UNITARY REPRESENTATIONS OF THE DIFFEOMORPHISM GROUP OF THE p-ADIC PROJECTIVE LINE

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In [3-5], the author has found several new series of unitary representations of the diffeomorphism group of the circle (at the points of degeneracy, these constructions give the representations with highest weight). I. M. Gel'fand has drawn the author's attention to the fact that these constructions must have p-adic analogue. The author expresses gratitude to I. M. Gel'fand and also to G. I. Ol'shanskii and A. V. Zelevinskii.

1. Bruhat–Tits Trees (see [1, 6]). A tree (a connected graph without loops), at each vertex of which p + 1 edges meet, is called a Bruhat–Tits tree J_p. Let $\mathrm{Ver}(J_p)$ be the vertex set of J_p with the natural metric $\rho(\cdot,\cdot)\in\mathbf{Z}$. Let a sequence of vertices $\alpha_1, \alpha_2, \ldots$ on J_p be such that α_i and α_{i+1} are connected by edges of the graph and $\alpha_i \neq \alpha_{i+2}$. Two paths $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ are equivalent if $\alpha_j = \beta_{j+n}$, starting from a certain j. The set of equivalence classes of paths is called the *Absolute* A1. The absolute is naturally identified with the p-adic projective line $\mathbf{Q}_p P^1$. A subgraph $L\subset J_p$ is said to be full if $L\setminus \partial L$ is connected. A full subgraph whose boundary consists of a single point is called a *branch* of J_p. The set of the paths that are contained in a branch M corresponds to an open subset of Ab(M).

We fix a point $w\in \mathrm{Ver}(J_p)$. Let $\ldots\alpha_{-1}, \alpha_0, \alpha_1, \ldots$ be a two-sided path that connects the points z_1 and z_2 of the absolute. Then the expression $|z_1, z_2| = p^{-\min\rho(w,\alpha_j)}$ defines a metric on Ab. We introduce a measure dz on the absolute. Let L be a branch of J_p that does not contain w. Then the measure of the set Ab(L) is equal to $p^{-\rho(w,\partial L)}$. We call each homeomorphism of Ab that is induced by the embedding of graphs $J_p\setminus M \to J_p$, where M is a finite full subgraph of J_p, a *local automorphism* of J_p. The group $\mathscr{L}\mathrm{Aut}(J_p)$ of local automorphisms of J_p is a totally disconnected topological group. It contains the group An of analytic transformations of $\mathbf{Q}_p P^1$ as well as the automorphism group $\mathrm{Aut}(J_p)$ of the tree J_p. The representation theory for the groups $\mathrm{Aut}(J_p)$ has been constructed by Cartier [6] and Ol'shanskii [1].

2. Infinite-Dimensional Pairs (G, K) (see [2]). Let $U(\infty)$, $O(\infty)$, $GL(\infty)$, etc. be, respectively, the full unitary, the full orthogonal, the full linear, etc. groups of operators of a Hilbert space.

Definition. Let G and K be groups of the above type such that $G\supset K$. Then the group (G, K) consists of the operators A that satisfy the following two conditions: 1. $A\in G$. 2. There exists a $V\in K$ such that A–V is a Hilbert–Schmidt operator.

Let us consider the space Ω with the Gaussian measure μ – the extension of the real Hilbert space H by means of the characteristic function $e^{-\langle x,x\rangle}$. We know that the measure μ is quasiinvariant with respect to the group $(GL(\infty, \mathbf{R}), O(\infty))$. Let us define the series Exp_ω of unitary representations of $(GL(\infty, \mathbf{R}), O(\infty))$:

$$\mathbf{Exp}_\omega(g)f(x) = f(g(x))\left(\frac{d\mu(gx)}{d\mu(x)}\right)^{1/2+i\omega},$$

where $d\mu(gx)/d\mu(x)$ is the Radon–Nikodym derivative of the transformation g. The Weil representation $(Sp(2\infty, \mathbf{R}), U(\infty))$ and the spinor representation $(O(2\infty), U(\infty))$ are well known. It has been shown in [2] that there exists a series of other pairs (G, K) that have interesting unitary representations, e.g., $(U(\infty), O(\infty))$ and $(U(2\infty), Sp(\infty))$.

