Duality for almost convex optimization problems via the perturbation approach

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Abstract. We deal with duality for almost convex finite dimensional optimization problems by means of the classical perturbation approach. To this aim some standard results from the convex analysis are extended to the case of almost convex sets and functions. The duality for some classes of primal-dual problems is derived as a special case of the general approach. The sufficient regularity conditions we need for guaranteeing strong duality are proved to be similar to the ones in the convex case.

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1 Introduction

Dealing with duality for a given optimization problem is one of the main features in mathematical programming and convex analysis both from theoretical and practical point of view.

There is a well developed theory of duality for convex optimization problems in finite dimensional spaces, as one can read for instance in [15]. Distinct dual problems have been investigated by using the so-called perturbation approach in [16]. This is based on the theory of conjugate functions and describes how a dual problem can be assigned to a primal one ([5]).

Generalized convex functions are those non-convex functions which possess at least one valuable property of convex functions. The growing interest in them during the last decades comes with no surprise since they are often more suitable than convex functions to describe practical problems originated from economics, management science, engineering, etc. (see for instance [10] and [12]). Therefore, the question concerning possible extensions of different optimality conditions and duality results also for non-convex programming problems arises naturally. Giannessi and Rapcsák in [9] and Mastroeni and Rapcsák in [13] have given statements on the solvability of generalized systems, which is an important tool for proving duality results. Kanniappan has considered

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in [11] a Fenchel-type duality theorem for non-convex and non-differentiable maximization problems, Beoni has considered in [1] the extension of the same result in the context of fractional programming, while Penot and Volle ([14]) have studied Fenchel duality for quasiconvex problems. In [4] an extension of Fenchel's duality theorem to so-called *nearly convex functions* is given. Regarding this generalized convexity concept, let us also mention our paper [2], where we deal with duality for an optimization problem with a nearly convex objective function subject to geometrical and inequality cone constraints also defined by nearly convex functions.

In this paper we consider another generalized convexity concept, called *almost convexity*, which is due to Frenk and Kassay ([6]). Almost convex sets are defined such that their closure is a convex set, and moreover, the relative interior of their closure is contained in the set itself. This concept leads to the so-called almost convex functions: those functions whose epigraphs are almost convex sets. We show first how standard results from the convex analysis may be extended to the case of almost convex sets and functions. Along with the nearly convex functions, the class of almost convex functions is another generalization of the class of convex functions which fulfills some of the important properties of the latter. The two classes of almost and nearly convex functions contain strictly the class of convex functions and do not coincide ([3]). By means of some counterexamples we also emphasize some basic properties of convex sets (functions) which do not hold for almost convex sets (functions). Among these, we mention that the intersection of almost convex sets may not be almost convex, and there are almost convex functions which are not quasiconvex.

Considering a general almost convex optimization problem we construct a dual to it by means of the classical perturbation approach and state some sufficient regularity conditions which guarantee strong duality. The duality for some classes of primal-dual problems is derived as a special case of the general approach. In this way we extend some results from [2] and [3].

By their definition it follows immediately that almost convex functions which are not convex fail to be lower semicontinuous. Thus the novelty of our results may be found exclusively within optimization problems with objectives which are not lower semicontinuous and/or feasible sets which are not closed.

The paper is organized as follows. Section 2 recalls the definitions of almost convex sets and functions as well as presents some basic facts and properties for them, necessary for the subsequent investigations. In Section 3 we deal with the duality in the general framework of the perturbation approach, by introducing to a primal optimization problem a conjugate dual problem. We are able to verify a strong duality assertion by replacing the classical convexity assumptions with almost convexity ones and assuming a general regularity condition. Finally, in Section 4 we get as a particular case strong duality results for the Lagrange and the so-called Fenchel-Lagrange dual problem of an optimization problem with an almost convex objective function and almost convex inequality cone constraints. The regularity condition we need here is a generalized Slater condition. Under the use of a classical regularity condition, the Fenchel duality in case of almost convex optimization problems is obtained as another application of the general results. The presentation is accompanied by several examples illustrating the theoretical considerations and results.

2 Almost convex sets and functions: basic properties

2.1 Almost convex sets

Definition 2.1 (cf. [6]) *A subset $C \subseteq \mathbb{R}^n$ is called almost convex, if $cl(C)$ is a convex set and $ri(cl(C)) \subseteq C$.*

It is obvious that any convex set $C \subseteq \mathbb{R}^n$ is also almost convex, but the converse is not true in general as the following example shows.

Example 2.1 *(Almost convex set which is not convex.) Let $C = ([0,1]\times[0,1]) \setminus \{(0,y) : y \in (\mathbb{R} \setminus \mathbb{Q})\} \subseteq \mathbb{R}^2$. It is easy to check that C is almost convex but not a convex set.*

Some properties which are specific for convex sets in $\mathbb{R}^n$ hold also for almost convex sets, as the following results show.

Lemma 2.1 *For any almost convex set $C \subseteq \mathbb{R}^n$ it follows that*

$$ri(cl(C)) = ri(C). \tag{1}$$

Proof: If C is empty, then (1) is trivial. Otherwise $cl(C)$ is nonempty and convex, and so $ri(cl(C)) \neq \emptyset$. This implies by Lemma 1.12, relations (1.19) and (1.24) in [8] that $aff(ri(cl(C))) = aff(cl(C)) = aff(C)$, which yields by almost convexity of C that $ri(cl(C)) \subseteq ri(C)$. Since the reverse inclusion is trivial, we obtain (1). ■

Notice that by the previous lemma, any nonempty almost convex set in $\mathbb{R}^n$ has a nonempty relative interior.

Lemma 2.2 *Let $C \subseteq \mathbb{R}^n$ be any almost convex set. Then*

$$\alpha cl(C) + (1-\alpha) ri(C) \subseteq ri(C), \quad \forall\, 0 \leq \alpha < 1. \tag{2}$$

Proof: Since $cl(C)$ is a convex set, by a well-known result (see for instance Rockafellar [15]) $\alpha cl(C) + (1-\alpha) ri(cl(C)) \subseteq ri(cl(C)), \quad \forall\, 0 \leq \alpha < 1$, and this, together with (1) proves the statement. ■

The proof of the next lemma is obvious taking into account the well-known properties of the operators cl and ri.

