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# OPTIMALITY CONDITIONS FOR PORTFOLIO OPTIMIZATION PROBLEMS WITH CONVEX DEVIATION MEASURES AS OBJECTIVE FUNCTIONS

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**Abstract.** In this paper we derive by means of the duality theory necessary and sufficient optimality conditions for convex optimization problems having as objective function the composition of a convex function with a linear mapping defined on a finite-dimensional space with values in a Hausdorff locally convex space. We use the general results for deriving optimality conditions for two portfolio optimization problems having as objective functions different convex deviation measures.

## 1. Introduction

The portfolio optimization theory is an important field of the optimization theory having its roots in the work [5] of Markowitz published in 1952. The problem considered there is to find an optimal portfolio in the sense of maximizing the expected profit of the portfolio while minimizing its risk, which leads actually to a multiobjective optimization problem. In this classical framework one can meet some quiet natural requirements: risk is measured by the classical variance or standard deviation, short sales are excluded and the sum of the portfolio fractions is equal 1.

Since this paper has been published by Markowitz many authors extended or changed the form of the feasible set in the classical case. Another direction of research consisted in using different objective functions, besides the variance and standard deviation, for measuring the risk of a portfolio. Some important functions used in the literature for this purpose are the so-called *risk* and *deviation measures*. The class of coherent risk measures was introduced in an axiomatic way in 1998 in [1]. Other papers written on this topic in the last time are due to Rockafellar,

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Uryasev and Zabarankin (cf. [9, 11]), Ruszczynski and Shapiro (cf. [12]), Pflug (cf. [6]) and Föllmer and Schied (cf. [4]).

The classical portfolio optimization problem is a vector optimization problem, one way to treat it being the use of different scalarization techniques. One can also avoid dealing with the expected profit of the portfolio as a component of the objective function, by including it as a constraint in the feasible set of the optimization problem (cf. [8], [10]). Thus besides the classical constraints for the portfolio optimization problem one can integrate different assumptions for feasible portfolios.

In this paper we consider the following optimization problem

$$(P) \qquad \inf_{\substack{g(x) \leqq 0, \\ x \in S}} f(Ax),$$

where $\mathcal{Z}$ is a Hausdorff locally convex space, $S \subseteq \mathbb{R}^n$ a nonempty convex set, $f : \mathcal{Z} \to \overline{\mathbb{R}}$ a convex function, $A : \mathbb{R}^n \to \mathcal{Z}$ is a linear mapping and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with the components being convex functions. To $(P)$ we assign a conjugate dual problem and prove weak and strong duality theorems, the latter under the fulfillment of a regularity condition. Furthermore, for the primal and dual problem we derive necessary and sufficient optimality conditions by means of strong duality.

Particular instances of the general optimization problem $(P)$ have been considered in the framework of the portfolio optimization theory, where $x$ has been interpreted as a portfolio vector of some given assets and $Ax$ has been providing its random return. Rockafellar and Uryasev considered in [8] the minimization of the variance, of the Value-at-Risk and of the Conditional-Value-at-Risk regarding the classical constraints. In [10] additional affine constraints have been introduced, sometimes under the claim of having positive portfolio fractions. Sufficient optimality conditions have been formulated by means of the so-called *risk envelope*. In the same paper portfolio optimization problems containing a riskless asset have been considered.

This paper is organized as follows. In the following section we introduce some definitions and notations from the convex analysis and stochastic theory we use within the paper. In section 3 we construct a conjugate dual for the optimization problem $(P)$ and prove the weak and strong duality theorems. By using the latter we derive necessary and sufficient optimality conditions. In the last section we present some special portfolio optimization problems with the objective function defined by means of two convex deviation measures, namely the *generalized variance* and the *generalized lower semivariance*, respectively. We introduce for these problems their conjugate duals and derive by using the general results developed in section 3 necessary and sufficient optimality conditions.

## 2. NOTATIONS AND PRELIMINARIES

Let $\mathcal{Z}$ be a Hausdorff locally convex space and $\mathcal{Z}^*$ its topological dual space which we endow with the weak$^*$ topology. We denote by $\langle x^*, x\rangle := x^*(x)$ the value of the linear continuous functional $x^* \in \mathcal{Z}^*$ at $x \in \mathcal{Z}$.

For a set $D \subseteq \mathcal{Z}$ we denote by $\delta_D : \mathcal{Z} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ the *indicator function* of the set $D$, that is defined by

$$\delta_D(x) = \begin{cases} 0, & x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

When $D$ is a non-empty subset of $\mathcal{Z}$ and $f : \mathcal{Z} \to \overline{\mathbb{R}}$ we denote by $f_D^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ the function defined by

$$f_D^*(x^*) = (f + \delta_D)^*(x^*) = \sup_{x\in D}\{\langle x^*, x\rangle - f(x)\}.$$

One can see that for $D = \mathcal{Z}$, $f_D^*$ becomes the (*Fenchel-Moreau*) *conjugate function of* $f$ which we denote by $f^*$. The *effective domain* of a function $f : \mathcal{Z} \to \overline{\mathbb{R}}$ is $dom(f) = \{x \in \mathcal{Z} : f(x) < +\infty\}$ and we say that $f$ is *proper* if $dom(f) \neq \emptyset$ and $f(x) > -\infty,\ \forall x \in \mathcal{Z}$.

For an optimization problem $(P)$ we denote by $v(P)$ its optimal objective value. We write $\min$ ($\max$) instead of $\inf$ ($\sup$) if the infimum (supremum) is attained.

The following result is the so-called Fenchel duality theorem (for a general version of this result one can consult [14]):

**Theorem 2.1.** *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $f : \mathcal{Z} \to \overline{\mathbb{R}}$ be proper and convex functions and $A : \mathbb{R}^n \to \mathcal{Z}$ a linear mapping. Assume that $\exists x' \in dom(h) \cap A^{-1}(dom(f))$ such that $f$ is continuous at $Ax'$. Then it holds:*

$$\inf_{x\in\mathbb{R}^n}\{f(Ax) + h(x)\} = \max_{x^*\in\mathcal{Z}^*}\{-h^*(-A^*x^*) - f^*(x^*)\}.$$

Consider now the *probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\Omega$ is a basic space, $\mathfrak{F}$ a $\sigma$-algebra on $\Omega$ and $\mathbb{P}$ a probability measure on the measurable space $(\Omega, \mathfrak{F})$. We assume later (cf. Subsection 4.2 and 4.3) that $\mathcal{Z}$ is a space of measurable real-valued random variables on $\Omega$, more precisely $\mathcal{Z} = L_p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R})$ (cf. Subsection 4.1).

Equalities and inequalities between random variables are to be viewed in the sense of holding almost surely (a.s.) regarding $\mathbb{P}$. Thus for $X, Y : \Omega \to \mathbb{R}$ when we write "$X = Y$" or "$X \geq Y$" we mean "$X = Y\ a.s.$" or "$X \geq Y\ a.s.$", respectively.