3. We introduce the scalar product

$$\langle f_1, f_2\rangle = \iint |z_1, z_2|^{-\lambda} f_1(z_1) f_2(z_2)\, dz_1 dz_2,$$

$(0 < \lambda < 1)$ in the space H of real-valued functions on $\mathbf{Q}_p P^1 = \mathrm{Ab}(J_p)$. Let $\mathscr{L}\mathrm{Aut}(J_p)$ act in H by the equality

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$$T_\lambda(q) f(z) = f(q(z)) \, |q'(z)|^{1-\lambda/2}.$$

where $|q'(z)|$ is the Radon–Nikodym derivative of the transformation $q \in \mathcal{L}\mathrm{Aut}(J_p)$. The restriction of the representation $T_\lambda(q)$ to $SL_2(\mathbf{Q}_p)$ is a unitary representation S_λ of an additional series (see, e.g., [7]).

THEOREM 1. The operators $T_\lambda(q)$ belong to $(GL(\infty, \mathbf{R}), O(\infty))$.

Thus, we obtain the series $\mathrm{Exp}_\omega(T_\lambda(q))$ of unitary representations of $\mathcal{L}\mathrm{Aut}(J_p)$.

THEOREM 2. The restriction of $\mathrm{Exp}_\omega(T_\lambda(q))$ to $SL_2(\mathbf{Q}_p)$ is $I \oplus \left(\bigoplus_{k\lambda<1} S_{k\lambda}\right) \oplus Q$, where I is the one-dimensional representation and Q is weakly contained in the regular representation of $SL_2(\mathbf{Q}_p)$.

COROLLARY. The cyclic hulls I of each of the subrepresentations $S_{k\lambda}$ are irreducible under the action of $\mathcal{L}\mathrm{Aut}(J_p)$.

4. Embedding of $\mathcal{L}\mathrm{Aut}(J_p)$ in $(U(\infty), O(\infty))$. Let $\mathcal{L}\mathrm{Aut}(J_p)$ act in $L^2(\mathrm{Ab})$ by the equality

$$T_s(q) f(z) = f(q(z)) \, |q'(z)|^{(1+is)/2}. \tag{1}$$

Then the operator

$$I_s f(z) = \int \frac{\overline{f(z)}\, dz}{|z_1, z_2|^{1+is}} + \sqrt{\frac{1-p^{is-1}}{1-p^{is}}}$$

is an antilinear projection on the real subspace of $L^2(\mathrm{Ab})$, which is "almost invariant" under the action (1).

THEOREM 3. Equation (1) defines an embedding of $\mathcal{L}\mathrm{Aut}(J_p)$ in $(U(\infty), O(\infty))$.

Restricting the representations of the Ol'shanskii group $(U(\infty), O(\infty))$ (see Sec. 2), we get a series of unitary representations of $\mathcal{L}\mathrm{Aut}(J_p)$ for which the analog of Theorem 2 is fulfilled.

5. Let $p = 4k + 3$ and K be a subgroup of index 2 in the multiplicative group $\mathbf{Q}_p$ that does not contain (–1). Let An^+ be the group of invertible analytic transformations of $\mathbf{Q}_pP^1$ with derivative that belongs to K. The group $SL_2(\mathbf{Q}_p)$ acts in the space L^2 of odd functions on M (the two-sheeted covering of $\mathbf{Q}_pP^1$) by Eq. (1) (a function on M is said to be odd if it changes sign under permutation of sheets). We can show that this space has an $SL_2(\mathbf{Q}_p)$-invariant quaternion structure. The action of SL_2 can be extended to the action of the semidirect product $An^+ \cdot C(\mathbf{Q}_pP^1, \mathbf{Z}_2)$, where $C(\mathbf{Q}_pP^1, \mathbf{Z}_2)$ is the set of the continuous functions on $\mathbf{Q}_pP^1$ with values in $\mathbf{Z}_2$. It turns out that this construction gives an embedding of $An^+ \cdot C(\mathbf{Q}_pP^1, \mathbf{Z}_2)$ in the unitary "almost quaternion" group $(U(2\infty), Sp(\infty))$. (In addition, the group A_n^+ can be extended to a group of transformations of a tree that "preserve orientation.")

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