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MATHEMATICAL LIFE

Grigori Iosifovich Olshanski (on his 70th birthday)

This paper is a survey of the mathematical work of Grigori Iosifovich Olshanski, the author of fundamental research papers concerning representations of infinite-dimensional groups, determinantal point processes, and multidimensional special functions. We shall also briefly discuss new directions and new opportunities that have arisen in connection with his discoveries.

Olshanski studied in the Faculty of Mechanics and Mathematics of Moscow State University in 1964–1969. In his diploma thesis (see [1]) he obtained Frobenius duality for spaces of type $L^2(G/\Gamma)$, where G is a nilpotent Lie group and Γ is a lattice. In 1972 he finished his postgraduate studies in the Department of the Theory of Functions and Functional Analysis of the faculty, with A. A. Kirillov as his advisor, and in 1973 he defended his Ph.D. thesis, *Representations of reductive groups over local non-Archimedean fields*. He then investigated *representations of the group of automorphisms of Bruhat–Tits trees* in [2], a paper which definitely attracted attention; on trees he also refined and perfected some of the methods that he later used in his investigations of classical groups [4].

Perhaps the main work of Olshanski was **development of the theory of representations of infinite-dimensional classical groups**. His further studies were connected with this work in one way or another, and in many respects it also determined the originality of his views on classical groups, special functions, random processes, and combinatorics. The work was published in a series of papers over the years 1978–1991 (beginning in [3], with the final publications[1] being [19] and [21], and moreover, this was the topic of his D.Sc. thesis [15], defended in 1990 at the Leningrad Branch of the Steklov Mathematical Institute).

[1]These results were partially republished in Neretin's 1996 book [62], but there have been no full republications as yet.

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By the time when Olshanski got involved in activities concerning representations of infinite-dimensional classical groups, many authors had already published significant works on this subject, including M. G. Krein, F. A. Berezin (whose 1965 book [57] was devoted to constructing infinite-dimensional symplectic and orthogonal spinors), D. Shale, F. Stinespring, G. E. Shilov, Fan Dyk Tin', Kirillov, Ş. V. Strătilă, and D. Voiculescu. Works of A. M. Vershik and S. V. Kerov on this topic date back to the early 1980s, and these works became an important point of support for Olshanski's investigations. There were also studies of related topics concerning representations of the infinite symmetric group by E. Toma, A. Liberman, Vershik, and Kerov. The asymptotic theory of symmetric groups was closely related to this subject, beginning with papers by V. L. Goncharov and W. Feller and obtaining new life with a 1977 paper by Vershik and A. A. Schmidt.

However, attempts to work out a coherent picture of representations of infinite-dimensional classical groups ran immediately into questions of what an infinite-dimensional classical group is and what its representations are. At first glance, it seems more natural to consider inductive limits, for example, the group $\mathrm{GL}(\infty, \mathbb{R}) = \lim_{n\to\infty} \mathrm{GL}(n, \mathbb{R})$ of all infinite real *finitary matrices*, that is, matrices g such that $g - 1$ has only finitely many non-zero matrix elements, and similar groups of real orthogonal matrices $\mathrm{O}(\infty)$, unitary matrices $\mathrm{U}(\infty)$, and so on (it is natural to treat these matrices as operators acting in ℓ^2). However, it soon became clear that one can readily construct innumerably many unitary representations of such groups (more than you would want, a lot more), while on the other hand it seems that one cannot say anything indisputably meaningful about these crowds of representations, nor can one find significant connections with other areas of mathematics. One might try to introduce different topologies on the inductive limits and consider the resulting classes of representations in their dependence on these topologies. But there are painfully many candidates for such topologies, and the question of constructing a representation theory depending on the topologies looks extremely uncomfortable. Olshanski found a way out in this very complicated and knotty situation. The key 'hint' was the notion of sphericity.

Let G be a group and let K be a subgroup of it. Recall that K is called a *spherical subgroup* if each irreducible unitary representation ρ of G contains at most one K-fixed vector (up to proportionality); ρ is called a *spherical representation* if such a non-zero vector v does exist. By a *spherical function* one means the corresponding matrix element $\Phi(g) := \langle \rho(g)v, v\rangle$; we assume that $\|v\| = 1$. By a theorem due to Gelfand the maximal compact subgroups of non-compact semisimple Lie groups are spherical (for example, $G = \mathrm{GL}(n, \mathbb{R})$ and $K = \mathrm{O}(n)$), and symmetric subgroups of compact Lie groups are also spherical (for example, $G = \mathrm{U}(n) \times \mathrm{U}(n)$ and $K = \operatorname{diag} \mathrm{U}(n)$, its diagonal subgroup). Altogether, the Gelfand theorem gives 20 series of classical semisimple spherical pairs $G(n) \supset K(n)$, and they are indexed by the semisimple Riemannian symmetric spaces ($G(n)$ is the group of isometries and $K(n)$ is the stabiliser of a point). It is natural to view spherical functions as multidimensional analogues of the Gauss hypergeometric functions (for the Lobachevskii spaces and their complex and quaternion analogues these are ordinary hypergeometric functions). The theory of hypergeometric functions interpolating the spherical functions was constructed by G. J. Heckman and E. M. Opdam, and a broader theory interpolating also p-adic objects was given by I. Macdonald. We note that the characters of compact Lie groups belong to the class of spherical functions and correspond to pairs of the type[2] $G = \mathrm{U}(n) \times \mathrm{U}(n)$, $K = \operatorname{diag}\mathrm{U}(n)$, and so on.