Having a random variable $X : \Omega \to \mathbb{R}$ which takes the constant value $c \in \mathbb{R}$, i.e. $X = c\ a.s.$, we identify it with the real number $c \in \mathbb{R}$.

For an arbitrary random variable $X : \Omega \to \mathbb{R}$, we also define $X_- : \Omega \to \mathbb{R}$ as being

$$X_-(\omega) := \max(-X(\omega), 0) \qquad \forall \omega \in \Omega.$$

## 3. Optimality Conditions Via Strong Duality

In this section we consider a conjugate dual problem to an optimization problem with the objective function being the composition of a convex function with a linear mapping with respect to convex inequality constraints. We prove weak and strong duality assertions and derive by means of the latter necessary and sufficient optimality conditions. The primal optimization problem we consider is

$$(P) \qquad \inf_{\substack{g(x) \leqq 0, \\ x \in S}} f(Ax),$$

where $S \subseteq \mathbb{R}^n$ is a non-empty convex set and $f : \mathcal{Z} \to \overline{\mathbb{R}}$ a proper and convex function. Further we assume that $A : \mathbb{R}^n \to \mathcal{Z}$ is a linear mapping and $g : \mathbb{R}^n \to \mathbb{R}^m$, $g = (g_1, \ldots, g_m)^T$ is such that $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ are convex functions. For $\lambda \in \mathbb{R}^m$ let $\lambda^T g : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $(\lambda^T g)(x) = \lambda^T g(x)$. By "$\leqq$" we denote the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$ on $\mathbb{R}^m$.

In the following we introduce a dual problem for $(P)$ and prove weak and strong duality theorems. The dual problem $(D)$ to $(P)$ we consider in this section is

$$(D) \qquad \sup_{\substack{\lambda \in \mathbb{R}^m_+, \\ x^* \in \mathcal{Z}^*}} \{-f^*(x^*) - (\lambda^T g)^*_S(-A^* x^*)\}. \tag{1}$$

For the beginning the following result can be easily proved.

**Theorem 3.1.** *Between $(P)$ and $(D)$ weak duality holds, namely $v(P) \geq v(D)$.*

*Proof.* Let $x$ be feasible to $(P)$ and $(\lambda, x^*)$ be feasible to $(D)$. By the Young-Fenchel-inequality one has:

$$f(Ax) \geq \langle x^*, Ax \rangle - f^*(x^*).$$

Since $g(x) \in -\mathbb{R}^m_+$ and $\lambda \in \mathbb{R}^m_+$, we get $\lambda^T g(x) \leq 0$. Thus one has $\forall x \in S$,

$$f(Ax) \geq -f^*(x^*) + \lambda^T g(x) + \langle x^*, Ax \rangle,$$

from which follows

$$f(Ax) \geq -f^*(x^*) + \inf_{x \in S} \{\lambda^T g(x) + \langle x^*, Ax \rangle\} = -f^*(x^*) - (\lambda^T g)^*_S(-A^* x^*).$$

Since $x$, $\lambda$ and $x^*$ have been arbitrary chosen, we get

$$v(P) = \inf_{\substack{x\in S,\\ g(x)\leq 0}} f(Ax) \geq \sup_{\substack{\lambda\in\mathbb{R}^m_+,\\ x^*\in\mathcal{Z}^*}} \{-f^*(x^*) - (\lambda^T g)^*_S(-A^*x^*)\} = v(D). \qquad \blacksquare$$

**Remark 3.1.** As the proof shows, the assertion of Theorem 3.1 applies without any convexity assumptions for the problem $(P)$.

In order to close the gap between the primal and the dual problem and to guarantee the existence of an optimal solution for the dual problem we need a so-called regularity condition. For $(P)$ the regularity condition we assume looks like

$$(CQ) \qquad \exists x' \in \mathrm{ri}(S)\cap A^{-1}(dom(f)): \quad \begin{cases} g_i(x') \leq 0, & i\in L,\\ g_i(x') < 0, & i \in N,\\ f \text{ is continuous at } Ax', \end{cases}$$

where $L = \{i \in \{1,\ldots,m\} : g_i \text{ is affine}\}$ and $N = \{1,\ldots,m\}\setminus L$. We can prove now the strong duality theorem.

**Theorem 3.2.** *Assume that $(CQ)$ is fulfilled and $v(P) > -\infty$. Then strong duality between $(P)$ and $(D)$ holds, namely $v(P) = v(D)$ and $(D)$ has an optimal solution.*

*Proof.* The Lagrange dual problem $(D_L)$ to $(P)$ is

$$(D_L) \qquad \sup_{\lambda\in\mathbb{R}^m_+}\inf_{x\in S}\{f(Ax) + \lambda^T g(x)\}.$$

Since $f\circ A$ and $g_i,\ i = 1,\ldots,m$, are convex functions defined on $\mathbb{R}^n$ and $(CQ)$ is fulfilled, strong duality between $(P)$ and $(D_L)$ follows (cf. Theorem 28.2 in [7]), i.e. $(D_L)$ has an optimal solution $\overline{\lambda}$ and we have

$$\begin{aligned} v(P) &= v(D_L) = \max_{\lambda\in\mathbb{R}^m_+}\inf_{x\in S}\{f(Ax)+\lambda^T g(x)\}\\ &= \inf_{x\in S}\{f(Ax) + \overline{\lambda}^T g(x)\} = \inf_{x\in\mathbb{R}^n}\{f(Ax)+\overline{\lambda}^T g(x) + \delta_S(x)\}. \end{aligned}$$

Let us consider the problem $\inf\limits_{x\in\mathbb{R}^n}\{f(Ax) + \overline{\lambda}^T g(x) + \delta_S(x)\}$. Since $dom(\overline{\lambda}^T g + \delta_S) = S$ and $(CQ)$ is fulfilled, by Theorem 2.1 there exists $\overline{x}^* \in \mathcal{Z}^*$ such that

$$\begin{aligned} \inf_{x\in\mathbb{R}^n}\{f(Ax)+\overline{\lambda}^T g(x)+\delta_S(x)\} &= \max_{x^*\in\mathcal{Z}^*}\{-f^*(x^*) - (\overline{\lambda}^T g + \delta_S)^*(-A^*x^*)\}\\ &= -f^*(\overline{x}^*) - (\overline{\lambda}^T g)^*_S(-A^*\overline{x}^*). \end{aligned}$$

In conclusion $(\overline{\lambda}, \overline{x}^*)$ is an optimal solution of $(D)$ and it holds

$$v(P) = \max_{\substack{\lambda \in \mathbb{R}^m_+, \\ x^* \in \mathcal{Z}^*}} \{-f^*(x^*) - (\lambda^T g)^*_S(-A^* x^*)\} = -f^*(\overline{x}^*) - (\overline{\lambda}^T g)^*_S(-A^* \overline{x}^*) = v(D).$$

This concludes the proof. ∎

By means of the strong duality we derive necessary and sufficient optimality conditions for the primal optimization problem $(P)$.

