Infinite symmetric group and bordisms of pseudomanifolds

Alexander A. Gaifullin[1], Yury A. Neretin[2]

We consider a category whose morphisms are bordisms of n-dimensional pseudomanifolds equipped with a certain additional structure (coloring). On the other hand, we consider the product G of $(n+1)$ copies of infinite symmetric group. We show that unitary representations of G produce functors from the category of n-dimensional bordisms to the category of Hilbert spaces and bounded linear operators.

1 Introduction

1.1. The statement. Denote by $S(k)$ the symmetric group of order k, by $S(\infty)$ the group of finitely supported permutations of $S(\infty)$. Consider the product $G = S(\infty)^{n+1}$ of $n+1$ copies of $\mathbb{N}$, consider the diagonal subgroup $K = \mathrm{diag}\,(\infty) \subset G$. Denote by $K(\alpha)$ the stabilizer of elements $1, \dots, \alpha \in \mathbb{N}$ in $K \simeq S(\infty)$. Denote by $K(\alpha) \setminus G/K(\beta)$ double coset spaces.

We show that for any α, β, $\gamma = 0, 1, 2, \dots$ there exists a natural operation ($\circ$-multiplication)

$$K(\alpha) \setminus G/K(\beta) \times K(\beta) \setminus G/K(\gamma) \to K(\alpha) \setminus G/K(\gamma).$$

The operation is associative, thus we get a category $\mathcal{K}$, whose objects are non-negative integers and sets of morphisms $\beta \to \alpha$ are $K(\alpha) \setminus G/K(\beta)$.

Remarks. a) Such phenomena are quite usual for infinite-dimensional ('large') groups, see, e.g., [16], [17], [11], [12]; apparently the first example was discovered by Ismagilov [10]. In particular, the object under the discussion was considered by Olshanski [16] for $n = 1$ and one of the authors [13] for $n = 2$.

b) Take two double cosets $\mathfrak{g} \in K(\alpha) \setminus G/K(\beta)$, $\mathfrak{h} \in K(\beta) \setminus G/K(\gamma)$. Choose their representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$. Obviously, double cosets $K(\alpha)ghK(\beta)$ depend on a choice of g, h. However, 'usually' we fall to one distinguished double coset, namely $\mathfrak{g} \circ \mathfrak{h}$. Precise sense of the word 'usually' is explained in Subsection 3.6. $\lozenge$

We obtain a geometric description of sets $K(\alpha) \setminus G/K(\beta)$, their elements are enumerated by n-dimensional pseudomanifolds equipped with special colorings. In particular, the set $K(0) \setminus G/K(0)$ is an obvious one-to-one correspondence with conjugacy classes of $S(\infty)^n$ with respect to the diagonal subgroup $S(\infty)$. So we get a geometric description of such classes.

We also obtain a geometric description of the product of double cosets, this is an operation similar to a product of bordisms.

[1]Supported by RFBR (project 14-01-92612), by grants of the President of the Russian Federation (NSh-4833.2014.1 and MD-2969.2014.1), and by Dmitri Zimin's "Dynasty" foundation.

[2]Supported by grants FWF, P22122, P25142.

Next, we construct a family of functors from our category to the category of Hilbert spaces and bounded operators. In fact, any unitary representation of G generates such a functor (and vice versa).

1.2. Structure of the paper. Section 2 contains preliminaries on pseudomanifolds and a description of a correspondence between the group $S(k)^{n+1}$ and colored n-dimensional pseudomanifolds with $2k$ cells. Equivalence of the category of double cosets and the category of bordisms is obtained in Section 3. In Section 4 we discuss representations of our category.

2 Pseudomanifolds and symmetric groups

First, we fix several definitions.

2.1. Simplcial cell complexes. Consider a disjoint union $\coprod \Xi_j$ of a finite collection of simplices Ξ_j. We consider a topological quotient space Σ of $\coprod \Xi_j$ with respect to certain equivalence relation. The quotient must satisfy the following properties

a) For any simplex Ξ_i, the tautological map $\xi_i : \Xi_i \to \Sigma$ is an embedding. Therefore we can think of Ξ_i as of a subset of Σ.

b) For any pair of simplices Ξ_i, Ξ_j, the intersection $\xi_i^{-1}\big(\xi_i(\Xi_i)\cap\xi_j(\Xi_j)\big) \subset \Xi_i$ is a union of faces of Ξ_i and the partially defined map

$$\Xi_i \xrightarrow{\xi_i} \Sigma \xrightarrow{\xi_j^{-1}} \Xi_j$$

is affine on each face.

We shall call such quotients *simplicial cell complexes*.

Remark on terminology. There are two similar (and more common) definitions of spaces composed from simplices (see, e.g., [9]). The first one is a more restrictive definition of "a simplicial complex". In this case, a nonempty intersection of two faces is a (unique) face. See examples of simplicial cell complexes, which are not simplicial complexes in Fig.2 and Fig.3.b. A more wide class of simplicial spaces are Δ-complexes, in this case gluing of a simplex with itself along faces is allowed (as for standard 1-vertex triangulations of two-dimensional surfaces), see Fig 1. $\lozenge$

2.2. Pseudomanifolds. A *pseudomanifold* of dimension n is a simplicial cell complex such that

a) Each face is contained in an n-dimensional face. We call n-dimensional faces *chambers.*

b) Each $(n-1)$-dimensional face is contained in precisely two chambers.

See, e.g., [19], [6].

Remark. Any cycle of singular $\mathbb{Z}$-homologies in a topological space can be realized as an image of a pseudo-manifold (this is more-or-less obvious). Recall that there are cycles in manifolds, which cannot be realized as images of manifolds. $\lozenge$

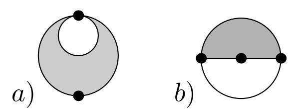

Figure 1: To the definition of simplicial cell complexes. The triangle a) is forbidden, the pair of triangles b) is allowed.

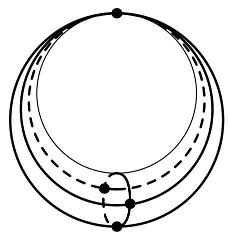

Figure 2: A non-normal two-dimensional pseudomanifold.

REMARK ON TERMINOLOGY. In literature, there exists another variant of a definition of a pseudomanifold. Seifert, Threlfall, [19] impose two additional requirements: a pseudomanifold must be a simplicial complex and must be 'strongly connected'. The latter conditions means that the complement of the union of faces of codimension 2 must be connected. $\lozenge$

2.3. Normal pseudomanifolds and normalization.

Links. Let Σ be a pseudomanifold, let Γ be its k-dimensional face. Consider all $(k+1)$-dimensional faces Φ_j of Σ containing Γ and choose a point φ_j in the relative interior of each face Φ_j. For each face $\Psi_m \supset \Gamma$ we consider the convex hull of all points φ_j that are contained in Ψ_m. The link of Γ is the simplicial cell complex whose faces are such convex hulls.

