Counting Level-1, Quaternionic Automorphic Representations on G_2

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange is background material I will only explain intuitively and imprecisely due to time constraints

Outline

- Background: Quaternionic Representations on G_2
- Background: Trace Formulas
- Background: Simple Trace Formula
- Selected Technical Difficulties

Details in [Dal21], *Counting Discrete, Level*-1, *Quaternionic Automorphic Representations on* G_2, ArXiv preprint

Relevant Perspective

Definition

Let G be a reductive group over a number field F. A discrete automorphic representation π for G is an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$.

- π_S: local component of π at some finite set of places S.
- π_∞: qualitative type of representation (modular vs. Maass, cohomological/algebraic, etc.),
- π_v's: specific representation of that type

Quaternionic G_2 reps

Question: Can we find nice examples of automorphic representations that don't correspond to forms which were discovered classically?

- Exceptional groups are good place to look
- Want to find nice class of π_∞—analogues to modular forms, not Maass forms

Simplest new example: $G = G_2$, π a quaternionic discrete series

- Quaternionic: puts a nice differential equation condition on functions, second-best to holomorphic
- Discrete series: Relevance here: studyable with trace formula
- Technicality: minimal K-type not a character $\implies$ automorphic forms are vector-valued functions
- One quaternionic discrete series π_k for each weight $k \geq 2$.

Applications

Where do these come up?

- Fourier coefficients encode information about cubic rings [GGS02]
- Partition functions in certain quantum models of black holes [FGKP18, Chap. 15]
- More in the future?

Main Question

Question: How do we describe the quaternionic-G_2 automorphic representations?
Example: Can we count them with some local conditions?

Answers

We can do both without too much trouble at level-1...

- level-1: π^∞ has a (necessarily 1d) subspace fixed by hyperspecial K^∞.

...in terms of compact form G_2^c

- $\cong G_2$ over all finite places, compact over ∞. In particular, $G_2^c(\mathbb{Z})$ defined.
- V_λ: finite-dimensional rep of $G_2^c(\mathbb{R})$ with highest weight λ, matrix coefficients in $L^2(G_2^c(\mathbb{R}))$.

Notation: β is the highest root of G_2

Formula

Theorem

Let $k > 2$. The number of discrete (equiv. cuspidal) level-1, quaternionic automorphic representations on G_2 of weight k is

$$|\mathcal{Q}_{n+2}(1)| =$$

$$\frac{1}{12096}\frac{1}{120}(n+1)(3n+4)(n+2)(3n+5)(2n+3) + \frac{1}{216}\frac{1}{6}(n+1)(n+2)(2n+3) + \frac{5}{192}\frac{1}{8}\begin{cases}(n+2)(3n+4) & n=0 \pmod 2\\ -(n+1)(3n+5) & n=1 \pmod 2\end{cases}$$

$$+\frac{1}{18}\begin{cases}\frac{2n}{3}+1 & n=0 \pmod 3\\ -\lfloor\frac{n}{3}\rfloor-1 & n=1,2 \pmod 3\end{cases} + \frac{1}{32}\begin{cases}\frac{3n}{2}+10 & n=0 \pmod 4\\ 6\lfloor\frac{n}{4}\rfloor-4 & n=1 \pmod 4\\ -2\lfloor\frac{n}{4}\rfloor-2 & n=2,3 \pmod 4\end{cases} + \frac{1}{24}\begin{cases}3\lfloor\frac{n}{6}\rfloor+5 & n=0,1 \pmod 6\\ 3\lfloor\frac{n}{6}\rfloor-2 & n=2,3 \pmod 6\\ 3\lfloor\frac{n}{6}\rfloor+3 & n=4,5 \pmod 6\end{cases}$$

