A COMPLEX SEMIGROUP THAT CONTAINS THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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Our aim is to construct a complex semigroup to which the highest weight representations of the Virasoro algebra (see [6, 5] extend.

0. Let G be a real semisimple Lie group, G_C the corresponding complex group Lie, and C an invariant convex cone in the Lie algebra $\mathfrak{g}$ of G. Then the subset $G\exp(iC)$ of G is an open semigroup.

Now let G be the group Diff of orientation-preserving diffeomorphisms of the circle $S^1 = R/2\pi Z$, let $\mathfrak{g}$ be the corresponding Lie algebra Vect of analytic vector fields on S^1, and let $C \subset$ Vect consist of the fields of the form $a(\varphi)\partial/\partial\varphi$, with $a(\varphi) > 0$. It is natural to expect that the set of formal products $g\cdot\exp(iv)$ with $g \in$ Diff and $v \in C$ will possess a structure of complex semigroup. It turns out that this is indeed the case, despite the fact that the group Diff_C does not exist!

1. Definition of the Semigroup. We realize S^1 as the set $|z| = 1$ in C. Let R(it) denote the map of C into itself which sends z into $e^{-t}z$.

We call element of the subgroup Γ a triplet $(p, R(it), q) \in \Gamma$, with $p, q \in$ Diff, $t > 0$, and $p(1) = 1$.

We let M denote the set of analytic maps ρ of the circle $|z| = 1$ into the disc $|z| < 1$, such that

1) $\rho'(e^{i\varphi})$ does not vanish;

2) $\rho(e^{i\varphi})$ is an anticlockwise oriented Jordan contour.

We call canonical decomposition of $\rho \in M$ a triplet $(p, R(it), q) \in \Gamma$, where p^{-1} maps the annular region bounded by the circle $|z| =1$ and the contour $\rho(e^{i\varphi})$ conformally onto an annulus of the form $\exp(-t) \le |z| \le 1$ (see [3, V, Sec. 1]), and q is determined from the equality of maps $\rho = pR(it)q$, which is meaningful on the circle $|z| = 1$.

LEMMA 1. Suppose $r \in$ Diff admits a univalent continuation to the annulus $\exp(-\alpha) \le |z| \le \exp(\alpha)$ and maps it into the interior of the disc $|z| < \exp(t)$, where $t > 0$. Let $0 < \mu < \alpha$. Put $\rho = R(it)rR(i\mu)$, and let $(p, R(is), q)$ be the canonical decomposition of ρ. Then

a) q admits a univalent continuation to the disc U; $\exp(-\alpha - \mu) \le |z| \le \exp(\alpha + \mu)$;

b) the domain q(U) is contained in the disc $|z| < \exp(s)$;

c) p admits a univalent continuation such that the equality $pR(is)q = R(it)rR(i\mu)$ becomes an equality of maps in the annulus $\exp(-\alpha + \mu) < |z| < \exp(\alpha + \mu)$.

Proof. It is obvious that ρ extends to the annulus $1 \le |z| \le \exp(\alpha + \mu)$. We continue it further by the symmetry principle.

The Exhaustic Procedure PI. Consider a triplet $\gamma = \{R(it), r, R(i\tau)\}$, where $r \in$ Diff, $t > 0$, and $\tau > 0$. We continue r to a univalent function in the annulus U: $\exp(-\alpha) \le |z| \le \exp(\alpha)$, where α is sufficiently small to guarantee that r(U) is contained in the disc $|z| < \exp(t)$. Let $\tau = \mu_1 + \ldots + \mu_k$, with all $\mu_j < \alpha$. Set $q_0 = r$ and $t_0 = t$. Let $(p_j, R(it_j), q_j)$ denote the canonical decomposition of $\rho_{j-1} = R(it_{j-1})q_{j-1}R(i\mu_j)$ (the correctness of this definition is guaranteed by Lemma 1). We correspond to γ the element $PI(\gamma) = (p_1p_2 \cdots p_k, R(it_k), q_k) \in \Gamma$.

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Definition of Multiplication. Let $\rho_1 = (p_1, R(it_1), q_1) \in \Gamma$ and $\rho_2 = (p_2, R(it_2), q_2) \in \Gamma$. We apply to $\{R(it_1), q_1p_2, R(it_2)\}$ an exhaustion procedure, and denote its result with $(u, R(is), v) \in \Gamma$. Then

$$\rho_1\rho_2 = (p_1u, R(is), vq_2).$$

2. Correctness of the Definition. LEMMA 2. Let U be a domain in $\mathbf{R}^n$. Let $\Omega(t) \subset \mathbf{C}$ be a variable two-connected domain depending on $t \in U$. Suppose the boundaries of $\Omega(t)$ are analytic Jordan contours $\varphi_t(s)$ and $\psi_t(s)$, where $s \in [0, 2\pi]$ and that $(\partial/\partial s)\varphi_t(s)$ and $(\partial/\partial_s)\cdot\psi_t(s)$ do not vanish. Suppose the functions $\Phi(t, s) = \varphi_t(s)$ and $\Psi(t, s) = \psi_t(s)$ are jointly continuous in the variables s, t. Let $f_t(z)$ denote the canonical univalent map of $\Omega(t)$ onto an annulus of the form $\lambda(t) \leq |z| \leq 1$, satisfying $f_t(\psi_t(0)) = 1$. Then the function $F(t, z) = f_t(z)$ is jointly analytic in the variables t and z.

Proof. The proof uses Theorem 1 of [3, V.1] and the procedure of reduction to the Dirichlet problem, given in [3, VI.4], and then one argues as in [4, 21.1].

It follows from Lemma 2 that the result of the exhaustion procedure PI depends real-analytically on the input parameters $t, \tau, r, \mu_1, \ldots, \mu_k$, and hence it suffices to verify the one-to-oneness of the output of PI for small τ. Analogously, it suffices to verify the associativity $\rho_1(\rho_2\rho_3) = (\rho_1\rho_2)\rho_3$ in a domain in which the product $\rho_1\rho_2\rho_3$ is meaningful as a composition of maps.

Let $\rho_j(t_1, \ldots, t_k) \in \Gamma$, $j = 1, 2$, and suppose that the ρ_j depend real-analytically on the parameters. Then $\rho_1\rho_2$ depends analytically on $t_1, \ldots, t_k$. In this way Γ is endowed with a structure of real-analytic diffeology in the sense of Souriau [7]. We remark that the chart M permits to introduce on Γ also a holomorphic diffeology.

3. Representations. Let V denote the Virasoro algebra (see [5, 6]), i.e., the algebra with the commutation relations

$$[u_n, u_m] = (m-n)\,u_{n+m} + \frac{1}{12}(n^3 - n)\,\delta_{n,-m}z, \quad [u_n, z] = 0,$$

where $n, m \in \mathbf{Z}$, and $\delta_{k,\ell}$ is the Kronecker symbol. Let L be a unitarizable highest-weight module over V. Let ρ denote the corresponding unitary projective representation of the group Diff (see [8, 6]). We let A_0 denote the self-adjoint operator corresponding to u_0.

THEOREM. Let $(p, R(it), q) \in \Gamma$. Then the formula

$$\hat{\rho}(p, R(it), q) = \rho(p)\exp(tA_0)\,\rho(q)$$

defines a projective representation of Γ by trace-class contractive operators in Hilbert space.

The proof is carried out for L(0, 1) in the construction of [6], and then the result is "multiplied" by means of tensor products in the spirit of [8].

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