Let us now replace n by ∞ and consider pairs of the type $G(\infty) \supset K(\infty)$. Olshanski suggested a simple and elegant construction using symmetric and orthogonal spinors that gives all spherical representations for all pairs of this kind. For example, note that the spherical representations for the pair $\mathrm{GL}(\infty) \supset \mathrm{O}(\infty)$ are indexed by finite families of real numbers $s_1, \dots, s_k$ (possibly with multiplicities), and the spherical functions are expressed in terms of the singular numbers λ_l of the matrix g by the formula

$$\Phi_{s;a}(g) = |\det(g)|^{ia} \prod_{j=1}^{k} \prod_{l=1}^{\infty} \Big(\frac{1}{2}(1 + is_j)\lambda_l + \frac{1}{2}(1 - is_j)\lambda_l^{-1}\Big)^{-1/2}. \tag{1}$$

Here $a \in \mathbb{R}$ is an additional (trivial and, in a certain sense, parasitic) parameter. The answer for other symmetric pairs $G(\infty) \supset K(\infty)$ has approximately the same form (a double product with respect to parameters and singular numbers with linear factors taken to powers $\pm 1/2$, ± 1, or ± 2). The set of parameters is countable for groups $G(\infty)$ of compact type, and it splits into four different families for, say, the pair $\mathrm{U}(\infty) \times \mathrm{U}(\infty) \supset \operatorname{diag}\mathrm{U}(\infty)$, which plays an important role in our subsequent presentation.[3]

The class of spherical representations, for all its obvious reasonableness, is not closed with respect to the simplest operations like tensor products and restrictions to subgroups, and it clearly had to be expanded. It turned out that every $\mathrm{O}(\infty)$-spherical representation ρ of the group $\mathrm{GL}(\infty, \mathbb{R})$ acting in a Hilbert space H automatically has the following properties.

1°. The restriction of ρ to K is a direct sum of tensor representations[4] of K.

2°. Let $K^\alpha \subset K$ be the subgroup leaving fixed the first α basis vectors in ℓ^2, and let H^α be the space of all K^α-fixed vectors in H. Then $\bigcup H^\alpha$ is dense in H.

3°. The restriction of ρ to K is continuous with respect to the uniform operator topology on K. (The properties 4°–6° below successively strengthen this property.)

4°. The restriction of ρ to K is continuous with respect to the weak operator topology on K.

5°. The representation ρ can be extended to the group of operators of the form $A(1 + T)$ by continuity, where A is an arbitrary orthogonal operator and T is of trace class.

[2]Let ρ be an irreducible unitary representation of the group $\mathrm{U}(n)$ on a space V. Let $\mathrm{End}(V)$ be the space of linear operators on V with the inner product $\langle A, B\rangle := (\dim V)^{-1} \operatorname{tr} AB^*$. The group $\mathrm{U}(n) \times \mathrm{U}(n)$ acts in $\mathrm{End}(V)$ by $(g_1, g_2)\colon A \mapsto \rho(g_1)A\rho(g_2^{-1})$. Then the identity operator is a spherical vector, and the spherical function has the form $\Phi(g_1, g_2) = \operatorname{tr}\rho(g_1 g_2^{-1})$, that is, it is equal to the value of the character at the point $g_1 g_2^{-1}$.

[3]Vershik and Kerov have shown how spherical functions on $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$ are obtained as limits of characters of $\mathrm{U}(n)$ as $n \to \infty$. See also footnotes 9 and 10 below.

[4]Take the standard representation π of the group $\mathrm{O}(\infty)$ on ℓ^2 and expand its tensor powers $\pi^{\otimes n}$ into irreducible subrepresentations. No interesting infinite-dimensional effects occur at this point, and the problem is actually the same as that of decomposing tensor powers for $\mathrm{GL}(N, \mathbb{C})$ (and simpler than the one for $\mathrm{O}(N)$). Representations occurring in these expansions are called *tensor representations*.

6°. If we restrict the representation ρ to the subgroup $\mathrm{SL}(\infty, \mathbb{R})$ of matrices with unit determinant, then the representation obtained can be extended to a representation of the group of operators $A(1+T)$, where A is an orthogonal operator and T is a Hilbert–Schmidt operator.[5,6]

The properties 1°–4° are proved in a rather simple way, while 5° and 6° are obtained as a result of a classification of spherical functions which, in contrast, is very complicated. However, it is more important that the properties 1°–4° turned out to be equivalent as abstract properties of unitary representations of the group G (without the condition of sphericity).[7]

This natural class of representations was called the *admissible representations* by Olshanski,[8] and it became the subject of his investigations. It is also important that the proposed approach enabled him to include in the general picture most of the interesting previous works on this topic.