Lemma 2.3 *Suppose that $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^m$ are almost convex sets. Then $C \times D$ is also almost convex in $\mathbb{R}^n \times \mathbb{R}^m$.*

Lemma 2.4 *Suppose that $C \subseteq \mathbb{R}^n$ is an almost convex set and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Then*
(i) The set $T(C)$ is almost convex;
(ii) $ri(T(C)) = T(ri(C))$.

Proof: (i). Since T is linear we have $T(C) \subseteq T(cl(C)) \subseteq cl(T(C))$ from which we obtain that $cl(T(C)) = cl(T(cl(C)))$. This proves that $cl(T(C))$ is a convex set. Taking the relative interior of both sides, then using the well-known relations $ri(cl(A)) = ri(A)$ and $ri(T(B)) = T(ri(B))$ for any convex sets A and B (see for instance [15]), and the fact that C is an almost convex set, we get the following relations

$$ri(cl(T(C))) = ri(cl(T(cl(C)))) = ri(T(cl(C))) = T(ri(cl(C))) \subseteq T(C), \tag{3}$$

which shows that $T(C)$ is an almost convex set.

(ii). By part (i), (3) and Lemma 2.1, we have that

$$ri(T(C)) = ri(cl(T(C))) = T(ri(cl(C))) = T(ri(C))$$

as claimed. ■

An immediate consequence of Lemma 2.3 and Lemma 2.4 using the linear operator $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $T(x, y) = \alpha x + \beta y$ is given by the observation that

$$ri(\alpha C_1 + \beta C_2) = \alpha ri(C_1) + \beta ri(C_2), \tag{4}$$

for any $\alpha, \beta \in \mathbb{R}$ and $C_i \subseteq \mathbb{R}^n, i = 1, 2$ almost convex sets.

The results above reveal that almost convex sets are in some sense "not so far" from convex sets. However, there are basic properties of convex sets like "the intersection of any family of convex sets is also convex" which almost convex sets fail to possess. The next example shows even more: the intersection of a linear subspace with an almost convex set may not be almost convex.

Example 2.2 *(The intersection of almost convex sets is not almost convex in general.) Take the set C as in Example 2.1 and let $D = \{(0, y) : y \in \mathbb{R}\} \subseteq \mathbb{R}^2$. Then both sets are almost convex (D is even convex, as being a linear subspace), $C \cap D = \{(0, y) : y \in [0, 1] \cap \mathbb{Q})\}$ has $cl(C \cap D)$ convex, but $ri(cl(C \cap D)) \nsubseteq C \cap D$. This shows that $C \cap D$ is not an almost convex set.*

A careful examination of the example above shows that the relative interiors of the two sets (nonempty for each of them) have no common point. As shown by the next result (which can be seen as an extension of Theorem 6.5 of [15]), the "intersection property" holds for an arbitrary family of almost convex sets too, provided their relative interior have a common point.

Theorem 2.1 *Let $C_i \subseteq \mathbb{R}^n$ $(i \in I)$ be almost convex sets satisfying $\cap_{i \in I} ri(C_i) \neq \emptyset$. Then*

(i) $cl(\cap_{i \in I} C_i) = \cap_{i \in I} cl(C_i)$;

(ii) $\cap_{i \in I} C_i$ is almost convex;

If the set I is finite, then also

(iii) $ri(\cap_{i \in I} C_i) = \cap_{i \in I} ri(C_i)$;

Proof: (i). Let $x \in \cap_{i \in I} ri(C_i)$ and take an arbitrary $y \in \cap_{i \in I} cl(C_i)$. Then by Lemma 2.2

$$(1 - \alpha)x + \alpha y \in \cap_{i \in I} ri(C_i), \quad \forall\, 0 \leq \alpha < 1,$$

thus, by letting $\alpha \to 1$ we obtain $y \in cl(\cap_{i \in I} ri(C_i))$. It follows that

$$\cap_{i \in I} cl(C_i) \subseteq cl(\cap_{i \in I} ri(C_i)) \subseteq cl(\cap_{i \in I} C_i) \subseteq \cap_{i \in I} cl(C_i), \tag{5}$$

hence, (i) holds.

(ii). By part (i) $cl(\cap_{i \in I} C_i) = \cap_{i \in I} cl(C_i)$, which shows that the set $cl(\cap_{i \in I} C_i)$ is convex. On the other hand, by (5) we obtain that $cl(\cap_{i \in I} ri(C_i)) = cl(\cap_{i \in I} C_i)$ and thus

$$ri(cl(\cap_{i \in I} ri(C_i))) = ri(cl(\cap_{i \in I} C_i)). \tag{6}$$

As a consequence of Lemma 2.1 and the almost convexity of C_i $(i \in I)$, the sets $ri(C_i)$ are convex. Thus $\cap_{i \in I} ri(C_i)$ is also convex (and, by the hypothesis, nonempty). Applying now again Lemma 2.1 for the convex set $\cap_{i\in I} ri(C_i)$ leads to

$$ri(cl(\cap_{i\in I} ri(C_i))) = ri(\cap_{i\in I} ri(C_i)) \subseteq \cap_{i\in I} ri(C_i) \subseteq \cap_{i\in I} C_i,$$

which, together with (6) proves the assertion (ii).

(iii) By Theorem 6.5 (part two) of [15] applied to $cl(C_i)$ instead of C_i we obtain

$$ri(\cap_{i\in I} cl(C_i)) = \cap_{i\in I} ri(cl(C_i)). \tag{7}$$

The right hand side of this relation equals $\cap_{i\in I} ri(C_i)$. Using (i), (ii) and Lemma 2.1 one gets $ri(\cap_{i\in I} cl(C_i)) = ri(cl(\cap_{i\in I} C_i)) = ri(\cap_{i\in I} C_i)$, and this together with (7) provides (iii). ■

Next we show another important property of almost convex sets. The following result can be seen as an extension of Theorem 6.7 of [15].