**Theorem 3.3.** *(a) If $(CQ)$ is fulfilled and $(P)$ has an optimal solution $\overline{x}$, then $(D)$ has an optimal solution $(\overline{\lambda}, \overline{x}^*)$ such that the following optimality conditions are fulfilled:*

$(i)$ $f(A\overline{x}) + f^*(\overline{x}^*) - \langle A\overline{x}, \overline{x}^* \rangle = 0,$

$(ii)$ $\langle A\overline{x}, \overline{x}^* \rangle + (\overline{\lambda}^T g)^*_S(-A^* \overline{x}^*) = 0,$

$(iii)$ $(\overline{\lambda}^T g)(\overline{x}) = 0.$

*(b) Let $\overline{x}$ be feasible to $(P)$ and $(\overline{\lambda}, \overline{x}^*)$ be feasible to $(D)$ fulfilling the optimality conditions $(i) - (iii)$. Then $\overline{x}$ is an optimal solution for $(P)$, $(\overline{\lambda}, \overline{x}^*)$ is an optimal solution for $(D)$ and $v(P) = v(D)$.*

*Proof.* (a) Since $(P)$ has an optimal solution and $(CQ)$ is fulfilled, Theorem 3.2 guarantees the existence of an optimal solution for $(D)$, $(\overline{\lambda}, \overline{x}^*)$, such that

$$\begin{aligned} v(P) = v(D) \Leftrightarrow \quad f(A\overline{x}) &= -f^*(\overline{x}^*) - (\overline{\lambda}^T g)^*_S(-A^* \overline{x}^*) \\ \Leftrightarrow \quad f(A\overline{x}) + f^*(\overline{x}^*) - \langle A\overline{x}, \overline{x}^* \rangle &= -\langle A\overline{x}, \overline{x}^* \rangle - (\overline{\lambda}^T g)^*_S(-A^* \overline{x}^*), \end{aligned}$$

which can be equivalently written as

$$\begin{aligned} &\left[f(A\overline{x}) + f^*(\overline{x}^*) - \langle A\overline{x}, \overline{x}^* \rangle\right] - \left[(\overline{\lambda}^T g)(\overline{x})\right] \\ &+ \left[\langle A\overline{x}, \overline{x}^* \rangle + (\overline{\lambda}^T g)(\overline{x}) - \inf_{x \in S}\{\langle x, A^* \overline{x}^* \rangle + (\overline{\lambda}^T g)(x)\}\right] = 0. \end{aligned} \tag{2}$$

On the other hand, by Young-Fenchel's inequality it holds

$$f(A\overline{x}) + f^*(\overline{x}^*) - \langle A\overline{x}, \overline{x}^* \rangle \geq 0. \tag{3}$$

Since $\overline{x}$ is feasible to $(P)$ and $\overline{\lambda} \in \mathbb{R}^m_+$ we have

$$-(\overline{\lambda}^T g)(\overline{x}) \geq 0. \tag{4}$$

Finally the following inequality is fulfilled:

$$\langle A\overline{x}, \overline{x}^* \rangle + (\overline{\lambda}^T g)(\overline{x}) - \inf_{x \in S} \{ \langle x, A^* \overline{x}^* \rangle + (\overline{\lambda}^T g)(x) \} \geq 0. \tag{5}$$

So formula (2) consists of 3 nonnegative terms, their sum being equal to zero. This means that (3), (4) and (5) are fulfilled with equality, which is nothing else than $(i) - (iii)$ are fulfilled if we drop $(\overline{\lambda}^T g)(\overline{x})$ in (5) based on $(iii)$.

(b) All calculations done within part (a) can be carried out in reverse direction. Thus it follows

$$f(A\overline{x}) = -f^*(\overline{x}^*) - (\overline{\lambda}^T g)^*_S(-A^*\overline{x}^*),$$

which together with Theorem 3.1 guarantees that $\overline{x}$ is an optimal solution to $(P)$, $(\overline{\lambda}, \overline{x}^*)$ is an optimal solution to $(D)$ and $v(P) = v(D)$.

Let us notice that for this statement no convexity assumption is needed. ■

In the following section we apply the results developed in this section to special portfolio optimization problems.

## 4. Applications to Portfolio Optimization

### 4.1. Convex risk and convex deviation measures

In this subsection we consider some particular instances of the general optimization problem $(P)$ coming from portfolio optimization with the objective function being a so-called deviation measure. To this end some explanatory notions are first necessary.

For a random variable $X : \Omega \to \mathbb{R}$ we define the *expectation value* with respect to $\mathbb{P}$ by

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega).$$

The *essential supremum* of $X$ is

$$\mathrm{essupX} = \inf\{\mathrm{a} \in \mathbb{R} : \mathbb{P}(\omega : \mathrm{X}(\omega) > \mathrm{a}) = 0\}.$$

Furthermore, for $p \in (1, +\infty)$ let $L_p$ be the following space of random variables:

$$L_p := L_p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ X : \Omega \to \mathbb{R},\ X \text{ measurable}, \int_{\Omega} |X(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\}.$$

The space $L_p$ equipped with the norm $||X||_p = (\mathbb{E}(|X|^p))^{\frac{1}{p}}$, $X \in L_p$, is a Banach space. It is well-known that the dual space of $L_p$ is $L_q := L_q(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R})$, where $q \in (1, +\infty)$ fulfills $\frac{1}{p} + \frac{1}{q} = 1$.

For $X \in L_p$ and $X^* \in L_q$ we have $\langle X^*, X\rangle := \int\limits_{\Omega} X^*(\omega)X(\omega)d\mathbb{P}(\omega) = \mathbb{E}(X^*X)$ as representation of the linear continuous functional (cf. [13]).

In the following we recall the notion of a *convex deviation measure* on $L_p$. The convex deviation measures have been introduced for the first time in [9] in connection to the convex risk measures. The latter are nothing else than extensions of the coherent risk measure which are widely used in practice. The following definition introduces the notion of a convex deviation measure in an axiomatic way. Examples of convex deviation measures are the *variance*, the *lower/upper semivariance*, the *standard (lower/upper) semideviation*, the *(lower/upper) semideviation* and the *Conditional Value-at-Risk Deviation (CVaRD)*.

**Definition 4.1.** The function $d : L_p \to \overline{\mathbb{R}}$ is called convex deviation measure if the following properties are fulfilled:

(D1) Translation invariance: $d(X + b) = d(X),\ \forall X \in L_p,\ \forall b \in \mathbb{R}$;

(D2) Strictness: $d(X) \geq 0,\ \forall X \in L_p$;

(D3) Convexity: $d(\lambda X+(1-\lambda)Y) \leq \lambda d(X)+(1-\lambda)d(Y),\ \forall \lambda \in [0, 1], \forall X, Y \in L_p$.