We have in fact already stated that admissible representations are those representations of the group G which have a 'good' restriction to the subgroup K, where one can take either of the properties 1° and 2° as the definition of a 'good' restriction. On the other hand, if desired, we may assume that we are considering representations of some topological group $\widehat{G} = \widehat{G}(K)$ obtained by completing G. In essence, this is the group described in 6°, but there are other topologies that lead to an equivalent representation theory.

Using this point of view, we obtain 20 classical groups (or group-subgroup pairs) of infinite rank in which one can see the groups of isometries of infinite-dimensional Riemannian symmetric spaces. For example, the infinite-dimensional unitary group $G = \mathrm{U}(\infty)$ appears in the list not by itself but rather together with a subgroup K; namely, we have the pairs

$$\mathrm{U}(\infty) \supset \mathrm{O}(\infty), \quad \mathrm{U}(2\infty) \supset \mathrm{U}(\infty) \times \mathrm{U}(\infty), \quad \mathrm{U}(2\infty) \supset \mathrm{Sp}(\infty),$$

and also $\mathrm{U}(\infty) \times \mathrm{U}(\infty) \supset \operatorname{diag} \mathrm{U}(\infty)$. Different spherical functions and different classes of unitary representations are associated with these pairs. If we agree that we are speaking about representations of topological groups, then these are just different topological groups. Concerning the group $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$, one would like to say that its representations are tensor products of representations of the factors, but an elementary theorem to which one would refer does not work in this case,[9] and the corresponding topological group is not a product of groups.

[5] For example, the representation connected with the spherical function (1) must be corrected for such an extension by setting $a = \sum s_j$; the divergent factor thus obtained extinguishes the divergence of the large product.

[6] Consider the homogeneous (symmetric) space $\mathrm{GL}(\infty, \mathbb{R})/\mathrm{O}(\infty)$ equipped with a natural and (in fact) unique invariant Riemannian metric. We complete it with respect to the geodesic distance and consider the group of isometries of the completion. It turns out that precisely this group is obtained.

[7] In this case the properties 5° and 6° are also equivalent to the properties 1°–4°; as for the statement in full generality (for *all* the pairs $G \supset K$ discussed below), this has not yet been completed.

[8] A problem of this kind at one time arose also for finite-dimensional groups. For example, an attempt to classify irreducible *non-unitary* representations of $\mathrm{SL}(2, \mathbb{R})$ immediately faces the problem of an invariant subspace, and, as W. Soergel noted in 1988, any counterexample to this problem produces a monstrous irreducible representation of $\mathrm{SL}(2, \mathbb{R})$. In the 1950s, Harish-Chandra singled out a class of *admissible representations* for which this kind of disaster does not arise.

Olshanski proved that all pairs (G, K) arising are type I groups (for example, this property of topological groups — which is hard to explain to non-specialists — implies the uniqueness of the decomposition of unitary representations into irreducible representations). He also proposed a simple mass construction of representations of groups (G, K) which is based on symplectic and orthogonal spinors and Howe duality (see the simple brief exposition in [9]). In this theory, spinors replace the parabolic induction used in the representation theory of semisimple Lie groups (introduced by Gelfand, M. A. Naimark, and Harish-Chandra). There is no doubt that the family thus constructed includes all irreducible representations of *all* (G, K)-pairs, but the far-advanced attempt at a proof is not yet complete, and the statement remains unproved in full generality.[10]

The paper [9] was immediately applied by Neretin to the construction of unitary representations of the group of diffeomorphisms of the circle, its combinatorial (tree) analogue, and loop groups. Olshanski [17] applied this approach successfully to the infinite symmetric group, and his student A. Yu. Okounkov carried out a complete classification of admissible representations of the pair $\mathrm{S}(\infty) \times \mathrm{S}(\infty) \supset \operatorname{diag} \mathrm{S}(\infty)$ in his thesis.

To work with infinite-dimensional classical groups, Olshanski developed the so-called **semigroup method**, which in the end led to the construction of algebraic structures of independent interest.