Theorem 2.2 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $C \subseteq \mathbb{R}^m$ be an almost convex set such that $T^{-1}(ri(C)) \neq \emptyset$. Then*

$$ri(T^{-1}(C)) = T^{-1}(ri(C)), \quad cl(T^{-1}(C)) = T^{-1}(cl(C)).$$

Proof: Let $D = \mathbb{R}^n \times C$, and let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be the graph of T. By the hypothesis the set $ri(G) \cap ri(D) = G \cap ri(D)$ is nonempty. Thus, by Theorem 2.1 (ii) $G \cap D$ is an almost convex set. We have $T^{-1}(C) = Pr_{\mathbb{R}^n}(G \cap D)$, where $Pr_{\mathbb{R}^n}$ is the projection operator of $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. Since this operator is linear, we obtain by Lemma 2.4

$$ri(T^{-1}(C)) = Pr_{\mathbb{R}^n}(ri(G \cap D)) = Pr_{\mathbb{R}^n}(G \cap ri(D)) = T^{-1}(ri(C)),$$

thus proving the first claim. For the second claim observe that $cl(T^{-1}(C)) \subseteq T^{-1}(cl(C))$ by the continuity of T. The reverse inclusion follows by the obvious relations (using that $G \cap ri(D) = ri(G) \cap ri(D) \neq \emptyset$)

$$cl(T^{-1}(C)) = cl(Pr_{\mathbb{R}^n}(G \cap D)) \supseteq Pr_{\mathbb{R}^n}(cl(G \cap D))$$

$$= Pr_{\mathbb{R}^n}(G \cap cl(D)) = T^{-1}(cl(C)).$$

■

By Theorem 2.2 we immediately obtain the following result.

Corollary 2.1 *If the linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $T^{-1}(ri(C)) \neq \emptyset$ for an almost convex set $C \subseteq \mathbb{R}^m$, then $T^{-1}(C)$ is an almost convex set.*

2.2 Almost convex functions

In this subsection we define the concept of almost convexity for extended real valued functions and for vector valued functions with respect to a set. Also, we show some important properties needed for establishing strong duality results.

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $M \subseteq \mathbb{R}^m$ a nonempty set. Recall that the epigraph of f is defined to be the set $epi(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$ and the effective domain as $dom(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The function f is called *proper* if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $dom(f) \neq \emptyset$.

Define the epigraph of g with respect to the set M as

$$epi_M(g) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y - g(x) \in M\}.$$

Definition 2.2 *The function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be almost convex if $epi(f)$ is an almost convex set (in $\mathbb{R}^n \times \mathbb{R}$). Moreover, the vector-valued function $g : \mathbb{R}^n \to \mathbb{R}^m$ is almost convex with respect to M (shortly M-almost convex) if $epi_M(g)$ is an almost convex set (in $\mathbb{R}^n \times \mathbb{R}^m$).*

We notice that $dom(f)) = Pr_{\mathbb{R}^n}(epi(f))$ with $Pr_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the linear projection operator, is almost convex if f is almost convex as a consequence of Lemma 2.4 (i).

Recall that in Example 2.1 we constructed an almost convex set $C \subseteq R^2$ which is not a convex set. Taking the indicator function δ_C of the set C it is immediate that this function is almost convex but not convex. With respect to vector-valued functions, the set M is usually a convex cone of K (e.g. in optimization theory) and the concept of K-convex functions, defined as having their epigraph a convex set, is widely used within the literature.

One might wonder whether there exist K-almost convex functions without being K-convex, or, in other words, the concept introduced in Definition 2.2 is a proper generalization of K-convexity? The next example provides such a function.

Example 2.3 *(K-almost convex function which is not K-convex) Let $g : \mathbb{R} \to \mathbb{R}^2$ given by*

$$g(x) = \begin{cases} (x, 0), & x \in \mathbb{Q}, \\ (0, 0), & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and let $K = \{(0, 0)\} \cup \{(s, t) \in \mathbb{R}^2 : t > 0\}$. It is obvious that $epi_K(g) = graph(g) + \{0\} \times K$ and

$$graph(g) = \{(x, x, 0) \in \mathbb{R}^3 : x \in \mathbb{Q}\} \cup \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R} \setminus \mathbb{Q}\}.$$

This leads to $epi_K(g) = graph(g) \cup \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. It can be easily seen that this set is almost convex without being convex.

It follows by Definition (2.2) and Lemma 2.1 that for an M-almost convex function $g : \mathbb{R}^n \to \mathbb{R}^m$ the set $ri(epi_M(g))$ is nonempty and convex. The next result establishes an exact formulation of this set.

Lemma 2.5 *Suppose that $g : \mathbb{R}^n \to \mathbb{R}^m$ is an M-almost convex function. Then one has*

$$ri((epi_M(g))) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y - g(x) \in ri(M)\}. \tag{8}$$

Proof: Consider the projection operators $Pr_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ on $\mathbb{R}^n$ and $Pr_{\mathbb{R}^m} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ on $\mathbb{R}^m$.

For an element $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, one has that $(x, y) \in ri((epi_M(g)))$ if and only if $x \in \mathbb{R}^n$ and $y \in Pr_{\mathbb{R}^m}(ri((epi_M(g))) \cap (\{x\} \times \mathbb{R}^m))$. Since by Lemma 2.4 we obtain $Pr_{\mathbb{R}^n}(ri((epi_M(g)) = ri(Pr_{\mathbb{R}^n}(epi_M(g))) = \mathbb{R}^n$, for all $x \in \mathbb{R}^n$ it holds

$$\emptyset \neq ri((epi_M(g))) \cap (\{x\} \times \mathbb{R}^m) = ri((epi_M(g))) \cap ri(\{x\} \times \mathbb{R}^m)$$

$$= ri(epi_M(g) \cap (\{x\} \times \mathbb{R}^m)).$$

Thus by Lemma 2.4 (ii) $(x, y) \in ri((epi_M(g)))$ if and only if $x \in \mathbb{R}^n$ and

$$y \in Pr_{\mathbb{R}^m}(ri((epi_M(g)) \cap (\{x\} \times \mathbb{R}^m)))$$

$$= ri\left(Pr_{\mathbb{R}^m}((epi_M(g)) \cap (\{x\} \times \mathbb{R}^m))\right).$$

Since for $x \in \mathbb{R}^n$, $Pr_{\mathbb{R}^m}((epi_M(g)) \cap (\{x\} \times \mathbb{R}^m)) = g(x) + M$, we get that $(x, y) \in ri((epi_M(g)))$ if and only if $x \in \mathbb{R}^n$ and $y \in ri(g(x)+M) = g(x)+ri(M)$. This concludes the proof. ■

Lemma 2.5 leads to the following result.