If $d$ is a convex deviation measure, a so-called *convex risk measure* $\rho : L_p \to \overline{\mathbb{R}}$, is defined by $\rho(X) = d(X) - \mathbb{E}(X)$. The most prominent example of convex risk measures is the *Conditional Value-at-Risk (CVaR)*.

In [3] Fischer provides a nice argument for considering risk measures in $L_p$ spaces for an arbitrary $p \in [1, +\infty]$. He proves that the well-known *Value-at-Risk (VaR)* $\mathrm{VaR}_\alpha(\mathrm{X}) = -\inf\{\mathrm{x} : \mathbb{P}(\mathrm{X} \leq \mathrm{x}) > \alpha)\}, \alpha \in [0, 1]$, can be represented by means of the risk measure defined by $\rho_p(X) = ||(X - \mathbb{E}(X))_-||_p - \mathbb{E}(X)$, in the sense that there exists an unique $p^* \in [1, +\infty]$ such that $\rho_{p^*}(X) = \mathrm{VaR}_\alpha(\mathrm{X})$. In case $1 < p^* < +\infty$ the risk measure is suitable for risk capital allocation ([3]). The objective function of the optimization problem treated in subsection 4.3 is nothing else than the deviation measure corresponding to $\rho_p$.

### 4.2. Minimization of the generalized variance

A natural aim in portfolio optimization is minimizing the risk of a portfolio while maximizing its expected return/profit (cf. [5]). In the classical literature on portfolio optimization risk is measured by the variance with respect to some classical constraints ensuring positive fractions in the portfolio with the sum equal 1. A classical strategy of dealing with this multiobjective optimization problem is to eliminate the maximizing function and to add a constraint for the expected return. Thus, in practical applications, a special benchmark must be chosen, which should be achieved by the return. In modern portfolio optimization different risk and deviation measures can be considered when dealing with this problem.

In [8] the portfolio optimization problem with the feasible set taken in the following calculation was considered for different objective functions, such as the classical variance, the Value-at-Risk and the Conditional Value-at-Risk.

In the first application we deal with the minimization of the generalized variance, a problem which contains the minimization of the variance as a special case. Let $(P_v)$ be the following primal problem:

$$(P_v) \quad \begin{array}{l} \inf\limits_{x\in G} \left|\left| \sum\limits_{i=1}^{n} x_i R_i - \mathbb{E}\left(\sum\limits_{i=1}^{n} x_i R_i\right) \right|\right|_p^a \\ G = \left\{ x \in \mathbb{R}^n : \ x \geqq 0, \ \ \sum\limits_{i=1}^{n} x_i - 1 = 0, \ \ B - \mathbb{E}\left(\sum\limits_{i=1}^{n} x_i R_i\right) \leq 0 \right\}, \end{array}$$

where $a > 1$ and $R_i : \Omega \to \mathbb{R}, \ R_i \in L_p, \ \forall i = 1, \ldots, n.$ We define by $R(\omega) = (R_1(\omega), \ldots, R_n(\omega)), \ \forall \omega \in \Omega$, the $n$-tuple containing the random returns for the considered assets and we assume that

$$B \leq \max_{i\in\{1,\ldots,n\}} \mathbb{E}(R_i). \tag{6}$$

Then we look for portfolios $x$ with expected return $\mathbb{E}\left(\sum\limits_{i=1}^{n} x_i R_i\right)$ of at least equal to $B$. The optimization variable $x = (x_1, \ldots, x_n)^T$ can be interpreted as the portfolio vector for $n$ given assets. The constraints taken in $(P_v)$ are most common in the context of portfolio optimization. They forbid short sales and force that the portfolio consists of the given assets.

Let us notice that $(P_v)$ can be written as

$$(P_v) \quad \inf_{x\in G} d_1(Ax),$$

where $d_1 : L_p \to \mathbb{R}$,

$$d_1(X) = ||X - \mathbb{E}(X)||_p^a \qquad (a > 1)$$

is the so-called *generalized variance* and $A : \mathbb{R}^n \to L_p, \ Ax = \sum\limits_{i=1}^{n} x_i R_i$ is a linear mapping. Let $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^{n+3}$ be the following vector function: $g(x) = \left(-x, \sum\limits_{i=1}^{n} x_i - 1, 1 - \sum\limits_{i=1}^{n} x_i, B - \mathbb{E}\big(\sum\limits_{i=1}^{n} x_i R_i\big)\right)$. For $a = p = 2$, $d_1$ becomes the classical *variance*. The generalized variance is a convex deviation measure and its conjugate function $d_1^* : L_q \to \overline{\mathbb{R}}$ is given by (cf. [2])

$$d_1^*(X^*) = \begin{cases} \min\limits_{c\in\mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(X^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\}, & \mathbb{E}(X^*) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can notice that for a portfolio vector $x$, $Ax$ provides the random portfolio return.

For calculating its adjoint operator $A^*: L_q \to \mathbb{R}^n$, take $x \in \mathbb{R}^n$ and $X^* \in L_q$. Then we have

$$\begin{aligned}\langle Ax, X^*\rangle &= \left\langle \sum_{i=1}^n x_i R_i, X^* \right\rangle = \mathbb{E}\left(\left(\sum_{i=1}^n x_i R_i\right) X^*\right)\\ &= \sum_{i=1}^n x_i \mathbb{E}(R_i X^*) = (\mathbb{E}(X^* R_1), \ldots, \mathbb{E}(X^* R_n))x = \langle x, A^* X^*\rangle\end{aligned}$$

and so we get

$$A^* X^* = (\mathbb{E}(X^* R_1), \ldots, \mathbb{E}(X^* R_n))^T. \tag{7}$$

For finding the dual of $(P_v)$ (cf. (1), where $\mathcal{Z} = L_p$, $\mathcal{Z}^* = L_q$ and $m = n+3$)

$$(D_v) \qquad \sup_{\lambda \in \mathbb{R}^{n+3}_+,\, X^* \in L_q} \{-d_1^*(X^*) - (\lambda^T g)^*(-A^* X^*)\},$$

we need to make first some calculations. For $\lambda = (\alpha, \beta^1, \beta^2, \gamma) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R}^{n+3}_+$ we have

$$\begin{aligned}&(\lambda^T g)^*(-A^* X^*) = \sup_{y \in \mathbb{R}^n} \{y^T(-A^* X^*) - (\lambda^T g)(y)\}\\ &= \sup_{y \in \mathbb{R}^n} \Bigg\{ \sum_{i=1}^n y_i(-A^* X^*)_i + \sum_{i=1}^n \alpha_i y_i - \beta^1\left(\sum_{i=1}^n y_i - 1\right) - \beta^2\left(1 - \sum_{i=1}^n y_i\right)\\ &\quad -\gamma\left(B - \sum_{i=1}^n y_i \mathbb{E}(R_i)\right)\Bigg\}\\ &= \sup_{y \in \mathbb{R}^n} \Bigg\{ \sum_{i=1}^n y_i(-A^* X^*)_i + \sum_{i=1}^n \alpha_i y_i - (\beta^1 - \beta^2)\left(\sum_{i=1}^n y_i - 1\right)\\ &\quad -\gamma\left(B - \sum_{i=1}^n y_i \mathbb{E}(R_i)\right)\Bigg\}\\ &= \sup_{y \in \mathbb{R}^n} \left\{ \sum_{i=1}^n y_i(-(A^* X^*)_i + \alpha_i - (\beta^1 - \beta^2) + \gamma \mathbb{E}(R_i)) \right\} + (\beta^1 - \beta^2) - \gamma B\end{aligned} \tag{8}$$