First approach. Consider a *unitary* representation ρ of some group G. Consider also the set of operators $\rho(g)$, where g ranges over G, and take the closure of this set in the weak operator topology.[11] It is easily seen that this closure is compact. If G is a semisimple Lie group, then as shown by R. Howe and C. Moore, we always (with some minor reservations) obtain a one-point compactification.[12] However, for infinite-dimensional groups the compactification should be very different from the group itself. A key object (and a stumbling block discussed for several years at Kirillov's seminar) turned out to be a semigroup arising in the weak closure of

[9] Restricting a spherical representation of this pair to one of the factors, we obtain a representation of the finitary group $\mathrm{U}(\infty)$ in the Murray–von Neumann factor of type II_1. The notion of trace is defined for operators in these factors, and the notion of character is defined for representations in the factors. The classification of characters of an infinite-dimensional symmetric group in factors of type II_1 was the subject of Toma's paper [64], which was one of the first papers on representations of infinite-dimensional groups. Later on, characters of infinite-dimensional classical groups of compact type, $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$, and $\mathrm{Sp}(\infty)$, were the object of investigations by Voiculescu, Vershik, and Kerov. As Olshanski noted, for any group K its characters of type II_1 are the same as the restrictions to one of the factors of spherical functions of $K \times K$ with respect to $\operatorname{diag} K$, so that we obtain equivalent theories.

[10] Over the years 1976–1991 many authors contributed to the proof that the list of spherical functions is complete, including Voiculescu [66], Vershik, Kerov, N. I. Nessonov, and D. Pickrell.

[11] We denote by $\mathscr{B}$ the set of all operators with norm $\leqslant 1$. A sequence in $\mathscr{B}$ converges weakly if all the sequences of matrix elements converge. This makes $\mathscr{B}$ a metrisable compact space, and operator multiplication is separately continuous on $\mathscr{B}$.

[12] There is an interesting classical analogue of the problem of the weak closure. Consider a finite-dimensional irreducible representation ρ of a semisimple Lie group. We take the set of all operators of the form $\mathbb{C} \cdot \rho(g)$ and consider its closure in the space of all linear transformations; this is a conic variety. Let us take its projectivisation. If ρ is in general position (namely, if all the numerical labels on the Dynkin diagram are non-zero), then the result does not depend on ρ. We obtain very funny and non-trivial smooth algebraic varieties known as *complete collineations* and *complete symmetric varieties*; J. G. Semple, C. de Conchini, and C. Procezi have devoted papers to these varieties.

the 'Weil representation' (of symplectic spinors). This is the semigroup of bounded Gaussian operators on the bosonic Fock space, that is, the semigroup of integral operators of the form

$$\mathscr{B}[S]f(z) = \int \exp\Big\{\frac{1}{2}(zKz^t + 2zL\overline{u}^t + \overline{u}M\overline{u}^t)\Big\} f(u)\, e^{-|u|^2}\, du\, d\overline{u},$$

where

$$S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix},$$

$z, u \in \ell_2(\mathbb{C})$, and $S = S^t$ is a symmetric matrix. As I. Segal explained in 1956, an invariant Gaussian measure exists and is concentrated on an extension of the Hilbert space, and f is a holomorphic function on ℓ_2. Olshanski found necessary conditions for the boundedness of these operators ($\|S\| \leqslant 1$, $\|K\| < 1$, $\|M\| < 1$, and K and M are Hilbert–Schmidt operators)[13] and conjectured that these conditions are sufficient. It is easy to see that

$$\mathscr{B}[S_1]\mathscr{B}[S_2] = \det(1 - M_1K_2)^{-1/2}\mathscr{B}[S_3],$$

where

$$S_3 := \begin{pmatrix} K_1 + L_1K_2(1 - M_1K_2)^{-1}L_1^t & L_1(1 - K_2M_2)^{-1}L_2 \\ L_2^t(1 - M_2K_1)^{-1}L_1^t & M_2 + L_2^t(1 - M_1K_2)^{-1}M_1L_2 \end{pmatrix}. \tag{2}$$

In the autumn of 1987, Olshanski and two of the present authors once discussed 12 unsuccessful approaches to the proof of the boundedness conditions for the operators $\mathscr{B}[S]$. The situation was soon clarified [16]: it turned out that the conditions above are nevertheless insufficient (however, already the trace-class condition for K and M is enough), that the elements of the semigroup Γ (that is, Gaussian integral operators) are indexed by Lagrangian linear relations[14] satisfying a certain 'positivity' condition, and that the formula (2) corresponds to multiplication of linear relations. It also turned out that the picture over p-adic and finite fields is parallel to the picture over the real numbers.

Second approach. For definiteness consider the same pair $G = \mathrm{GL}(\infty, \mathbb{R})$, $K = \mathrm{O}(\infty)$. Let $K^\alpha \subset K$ be the same subgroups as above, in the definition of admissibility. Consider the set of double cosets $\Gamma^\alpha := K^\alpha \backslash G / K^\alpha$. It turns out that the following associative multiplication $*$ is defined on Γ^α. We represent g as a block matrix of size $(\alpha + \infty) \times (\alpha + \infty)$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

these matrices are defined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & bv \\ uc & udv \end{pmatrix},$$

[13] In the case when the number of variables is finite, these conditions are sufficient. Olshanski also found formulae for the norms of the operators [25].