Lemma 2.6 *Suppose that $X \subseteq \mathbb{R}^n$ is an almost convex set and $g : \mathbb{R}^n \to \mathbb{R}^m$ is an M-almost convex function. Then one has $ri(g(X)+M) = g(ri(X))+ri(M)$.*

Proof: By using the projection operator $Pr_{\mathbb{R}^m} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, we can write $ri(g(X) + M)$ equivalently as

$$ri(g(X) + M) = riPr_{\mathbb{R}^m}\left(epi_M(g) \cap (X \times \mathbb{R}^m)\right).$$

As shown by the proof of Lemma 2.5 $ri((epi_M(g))) \cap (\{x\} \times \mathbb{R}^m) \neq \emptyset$ for every $x \in \mathbb{R}^n$, thus $ri(epi_M(g)) \cap ri(X \times \mathbb{R}^m)$ is nonempty and by Theorem 2.1 and Lemma 2.4 we get that $epi_M(g) \cap (X \times \mathbb{R}^m)$ is almost convex and moreover

$$riPr_{\mathbb{R}^m}\left(epi_M(g) \cap (X \times \mathbb{R}^m)\right) = Pr_{\mathbb{R}^m}\left(ri(epi_M(g) \cap (X \times \mathbb{R}^m))\right)$$

$$= Pr_{\mathbb{R}^m}\left(ri(epi_M(g)) \cap (ri(X) \times \mathbb{R}^m)\right).$$

But, by the previous lemma it holds $ri(epi_M(g)) \cap (ri(X) \times \mathbb{R}^m) = \{(x, y) : x \in ri(X), y \in g(x) + ri(M)\}$ and so $Pr_{\mathbb{R}^m}\left(ri(epi_M(g)) \cap (ri(X) \times \mathbb{R}^m)\right) = g(ri(X)) + ri(M)$. In conclusion, $ri(g(X) + M) = g(ri(X)) + ri(M)$. ■

It is well-known that any local minimum point of a convex function is also a global minimum point. One might wonder whether this important property still holds for almost convex functions. The next result shows that it is indeed the case.

Theorem 2.3 *Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper almost convex function. If $\bar{x} \in dom(f)$ is a local minimum point of f then it is also a global minimum point of f.*

Proof: Our assumption means that there exists an $\varepsilon > 0$ such that $f(\bar{x}) \leq f(x)$ for every $x \in dom(f) \cap B(\bar{x}, \varepsilon)$, where $B(\bar{x}, \varepsilon)$ is the open ball centered at $\bar{x}$ with radius ε. Supposing the contrary, there exists an element $\bar{y} \in dom(f)$ such that

$$f(\bar{y}) < f(\bar{x}) \tag{9}$$

We have $\bar{y} \in dom(f) \subseteq dom(\bar{f})$, where $\bar{f}$ is the so called *lower semicontinuous hull function of* f, which - in case f is almost convex - is a convex function (see for instance [6]). Therefore $ri(dom(\bar{f})) \neq \emptyset$, thus by choosing an element $\bar{z} \in ri(dom(\bar{f}))$ we obtain by Theorem 6.1 of [15] $t\bar{z}+(1-t)\bar{y} \in ri(dom(\bar{f}))$, $\forall\, 0 < t \leq 1$. Thus, by Theorem 1 of [3] (see also [6]) and the convexity of $\bar{f}$ we obtain

$$f(t\bar{z} + (1-t)\bar{y}) = \bar{f}(t\bar{z} + (1-t)\bar{y}) \leq t\bar{f}(\bar{z}) + (1-t)\bar{f}(\bar{y})$$

$$= \bar{f}(\bar{y}) + t(\bar{f}(\bar{z}) - \bar{f}(\bar{y})) \leq f(\bar{y}) + t(\bar{f}(\bar{z}) - \bar{f}(\bar{y})), \quad \forall\, 0 < t \leq 1. \tag{10}$$

Due to (9) we may choose a (sufficiently small) $\bar{t} > 0$ such that $f(\bar{y}) + \bar{t}(\bar{f}(\bar{z}) - \bar{f}(\bar{y})) < f(\bar{x})$ and such, denoting $z(\bar{t}) = \bar{t}\bar{z} + (1-\bar{t})\bar{y} \in ri(dom(\bar{f}))$ we obtain by (10) that

$$f(z(\bar{t})) < f(\bar{x}). \tag{11}$$

Now since $\bar{x} \in dom(f) \subseteq dom(\bar{f})$, again by Theorem 6.1 of [15] we have $\lambda z(\bar{t}) + (1-\lambda)\bar{x} \in ri(dom(\bar{f})), \quad \forall\, 0 < \lambda \leq 1$, hence by (11)

$$f(\lambda z(\bar{t}) + (1-\lambda)\bar{x}) = \bar{f}(\lambda z(\bar{t}) + (1-\lambda)\bar{x})$$

$$\leq \lambda \bar{f}(z(\bar{t})) + (1-\lambda)\bar{f}(\bar{x}) < \bar{f}(\bar{x}) \leq f(\bar{x}), \quad \forall\, 0 < \lambda \leq 1. \tag{12}$$

Since $\lambda z(\bar{t}) + (1-\lambda)\bar{x} \to \bar{x}$ as $\lambda \to 0$, one may choose $0 < \bar{\lambda} < 1$ such that $x(\bar{\lambda}) = \bar{\lambda} z(\bar{t}) + (1-\bar{\lambda})\bar{x} \in B(\bar{x}, \varepsilon)$ and such, by (12) we obtain that $f(x(\bar{\lambda})) < f(\bar{x})$, contradicting the hypothesis. This completes the proof. ■

As well-known, the property discussed in Theorem 2.3 i.e, "local minima coincide with global minima" is satisfied by quasiconvex functions as well. One might wonder what is the relationship between the classes of almost convex and quasiconvex functions. The next two examples show that none of them is included in the other. First we construct an almost convex function whose domain is a convex set, which is not quasiconvex.