Normal pseudomanifolds. A pseudomanifold is *normal* if the link of any face of codimension $\geqslant 2$ is connected.

EXAMPLE. Consider a triangulated compact two-dimensional surface Σ. Let a, b be two vertices that are not connected by an edge. Gluing together a and b we get a pseudomanifold which is not normal, see Fig.2. $\lozenge$

Normalization. For any pseudomanifold Σ there is a unique *normalization* ([8]), i.e. a normal pseudomanifold $\widetilde{\Sigma}$ and a map $\pi : \widetilde{\Sigma} \to \Sigma$ such that

— restriction of π to any face of $\widetilde{\Sigma}$ is an affine bijective map of faces.

— the map π send different n-dimensional and $(n-1)$-dimensional faces to different faces.

A construction of the normalization. To obtain a normalization of Σ we cut a pseudomanifold Σ into a disjoint collection of chambers Ξ_i. As above, denote by $\xi_i : \Xi_i \to \Sigma_j$ the embedding of Ξ_i to Σ. Let $x \in \Xi_i$, $y \in \Xi_j$. We say that $x \sim y$ if $\xi_i(x) = \xi_j(y)$ and this point is contained in a common $(n-1)$-dimensional

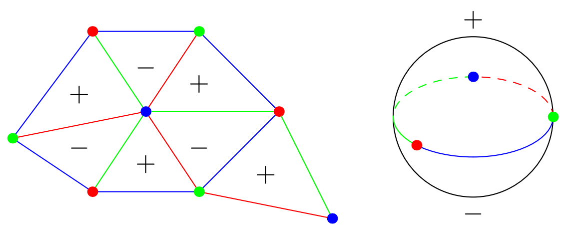

Figure 3: Reference to the definition of colored pseudomanifolds:
a) a colored two-dimensional pseudomanifold;
b) a double chamber.

face of the chambers $\xi_i(\Xi_i)$ and $\xi_j(\Xi_j)$. We extend $\sim$ to an equivalence relation by the transitivity. The quotient of $\coprod \Xi_i$ is the normalization of Σ.

The following way of normalization is more visual. Let Σ be non-normal. Let Ξ be a face of codimension 2 with link consisting of m connected components. Consider a small closed neighborhood $\mathcal{O}$ of Ξ in Σ. Then $\mathcal{O} \setminus \Xi$ is disconnected and consists of m components, say $\mathcal{O}_1, \dots, \mathcal{O}_m$. Let $\overline{\mathcal{O}}_j$ be the closure of $\mathcal{O}_j$ in Σ, $\overline{\mathcal{O}}_j = \mathcal{O}_j \cup \Xi_j$. We replace $\mathcal{O}$ by the disjoint union of $\overline{\mathcal{O}}_j$ and get a new pseudomanifold Σ' (in Fig.2, we duplicate the upper vertex). Then we repeat the same operation to another stratum with disconnected link. These operation enlarges number of strata of codimension $\geqslant 2$, the strata of dimension n and $(n-1)$ remain the same (and the incidence of these strata is preserved). Therefore the process is finite and we get a normal pseudomanifold. $\Diamond$

2.4. Colored pseudomanifolds. Choose $n+1$ colors (say, red, blue, green, orange, etc.). Consider an n-dimensional *normal* pseudomanifold Σ. A coloring of Σ is the following structure

a) To any chamber we assign a sign $(+)$ or $(-)$. Chambers adjacent to plus-chambers are minus-chamber and vise versa.

b) Each vertex of the complex is colored in such a way that the colors of vertices of each chamber are pairwise different.

c) All $(n-1)$-dimensional faces are colored, in such a way that colors of faces of a chamber are pairwise different, and a color of a face coincides with a color of the opposite vertex of any chamber containing this face.

We say that a *double-chamber* is a colored n-dimensional pseudomanifold obtained from two identical copies Δ_1, Δ_2 of an n-dimensional simplex by identification of the corresponding $x \in \Delta_1$, $x \in \Delta_2$ of the boundaries of Δ_1, Δ_2.

Remark. Colored pseudomanifolds were introduced by Pezzana and Ferri in 1975-1976, see [18], [3], [4], [5]. $\Diamond$

2.5. A correspondence between pseudomanifolds and symmetric groups. Denote by $S(L)$ the symmetric group of order L. Denote by

$$S(L)^{n+1} := S(L) \times \cdots \times S(L)$$

the direct product of $n+1$ copies of $S(L)$, we assign $n+1$ colors, say, red, blue, orange, etc., to copies of $S(L)$.

Consider a colored pseudomanifold Σ with $2L$ chambers. We say that a labeling of Σ is a bijection of the set $\{1,2,\dots,L\}$ with the set of plus-chambers of Σ and a bijection of $\{1,2,\dots,L\}$ with the set of minus-chambers of Σ.

Theorem 2.1 *There is a canonical one-to-one correspondence between the group $S(L)^{n+1}$ and the set of all labeled colored normal n-dimensional pseudo-manifolds with $2L$ chambers.*

Remark. This correspondence for $n=2$ was proposed in [13]. Earlier there was a construction of Pezzana–Ferri (1975-1976), [18], [3], [4]. They considered bipartite $(n+1)$-valent graphs whose edges are colored in $(n+1)$ colors, edges adjacent to a given vertex have pairwise different colors. Such graphs correspond to colored pseudomanifolds. In [5]–[7] there was considered an action of free product $\mathbb{Z}_2 * \dots * \mathbb{Z}_2$ of n copies of $\mathbb{Z}_2$ on the set of chambers of a colored pseudomanifold. A construction relative to the present construction was considered in [1]. $\lozenge$

Construction of the correspondence. Indeed, consider a labeled colored normal pseudomanifold Σ with $2L$ chambers. Fix a color (say, blue). Consider all blue $(n-1)$-dimensional faces A_1, A_2, ... Each blue face A_j is contained in the plus-chamber with some label $p(j)$ and in the minus-chamber with some label $q(j)$. We take an element of the symmetric group $S(L)$ setting $p(j) \mapsto q(j)$ for all blue faces A_j. We repeat the same construction for all colors and obtain a tuple $(g^{(1)},\dots,g^{(n+1)}) \in S(L)^{n+1}$.