[14] A *linear relation* in a linear space V is a subspace of $V \oplus V$; this subspace can be treated as a 'linear operator', which is however not necessarily everywhere defined and can be multivalued.

where u and v are orthogonal matrices. In this notation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} p & q \\ r & t \end{pmatrix} := \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 & q \\ 0 & 1 & 0 \\ r & 0 & t \end{pmatrix} = \begin{pmatrix} ap & b & aq \\ cp & d & cq \\ r & 0 & t \end{pmatrix}. \tag{3}$$

The resulting matrix has size $\alpha + (\infty + \infty) = \alpha + \infty$ and is viewed as an element of the set Γ^α. Moreover, for every admissible representation ρ on a space H the semigroup Γ^α acts on the subspace H^α of all K^α-fixed vectors. Namely, let P^α denote the projection onto H^α. For any $g \in \mathrm{GL}(\infty, \mathbb{R})$ we define the following operator on the space H^α:

$$\widetilde{\rho}_\alpha(g) := P^\alpha \rho(g) P^\alpha.$$

This operator depends only on the double coset (which is obvious), and moreover, the following identity holds:

$$\widetilde{\rho}_\alpha(g_1 * g_2) = \widetilde{\rho}_\alpha(g_1)\widetilde{\rho}_\alpha(g_2)$$

(which is surprising).

Now we make some general comments. The classical representation theory uses several objects of the following type (algebras of Hecke–Iwahori type). Let G be a locally compact group and let L be a compact subgroup of it. Then the convolution algebra of compactly supported *L-bi-invariant functions* on G is well defined (that is, $f(l_1 g l_2) = f(g)$ for any $l_1, l_2 \in L$), in other words, we take functions that are constant on the double cosets of G with respect to L. It is easy to see that for every unitary representation ρ of G the convolution algebra acts on the space of L-invariant vectors (moreover, in the case of an irreducible representation, the initial representation ρ can be recovered from the action if it is non-zero). For this reason, such convolution algebras are a tool (insofar as they are comprehensible, which is not often) for investigating unitary representations; in some cases such algebras live their lives independently of the groups that generated them (like Hecke algebras and affine Hecke algebras). It turned out that in the infinite-dimensional limit the convolution algebras can degenerate into semigroups, and these semigroups often prove to be interesting comprehensible objects. The semigroup structure on the set of double cosets was first discovered by R. S. Ismagilov in 1967, as he considered spherical functions on the group $\mathrm{SL}(2, P)$, where P is a non-Archimedean not locally compact field. Olshanski [4] interpreted Ismagilov's results in terms of the automorphism group of a tree in which countably many edges go out of each vertex. For all the bizarreness of this graph, in a certain sense it is a ('tiny') analogue of the infinite-dimensional Lobachevskii space $\mathrm{SO}_0(1, \infty)/\mathrm{SO}(\infty)$. In any case, the detected effect has already been applied to more familiar objects.

Let us return to the pair $\mathrm{GL} \supset \mathrm{O}$. Olshanski discovered connections between multiplication of double cosets and certain constructions in operator theory. Over the period 1946–1955, M. S. Livshits and V. P. Potapov treated *characteristic functions* as spectral data of operators close to unitary operators (similar investigations were also carried out simultaneously by C. Foiaş and B. Szőkefalvi-Nagy, J. Helton, and others). For simplicity let d be an operator with norm 1 which differs from a unitary operator by a finite-dimensional operator of rank α. Then d can be embedded into some unitary matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $\alpha + \infty$ as a block (this is a simple operation, which can be performed canonically). The *characteristic operator function*

$$\chi(\lambda) := a + \lambda b(1 - \lambda d)^{-1} c \tag{4}$$

is defined in the unit disk $|\lambda| < 1$ in $\mathbb{C}$.[15] Then $\|\chi(\lambda)\| \leqslant 1$, and the divisors of $\chi(\lambda)$ in the class of such holomorphic functions are in a one-to-one correspondence with the invariant subspaces of the operator d.

In operator theory the operation of '*multiplication of operator colligations*' is known, where a 'colligation' is understood to be a conjugacy class of the full infinite-dimensional unitary group with respect to a smaller (but infinite-dimensional) unitary group, that is, the equivalence of unitary matrices has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & bu^{-1} \\ uc & udu^{-1} \end{pmatrix},$$

where u is unitary. Multiplication of 'colligations' is defined by (3). This operation was defined (in final form apparently by V. M. Brodskii in 1971) as an answer to the question of how to interpret multiplication of characteristic functions in the language of operators. Nowadays it is more convenient to understand everything in the reverse order: corresponding to a 'colligation', that is, a conjugacy class, is the characteristic function (4), and corresponding to multiplication of colligations is multiplication of characteristic functions.