$$+\frac{1}{7}\begin{cases}1 & n=0 \pmod 7\\ -1 & n=4 \pmod 7\\ 0 & n=1,2,3,5,6 \pmod 7\end{cases} + \frac{1}{4}\begin{cases}1 & n=0 \pmod 8\\ -1 & n=5 \pmod 8\\ 0 & n=1,2,3,4,6,7 \pmod 8\end{cases} + \begin{cases}\lfloor\frac{n+2}{4}\rfloor\left(\lfloor\frac{n+2}{12}\rfloor-1\right) & n=0 \pmod{12}\\ \lfloor\frac{n+2}{4}\rfloor\lfloor\frac{n+2}{12}\rfloor & n=2,4,6,8,10 \pmod{12}\\ -\left(\lfloor\frac{3n+5}{12}\rfloor-1\right)\left(\lfloor\frac{n+3}{12}\rfloor-1\right) & n=11 \pmod{12}\\ -\left(\lfloor\frac{3n+5}{12}\rfloor-1\right)\lfloor\frac{n+3}{12}\rfloor & n=3,7 \pmod{12}\\ -\lfloor\frac{3n+5}{12}\rfloor\lfloor\frac{n+3}{12}\rfloor & n=1,5,9 \pmod{12}\end{cases}.$$

- $G_c^2(\mathbb{Z})$-fixed space in V_λ—Weyl character formula
- Endoscopic correction: counts of classical modular forms

A Jacquet-Langlands-style result

Theorem

Let $k > 2$. If k is even:

- *the discrete (equiv. cuspidal) level-1, weight k quaternionic representations of G_2 are the exactly the unramified representations of $G_2(\mathbb{A})$ with infinite component π_k and Satake parameters coming from weight $(k-2)\beta$ algebraic modular forms on G_2^c* **in addition to** *those coming from pairs of classical cupsidal newforms in $\mathcal{S}_{3k-2}(1) \times \mathcal{S}_{k-2}(1)$.*

If k is odd:

- *such representations of G_2 are the exactly those coming from weight $(k-2)\beta$ algebraic modular forms on G_2^c* **except for** *those also coming from pairs of classical cupsidal newforms in $\mathcal{S}_{3k-3}(1) \times \mathcal{S}_{k-1}(1)$.*

Table

Table: Counts of discrete, quaternionic automorphic representations of level 1 on G_2.

k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$
3	0	13	5	23	76	33	478	43	1792
4	0	14	13	24	126	34	610	44	2112
5	0	15	8	25	121	35	637	45	2250
6	**1**	16	23	26	175	36	807	46	2619
7	0	17	17	27	173	37	849	47	2790
8	2	18	37	28	248	38	1037	48	3233
9	1	19	30	29	250	39	1097	49	3447
10	4	20	56	30	341	40	1332	50	3938
11	1	21	50	31	349	41	1412	51	4201
12	9	22	83	32	460	42	1686	52	4780

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Method

First trick to try for studying subreps: look at traces

- Assume for a moment

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi) = \bigoplus_{\pi \text{ d.a.}} \pi$$

- Then if R is an operator on L^2

$$\operatorname{tr}_{L^2} R = \sum_{\pi \text{ d.a.}} \operatorname{tr}_\pi R$$

- Source of R? Convolution: f cmpct. support smooth/$G(\mathbb{A})$:

$$f(v) := R_f(v) = \int_{G(\mathbb{A})} f(g) g \cdot v \, dg$$

Test Functions Example

Want: f such that

$$\mathrm{tr}_{L^2}(f) = \#\{G_2\text{-quat, lv. 1, wt. } k\}$$

Idea: $f = \prod_v f_v$ so $\mathrm{tr}_\pi(f) = \prod_v \mathrm{tr}_{\pi_v}(f_v)$

- Find f_∞ so that

$$\mathrm{tr}_{\pi_\infty}(f_\infty) = \mathbf{1}_{\pi_\infty \text{ is the weight-}k\text{, quaternionic discrete series}}$$

- If K^∞ is a maximal compact in $G_2(\mathbb{A}^\infty)$ note that

$$\mathrm{tr}_{\pi^\infty}(\mathbf{1}_{K^\infty}) = \mathrm{vol}(K^\infty)\mathbf{1}_{\pi^\infty \text{ is unramified}}$$

Therefore, plug in $f = f_\infty \mathbf{1}_{K^\infty}$

Trace Formula

How do we compute $\mathrm{tr}_{L^2}(f)$?