Example 2.4 *(Almost convex function which is not quasiconvex) Let* $C = [0,1] \times [0,1] \in \mathbb{R}^2$ *and define* $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ *as*

$$f(x,y) = \begin{cases} 1, & (x,y) = (0,1/2) \\ 0, & (x,y) \in C \setminus \{(0,1/2)\} \\ +\infty, & (x,y) \notin C. \end{cases}$$

It is easy to see that f *is almost convex. On the other hand,*

$$1 = f\left(\frac{(0,0)+(0,1)}{2}\right) > \max\{f(0,0), f(0,1)\} = 0,$$

showing that f *is not quasiconvex.*

It is obvious that not any quasiconvex function is almost convex. For instance, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{x}, x \geq 0 \\ 0, \quad x < 0 \end{cases}$$

is increasing, and such quasiconvex, but clearly not almost convex.

3 Strong duality for almost convex optimization problems

One of the most fruitful approaches in the duality theory is the one based on the so-called *perturbation theory.* The main idea is to attach to a general optimization problem (notice that every constrained optimization problem may be equivalently written as an optimization problem without constraints, but with a different objective function)

$$(P) \qquad \inf_{x \in \mathbb{R}^n} F(x),$$

where $F : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, a dual one by using the *perturbation function* $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$. We call $\mathbb{R}^m$ the space of the perturbation variables and Φ

has to fulfill the following relation $\Phi(x,0) = F(x), \forall x \in \mathbb{R}^n$. A dual problem to (P) may be defined as follows ([5], [15])

$$(D) \qquad \sup_{y^* \in \mathbb{R}^m} \{-\Phi^*(0, y^*)\},$$

where by Φ^* we denote the conjugate of the function Φ.

From this generalized dual one can obtain for constrained primal problems in particular three dual problems (i.e., Lagrange, Fenchel and Fenchel-Lagrange) by choosing the perturbation function Φ in an appropriate way as done in [16] and [2].

In connection with the perturbation function Φ define the so called *infimal value function* $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ by

$$h(y) = \inf\{\Phi(x, y) : x \in \mathbb{R}^n\}. \tag{13}$$

Obviously, the primal problem (P) can be written as $h(0) = \inf\{\Phi(x, 0) : x \in \mathbb{R}^n\}$, while an easy calculation shows that the dual (D) is the problem

$$\sup_{y^* \in \mathbb{R}^m} \{-h^*(y^*)\},$$

where h^* denotes the conjugate function of h. If we denote by $v(P)$ and $v(D)$ the optimal objective values of the primal and the dual problems, respectively, then it is immediate that $v(D) \leq v(P)$ (*weak duality*). It is well-known that under usual convexity and regularity assumptions the *strong duality* also holds, i.e., $v(P) = v(D)$ and the dual problem admits at least one solution (see for instance [15]). It comes out naturally to investigate whether the strong duality holds for the general problems (P) and (D) if one is weakening the convexity assumptions usually considered in the literature. Next we show that the strong duality result for the above mentioned problems by replacing the convexity with almost convexity still holds.

Theorem 3.1 *Suppose that the function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is almost convex and $0 \in ri(dom(h))$. Then there exists a vector $\overline{y^*} \in \mathbb{R}^m$ such that*

$$h(0) = -h^*(\overline{y^*}). \tag{14}$$

Proof: In case $h(0) = -\infty$ (14) holds trivially for every $y^* \in \mathbb{R}^m$. Therefore, we can assume that $h(0) > -\infty$ and so, by $0 \in ri(dom(h))$ it follows that $h(0)$ is finite. As mentioned within the proof of Theorem 2.3, the function $\bar{h}$ is convex. Moreover (cf. [6] or [3]) $ri(dom(h)) = ri(dom(\bar{h}))$ and $h(0) = \bar{h}(0)$, thus $\bar{h}(0)$ is also finite. It follows by Corollary 7.2.1 of [15] that $\bar{h}$ is proper and since $0 \in ri(dom(\bar{h}))$ we obtain from Theorem 23.4 of [15] that $\partial\bar{h}(0) \neq \emptyset$, which implies the existence of a vector $\overline{y^*} \in \partial\bar{h}(0)$ meaning that

$$\bar{h}(0) + \bar{h}^*(\overline{y^*}) = 0. \tag{15}$$

Since $h(0) = \bar{h}(0)$ and $(\bar{h})^* = h^*$ (15) reduces to (14). ■

Observe that Theorem 3.1 provides strong duality between the primal problem (P) and its dual (D). Indeed, relation (14) implies $v(P) = v(D)$ with $\overline{y^*}$ being a solution of (D).

In the last result almost convexity of the function h plays a crucial role. Therefore it is natural to ask which condition on the function Φ guarantees the almost convexity of h. The next result gives an answer to this question.

Theorem 3.2 *If the function Φ is almost convex, then h is also almost convex.*

Proof: First we show that the set $cl(epi(h))$ is convex. To do so, let us denote by $Pr_{\mathbb{R}^m \times \mathbb{R}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}$ the projection operator of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ on $\mathbb{R}^m \times \mathbb{R}$. Clearly this is a linear operator. Denoting by $epi_S(h)$ the strict epigraph of h, i.e. the set $\{(y, r) \in \mathbb{R}^m \times \mathbb{R} : h(y) < r\}$, it is immediate to check that

$$epi_S(h) \subseteq Pr_{\mathbb{R}^m \times \mathbb{R}}(epi(\Phi)) \subseteq epi(h). \tag{16}$$

Taking into consideration that $cl(epi_S(h)) = cl(epi(h))$, relation (16) leads to $cl(epi(h)) = cl(Pr_{\mathbb{R}^m \times \mathbb{R}}(epi(\Phi)))$. Since $epi(\Phi)$ is almost convex, it follows by Lemma 2.4 (i) that $Pr_{\mathbb{R}^m \times \mathbb{R}}(epi(\Phi)$ is almost convex, hence $cl(epi(h))$ is a convex set. In order to prove the relation $ri(cl(epi(h)) \subseteq epi(h)$, observe that $ri(cl(epi(h))) = ri(cl(Pr_{\mathbb{R}^m \times \mathbb{R}}(epi(\Phi)))) \subseteq Pr_{\mathbb{R}^m \times \mathbb{R}}(epi(\Phi)) \subseteq epi(h)$. This completes the proof. ■

The next result is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.1 *Suppose that the function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is almost convex and $0 \in riPr_{\mathbb{R}^m}(dom(\Phi))$. Then we have strong duality between the problems (P) and (D).*

Proof: By Theorem 3.2 the function h given by (13) is almost convex. Moreover, the obvious equality $dom(h) = Pr_{\mathbb{R}^m}(dom(\Phi))$ implies that $ri(dom(h)) = riPr_{\mathbb{R}^m}(dom(\Phi))$. Thus by Theorem 3.1 we obtain $h(0) = -h^*(\overline{y^*})$. This implies by $h(0) = v(P)$, $h^*(\overline{y^*}) = \Phi^*(0, \overline{y^*})$ and weak duality ($v(D) \leq v(P)$) that $v(D) = v(P)$ and $\overline{y^*}$ is a solution of the dual problem. ■

4 Applications: Lagrange, Fenchel and Fenchel-Lagrange duality

In this section we apply the results of Section 3 to obtain strong duality for almost convex optimization problems in case of different types of dual problems considered within the literature (see for instance [16] and [2]).