In any case, the similarity of the multiplication formulae (3) was glaring,[16] and by calculating the operators $\widetilde{\rho}_\alpha(g)$ explicitly (they are Gaussian integral operators on the bosonic Fock space), Olshanski discovered that their kernels can be written in terms of expressions of the type (4) for certain block matrices a, b, c, and d. Correspondingly, multiplication of the integral operators was rewritten as an operation on the characteristic functions. This multiplication was given by a very cumbersome formula, but a little later the construction was modified by Neretin. Now we describe what is obtained as a result of a natural simplification. Consider the space $\mathbb{C}^{2\alpha}$ with the natural skew-symmetric form $\{\,\cdot\,,\cdot\,\}$ and the natural complex conjugation. Then the indefinite inner product $M(v,w) := i\{v, \overline{w}\}$ is also defined. It turned out that corresponding to each double coset is a meromorphic function $\chi(\lambda)$ on the Riemann sphere with values in the symplectic group $\mathrm{Sp}(2\alpha, \mathbb{C})$, and moreover, for real λ the matrix $\chi(\lambda)$ lies in the real group $\mathrm{Sp}(2\alpha, \mathbb{R})$, while for $\operatorname{Im}\lambda > 0$ it is M-contracting, that is,

$$M(\chi(\lambda)v, \chi(\lambda)v) \leqslant M(v,v) \quad \text{for all } v, \tag{5}$$

and for $\operatorname{Im}\lambda < 0$ it is M-stretching. Corresponding to multiplication in the semigroup Γ^α is pointwise multiplication of the characteristic functions.[17] Then Γ^α becomes an intelligible object and is transformed into a semigroup of holomorphic matrix-valued functions.

[15] Bearing in mind the subsequent discussion, we note that $\chi(\lambda)$ is unitary on the circle $|\lambda| = 1$ for finitary matrices (and of course is meromorphic on the whole of the Riemann sphere).

[16] It should be noted that, while the formulae look outwardly the same, the equivalences are significantly different, and hence the multiplication operations are also different. But similar. . . .

It remains to say how the two approaches to constructing semigroup envelopes are connected: the weak closure can be interpreted as the limit of the semigroups Γ^α as $\alpha \to \infty$.

It turns out that the weak closure of a group can be very unlike the group itself, and semigroup envelopes can be new and interesting algebraic structures. In particular, Neretin discovered the complex envelope of the diffeomorphism group of the circle (the existence of such an envelope was conjectured by Olshanski) and, using the semigroup technique and [16], constructed representations of that group and models of conformal quantum field theory. The semigroup technique can also be applied to some other classes of infinite-dimensional groups: infinite symmetric groups, transformation groups of measure spaces, and infinite-dimensional classical groups over p-adic and finite fields.

Here we mention the associated papers [5], [18], [26], [6], [7], and [8] of Olshanski on semisimple Lie groups, papers which are related to his great work on infinite-dimensional groups. We cannot discuss all these papers and shall only mention briefly those concerned with '**Olshanski semigroups**' [6]–[8] (it is interesting that, in the 1980s and the beginning of the 1990s, he was mainly known as the author of these papers). For a (semisimple) Lie algebra $\mathfrak{g}$ together with a convex cone C in $\mathfrak{g}$ that is invariant under conjugation by elements of the Lie group G, consider all possible curves $\gamma(t)$ in G such that $\gamma(t)^{-1}\gamma'(t) \in C$, together with the points in the group that can be reached from the identity element by motion along these curves. It is easy to see that, locally, a subsemigroup of G is obtained, and the tangent cone to this semigroup at the identity element coincides with C. Olshanski obtained a classification of all such convex cones in semisimple algebras, all global subsemigroups in G generated by convex cones, and also of representations of these semigroups by contracting linear operators on a Hilbert space.[18] He also used these semigroups to separate holomorphic series in L^2-spectra on groups. He became interested in these objects as a tool for classifying admissible unitary representations of groups of 'finite rank' $\mathrm{O}(p,\infty)$, $\mathrm{U}(p,\infty)$, and $\mathrm{Sp}(p,\infty)$, which he accomplished in [10].

Now we return to the picture that existed in infinite-dimensional classical groups after Olshanski's papers. It is perhaps natural to extend the class of representations[19] he introduced. The point is that sphericity, which is a rare phenomenon for Lie groups, is rather ordinary for infinite-dimensional groups. For some time it was unclear whether or not it is possible to work in an extended generality; however,

[17] A rational function $\mathbb{C} \to \mathrm{Sp}(2\alpha,\mathbb{C})$ can be treated as a holomorphic map Ξ of the Riemann sphere into the Lagrangian Grassmannian in $\mathbb{C}^{4\alpha}$ (the graph of an element of the group $\mathrm{Sp}(2\alpha,\mathbb{C})$ is a Lagrangian subspace, and all the singularities are removable). Then for $\operatorname{Im}\lambda > 0$ the form M is non-negative on the subspace $\Xi(\lambda)$, and for $\operatorname{Im}\lambda < 0$ it is non-positive.

[18] The case (to which the term 'Olshanski semigroup' actually refers) when the Lie algebra $\mathfrak{g}$ is a complexification $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ of a semisimple Lie algebra $\mathfrak{h}$ and the cone (wedge) contains $\mathfrak{h}$ is of particular interest. An example of such a semigroup is the subsemigroup of M-contracting matrices in $\mathrm{Sp}(2\alpha,\mathbb{C})$ (see (5)).