- Tool: Arthur-Selberg trace formula

$$\sum_{\pi \in \mathcal{AR}(G)} m_\pi \,\mathrm{tr}_\pi(f) \approx$$

$$\sum_{\gamma \in [G(F)]} \mathrm{Vol}(G_\gamma(F) \backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, dg$$

- Interested in spectral side I_{spec}: averages over aut. reps.
- Try to compute geometric side I_{geom}
  - rational conjugacy classes, volumes of adelic quotients, orbital integrals

Discrete Series

Infinite component discrete series $\Longrightarrow$ make the $\approx$ explicit:

- Discrete series: appear discretely in $L^2(G(F_\infty))$.
- Classified into L-packets Π_λ
- $G = \mathrm{GL}_2/\mathbb{Q}$
  - L-packets singletons parameterized by $k \geq 2$.
  - Regular when $k > 2$.
  - $\pi_\infty \in \Pi_k$ means π a holomorphic modular form of weight k.
- $G = G_2$
  - L-packets are *triples* parameterized by dominant weights λ of G_2
  - Regular if λ is
  - $\Pi_{(k-2)\beta}$ for β the highest root contains the (sole) quaternionic discrete series π_k of weight k (the one with minimal K-type trivial on one SU_2-component)

"Simple" trace formula

Theorem ([Art89])

Let G/F be a cuspidal reductive group and let Π_λ be a regular discrete series L-packet. Let $\mathcal{A}_\lambda$ be the set of automorphic representations π of G with $\pi_\infty \in \Pi_\lambda$. Then for any compactly supported smooth test function f on $G(\mathbb{A}^\infty)$

$$\sum_{\pi \in \mathcal{A}_\lambda} \mathrm{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{ell}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- "Conjugacy classes" counted with principle of inclusion-exclusion
- "Volume term"
- "Orbital integral" factored into infinite and finite places

Test Function At Infinity

- Discrete Series π come with pseudocoefficients φ_π. For ρ a basic representation, $\mathrm{tr}_\rho(\varphi_\pi) = \mathbf{1}_{\pi=\rho}$
- η_λ Euler-Poincaré function

$$\eta_\lambda = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|} \sum_{\pi \in \Pi_\lambda} \varphi_\pi$$

- When λ regular, for ρ any unitary representation: $\mathrm{tr}_\rho(\eta_\lambda) = |\Pi_{\mathrm{disc}}(\lambda)|^{-1} \mathbf{1}_{\pi \in \Pi_\lambda}$
- Simple trace formula: use Euler-Poincaré's as infinite component of test function: $\eta_\lambda f^\infty$, the above computes spectral side, geometric side harder

This doesn't quite work for us

Problem 1: counts all reps with $\pi_\infty \in \Pi_{(k-2)\beta}$ instead of all with $\pi_\infty = \pi_k$

- Solution Idea: Use pseudocoefficient at ∞ instead of EP-function.
- Geometric side doesn't simplify nicely then!
- Stabilization resolves this

Problem 2: $(k-2)\beta$ not regular!

- Spectral side may not simplify nicely w/ $f_\infty = \eta_{(k-2)\beta}$ or φ_{π_k}.
- Solution: Facts from real representation theory $\implies$ not an issue for specifically quaternionic ds

Problem 3: Terms on geometric side explicit but very hard

- Solution: Chenevier/Taïbi have tricks to simplify—only level 1

Summary of full solution

Let H be the endoscopic group $\mathrm{SO}_4 \cong \mathrm{SL}_2 \times \mathrm{SL}_2 / \pm 1$:

$$I^{G_2}(\varphi_{\pi_k} \mathbf{1}_{K^\infty}) = I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta} \mathbf{1}_{K^\infty_{G_2^c}}) - \frac{1}{2} I^H((\eta^{G_2^c}_{(k-2)\beta})^H \mathbf{1}_{K^\infty_H}) - \frac{1}{2} I^H((\varphi_{\pi_k})^H \mathbf{1}_{K^\infty_H})$$

- I^{G_2} term : The count we want by problem 2 solution
- $I^{G_2^c}$ term: G_2^c compact $\implies \# G_2^c(\mathbb{Z})$-fixed vectors in V_λ.
- $(\eta_\lambda^{G_2^c})^H$ terms: endoscopic transfers, explicit linear combinations of EP-functions
- $I^H(\eta_\lambda)$ terms: H isogenous to $\mathrm{SL}_2 \times \mathrm{SL}_2$ so products of counts of modular forms at level 1, weights from above

Papers Mentioned

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