Conversely, consider an element of the group $S(L)^{n+1}$. Consider L labeled copies of a colored chamber (plus-chambers) and another collection of L labeled copies of the same chamber with another orientation (minus-chambers). Let the blue permutation send $\alpha \mapsto \beta$. Then we glue the the plus-chamber with label α with the minus-chamber with label β along the blue face (preserving colorings of vertices). The same is done for all colors. The obtained pseudomanifold Σ is normal because the normalization procedure from Subsection 2.3 applied to Σ produces Σ itself. $\square$

2.6. The multiplication in symmetric group and pseudomanifolds. Describe the multiplication in $S(L)^{n+1}$ in a geometric language. Consider two labeled colored pseudomanifolds Σ, Ξ. Remove interiors of minus-chambers of Σ remembering a minus-label on each face of a removed chamber, denote the topological space obtained in this way by Σ_-. All $(n-1)$-faces of Σ_- are colored and labeled. In the same way, we remove plus-chambers from Ξ and get a complex Ξ_+. Next, we glue the corresponding faces of Σ_- and Ξ_+ (with coinciding colors and labels according coloring of vertices). In this way, we get a pseudomanifold and consider its normalization.

2.7. Colored quasibordisms. Fix $n \geqslant 1$. We define a category Bor of quasibordisms. Its objects are nonnegative integers. A morphism $\beta \to \alpha$ is the following collection of data

1) A colored n-dimensional normal pseudomanifold (generally, disconnected).

2) An injective map of the set $\{1, 2, \dots, \alpha\}$ to the set of plus-chambers and an injective map of the set $\{1, 2, \dots, \beta\}$ to the set of minus-chambers In other words, we assign labels 1, …, α to some plus-chambers. and labels 1, …, β to some minus-chambers.

We require that each double-chamber has at least one label.

Composition. Let $\Sigma \in \mathrm{Mor}(\beta, \alpha)$, $\Lambda \in \mathrm{Mor}(\gamma, \beta)$. We define their composition $\Sigma \diamond \Lambda$ as follows. Remove interiors of labeled minus-chambers of Σ and interiors of labeled plus-chambers of Λ. Next, for each $s \leqslant \beta$, we glue boundaries of the minus-chamber of Σ with label s with the boundary of the plus-chamber of Λ with label s according the simplicial structure of boundaries and colorings of $(n-1)$-simplices. Next, we normalize the resulting pseudomanifold.

Finally we remove label-less double chambers (such components can arise as a result of gluing of two label-keeping double chambers).

The identity morphism in $\mathrm{Mor}(\alpha, \alpha)$ is a union of α double chambers with coinciding labels on its sides.

Involution. For a morphism $\Sigma \in \mathrm{Mor}(\beta, \alpha)$ we define the morphism $\Sigma^* \in \mathrm{Mor}(\alpha, \beta)$ by changing of signs on chambers. Thus we get an *involution* in the category Bor. For any $T \in \mathrm{Mor}(\beta, \alpha)$, $S \in \mathrm{Mor}(\gamma, \beta)$ we have

$$(S \diamond T)^* = T^* \diamond S^*$$

In the next section we show that this category is equivalent to the category of double cosets.

3 Multiplication of double cosets and quasibordisms

3.1. Symmetric groups. Notation. Denote by $K = S(\infty)$ the group of finitely supported permutations of $\mathbb{N}$. By $\overline{K} = \overline{S}(\infty)$ we denote the group of all permutations of $\mathbb{N}$. Denote by $K(\alpha) \subset K$, $\overline{K}(\alpha) \subset \overline{K}$ the stabilizers of points 1, …, α. We equip $\overline{S}(\infty)$ with a natural topology assuming that the subgroups $K(\alpha)$ are open.

Sometimes we will represent elements of symmetric groups as $0-1$-matrices.

3.2. Multiplication of double cosets. Denote the product of $(n+1)$ copies of $S(\infty)$ by G. Denote by $K \simeq S(\infty)$ the diagonal subgroup in G, its elements have the form $(g, g, \dots, g)$.

Consider double cosets $K(\alpha) \setminus G / K(\beta)$, i.e., elements of G defined up to the equivalence

$$g \sim k_1 g k_2, \qquad k_1 \in K(\alpha),\, k_2 \in K(\beta)$$

We wish to define product of double cosets

$$K(\alpha) \setminus G/K(\beta) \;\times\; K(\beta) \setminus G/K(\gamma) \;\to\; K(\alpha) \setminus G/K(\gamma).$$

For this purpose, define elements $\theta_\sigma[j] \in K(\sigma)$ by

$$\theta_\sigma[j] := \begin{pmatrix} \mathbf{1}_\sigma & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_j & 0 \\ 0 & \mathbf{1}_j & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix},$$

where $\mathbf{1}_j$ denotes the unit matrix of order j.

Proposition 3.1 *Let*

$$\mathfrak{g} \in K(\alpha) \setminus G/K(\beta), \quad \mathfrak{h} \in K(\beta) \setminus G/K(\gamma)$$

be double cosets. Let g, $h \in G$ be their representatives. Then the sequence

$$\mathfrak{r}_j := K(\alpha) \cdot g\theta_\beta[j]h \cdot K(\gamma) \ \in K(\alpha) \setminus G/K(\gamma) \tag{3.1}$$

is eventually constant. The limit value of $\mathfrak{r}_j$ does not depend on a choice of representatives $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Moreover, if g, $h \in S(L)^{n+1} \subset S(\infty)^{n+1}$, then it is sufficient to consider $j = L - \beta$.

We define the product

$$\mathfrak{g} \circ \mathfrak{h} \in K(\alpha) \setminus G/K(\gamma)$$

of double cosets as the limit value of the sequence (3.1).

Proposition 3.2 *The $\circ$-product is associative.*

Remark. Take two double cosets $\mathfrak{g} \in K(\alpha) \setminus G/K(\beta)$, $\mathfrak{h} \in K(\beta) \setminus G/K(\gamma)$. Choose their representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$. Obviously, the double cosets $K(\alpha)ghK(\beta)$ depend on a choice of g, h. However, in a certain sense, 'for almost all' pairs of representatives we get $\mathfrak{g} \circ \mathfrak{h}$. A precise sense of this statement is explained below in Subs. 3.6. Before this, we give an operational formula for the $\circ$-product (which can be regarded as an alternative definition) and formal proofs of Propositions 3.1-3.2. $\lozenge$

3.3. Formula for the product. Represent g as a collection of block matrices $\bigl(g^{(1)}, \dots, g^{(n+1)}\bigr)$ of size

$$\bigl(\alpha + (L-\alpha) + (L-\beta) + \infty\bigr) \times \bigl(\beta + (L-\beta) + (L-\beta) + \infty\bigr),$$

represent h as a collection of block matrices $\bigl(h^{(1)}, \dots, h^{(n+1)}\bigr)$ of size

$$\bigl(\beta + (L-\beta) + (L-\beta) + \infty\bigr) \times \bigl(\gamma + (L-\gamma) + (L-\beta) + \infty\bigr)$$

$$g^{(k)} = \begin{pmatrix} a^{(k)} & b^{(k)} & 0 & 0 \\ c^{(k)} & d^{(k)} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{L-\beta} & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}, \qquad h^{(k)} = \begin{pmatrix} p^{(k)} & q^{(k)} & 0 & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{L-\beta} & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}. \tag{3.2}$$