[19] It should be mentioned that Olshanski found a surprisingly general abstract theorem [12] about inductive limits $\lim_{\to} G(n) =: G(\infty)$ of locally compact groups. Namely, every matrix element $\langle T(g)v, v\rangle$ of an *irreducible* unitary representation of the group $G(\infty)$ can be approximated, uniformly on compact subsets, by matrix elements $\langle T_n(g)v_n, v_n\rangle$ of *irreducible* representations T_n of the groups $G(n)$. This assertion gives, for example, a new method for proving results in exchangeability theory (see [29]).

Nessonov obtained a series of classification theorems without classical analogues. For example (see [63]), let us take a countable set Ω partitioned into n countable subsets $\Omega_1, \dots, \Omega_n$, the group $G = \mathrm{S}(\Omega)$ of all permutations of Ω with finite support, and the Young subgroup K of it consisting of permutations that take each Ω_j to itself. Then the K-spherical functions on G are indexed by complex positive-definite matrices $A = \{a_{kl}\}$ of size $n \times n$ with 1s on the diagonal, and the spherical functions are given by the formula

$$\Phi_A(g) = \prod_{k,l} a_{kl}^{\tau_{kl}(g)},$$

where $\tau_{kl}(g)$ is the number of elements that g takes from Ω_k to Ω_l. Anyway, going to a more general theory is possible (and takes place step-by-step), but this also means broadening the scope of application of the techniques proposed by Olshanski.

Various problems in harmonic analysis connected with non-compact Lie groups (decomposition of regular and quasi-regular representations, restrictions to subgroups, tensor products) usually involve continuous spectra. In the course of the 1960s–1980s only discrete spectra or representations not of type I were encountered in all problems of any interest associated with infinite-dimensional groups. However, an **infinite-dimensional non-commutative harmonic analysis** does exist, and Olshanski made decisive contributions to its discovery.

It turns out that there is a natural canonical map Υ^n_{n-1} from a symmetric group $\mathrm{S}(n)$ to the *smaller* (!) symmetric group $\mathrm{S}(n-1)$. One of many possible definitions of this map is as follows. Factorize $g \in \mathrm{S}(n)$ into a product of independent cycles, $g = (i_1 i_2 \dots)(j_1 j_2 \dots) \cdots$, and just delete n from this. It is clear that the resulting map is not a homomorphism, but it is in good agreement with the group structure: namely, for any $h_1, h_2 \in \mathrm{S}(n-1)$

$$\Upsilon^n_{n-1}(h_1 g h_2) = h_1 \Upsilon^n_{n-1}(g) h_2. \tag{6}$$

We can now consider the infinite increasing chain

$$\mathrm{S}(1) \xleftarrow{\Upsilon^2_1} \cdots \xleftarrow{\Upsilon^{n-1}_{n-2}} \mathrm{S}(n-1) \xleftarrow{\Upsilon^n_{n-1}} \mathrm{S}(n) \xleftarrow{\Upsilon^{n+1}_n} \cdots, \tag{7}$$

and the projective limit $\mathfrak{S}$ of this chain.[20] It is obvious that for any $p \in \mathrm{S}(n-1)$ its pre-image $(\Upsilon^n_{n-1})^{-1} p \subset \mathrm{S}(n)$ consists of precisely n points, and thus our maps are consistent with the uniform probability distributions on the groups $\mathrm{S}(n)$. Therefore, by the Kolmogorov theorem on inverse limits, there is a probability measure μ_0 on $\mathfrak{S}$ whose projections onto all the $\mathrm{S}(n)$ coincide with the uniform distributions (in fact, there is a natural one-parameter family of measures μ_t on $\mathfrak{S}$, the *Ewens measures*).

This construction arose in connection with W. J. Ewens's works in the 1970s on the distribution of alleles in biological populations. From the late 1970s to the early 1980s it attracted the attention of some mathematicians (J. F. C. Kingman, J. Pitman, D. Aldus, and others). Vershik, Kerov, and Olshanski [23] found that, although the inverse limit is not a group, the infinite symmetric group $\mathrm{S}(\infty)$ still acts on it by left and right multiplications; this can readily be verified by looking at the

[20] This limit (the Chinese restaurant process) admits various constructive descriptions; see [41] and also [59] by Kerov and Tsilevich.

formula (6). The measure μ_0 is invariant under these actions, and the measures μ_t are quasi-invariant. Therefore, we obtain a family of unitary representations of the group[21] $\mathrm{S}(\infty) \times \mathrm{S}(\infty)$ on the spaces $L^2(\mathfrak{S}, \mu_t)$. It is natural to look at these representations as analogues of a regular representation.[22]

Consider now a unitary block matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $m + (n - m)$. We write

$$\Upsilon^n_m(g) := a - b(1 + d)^{-1} c. \tag{8}$$

It turns out that we obtain a map $\mathrm{U}(n) \to \mathrm{U}(m)$ defined almost everywhere (this is not obvious[23]), it commutes with left and right multiplications by elements of $\mathrm{U}(n)$, the image of the Haar measure with respect to Υ^n_m turns out to be the Haar measure, and therefore we can consider the inverse limit of the chain $\mathfrak{U} := \lim_{\leftarrow n} \mathrm{U}(n)$. Neretin constructed a natural two-parameter family of probability measures on $\mathfrak{U}$. In this connection, the problem arises of decomposing the space L^2 on $\mathfrak{U}$ into an integral of spherical representations.

