motivation
- Prototype Case: GL_2
- General Case Key Points

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- Powerful tool for studying—trace formula

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- arithmetic geometry interpretation confusing??

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- regular integral: definition misleading: half-integral or integral depending on orth. vs. symp.
- weight k to ∞ $\leftrightarrow$ min. difference b/w Hodge weights to ∞.
- Langlands correspondence conjectural!

Final Question

Fix N, regular integral infinitesimal character λ at ∞ and conductor $\mathfrak{n}$. Consider the set $S_{2N}(\lambda, \mathfrak{n})$ of automorphic representations on $\mathrm{GL}_{2N}\mathbb{Q}$ that:

- have inf. char. λ at ∞,
- have conductor $\mathfrak{n}$,
- are symplectic self-dual.

What fraction of $\pi \in S_{2N}(\lambda, \mathfrak{n})$ have root number $\epsilon(1/2, \pi) = \pm 1$ as $\lambda \to \infty$?

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- $N = 1$: original modular form question Q3

Final Question: SO_{2N+1}-version

Fix N, regular integral infintesimal character at ∞, and conductor $\mathfrak{n}$. Consider the set $S'_{2N}(\lambda, \mathfrak{n})$ of "newforms" (simple generic A-packets) on SO_{2N+1} that

- have inf. char. λ at ∞,
- have standard L-function with conductor $\mathfrak{n}$.

What fraction of $\pi \in S'_{2N}(\lambda, \mathfrak{n})$ have root number

$$\epsilon(1/2, \pi, \mathrm{std}) = \pm 1$$

as $\lambda \to \infty$?

Result

Theorem

Exactly $1/2$ *of* $\pi \in S_{2N}(\lambda, \mathfrak{n})$ *have* $\epsilon(1/2, \pi) = +1$ *as* $\lambda \to \infty$ *if and only if there is* p *such that* $v_p(\mathfrak{n})$ *is either odd or* $> 2N$*,*

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- Works over arbitrary totally-real number field F
- independent of weighting by Satake parameters for $p \nmid \mathfrak{n}$.
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Work-in-progress:

- Conjugate self-dual case for E/F CM quadratic (confusing: appears to fail even for large $\mathfrak{n}$ when $E \cong F[i]$??)
- Speculative: level-aspect $\mathfrak{n} \to \infty$

Method of Proof

Step 1: Understand the classical GL_2-proof in great detail (Yamauchi, Skoruppa-Zagier, Martin)

Step 2: See if modern trace formula technology can generalize the individual steps

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- Factors as product $\mathrm{GL}_2\mathbb{R} \times \prod'_v \mathrm{GL}_2(\mathbb{Q}_v)$
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Intuition: Work with all levels at once, things become cleaner

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- vectors $x \in \pi$ $\leftrightarrow$ modular forms on upper-half plane
- The newform: $x = x_\infty \otimes \bigotimes_v x_v \in \pi$: unique subspace s.t.
    - weight-k: π_∞: $\mathrm{SO}_2\mathbb{R}$ acts by $\theta \cdot x_\infty = e^{2\pi i k\theta} x_\infty$
    - level-N: x^∞ fixed by $K_1(N)$.

Trace Formula

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$$\approx \sum_{[\gamma] \in [\mathrm{GL}_2(\mathbb{Q})]} \mathrm{Vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx$$

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Usage: Pick clever test functions to make left be the statistic you want and compute right

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Key Idea: As $k \to \infty$, term for $\gamma = 1$ dominates (When $Z_G = 1$):

$$\tau(\mathrm{GL}_2) \dim(k) f^\infty(1)$$

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Equidistribution: $E_N^{\infty}(1) = 0$ and $C_N^{\infty}(1) \neq 0$.

The function C_N^∞

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- $C_{\mathfrak{n}}^\infty(1) \neq 0$

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- $\mathrm{GL}_2 \times \mathrm{GL}_1$ Rankin-Selberg: $\iota_{\mathfrak{n}} := \begin{pmatrix} 0 & 1 \\ -\mathfrak{n} & 0 \end{pmatrix}$ acts on newvector with eigenvalue $\approx \epsilon(1/2, \pi)$.

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- $E_\mathfrak{n}^\infty(1) = 0$ whenever $\mathbf{1}_{\iota_1 K_1(1)}$ doesn't appear in combination $\iff$ $\mathfrak{n}$ not a perfect square

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- Issue: $\mathrm{tr}_{\pi}(\widetilde{E}_{\mathfrak{n}}^{\infty}) = \epsilon(1/2, \pi)$ only works for generic π

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Conclusion:

$$\sum_{\pi \in S_{2N}(\lambda, \mathfrak{n})} \epsilon(1/2, \pi) \asymp (\dim_{\mathrm{SO}_{2N+1}} \lambda)(\widetilde{E}_{\mathfrak{n}}^{\infty})^{\mathrm{SO}_{2N+1}}(1)$$

in terms of Endoscopic transfer

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- ....but this also changes the conductor correspondingly

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- need to develop stable Fourier inversion formula
- Technical issue!