Then we write a representative of the double coset $\mathfrak{g} \circ \mathfrak{h}$ as

$$(g \circ h)^{(k)} := g \cdot \theta_\beta[L-\beta] \cdot h = \begin{pmatrix} a^{(k)}p^{(k)} & a^{(k)}q^{(k)} & b^{(k)} & 0 \\ c^{(k)}p^{(k)} & c^{(k)}q^{(k)} & d^{(k)} & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}.$$

3.4. Proof of Proposition 3.1. First, we show that the result does not depend on a choice of j Denote

$$\mu = L-\beta, \quad \nu = L-\alpha, \quad \varkappa = L-\gamma.$$

Preserving the previous notation for $g^{(k)}$, $h^{(k)}$, we write

$$(g \cdot \theta_\beta[\mu+j] \cdot h)^{(k)} = \begin{pmatrix} a^{(k)}p^{(k)} & a^{(k)}q^{(k)} & 0 & b^{(k)} & 0 & 0 \\ c^{(k)}p^{(k)} & c^{(k)}q^{(k)} & 0 & d^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_j & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}.$$

This coincides with

$$\begin{pmatrix} \mathbf{1}_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_j & 0 & 0 \\ 0 & 0 & \mathbf{1}_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_j & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \begin{pmatrix} a^{(k)}p^{(k)} & a^{(k)}q^{(k)} & b^{(k)} & 0 & 0 & 0 \\ c^{(k)}p^{(k)} & c^{(k)}q^{(k)} & d^{(k)} & 0 & 0 & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_j & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_j & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \times$$

$$\times \begin{pmatrix} \mathbf{1}_\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_\varkappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_j & 0 \\ 0 & 0 & \mathbf{1}_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}.$$

Next, we show that (3.1) does not depend on the choice of representatives of double cosets. To be definite, replace a collection $\{g^{(k)}\}$ in (3.2) by

$$\begin{pmatrix} \mathbf{1}_\alpha & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \begin{pmatrix} a^{(k)} & b^{(k)} & 0 \\ c^{(k)} & d^{(k)} & 0 \\ 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \begin{pmatrix} \mathbf{1}_\alpha & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \mathbf{1}_\infty \end{pmatrix} = \begin{pmatrix} a^{(k)} & b^{(k)}v & 0 \\ uc^{(k)} & ud^{(k)}v & 0 \\ 0 & 0 & \mathbf{1}_\infty \end{pmatrix}.$$

Then $(g \circ h)^{(k)}$ is

$$\begin{pmatrix} a^{(k)}p^{(k)} & a^{(k)}q^{(k)} & b^{(k)}v & 0 \\ uc^{(k)}p^{(k)} & uc^{(k)}q^{(k)} & ud^{(k)}v & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{1}_\alpha & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & \mathbf{1}_\mu & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \begin{pmatrix} a^{(k)}p^{(k)} & a^{(k)}q^{(k)} & b^{(k)} & 0 \\ c^{(k)}p^{(k)} & c^{(k)}q^{(k)} & d^{(k)} & 0 \\ r^{(k)} & t^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix} \begin{pmatrix} \mathbf{1}_\gamma & 0 & 0 & 0 \\ 0 & \mathbf{1}_\varkappa & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}$$

This completes the proof. □

3.5. Proof of Proposition 3.2. PROOF. Let g, $h \in G$ be as above, and let $w = (w^{(1)}, \dots, w^{(n+1)}) \in G$ be given by

$$w^{(k)} = \begin{pmatrix} x^{(k)} & z^{(k)} & 0 \\ y^{(k)} & u^{(k)} & 0 \\ 0 & 0 & \mathbf{1}_\infty \end{pmatrix}.$$

Evaluating $(g \circ h) \circ w$ and $g \circ (h \circ w)$ we get the matrices

$$\begin{pmatrix} a^{(k)}p^{(k)}x^{(k)} & a^{(k)}p^{(k)}y^{(k)} & a^{(k)}q^{(k)} & b^{(k)} & 0 & 0 \\ c^{(k)}p^{(k)}x^{(k)} & c^{(k)}p^{(k)}y^{(k)} & c^{(k)}q^{(k)} & d^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ r^{(k)}x^{(k)} & r^{(k)}y^{(k)} & t^{(k)} & 0 & 0 & 0 \\ z^{(k)} & u^{(k)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}, \tag{3.3}$$

$$\begin{pmatrix} a^{(k)}p^{(k)}x^{(k)} & a^{(k)}p^{(k)}y^{(k)} & a^{(k)}q^{(k)} & 0 & b^{(k)} & 0 \\ c^{(k)}p^{(k)}x^{(k)} & c^{(k)}p^{(k)}y^{(k)} & c^{(k)}q^{(k)} & 0 & d^{(k)} & 0 \\ r^{(k)}x^{(k)} & r^{(k)}y^{(k)} & t^{(k)} & 0 & 0 & 0 \\ z^{(k)} & u^{(k)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}. \tag{3.4}$$

Both matrices are elements of the double coset containing

$$\begin{pmatrix} a^{(k)}p^{(k)}x^{(k)} & a^{(k)}p^{(k)}y^{(k)} & a^{(k)}q^{(k)} & b^{(k)} & 0 & 0 \\ c^{(k)}p^{(k)}x^{(k)} & c^{(k)}p^{(k)}y^{(k)} & c^{(k)}q^{(k)} & d^{(k)} & 0 & 0 \\ r^{(k)}x^{(k)} & r^{(k)}y^{(k)} & t^{(k)} & 0 & 0 & 0 \\ z^{(k)} & u^{(k)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix}, \tag{3.5}$$

matrix (3.3) is obtained from (3.5) by a permutation of rows, and matrix (3.4) is obtained from (3.5) by a permutation of columns. □

3.6. Concentration of convolutions. A phenomenon of concentration of convolutions for 'large' groups firstly was observed by Olshanski in [15].

Fix α. Let $L \geqslant \alpha$. Let $L < L'$. We regard the symmetric group $S(L)$ as a subgroup in $S(L')$ embedded as $h \mapsto \begin{pmatrix} h & 0 \\ 0 & \mathbf{1} \end{pmatrix}$. Denote by G_L the group $S(L)^{n+1}$; for $L < L'$ we have a canonical embedding

$$i_{L,L'} : G_L \to G_{L'}.$$

To simplify notation, we denote $i_{L,L'}(g)$ by the same symbol g.

Next, we consider a subgroup $K_L(\alpha) \subset G(L)$ defined as a subgroup of the diagonal $S(L) \subset G_L$ consisting of $(\alpha + (L-\alpha)) \times (\alpha + (L-\alpha))$-matrices of the form $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & z \end{pmatrix}$.