4.1 Lagrange and Fenchel-Lagrange duality for almost convex optimization problems

Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $K \subseteq \mathbb{R}^k$ a nonempty convex cone with $K^* := \{k^* \in \mathbb{R}^k : k^{*T}k \geq 0, \forall k \in K\}$ its dual cone. Consider the partial ordering $\leq_K$ induced by K in $\mathbb{R}^k$, namely for $y, z \in \mathbb{R}^k$ we have that $y \leq_K z$, iff $z - y \in K$. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$. The optimization problem which we investigate in this subsection is the following

$$(P^1) \qquad \inf_{x \in G} f(x),$$

where

$$G = \{x \in X : g(x) \leq_K 0\}.$$

In what follows we always suppose that the set $G \cap dom(f)$ is nonempty. We denote by $v(P^1)$ the optimal objective value of (P^1). It is easy to see that in fact (P^1) is a particular case of (the general) primal problem (P): take $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ given by $F(x) = f(x) + \delta_G(x)$, with δ_G the indicator function of the set G.

By giving particular forms for the perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ introduced in Section 3 we obtain two types of dual problems attached to the primal optimization problem (P^1): the Lagrange and the Fenchel-Lagrange dual problem. Using the general duality theorem established in Section 3 (Corollary 3.1), we obtain strong duality results for these types of dual problems. Let us first start with Lagrange duality.

Consider the function $\Phi_L : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ defined by

$$\Phi_L(x, y) = \begin{cases} f(x), & \text{if} \quad x \in X, \quad g(x) \leq_K y, \\ +\infty, & \text{otherwise.} \end{cases} \tag{17}$$

Notice that evaluating $\Phi_L^*(0, y^*)$ yields (cf. [16], [2]) with the definition of the dual problem (D) introduced in Section 3 (with $m = k$ and $\Phi = \Phi_L$) the well-known Lagrange dual problem

$$(D_L) \qquad \sup_{y^* \in K^*} \inf_{x \in X} \{f(x) + (y^*)^T g(x)\}.$$

The next result guarantees the almost convexity of the function Φ_L under some suitable conditions upon X, f and g.

Theorem 4.1 *If $X \subseteq \mathbb{R}^n$ is an almost convex set, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an almost convex function, $g : \mathbb{R}^n \to \mathbb{R}^k$ is a K-almost convex (vector -valued) function and*

$$ri(X) \cap ri(dom(f)) \neq \emptyset, \tag{18}$$

then Φ_L given by relation (17) is an almost convex function.

Proof: Define the linear operator $T : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ given by $T(x, r, y) = (x, y, r)$. Then it is easy to verify that

$$epi(\Phi_L) = T(epi(f) \times \mathbb{R}^k) \cap ((epi_K(g) \times \mathbb{R}) \cap \Big(X \times \mathbb{R}^k \times \mathbb{R}\Big). \tag{19}$$

By Lemmas 2.3 and 2.4 (i) the sets $T(epi(f) \times \mathbb{R}^k)$, $epi_K(g) \times \mathbb{R}$ and $X \times \mathbb{R}^k \times \mathbb{R}$ are almost convex. If we show that

$$ri\left(T(epi(f) \times \mathbb{R}^k)\right) \cap ri\,((epi_K(g) \times \mathbb{R}) \cap ri\left(X \times \mathbb{R}^k \times \mathbb{R}\right) \neq \emptyset$$

or, equivalently,

$$T(ri(epi(f)) \times \mathbb{R}^k) \cap (ri(epi_K(g)) \times \mathbb{R}) \cap \Big(ri(X) \times \mathbb{R}^k \times \mathbb{R}\Big) \neq \emptyset, \tag{20}$$

then the assertion follows by Theorem 2.1 (ii).

To do so, we consider $x' \in ri(dom(f)) \cap ri(X)$, $k' \in ri(K)$ and $y' := g(x') + k'$.

Since $x' \in ri(dom(f)) \subseteq dom(f)$ we may choose a number $r' \in \mathbb{R}$ with $f(x') < r'$. The function f is almost convex, hence one has $ri(dom(f)) = ri(dom(\bar{f}))$ and $f(x') = \bar{f}(x')$ (see [3] or [6]). It is also known (cf. [15]) that $ri(epi(\bar{f})) = \{(x, r) : \bar{f}(x) < r, x \in ri(dom(\bar{f}))\}$, therefore $(x', r') \in ri(epi(\bar{f})) = ri(epi(f))$. Thus $(x', r', y') \in ri(epi(f)) \times \mathbb{R}^k$ and $(x', y', r') \in T(ri(epi(f)) \times \mathbb{R}^k)$.

More than that, by Lemma 2.5 one has $(x', y', r') \in (ri(epi_K(g)) \times \mathbb{R}) \cap \big(ri(X) \times \mathbb{R}^k \times \mathbb{R}\big)$. showing that (20) holds and concluding the proof. ■

Notice that the regularity condition (18) in Theorem 4.1 is essential: if we drop it, the almost convexity of Φ_L cannot be guaranteed, as the next example shows.

Example 4.1 *Consider the set* $C = ([0,2] \times [0,2]) \setminus (\{0\} \times]0,1[)$*, let* $f = \delta_C$*,* $g : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $g(x,y) = -1$ *for all* $(x,y) \in \mathbb{R}^2$*,* $K = \mathbb{R}_+$ *and* $X = \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$*. Then all the assumptions of Theorem 4.1 are satisfied, except (18). Evaluating* $epi(\Phi_L)$ *by formula (19) we obtain*

$$epi(\Phi_L) = (\{(0,0)\} \cup (\{0\} \times [1,2])) \times [-1, +\infty[\times \mathbb{R}_+,$$

which is not an almost convex set.