$$\begin{aligned}&= \begin{cases} \beta^1 - \beta^2 - \gamma B, & \text{if } -(A^* X^*)_i + \alpha_i - (\beta^1 - \beta^2) + \gamma \mathbb{E}(R_i) = 0, i = 1, \ldots, n,\\ +\infty, & \text{otherwise,} \end{cases}\\ &= \begin{cases} \beta^1 - \beta^2 - \gamma B, & \text{if } \alpha_i = \beta^1 - \beta^2 + \mathbb{E}(X^* R_i) - \gamma \mathbb{E}(R_i), i = 1, \ldots, n,\\ +\infty, & \text{otherwise.} \end{cases}\end{aligned} \tag{9}$$

So one has the following dual problem:

$(D_v)$

$$\sup_{\substack{X^*\in L_q, \mathbb{E}(X^*)=0,\\ \alpha\in\mathbb{R}^n_+, \beta^1\in\mathbb{R}_+, \beta^2\in\mathbb{R}_+, \gamma\in\mathbb{R}_+,\\ \alpha_i=\beta^1-\beta^2+\mathbb{E}(X^*R_i)-\gamma\mathbb{E}(R_i),\\ i=1,\dots,n}} \left\{-\min_{c\in\mathbb{R}}\left\{(a-1)\left\|\tfrac{1}{a}(X^*-c)\right\|_q^{\frac{a}{a-1}}\right\}-\beta^1+\beta^2+\gamma B\right\}$$

$$\Leftrightarrow \sup_{\substack{X^*\in L_q, \mathbb{E}(X^*)=0,\\ c\in\mathbb{R}, \beta:=\beta^1-\beta^2\in\mathbb{R}, \gamma\in\mathbb{R}_+,\\ -\beta\leq\mathbb{E}(X^*R_i)-\gamma\mathbb{E}(R_i), i=1,\dots,n}} \left\{(1-a)\left\|\frac{1}{a}(X^*-c)\right\|_q^{\frac{a}{a-1}}-\beta+\gamma B\right\}$$

$$\Leftrightarrow \sup_{\substack{X^*\in L_q, \mathbb{E}(X^*)=0,\\ c\in\mathbb{R}, \gamma\in\mathbb{R}_+}} \left\{(1-a)\left\|\frac{1}{a}(X^*-c)\right\|_q^{\frac{a}{a-1}}+\min_{i\in\{1,\dots,n\}}(\mathbb{E}(X^*R_i)-\gamma\mathbb{E}(R_i))+\gamma B\right\}.$$

**Remark 4.1.** For $X^*\in L_q, Y\in L_p$ we denote by $\mathrm{cov}(X^*, Y)=\mathbb{E}((X^*-\mathbb{E}(X^*))(Y-\mathbb{E}(Y)))$ the covariance of the random variables $X^*$ and $Y$. If for $X^*\in L_q,\ \mathbb{E}(X^*)=0$ then one has for $i=1,\dots,n$

$$\begin{aligned}\mathrm{cov}(X^*, R_i) &= \mathbb{E}\big((X^*-\mathbb{E}(X^*))(R_i-\mathbb{E}(R_i))\big)\\ &= \mathbb{E}(X^*R_i)-\mathbb{E}(X^*)\mathbb{E}(R_i)=\mathbb{E}(X^*R_i).\end{aligned}$$

Thus the dual problem can be written as

$$(D_v) \sup_{\substack{X^*\in L_q, \mathbb{E}(X^*)=0,\\ c\in\mathbb{R}, \gamma\in\mathbb{R}_+}} \left\{(1-a)\left\|\frac{1}{a}(X^*-c)\right\|_q^{\frac{a}{a-1}}+\min_{i\in\{1,\dots,n\}}(\mathrm{cov}(X^*, R_i)-\gamma\mathbb{E}(R_i))+\gamma B\right\}.$$

Since the inequality constraints of $(P_v)$ are affine (the feasible set $G$ is in fact a polyhedral set) and $x'=(1,0,\dots,0)^T$ is a feasible point to $(P_v)$, $(CQ)$ is fulfilled. Moreover, one can easily see that $v(P_v)\geq 0$.

By Theorem 3.2 we can state now the following strong duality theorem.

**Theorem 4.1.** *Between $(P_v)$ and $(D_v)$ strong duality holds, i.e. $v(P_v)=v(D_v)$ and the dual problem $(D_v)$ has an optimal solution.*

**Remark 4.2.** Since the problem $(P_v)$ has a compact feasible set and a continuous objective function, the existence of an optimal solution $\overline{x}$ for it is guaranteed.

Next we derive necessary and sufficient optimality conditions for $(P_v)$ and $(D_v)$ by using Theorem 3.3.

**Theorem 4.2.** *(a) Let $\overline{x}$ be an optimal solution for $(P_v)$, then $(D_v)$ has an optimal solution $(\overline{X}^*, \overline{c}, \overline{\gamma})\in L_q\times\mathbb{R}\times\mathbb{R}_+$ such that the following optimality conditions are fulfilled:*

(1) $\left\|\sum\limits_{i=1}^n \overline{x}_i R_i - \mathbb{E}\left(\sum\limits_{i=1}^n \overline{x}_i R_i\right)\right\|_p^a + (a-1)\left\|\frac{1}{a}(\overline{X}^* - \overline{c})\right\|_q^{\frac{a}{a-1}}$
$- \operatorname{cov}\left(\overline{X}^*, \sum\limits_{i=1}^n \overline{x}_i R_i\right) = 0,$

(2) $\mathbb{E}(\overline{X}^*) = 0,$

(3) $\min\limits_{c\in\mathbb{R}}\left\{(a-1)\left\|\frac{1}{a}(\overline{X}^* - c)\right\|_q^{\frac{a}{a-1}}\right\} = (a-1)\left\|\frac{1}{a}(\overline{X}^* - \overline{c})\right\|_q^{\frac{a}{a-1}},$

(4) $\sum\limits_{i=1}^n \operatorname{cov}(\overline{X}^*, R_i)\overline{x}_i - \overline{\gamma}B \leq \min\limits_{i\in\{1,\dots,n\}}\{\operatorname{cov}(\overline{X}^*, R_i) - \overline{\gamma}\mathbb{E}(R_i)\}.$