Consider the group algebras $\mathbb{C}[G_L]$, equip them by ℓ_1-norms,

$$\| \sum_{g \in G_L} c_g g \|_{\mathbb{C}[G_L]} := \sum_{g \in G_L} |c_g|.$$

Denote by $*$ the convolution in the group algebra. Evidently,

$$\|\psi * \theta\|_{\mathbb{C}[G_L]} \leqslant \|\psi\|_{\mathbb{C}[G_L]} \|\theta\|_{\mathbb{C}[G_L]}.$$

For any L we define an element π_L of the group algebra $\mathbb{C}[K_L(\alpha)]$ by

$$\pi_L := \pi_L^\alpha := \frac{1}{\# K_L(\alpha)} \sum_{k \in K_L(\alpha)} k, \tag{3.6}$$

where $\# X$ denotes the number of elements of a set X.

For any $L \geqslant N$ we define an element $\varphi_L(g)$ of the group algebra $\mathbb{C}[G_L]$ by

$$\varphi_L(g) := \pi_L * g * \pi_L = \frac{1}{\#\bigl(K_L(\alpha)\, g\, K_L(\alpha)\bigr)} \sum_{r \in K_L(\alpha)\, g\, K_L(\alpha)} r. \tag{3.7}$$

Theorem 3.3 *Fix g, $h \in G_N$. For any $\varepsilon > 0$ there exists L_0 such that for any $L > L_0$, we have*

$$\|\varphi_L(g) * \varphi_L(h) - \varphi_L(g \circ h)\|_{\mathbb{C}[G_L]} < \varepsilon.$$

REMARK. Formally, the $\circ$-operation was defined for double cosets. Let us define it for elements of G_N. Let $g = (g^{(1)}, \dots, g^{(n+1)})$, $h = (h^{(1)}, \dots, h^{(n+1)}) \in G_N$. Represent $i_{N,L}(g)$, $i_{N,L}(h)$ as block matrices of size $\alpha + (N-\alpha) + (N-\alpha) + (L - 2N + \alpha)$:

$$g^{(j)} = \begin{pmatrix} a^{(j)} & b^{(j)} & 0 & 0 \\ c^{(j)} & d^{(j)} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \qquad h^{(j)} = \begin{pmatrix} p^{(j)} & q^{(j)} & 0 & 0 \\ r^{(j)} & t^{(j)} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}. \tag{3.8}$$

Then $g \circ h$ is the collection of matrices

$$\begin{pmatrix} a^{(j)}p^{(j)} & a^{(j)}q^{(j)} & b^{(j)} & 0 \\ c^{(j)}p^{(j)} & c^{(j)}q^{(j)} & d^{(j)} & 0 \\ r^{(j)} & t^{(j)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

PROOF. Represent

$$\varphi_L(g) * \varphi_L(h) = \pi_L * g * \pi_L * h * \pi_L = \frac{1}{\#K_L(\alpha)} \sum_{k \in K_L(\alpha)} \pi_L * gkh * \pi_L.$$

We wish to show that a large majority of summands of this sum coincide with $\pi_L(g \circ h)\pi_L$. We write $k \in K_L(\alpha)$ as

$$k = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & u & v_1 & v_2 \\ 0 & w_1 & x_{11} & x_{12} \\ 0 & w_2 & x_{21} & x_{22} \end{pmatrix}. \tag{3.9}$$

Denote by R the set of all matrices $k \in K_L(\alpha)$ such that $u = 0$. Notice that the block u has a fixed size $N - \alpha$, the whole 0–1-matrix $\begin{pmatrix} u & v \\ w & x \end{pmatrix}$ has size $L - \alpha$. Therefore

$$\frac{\#R}{\#K_L(\alpha)} \to 1 \qquad \text{as } L \to \infty.$$

Next, we show that for all $k \in R$ elements gkh are contained in one double coset. Thus we set $u = 0$ and evaluate gkh:

$$g^{(j)}kh^{(j)} = \begin{pmatrix} a^{(j)}p^{(j)} & a^{(j)}q^{(j)} & b^{(j)}v_1 & b^{(j)}v_2 \\ c^{(j)}p^{(j)} & c^{(j)}q^{(j)} & d^{(j)}v_1 & d^{(j)}v_2 \\ w_1r^{(j)} & w_1t^{(j)} & x_{11} & x_{12} \\ w_2r^{(j)} & w_2t^{(j)} & x_{21} & x_{22}. \end{pmatrix} \tag{3.10}$$

We are interested in a double coset containing ghk. For this purpose, consider a tuple $y \cdot gkh \cdot z$, where y, $z \in K_L(\alpha)$ have the form

$$y = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & y_{11} & y_{12} \\ 0 & 0 & y_{21} & y_{22} \end{pmatrix}, \qquad z = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & z_{11} & z_{12} \\ 0 & 0 & z_{21} & z_{22} \end{pmatrix}.$$

Then the tuple (3.10) transforms to a tuple of the same form with new v_1, v_2, w_1, w_2, x_{ij}, namely

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.$$

In this way we can get

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \end{pmatrix}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix}.$$

After this, in the $0-1$-matrix k (see (3.9)), we have $u = 0$, $v_1 = w_1 = \mathbf{1}$, $v_2 = 0$, $w_2 = 0$. This implies $x_{11} = 0$, $x_{12} = 0$, $x_{21} = 0$, and we get

$$y \cdot g^{(j)} k h^{(j)} \cdot z = \begin{pmatrix} a^{(j)}p^{(j)} & a^{(j)}q^{(j)} & b^{(j)} & 0 \\ c^{(j)}p^{(j)} & c^{(j)}q^{(j)} & d^{(j)} & 0 \\ r^{(j)} & t^{(j)} & 0 & 0 \\ 0 & 0 & 0 & x_{22}. \end{pmatrix}$$

Clearly, we can also can make $x_{22} = \mathbf{1}$. Thus for all k with $u = 0$ the product gkh is contained in $K_L(\alpha) \cdot (g \circ h) \cdot K_L(\alpha)$. □

Next, fix α, β, N and $L \geqslant \max(\alpha, \beta, N)$. For any $g \in G_N$ we define an element $\varphi_L^{\alpha,\beta}(g)$ of $\mathbb{C}[G(L)]$ by

$$\varphi_L^{\alpha,\beta}(g) = \pi_L^\alpha * g * \pi_L^\beta.$$

Proposition 3.4 *Fix α, β, γ. Fix g, $h \in G_N$. For any $\varepsilon > 0$ there exists L_0 such that for any $L > L_0$, we have*

$$\|\varphi_L^{\alpha,\beta}(g) * \varphi_L^{\beta,\gamma}(h) - \varphi_L^{\alpha,\gamma}(g \circ h)\|_{\mathbb{C}[G_L]} < \varepsilon.$$