Assuming relation (18) fulfilled, we say that the problem (P^1) satisfies the *generalized Slater condition* if

$$0 \in ri[g(X \cap dom(f)) + K]$$

or, equivalently, (cf. Lemma 2.6 and Theorem 2.1)

$$0 \in g(ri(X) \cap ri(dom(f))) + ri(K). \tag{21}$$

The result below states strong Lagrangian duality for almost convex functions under suitable assumptions.

Theorem 4.2 *Suppose that the assumptions of Theorem 4.1 and (21) hold. Then the strong Lagrangian duality holds, i.e.* $v(P^1) = v(D_L)$ *and the dual problem admits a solution.*

Proof: We show that the assumptions of Corollary 3.1 hold for $\Phi = \Phi_L$. Almost convexity of Φ_L is guaranteed by Theorem 4.1, so we only have to verify the regularity condition

$$0 \in ri Pr_{\mathbb{R}^k}(dom(\Phi_L)). \tag{22}$$

It is immediate to show that $Pr_{\mathbb{R}^k} dom(\Phi_L) = g(X \cap dom(f)) + K$, and so, (21) is equivalent to (22). ■

Notice that in case the cone K has a nonempty interior (as for instance when $K = \mathbb{R}^k_+$ (the positive orthant of $\mathbb{R}^k$)), the generalized Slater condition (21) reduces to the (usual) *Slater condition*, namely

$$0 \in g(X \cap dom(f)) + int(K), \tag{23}$$

or, equivalently, there exists an element $\hat{x} \in X \cap dom(f)$ such that $g(\hat{x}) \in -int(K)$. Indeed, since in this case $g(X \cap dom(f)) + int(K)$ is an open set we have by Theorem 3.2 of [7] that

$$ri[g(X \cap dom(f)) + K] = int[g(X \cap dom(f)) + K]$$

$$= int[g(X \cap dom(f)) + int(K)] = g(X \cap dom(f)) + int(K).$$

Let us turn now to study Fenchel-Lagrange duality.

Consider the function $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ given by

$$\Phi_{FL}(x,u,y) = \begin{cases} f(x+u), & \text{if} \quad x \in X, \quad g(x) \leq_K y, \\ +\infty, & \text{otherwise.} \end{cases} \tag{24}$$

Observe that evaluating $\Phi^*_{FL}(0, u^*, y^*)$ yields (cf. [16], [2]) with the definition of the dual problem (D) introduced in Section 3 (with $m = n + k$ and $\Phi = \Phi_{FL}$) the Fenchel-Lagrange dual problem

$$(D_{FL}) \qquad \sup_{u^* \in \mathbb{R}^n, y^* \in K^*} \{-f^*(u^*) + \inf_{x \in X} [(u^*)^T x + (y^*)^T g(x)]\}.$$

First we give sufficient conditions for the almost convexity of the function Φ_{FL}.

Theorem 4.3 *If $X \subseteq \mathbb{R}^n$ is an almost convex set, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an almost convex function, $g : \mathbb{R}^n \to \mathbb{R}^k$ is a K-almost convex function and (18) holds, i.e., $ri(X) \cap ri(dom(f)) \neq \emptyset$, then Φ_{FL} given by relation (24) is an almost convex function.*

Proof: Consider the linear operators $V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$ and $W : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ given by $V(x, u, y, r) = (x + u, r, y)$ and $W(x, y, u, r) = (x, u, y, r)$, respectively. Then it can be easily checked that

$$epi(\Phi_{FL}) = V^{-1}(epi(f) \times \mathbb{R}^k) \cap W(epi_K(g) \times \mathbb{R}^n \times \mathbb{R}) \cap (X \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}). \quad (25)$$

By the same arguments as in the proof of Theorem 4.1 it follows that the sets $W(epi_K(g) \times \mathbb{R}^n \times \mathbb{R})$ and $X \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ are almost convex. Since f is almost convex, the set $ri(epi(f)) \times \mathbb{R}^k$ is nonempty, and so, if $(x, r, y) \in ri(epi(f)) \times \mathbb{R}^k$, then $(x, 0, y, r) \in V^{-1}(ri(epi(f) \times \mathbb{R}^k))$, hence $V^{-1}(ri(epi(f) \times \mathbb{R}^k)) \neq \emptyset$. By Corollary 2.1 one gets that the set $V^{-1}(epi(f) \times \mathbb{R}^k)$ is almost convex, too. Therefore the assertion follows by Theorem 2.1 (ii) if we show that

$$ri(V^{-1}(epi(f) \times \mathbb{R}^k)) \cap ri(W(epi_K(g) \times \mathbb{R}^n \times \mathbb{R})) \cap ri(X \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}) \neq \emptyset.$$

Let us choose $x' \in ri(X) \cap ri(dom(f))$, $k' \in ri(K)$, $r' > f(x')$ and define $y' := g(x') + k'$. Then, by the same argument as in the proof of Theorem 4.1, $(x', r') \in ri(epi(f))$. By Lemma 2.5, $(x', y', 0, r') \in ri(epi_K(g)) \times \mathbb{R}^n \times \mathbb{R}$ and so, by Lemma 2.4 (ii) it follows that $(x', 0, y', r') \in W(ri(epi_K(g)) \times \mathbb{R}^n \times \mathbb{R}) = ri(W(epi_K(g) \times \mathbb{R}^n \times \mathbb{R}))$. It is obvious that $(x', 0, y', r') \in ri(X \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R})$. Finally, since $(x', 0, y', r') \in V^{-1}ri((epi(f) \times \mathbb{R}^k))$, by Theorem 2.2 we obtain that $(x', 0, y', r') \in ri(V^{-1}(epi(f) \times \mathbb{R}^k))$. Thus we have found an element belonging to

$$ri(V^{-1}(epi(f) \times \mathbb{R}^k)) \cap ri(W(epi_K(g) \times \mathbb{R}^n \times \mathbb{R})) \cap ri(X \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}),$$

showing that this set is nonempty. This completes the proof. ■

Comparing Theorems 4.1 and 4.3 it can be seen that the same conditions guarantee the almost convexity of Φ_L and Φ_{FL}. As the next result shows, the same conditions guarantee the strong Lagrange and strong Fenchel-Lagrange duality for almost convex functions.