*(b) Let $\overline{x}$ be feasible to $(P_v)$ and $(\overline{X}^*, \overline{c}, \overline{\gamma})$ be feasible to $(D_v)$ fulfilling the optimality conditions (1)-(4). Then $\overline{x}$ is an optimal solution for $(P_v)$, $(\overline{X}^*, \overline{c}, \overline{\gamma})$ is an optimal solution for $(D_v)$ and $v(P_v) = v(D_v)$.*

*Proof.* (a) In order to prove the theorem we just have to particularize the conditions $(i)-(iii)$ in Theorem 3.3. By the latter, for $\overline{x} \in G$ there exists $\overline{X}^* \in L_q$ and $\overline{\lambda} = (\overline{\alpha}, \overline{\beta}^1, \overline{\beta}^2, \overline{\gamma}) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and $\overline{\beta} := \overline{\beta}^1 - \overline{\beta}^2 \in \mathbb{R}$ such that (cf. (8))

(i) $\left\|\sum\limits_{i=1}^n \overline{x}_i R_i - \mathbb{E}\left(\sum\limits_{i=1}^n \overline{x}_i R_i\right)\right\|_p^a + \min\limits_{c\in\mathbb{R}}\left\{(a-1)\left\|\frac{1}{a}(\overline{X}^* - c)\right\|_q^{\frac{a}{a-1}}\right\}$
$- \mathbb{E}\left(\overline{X}^* \sum\limits_{i=1}^n \overline{x}_i R_i\right) = 0,$
$\mathbb{E}(\overline{X}^*) = 0,$

(ii) $\mathbb{E}\left(\overline{X}^* \sum\limits_{i=1}^n \overline{x}_i R_i\right) + \overline{\beta} - \overline{\gamma}B = 0,$
$\overline{\alpha}_i = \overline{\beta} + \mathbb{E}(\overline{X}^* R_i) - \overline{\gamma}\mathbb{E}(R_i),\ \forall i = 1, \dots, n,$

(iii) $-\sum\limits_{i=1}^n \overline{\alpha}_i \overline{x}_i + \overline{\beta}\left(\sum\limits_{i=1}^n \overline{x}_i - 1\right) + \overline{\gamma}\left(B - \sum\limits_{i=1}^n \overline{x}_i \mathbb{E}(R_i)\right) = 0.$

Let now $\overline{c} \in \mathbb{R}$ be such that

$$\min_{c\in\mathbb{R}}\left\{(a-1)\left\|\frac{1}{a}(\overline{X}^* - c)\right\|_q^{\frac{a}{a-1}}\right\} = (a-1)\left\|\frac{1}{a}(\overline{X}^* - \overline{c})\right\|_q^{\frac{a}{a-1}}.$$

Since $\overline{\alpha}_i \geq 0,\ i = 1, \dots, n,$ conditions $(i)-(iii)$ can be equivalently written as

(i) $\left\| \sum_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left( \sum_{i=1}^{n} \overline{x}_i R_i \right) \right\|_p^a + (a-1) \left\| \frac{1}{a}(\overline{X}^* - \overline{c}) \right\|_q^{\frac{a}{a-1}}$
$- \mathbb{E}\left( \overline{X}^* \sum_{i=1}^{n} \overline{x}_i R_i \right) = 0,$
$\mathbb{E}(\overline{X}^*) = 0,$
$\min_{c \in \mathbb{R}} \left\{ (a-1) \left\| \frac{1}{a}(\overline{X}^* - c) \right\|_q^{\frac{a}{a-1}} \right\} = (a-1) \left\| \frac{1}{a}(\overline{X}^* - \overline{c}) \right\|_q^{\frac{a}{a-1}},$

(ii) $\mathbb{E}\left( \overline{X}^* \sum_{i=1}^{n} \overline{x}_i R_i \right) + \overline{\beta} - \overline{\gamma} B = 0,$
$-\overline{\beta} \leq \min_{i \in \{1,\ldots,n\}} \{\mathbb{E}(\overline{X}^* R_i) - \overline{\gamma}\mathbb{E}(R_i)\},$

(iii) $-\sum_{i=1}^{n} \mathbb{E}(\overline{X}^* R_i)\overline{x}_i - \overline{\beta} + \overline{\gamma} B = 0.$

Observe that in $(iii)$ we have substituted $\overline{\alpha}_i$ by $\overline{\beta} + \mathbb{E}(\overline{X}^* R_i) - \overline{\gamma}\mathbb{E}(R_i),\ i = 1, \ldots, n.$

Using that $\sum_{i=1}^{n} \mathbb{E}(\overline{X}^* R_i)\overline{x}_i = \mathbb{E}\left( \overline{X}^* \sum_{i=1}^{n} \overline{x}_i R_i \right)$, the relations $(i) - (iii)$ become equivalently

(1) $\left\| \sum_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left( \sum_{i=1}^{n} \overline{x}_i R_i \right) \right\|_p^a + (a-1) \left\| \frac{1}{a}(\overline{X}^* - \overline{c}) \right\|_q^{\frac{a}{a-1}}$
$- \mathbb{E}\left( \overline{X}^* \sum_{i=1}^{n} \overline{x}_i R_i \right) = 0,$

(2) $\mathbb{E}(\overline{X}^*) = 0,$

(3) $\min_{c \in \mathbb{R}} \left\{ (a-1) \left\| \frac{1}{a}(\overline{X}^* - c) \right\|_q^{\frac{a}{a-1}} \right\} = (a-1) \left\| \frac{1}{a}(\overline{X}^* - \overline{c}) \right\|_q^{\frac{a}{a-1}},$

(4) $\sum_{i=1}^{n} \mathbb{E}(\overline{X}^* R_i)\overline{x}_i - \overline{\gamma} B \leq \min_{i \in \{1,\ldots,n\}} \{\mathbb{E}(\overline{X}^* R_i) - \overline{\gamma}\mathbb{E}(R_i)\}.$

Using Remark 4.1, this leads to the desired solution.

(b) The calculations given in part (a) can be done in reverse order and the conclusion follows. ■

### 4.3. Minimization of the generalized lower semivariance

The minimization problem we treat in this subsection has the following formulation

$$(P_s) \qquad \inf_{x\in G} \left|\left| \left( \sum_{i=1}^n x_iR_i - \mathbb{E}\left(\sum_{i=1}^n x_iR_i\right)\right)_- \right|\right|_p^a$$

$$G = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i - 1 = 0,\ B - \mathbb{E}\left(\sum_{i=1}^n x_iR_i\right) \leq 0 \right\},$$

where $a > 1$, $R_i : \Omega \to \mathbb{R}$, $R_i \in L_p (i = 1, \ldots, n)$, $R(\omega) = (R_1(\omega), \ldots, R_n(\omega))$ is the $n$-tuple of random returns for the considered assets and $B \in \mathbb{R}$ is a constant *benchmark* for the expected return of the portfolio represented by $x \in G$.

Further, the first asset is assumed to be riskless, namely $0 < \mathbb{E}(R_1) = R_1 =$ const. This assumption is not restricting the generality of the approach. We consider it because in real capital markets besides risky assets like stocks and investment funds etc. there are always available riskless securities e.g. fixed-interest securities like bonds. And this has an important influence on the expected return and risk of the portfolios that are combined with such a riskless asset and therefore on the investment behavior of the different investors acting at the financial markets.