Proof is the same. We must indicate sizes of matrices and write more superscripts and subscripts. □

Another proof of Proposition 3.2 (associativity). Consider two products

$$\varphi_L(g) * \big(\varphi_L(h) * \varphi_L(z)\big) = \big(\varphi_L(g) * \varphi_L(h)\big) * \varphi_L(z).$$

For large L the first convolution is concentrated (up to ε) on $K_L(\alpha) \cdot g \circ (h \circ z) \cdot K_L(\alpha)$ and the second convolution is concentrated on $K_L(\alpha) \cdot (g \circ h) \circ z \cdot K_L(\alpha)$. Thus the two $\circ$-products coincide. □

3.7. Involution. The map $g \mapsto g^{-1}$ induces the map $\mathfrak{g} \mapsto \mathfrak{g}^*$ of double cosets

$$K(\alpha) \setminus G / K(\beta) \to K(\beta) \setminus G / K(\alpha).$$

Evidently, $(\mathfrak{g} \circ \mathfrak{h})^* = \mathfrak{h}^* \circ \mathfrak{g}^*$.

3.8. Correspondence between symmetric groups and pseudomanifolds. Infinite case. We say that an *infinite pseudo-manifold* is a disjoint union of a countable collection of compact pseudomanifolds such that all but a finite number of its components are double-chambers.

We define a colored infinite pseudo-manifold as above. A labeled pseudomanifold is a colored pseudomanifold with a numbering of plus-chambers by natural numbers and a numbering of minus-chambers by natural numbers such that all but a finite number of double-chambers have the same labels on both chambers.

Theorem 3.5 *There is a canonical one-to-one correspondence between the group* $S(\infty)^{n+1}$ *and the set of all labeled colored normal infinite pseudomanifolds.*

The correspondence is given by the same construction obtained as in Subsection 2.5.

3.9. Equivalence of categories.

Theorem 3.6 *The category* $\mathcal{K}$ *of double cosets and the category* Bor *of quasibordisms are equivalent. The equivalence is given by the following construction.*

CORRESPONDENCE $\mathrm{Mor}_{\mathcal{K}}(\beta,\alpha) \longleftrightarrow \mathrm{Mor}_{\mathrm{Bor}}(\beta,\alpha)$. Let $\mathfrak{g} \in K(\alpha) \setminus G/K(\beta)$ be a double coset. Let $g \in \mathfrak{g}$ be its representative. Consider the corresponding labeled colored pseudomanifold. A left multiplication $g \mapsto ug$ by an element $u \in K(\alpha)$ is equivalent to a permutation u of labels $\alpha+1$, $\alpha+2$, ... on plus-chambers. A right multiplication $g \mapsto gv$ by an element $v \in K(\beta)$ is equivalent to a permutation of labels $\beta+1$, $\beta+2$, ... on minus-chambers.

Thus passing to double cosets is equivalent to forgetting labels $> \alpha$ on plus-chambers and labels $> \beta$ on minus-chambers. Notice that all but a finite number of double-chambers are label-less. Such label-less double chambers can be forgotten. Thus we get a quasibordism.

CORRESPONDENCE OF PRODUCTS. Let g, h be representatives of double cosets. Let Σ, Ξ be the corresponding infinite labeled colored pseudomanifolds. Let Σ' correspond to $g\theta_\beta[j]$, where j is large. We multiply $g\theta_\beta[j]$ by h according to the rule in Subsection 2.5.

Notice that minus-chambers of Σ' with labels $> \beta$ are glued with double-chambers. Plus-chambers of Ξ with labels $> \beta$ are also glued with double chambers. Both operations yield a changing of labels on chambers. This means that in fact we glue together only chambers with labels $\leqslant \beta$, in remaining cases we change labels on chambers only. Afterwards we forget all labels which are grater than β and get the operation described in Subsection 2.7. □

4 Representations

Here we construct a family of representations of the group G. This produces representations of the category of double cosets and therefore representations of the category of quasibordisms. The construction is an extension of [13] (where the case $n = 2$ was considered), more ways of constructions of representations of the group G, see in [12], [13].

4.1. The group $\mathbb{G}$**.** We define an 'intermediate' group $\mathbb{G}$,

$$S(\infty)^{n+1} \subset \mathbb{G} \subset \overline{S}(\infty)^{n+1},$$

consisting of tuples $(g_1, \dots, g_{n+1}) \in \overline{S}(\infty)^{n+1}$ such that $g_i g_j^{-1} \in S(\infty)$ for all i, j. Denote by $\mathbb{K} \simeq \overline{S}(\infty)$ the diagonal subgroup consisting of tuples $(g, \dots, g)$. Define the subgroup $\mathbb{K}(\alpha)$ to be the group of all $(h, \dots, h)$, where h fixes $1, \dots, \alpha$. Define the topology on $\mathbb{G}$ assuming that subgroups $\mathbb{K}(\alpha)$ are open.

Obviously, there is the identification of double cosets

$$K(\alpha)\setminus G/K(\beta)\simeq \mathbb{K}(\alpha)\setminus \mathbb{G}/\mathbb{K}(\beta).$$

4.2. A family of representation of $\mathbb{G}$. Consider $(n+1)$ Hilbert spaces[3] V_{red}, V_{orange}, V_{blue}, ... Consider their tensor product

$$\mathcal{V}=V_{red}\otimes V_{blue}\otimes V_{green}\otimes\dots$$

Fix a unit vector $\xi\in\mathcal{V}$. Consider a countable tensor product of Hilbert spaces

$$\mathfrak{V}=(\mathcal{V},\xi)\otimes(\mathcal{V},\xi)\otimes(\mathcal{V},\xi)\otimes\dots= \\ =(V_{red}\otimes V_{blue}\otimes\dots\ ,\xi)\otimes(V_{red}\otimes V_{blue}\otimes\dots\ ,\xi)\otimes\dots \qquad (4.1)$$

(for a definition of tensor products, see [20]). Denote

$$\mathfrak{v}=\xi\otimes\xi\otimes\dots\in\mathfrak{V}.$$

We define a representation ν of $\mathbb{G}$ in $\mathfrak{V}$ in the following way. The 'red' copy of $S(\infty)$ acts by permutations of factors V_{red}. The 'blue' copy S_∞ acts by permutation of factors V_{blue}, etc. Thus we get an action of the group $S(\infty)^{n+1}$. The diagonal $\mathbb{K}=\overline{S}(\infty)$ acts by permutations of factors $\mathcal{V}$.