Theorem 4.4 *Suppose that the assumptions of Theorem 4.2 hold. Then the strong Fenchel-Lagrange duality holds, i.e., $v(P^1) = v(D_{FL})$ and the dual problem admits a solution.*

Proof: We show that the assumptions of Corollary 3.1 hold for $\mathbb{R}^n \times \mathbb{R}^k$ instead of $\mathbb{R}^m$ and for $\Phi = \Phi_{FL}$. Almost convexity is guaranteed by Theorem 4.3, so we only have to verify the regularity condition

$$(0, 0) \in riPr_{\mathbb{R}^n \times \mathbb{R}^k}(dom(\Phi_{FL})). \quad (26)$$

To this aim consider the function $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^k$ given by $F(x) = (-x, g(x))$. It is immediate to check that $Pr_{\mathbb{R}^n \times \mathbb{R}^k}(dom(\Phi_{FL})) = F(X) + dom(f) \times K$.

Let us show that F is a $dom(f) \times K$-almost convex function. Indeed,

$$epi_{dom(f) \times K}(F) = \{(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k : (u, y) - F(x) \in dom(f) \times K\}$$

$$= \{(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k : x + u \in dom(f), y - g(x) \in K\},$$

which is nothing else than the domain of Φ_{FL} for the particular case $X = \mathbb{R}^n$. We have shown in Theorem 4.3 that Φ_{FL} is almost convex (for any almost convex set $X \subseteq \mathbb{R}^n$) and its domain being the projection of its epigraph, it follows that $epi_{dom(f)\times K}(F)$ is an almost convex set, i.e., F is a $dom(f) \times K$-almost convex function. Now applying Lemma 2.6 for $\mathbb{R}^n \times \mathbb{R}^k$ instead of $\mathbb{R}^k$, F instead of g and $dom(f) \times K$ instead of M, we obtain that (26) is equivalent to $(0,0) \in F(ri(X)) + ri(dom(f)) \times ri(K)$, which is nothing else than there exists a vector $x' \in ri(X) \cap ri(dom(f))$ such that $g(x') \in -ri(K)$. Since the latter is equivalent to (21), the proof is complete. ■

4.2 Fenchel duality for almost convex optimization problems

Consider the functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^k \to \overline{\mathbb{R}}$ and a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$. Define $\Phi_F : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ by

$$\Phi_F(x, y) = f(x) + g(Ax + y). \tag{27}$$

The primal problem we deal with in this subsection is

$$(P^2) \qquad \inf_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\}.$$

Notice that evaluating $\Phi_F^*(0, y^*)$ yields with the definition of the dual problem (D) introduced in Section 3 (with $m = k$ and $\Phi = \Phi_F$) the well-known Fenchel dual problem

$$(D_F) \qquad \sup_{y^* \in \mathbb{R}^k} \{-f^*(-A^*y^*) - g^*(y^*)\},$$

where A^* denotes the adjoint operator of A. The next result, needed for Fenchel duality, provides sufficient conditions for almost convexity of Φ_F.

Theorem 4.5 *If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^k \to \overline{\mathbb{R}}$ are proper almost convex functions, than the function Φ_F given by (27) is almost convex.*

Proof: Let us consider the linear operators $V : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ and $W : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ defined by $V(x, y, r) = (x, Ax + y, r)$ and $W(x, r, y, s) = (x, y, r + s)$, respectively. Then a simple calculation shows that the epigraph of Φ_F can be evaluated as $epi(\Phi_F) = V^{-1}(W(epi(f) \times epi(g)))$.

Indeed, $(x, y, r) \in epi(\Phi_F) \Leftrightarrow f(x) + g(Ax + y) \leq r \Leftrightarrow ((x, f(x)), (Ax + y, r - f(x))) \in epi(f) \times epi(g) \Leftrightarrow (x, Ax + y, r) \in W(epi(f) \times epi(g)) \Leftrightarrow (x, y, r) \in V^{-1}(W(epi(f) \times epi(g)))$.

By Lemma 2.3 and Lemma 2.4 (i) it follows that $W(epi(f) \times epi(g))$ is an almost convex set, and by Corollary 2.1 we conclude the proof if we show that $V^{-1}(ri(W(epi(f) \times epi(g)))) \neq \emptyset$. Since f and g are almost convex, $ri(epi(f)) \times ri(epi(g)) \neq \emptyset$, thus, by Lemma 2.4 (ii) we obtain $ri(W(epi(f) \times epi(g))) = W(ri(epi(f) \times epi(g))) \neq \emptyset$. Choose an element $(x', y', r') \in ri(W(epi(f) \times epi(g)))$. Then $(x', y' - Ax', r') \in V^{-1}(ri(W(epi(f) \times epi(g))))$ and we are done. ■

Notice that differently to Theorems 4.1 and 4.3 in Theorem 4.5 no regularity condition is needed.

Now let us give sufficient conditions for the Fenchel duality in case of almost convex optimization problems.

Theorem 4.6 *If the assumptions of Theorem 4.5 are satisfied and*

$$ri(dom(g)) \cap A(ri(dom(f))) \neq \emptyset, \tag{28}$$

the strong Fenchel duality holds, i.e., $v(P^2) = v(D_F)$ *and the dual problem admits a solution.*

Proof: By Theorem 4.5 we obtain that Φ_F is almost convex. The result follows by Corollary 3.1 if we show that

$$0 \in riPr_{\mathbb{R}^k}(dom(\Phi_F)). \tag{29}$$

To do this, let us observe that $y \in Pr_{\mathbb{R}^k}(dom(\Phi_F)) \Leftrightarrow \exists x \in \mathbb{R}^n : f(x) + g(Ax + y) < +\infty \Leftrightarrow \exists x \in \mathbb{R}^n : x \in dom(f), Ax + y \in dom(g) \Leftrightarrow \exists x \in \mathbb{R}^n : x \in dom(f), y \in dom(g) - Ax \Leftrightarrow y \in dom(g) - A(dom(f))$. This shows that (29) is equivalent to $0 \in ri(dom(g) - A(dom(f)))$, which is equivalent to (28). ■

Let us finally notice that in Theorem 4.6 we have rediscovered the strong Fenchel duality result for almost convex optimization problems presented in [3] by using a different approach.

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