Usually, the expected values $\mathbb{E}(R_1), \ldots, \mathbb{E}(R_n)$ of the components of $R$ are assumed to be positive, i.e. $\mathbb{E}(R_i) = \langle 1, R_i\rangle > 0, \quad \forall i = 1, \ldots, n$. Moreover, we assume that $G$ is nonempty. Let us remark that this could be guaranteed by assuming that (6) holds. The portfolio fractions have the sum equal to 1, but this time we permit short sales, that arises for $x_i < 0$. Optimization problems where one can find similar formulations of the feasible set have been treated in [10].

Let us notice that $(P_s)$ can be written as

$$(P_s) \qquad \inf_{x\in G} d_2(Ax),$$

where $d_2 : L_p \to \mathbb{R}$,

$$d_2(X) = ||(X - \mathbb{E}(X))_-||_p^a \qquad (a > 1)$$

is the so-called *generalized lower semivariance* and the linear mapping $A : \mathbb{R}^n \to L_p$ is given by $Ax = \sum_{i=1}^n x_iR_i$. As we have seen in the previous section, for $X^* \in L_p$, $A^*X^* = (\mathbb{E}(X^*R_1), \ldots, \mathbb{E}(X^*R_n))^T$. Further we define $g : \mathbb{R}^n \to \mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3$, $g(x) = \left(\sum_{i=1}^n x_i - 1, 1 - \sum_{i=1}^n x_i, B - \mathbb{E}\big(\sum_{i=1}^n x_iR_i\big)\right)$.

The generalized variance as treated in the previous subsection has the disadvantage of measuring both positive and negative deviation. In financial application often only loss, represented by negative deviation, is important. This is the reason why we use the *generalized lower semivariance* as deviation measure. For $a = p = 2$, $d_2$ becomes the classical *lower semivariance*.

The generalized lower semivariance is a convex deviation measure and its conjugate $d_2^*: L_q \to \overline{\mathbb{R}}$ is given by (cf. [2])

$$d_2^*(X^*) = \begin{cases} (a-1)\left|\left|\frac{1}{a}(\mathrm{essupX^*} - \mathrm{X^*})\right|\right|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}}, & \mathbb{E}(X^*) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For finding the dual of $(P_s)$ we also need to make some calculations. For $\lambda = (\alpha^1, \alpha^2, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R}^3_+$ we have

$$(\lambda^T g)^*(-A^*X^*) = \sup_{y \in \mathbb{R}^n} \{y^T(-A^*X^*) - (\lambda^T g)(y)\}$$

$$= \sup_{y \in \mathbb{R}^n} \left\{ \sum_{i=1}^n y_i \left(-A^*X^*\right)_i - \alpha^1 \left(\sum_{i=1}^n y_i - 1\right) - \alpha^2 \left(1 - \sum_{i=1}^n y_i\right) \right.$$
$$\left. -\beta \left(B - \sum_{i=1}^n y_i \mathbb{E}(R_i)\right) \right\}$$

$$= \sup_{y \in \mathbb{R}^n} \left\{ \sum_{i=1}^n y_i \left(-A^*X^*\right)_i - (\alpha^1 - \alpha^2) \left(\sum_{i=1}^n y_i - 1\right) - \beta \left(B - \sum_{i=1}^n y_i \mathbb{E}(R_i)\right) \right\}$$

$$= \sup_{y \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \big( - (A^*X^*)_i - \alpha^1 + \alpha^2 + \beta \mathbb{E}(R_i) \big) y_i \right\} + \alpha^1 - \alpha^2 - \beta B$$

$$= \begin{cases} \alpha^1 - \alpha^2 - \beta B, & \text{if } -(A^*X^*)_i - \alpha^1 + \alpha^2 + \beta \mathbb{E}(R_i) = 0,\ i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \alpha^1 - \alpha^2 - \beta B, & \text{if } \alpha^1 - \alpha^2 = -\mathbb{E}(X^* R_i) + \beta \mathbb{E}(R_i),\ \ i = 1, \dots, n, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now the dual problem turns out to be (cf. (1))

$$(D_s) \sup_{\substack{(X^*, \alpha^1, \alpha^2, \beta) \in L_q \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \mathbb{E}(X^*) = 0, \\ \alpha^1 - \alpha^2 = -\mathbb{E}(X^* R_i) + \beta \mathbb{E}(R_i), \\ i = 1, \dots, n}} \left\{ (1-a) \left|\left| \frac{1}{a} (\mathrm{essupX^*} - \mathrm{X^*}) \right|\right|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}} - \alpha^1 + \alpha^2 + \beta \mathrm{B} \right\}.$$

Since for $i = 1$, $R_i = R_1$ is constant and $\alpha := \alpha^1 - \alpha^2 \in \mathbb{R}$, if $\mathbb{E}(X^*) = 0$ we get

$$\beta = \frac{\alpha + \mathbb{E}(X^* R_1)}{\mathbb{E}(R_1)} = \frac{\alpha}{R_1}.$$

Because of $\beta \geq 0$ we must have $\alpha \geq 0$ and as $\mathbb{E}(X^* R_i) = \mathrm{cov}(\mathrm{X}^*, \mathrm{R_i})$, for $X^* \in L_q$, $\mathbb{E}(X^*) = 0$, the dual can be equivalently written as follows

$$(D_s) \qquad \sup_{\substack{(X^*,\alpha)\in L_q\times\mathbb{R}_+,\\ \mathbb{E}(X^*)=0,\\ \alpha\left(\frac{\mathbb{E}(R_i)}{R_1}-1\right)=\mathrm{cov}(\mathrm{X}^*,\mathrm{R_i}),\mathrm{i}=2,\ldots,\mathrm{n}}} \left\{(1-a)\left\|\frac{1}{a}(\mathrm{essup}\mathrm{X}^* - \mathrm{X}^*)\right\|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}} - \alpha\left(1-\frac{\mathrm{B}}{\mathrm{R_1}}\right)\right\}.$$

**Remark 4.3.** A special case of the considered example arises if we choose $B = R_1$, i.e. the expected portfolio return should at least achieve the riskless return.

Optimization problems with the feasible set being defined like for $(P_s)$ (for $B = R_1 + \Delta$, $\Delta > 0$) have been treated in [10].

Since the constraints in $(P_s)$ are affine and the feasible set is nonempty, $(CQ)$ is fulfilled. Moreover, let us notice that $v(P_s) \geq 0$. By Theorem 3.2 we can state now the following strong duality theorem.

**Theorem 4.3.** *Between $(P_s)$ and $(D_s)$ strong duality holds, i.e. $v(P_s) = v(D_s)$ and the dual problem $(D_s)$ has an optimal solution.*

Now, by Theorem 3.3, we can derive necessary and sufficient optimality conditions.