REMARK. For type I groups H_1, H_2 irreducible unitary representations of $H_1\times H_2$ are tensor products of representations of H_1 and H_2 (see, e.g., [2] 13.1.8). However, $S(\infty)$ is not a type I group. *Representations of $S(\infty)^{n+1}$ constructed above are not tensor products of representations of $S(\infty)$.* ◊

4.3. Representations of the category $\mathcal{K}$. Consider a unitary representation ρ of the group $\mathbb{G}$ in a Hilbert space H. For $\alpha=0,1,2,\dots$ consider the subspace H_α of $\mathbb{K}(\alpha)$-fixed vectors in H. Denote by P_α the operator of orthogonal projection to H_α. Let $\mathfrak{g}\in\mathbb{K}(\alpha)\setminus\mathbb{G}/\mathbb{K}(\beta)$ be a double coset, and let $g\in\mathbb{G}$ be its representative. We define an operator

$$\overline{\rho}(\mathfrak{g}):H_\beta\to H_\alpha$$

by

$$\overline{\rho}(\mathfrak{g})=P_\alpha\rho(g)\Big|_{H_\beta}$$

Theorem 4.1 *The operator $\overline{\rho}(\mathfrak{g})$ does not depend on the choice of a representative $g\in\mathfrak{g}$. For any α, β, γ,*

$$\mathfrak{g}\in\mathbb{K}(\alpha)\setminus\mathbb{G}/\mathbb{K}(\beta),\quad \mathfrak{h}\in\mathbb{K}(\beta)\setminus\mathbb{G}/\mathbb{K}(\gamma)$$

we have

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(\mathfrak{h})=\overline{\rho}(\mathfrak{g}\circ\mathfrak{h})$$

[3] We admit arbitrary, finite-dimensional or infinite-dimensional, separable Hilbert spaces.

See a proof for $n=2$ in [13], the general case is similar. In the next subsection we present an independent proof.

Theorem 4.2 *Let π be a representation of the category $\mathcal{K}$ in Hilbert spaces compatible with the involution and satisfying $\|\pi(\mathfrak{g})\| \leqslant 1$ for all $\mathfrak{g}$. Then π is equivalent to some representation $\overline{\rho}$, where ρ is a unitary representation of $\mathbb{G}$.*

This is a special case[4] of [11], Theorem VIII.1.10.

In particular, for any representation of $\mathbb{G}$ constructed above we obtain a representation of the category of colored quasibordisms.

4.4. Proof of Theorem 4.1. We use notation and statements of Subsection 3.6. The group $\mathbb{G}$ contains a family of subgroups $S(L)^{n+1}$. Therefore the group algebras $\mathbb{C}\bigl[S(L)^{n+1}\bigr]$ act in H. Since the representation ρ is unitary, for any element $\psi \in \mathbb{C}\bigl[S(L)^{n+1}\bigr]$ we have the following upper bound for the operator norm of $\rho(\psi)$:

$$\|\rho(\psi)\| \leqslant \|\psi\|_{\mathbb{C}[S(L)^{n+1}]} \tag{4.2}$$

Denote by $H_{\alpha,L}$ the subspace of $K_L(\alpha)$-fixed vectors. Denote by $P_{\alpha,L}$ the operator of orthogonal projection to $H_{\alpha,L}$. Evidently

$$H_{\alpha,L} \supset H_{\alpha,L+1}, \qquad P_{\alpha,L}\, P_{\alpha,L+1} = P_{\alpha,L+1}.$$

Also,

$$H_\alpha = \cap_L H_\alpha^L.$$

Hence the sequence $P_{\alpha,L}$ tends to P_α in the strong operator topology as $L \to \infty$. Therefore for any $g \in \mathbb{G}$, α, β we have a strong convergence

$$P_{\alpha,L}\rho(g)P_{\beta,L} \xrightarrow{s} P_\alpha\rho(g)P_\beta \qquad \text{as } L \to \infty.$$

Hence, for any g, $h \in \mathbb{G}$ and α, β, γ we have a weak operator convergence

$$P_{\alpha,L}\rho(g)P_{\beta,L} \cdot P_{\beta,L}\rho(h)P_{\gamma,L} \xrightarrow{w} P_\alpha\rho(g)P_\beta \cdot P_\beta\rho(h)P_\gamma \qquad \text{as } L \to \infty. \tag{4.3}$$

On the other hand,

$$P_{\alpha,L} = \rho\bigl(\pi_L^\alpha\bigr)$$

(see 3.6) and we can write (4.3) as

$$\rho\bigl(\varphi_L^{\alpha,\beta}(g)\bigr)\rho\bigl(\varphi_L^{\beta,\gamma}(h)\bigr) \longrightarrow P_\alpha\rho(g)P_\beta \cdot P_\beta\rho(h)P_\gamma \qquad \text{as } L \to \infty,$$

see (3.7).

[4]Consider a representation of $\mathcal{K}$, for each α we have a Hilbert space $H[\alpha]$. The semigroup $\mathrm{Mor}(\alpha,\alpha)$ contains the group $S(\alpha)^{n+1}$ and therefore we have a representation of $S(\alpha)^{n+1}$ in $H[\alpha]$. On the other hand, $\mathrm{Mor}(\alpha,\alpha)$ admits a natural embedding to $\mathrm{Mor}(\alpha+1,\alpha+1)$ (we add two double chambers, one contains the label $\alpha+1$ on plus-side, another contains the label $\alpha+1$ on minus side). It is possible to construct an inductive limit of spaces $H[\alpha]$ and a limit of representations of $S(\alpha)^{n+1}$. A proof is not difficult but a verification of details is long.

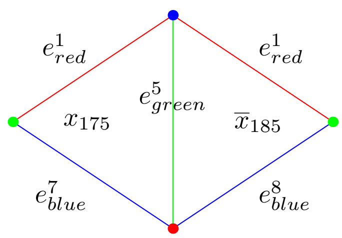

Figure 4: Arrangement of basis elements on a pseudomanifold

Keeping in mind Proposition 3.4 and (4.2) we get

$$\left\|\rho\bigl(\varphi_L^{\alpha,\beta}(g)\bigr)\rho\bigl(\varphi_L^{\beta,\gamma}(h)\bigr)-\rho\bigl(\varphi_L^{\alpha,\gamma}(g\circ h)\bigr)\right\|\longrightarrow 0 \qquad \text{as } L\to\infty.$$

This implies a weak convergence

$$\rho\bigl(\varphi_L^{\alpha,\gamma}(g\circ h)\bigr)\xrightarrow{w} P_\alpha\rho(g)P_\beta\cdot P_\beta\rho(h)P_\gamma.$$

On the other hand,

$$\rho\bigl(\varphi_L^{\alpha,\gamma}(g\circ h)\bigr)\xrightarrow{w} P_\alpha\rho(g\circ h)P_\beta.$$

Comparing the last two convergences, we get the desired statement. $\square$

REMARK. A proof does not use the continuity of a representation ρ with respect to the topology of $\mathbb{G}$, and formally the conclusion of the theorem holds for all unitary representations of $S(\infty)^{n+1}$. However, the continuity on $\mathbb{K}=\overline{S(\infty)}$ is equivalent to the condition: $\cup H_\alpha$ is dense in H (see [15]). In a general case, the space H splits as

$$H=\overline{\cup_\alpha H_\alpha}\oplus\bigl(\cup_\alpha H_\alpha\bigr)^\perp.$$

It is easy to show that these summand are $S(\infty)^{n+1}$-invariant. In the first summand the representation admits a continuation to the group $\mathbb{G}$, in the second summand all operators $\overline{\rho}(\cdot)$ are 0. Thus an extension of a generality makes no sense. $\lozenge$.