**Theorem 4.4.** *(a) Let $\overline{x}$ be an optimal solution of $(P_s)$, then $(D_s)$ has an optimal solution $(\overline{X}^*, \overline{\alpha}) \in L_q \times \mathbb{R}_+$ such that the following optimality conditions are fulfilled:*

(1) $\left\|\left(\sum\limits_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left(\sum\limits_{i=1}^{n} \overline{x}_i R_i\right)\right)_-\right\|_p^a + (a-1)\left\|\frac{1}{a}(\mathrm{essup}\overline{\mathrm{X}}^* - \overline{\mathrm{X}}^*)\right\|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}}$
$- \mathrm{cov}\left(\overline{\mathrm{X}}^*, \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R_i}\right) = 0,$

(2) $\mathbb{E}(\overline{X}^*)=0,$

(3) $\overline{\alpha}\left(1 + \frac{B}{R_1}\right) = \mathrm{cov}\left(\overline{\mathrm{X}}^*, \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R_i}\right),$

(4) $\overline{\alpha}\left(\frac{\mathbb{E}(R_i)}{R_1} - 1\right) = \mathrm{cov}(\overline{\mathrm{X}}^*, \mathrm{R_i}),\ \mathrm{i} = 2, \ldots, \mathrm{n},$

(5) $\overline{\alpha} \geq 0,$

(6) $\overline{\alpha}\left(B - \sum\limits_{i=1}^{n} \overline{x}_i \mathbb{E}(R_i)\right) = 0.$

*(b) Let $\overline{x}$ be feasible to $(P_s)$ and $(\overline{X}^*, \overline{\alpha})$ be feasible to $(D_s)$ fulfilling the optimality conditions (1)-(6). Then $\overline{x}$ is an optimal solution for $(P_s)$, $(\overline{X}^*, \overline{\alpha})$ is an optimal solution for $(D_s)$ and $v(P_s) = v(D_s)$*

*Proof.* (a) By Theorem 3.3, for $\overline{x} \in G$ we get the existence of $\overline{X}^* \in L_q$ and $\overline{\lambda} = (\overline{\alpha}^1, \overline{\alpha}^2, \overline{\beta}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ such that the following conditions hold (cf. (10), where $\overline{\alpha} = \overline{\alpha}^1 - \alpha^2 \in \mathbb{R}$)

$$\text{(i)}\ \left\| \left( \sum_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left( \sum_{i=1}^{n} \overline{x}_i R_i \right) \right)_- \right\|_p^a + (a-1) \left\| \frac{1}{a} (\mathrm{essup} \overline{\mathrm{X}}^* - \overline{\mathrm{X}}^*) \right\|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}} - \mathrm{cov}\left( \overline{\mathrm{X}}^*, \sum_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R}_{\mathrm{i}} \right) = 0, \quad \mathbb{E}(\overline{\mathrm{X}}^*) = 0,$$

$$\text{(ii)}\ \mathrm{cov}\left( \overline{\mathrm{X}}^*, \sum_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R}_{\mathrm{i}} \right) + \overline{\alpha} - \overline{\beta}\mathrm{B} = 0,$$
$$\overline{\alpha} = -\mathrm{cov}(\overline{\mathrm{X}}^*, \mathrm{R}_{\mathrm{i}}) + \overline{\beta}\mathbb{E}(\mathrm{R}_{\mathrm{i}}),\ \mathrm{i} = 1, \ldots, \mathrm{n},$$

$$\text{(iii)}\ \overline{\alpha} \left( \sum_{i=1}^{n} \overline{x}_i - 1 \right) + \overline{\beta} \left( B - \sum_{i=1}^{n} \overline{x}_i \mathbb{E}(R_i) \right) = 0.$$

For $i = 1$, from $(ii)$ we get $\overline{\beta} = \frac{\overline{\alpha}}{R_1}$ and the optimality conditions become equivalently

$$\text{(i)}\ \left\| \left( \sum_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left( \sum_{i=1}^{n} \overline{x}_i R_i \right) \right)_- \right\|_p^a + (a-1) \left\| \frac{1}{a} (\mathrm{essup} \overline{\mathrm{X}}^* - \overline{\mathrm{X}}^*) \right\|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}} - \mathrm{cov}\left( \overline{\mathrm{X}}^*, \sum_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R}_{\mathrm{i}} \right) = 0, \quad \mathbb{E}(\overline{\mathrm{X}}^*) = 0,$$

$$\text{(ii)}\ \overline{\alpha} + \overline{\beta} B = \mathrm{cov}\left( \overline{\mathrm{X}}^*, \sum_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R}_{\mathrm{i}} \right),$$
$$\overline{\beta} = \frac{\overline{\alpha}}{R_1}, \quad \overline{\beta} = \frac{\overline{\alpha} + \mathrm{cov}(\overline{\mathrm{X}}^*, \mathrm{R}_{\mathrm{i}})}{\mathbb{E}(R_i)},\ i = 2, \ldots, n,$$

$$\text{(iii)}\ \overline{\alpha} + \overline{\beta} B = \overline{\alpha} \sum_{i=1}^{n} \overline{x}_i + \overline{\beta} \sum_{i=1}^{n} \overline{x}_i \mathbb{E}(R_i).$$

Since $\overline{\beta} \geq 0$ it follows that $\overline{\alpha} \geq 0$ and the following conditions are equivalent to the previous one:

$$\text{(1)}\ \left\| \left( \sum_{i=1}^{n} \overline{x}_i R_i - \mathbb{E}\left( \sum_{i=1}^{n} \overline{x}_i R_i \right) \right)_- \right\|_p^a + (a-1) \left\| \frac{1}{a} (\mathrm{essup} \overline{\mathrm{X}}^* - \overline{\mathrm{X}}^*) \right\|_{\mathrm{q}}^{\frac{\mathrm{a}}{\mathrm{a}-1}} - \mathrm{cov}\left( \overline{\mathrm{X}}^*, \sum_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}} \mathrm{R}_{\mathrm{i}} \right) = 0,$$

(2) $\mathbb{E}(\overline{X}^*) = 0$,

(3) $\overline{\alpha}\left(1 + \frac{B}{R_1}\right) = \mathrm{cov}\left(\overline{\mathrm{X}}^*, \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} \overline{\mathrm{x}}_{\mathrm{i}}\mathrm{R}_{\mathrm{i}}\right)$,

(4) $\overline{\alpha}\left(\frac{\mathbb{E}(R_i)}{R_1} - 1\right) = \mathrm{cov}(\overline{\mathrm{X}}^*, \mathrm{R}_{\mathrm{i}}),\ \mathrm{i} = 2, \ldots, \mathrm{n}$,

(5) $\overline{\alpha} \geq 0$,

(6) $\overline{\alpha}\left(B - \sum\limits_{i=1}^{n} \overline{x}_i \mathbb{E}(R_i)\right) = 0$.

(b) The calculations given in part (a) can be done in reverse order and the conclusion follows.


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