4.5. Spherical functions. In the above example we have

$$\mathfrak{V}_\alpha=\underbrace{(\mathcal{V},\xi)\otimes\dots(\mathcal{V},\xi)}_{\alpha\text{ times}}\otimes\xi\otimes\xi\dots\simeq\mathcal{V}^{\otimes\alpha},$$

in particular

$$\mathfrak{V}_0=\mathfrak{v}.$$

We wish to write an explicit formula for the spherical function

$$\Phi(g)=\langle\nu(g)\mathfrak{v},\mathfrak{v}\rangle.$$

Choose an orthonormal basis in each space V_{red}, V_{blue}, V_{green}, etc.

$$e^i_{red} \in V_{red}, \quad e^j_{blue} \in V_{blue}, \quad e^k_{green} \in V_{green}, \ldots$$

This determines the basis

$$e^i_{red} \otimes e^j_{blue} \otimes e^k_{green} \otimes \ldots$$

in $\mathcal{V}$. Expand ξ in this basis,

$$\xi = \sum x_{ijk\ldots} e^i_{red} \otimes e^j_{blue} \otimes e^k_{green} \otimes \ldots \tag{4.4}$$

Consider the double coset $\mathfrak{g}$ containing g and the corresponding colored pseudomanifold Σ. Assign to each $(n-1)$-face an element of the basis of the corresponding color (in arbitrary way). Fix such arrangement. Consider a chamber Δ, on its faces we have certain basis vectors e^i_{red}, e^j_{blue}, e^k_{green}, ... Then we assign the number $x(\Delta) := x_{ijk\ldots}$ (see the last formula) to Δ.

Proposition 4.3

$$\Phi(g) = \sum_{\substack{arangements \\ of\ basis\ elements}} \prod_{plus\text{-}chambers\ \Delta} x(\Delta) \cdot \prod_{minus\text{-}chambers\ \Gamma} \overline{x(\Gamma)}$$

Proof coincides with proof of Proposition 4.2 in [13].

References

[1] Bukhshtaber, V. M.; Gaifullin, A. A. *Representations of m-valued groups on triangulations of manifolds.* Russian Math. Surveys 61 (2006), no. 3, 560-562

[2] Dixmier, J. *Les C^*-algebrés et leurs representations.* Gauthier Villars, 1964.

[3] Ferri, M. *Una rappresentazione delle n-varieta topologiche triangolabili mediante grafi* (n+1)*-colorati.* Boll. Un. Mat. Ital. B (5) 13 (1976), no. 1, 250-260.

[4] Ferri, M.; Gagliardi, C.; Grasselli, L. *A graph-theoretical representation of PL-manifolds: a survey on crystallizations.* Aequationes Math. 31 (1986), no. 2-3, 121-141.

[5] Gaifullin, A. A. *The Manifold of Isospectral Symmetric Tridiagonal Matrices and Realization of Cycles by Aspherical Manifolds*, Proc. Steklov Inst. Math., 263 (2008), 38-56

[6] Gaifullin, A. *Universal realisators for homology classes.* Geom. Topol. 17 (2013), no. 3, 1745-1772.

[7] Gaifullin, A. *Combinatorial realisation of cycles and small covers.* European Congress of Mathematics (Kraków, 2–7 July, 2012), eds. R. Latala et al., European Mathematical Society, 2013, 315–330,

[8] Goresky, M.; MacPherson, R., *Intersection homology theory*, Topology 19 (1980), no. 2, 135-162.

[9] Hatcher, A. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[10] Ismagilov, R.S., *Elementary spherical functions on the groups* $\mathrm{SL}(2, P)$ *over a field* P*, which is not locally compact with respect to the subgroup of matrices with integral elements.* Math. USSR-Izvestiya, 1967, 1:2, 349-380.

[11] Neretin, Yu. A. *Categories of symmetries and infinite-dimensional groups.* Oxford University Press, New York, 1996.

[12] Neretin, Yu. A. *Infinite symmetric group and combinatorial descriptions of semigroups of double cosets.* arXiv:1106.1161

[13] Neretin, Yu. *Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories.* Int. Math. Res. Not. IMRN 2012 (2012), no. 3, 501-523.

[14] Neretin, Yu.A. *A remark on representations of infinite symmetric groups.* J. Math. Sci., New York 190, No. 3, 464-467 (2013); translation from Zap. Nauchn. Semin. POMI 403, 103-109 (2012).

[15] Olshansky, G. I. *Unitary representations of the infinite symmetric group: a semigroup approach.* in *Representations of Lie groups and Lie algebras* (ed. A.A. Kirillov), 181-197, Akad. Kiadó, Budapest, 1985.

[16] Olshanski, G. I. *Unitary representations of* (G, K)*-pairs that are connected with the infinite symmetric group* $S(\infty)$*.* Leningrad Math. J. 1 (1990), no. 4, 983-1014.

[17] Olshanski, G. I. *On semigroups related to infinite-dimensional groups.* Topics in representation theory, 67-101, Adv. Soviet Math., 2, Amer. Math. Soc., Providence, RI, 1991.

[18] Pezzana, M. *Diagrammi di Heegaard e triangolazione contratta.* (Italian) Collection in memory of Enrico Bompiani. Boll. Un. Mat. Ital. (4) 12 (1975), no. 3, suppl., 98-105.

[19] Seifert, H.; Threlfall, W. *Lehrbuch der Topologie.* Leipzig und Berlin: B. G. Teubner. VII, 353., 1934; English transl. Academic Press, 1980.

[20] von Neumann, J. *On infinite direct products.* Compos. Math. 6, 1-77 (1938). Reprinted in von Neumann *Collected works, V.3*, Pergamon Press, 1963, 323-399.

A.Gaifullin: Steklov Mathematical Institute, Moscow, Russia;
MechMath Dept., Moscow State University
Kharkevich Institute for Information Transmission Problems, Moscow, Russia
agaif@mi.ras.ru
Yu.Neretin: Math. Dept., University of Vienna;
Institute for Theoretical and Experimental Physics;
MechMath Dept., Moscow State University
neretin(at)mccme.ru