The final expressions for spectral measures are not at all similar to the classical 'Plancherel formulae' for semisimple Lie groups (like the long product of Γ functions obtained by S. G. Gindikin and F. I. Karpelevich). As we saw above, spherical representations depend on a countable family of parameters, and it was necessary to find a measure on some space of countable subsets of the real line. This problem turned out to be very difficult and new approaches had to be found. In trying to calculate the spectral measure, Olshanski understood that the expressions arising in his work resembled formulae known in theoretical physics, in the theory of random matrices. Finally, the problem for the infinite symmetric group was solved in a series of preprints by Olshanski and his student A. M. Borodin in 1998; the final publications are [33] and [39]. The problem for the infinite-dimensional unitary group was also solved by Olshanski and Borodin in 2001 (see [41]).

In both cases the answer was given in the language of *determinantal random point processes*. Such a process (for definiteness, on the line) is given by a kernel $K(x, y)$. The measure on the set Ξ of countable subsets of the line is determined by the following property: the probability that infinitesimal intervals $[x_1, x_1 + dx_1], \dots, [x_n, x_n +$

[21]We mean the group of infinite finitely supported permutations, though in fact it is the completion of the group $\mathrm{S}(\infty) \times \mathrm{S}(\infty)$ (consisting of pairs of permutations (g_1, g_2) such that $g_1 g_2^{-1}$ is finitely supported) that acts on $\mathfrak{S}$.

[22]The representation of the group $\mathrm{S}(\infty) \times \mathrm{S}(\infty)$ on $\ell^2(\mathrm{S}(\infty))$ which seems to be a natural candidate for a regular representation is irreducible, and the restriction to one of the factors gives the Murray–von Neumann factor of type II_1. Formally, such representations can be decomposed into an integral of irreducible representations, but this problem is drastically pathological. Nothing like the classical statements about regular representations of finite or compact groups (G. Frobenius, F. E. Molin, F. Peter, and H. Weyl) is obtained, and nothing interesting occurs in general. For countable discrete groups these facts were known for a long time.

[23]However, it is obvious that the expression (8) is a special case of (4), and the statement formulated above is a special case of the result mentioned in footnote 15 and discovered by Livshits in the 1940s. On the other hand, the graph of an element $g \in \mathrm{U}(n)$ is a subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$ which is isotropic with respect to the Hermitian form with matrix $i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One can see that Υ^n_m is a map of Grassmannians that is given by some linear relation (see footnote 14). We also note that the strange calculations of integrals in Hua Loo-Keng's well-known book [58] are explained by such 'degenerate symmetries'.

$dx_n]$ have non-empty intersections with a set $\xi \in \Xi$ is equal to

$$\det_{\alpha,\beta} K(x_\alpha, x_\beta)\, dx_1 \cdots dx_n.$$

The kernels are expressed in terms of degenerate hypergeometric functions for the symmetric group and in terms of the Gauss hypergeometric functions for the unitary group.

Currently we know a certain zoo of natural problems in infinite-dimensional harmonic analysis. The inverse limits of the type $\lim_{\leftarrow n} \mathrm{U}(n)$ exist for all the ten series of classical compact symmetric spaces. To see them from the formula (8) one should consider unitary matrices over $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$ and look through all the possible symmetry conditions $g = \pm g^t$ and $g = \pm g^*$ for matrices. The first inverse limit of compact Riemannian symmetric spaces was constructed by Pickrell in 1987 (complex Grassmannians). There are variations of the above construction with the symmetric group (a harmonic analysis problem on one of these spaces, an inverse limit of combinations of pairs, was considered by E. Strakhov). Neretin also constructed inverse limits of Grassmannians over p-adic fields and similar constructions for infinite-dimensional Grassmannians and flag varieties over finite fields. So far it is not clear whether the possibilities of harmonic analysis are limited to calculating spectral measures or whether they will go beyond that. In any case, a new field of activity has arisen with the solution of problems connected with $\mathfrak{S}$ and $\mathfrak{U}$.

At the end of the 1990s, the interests of Olshanski himself turned towards probability theory and **determinantal processes**, which he began to investigate building on representation theory. We recall that an example of a determinantal random point process was first discovered in 1962 by the physicist F. Dyson. He considered the distribution of eigenvalues of unitary matrices of order N as $N \to \infty$. It is clear that, in the limit, the eigenvalues 'sit' on the unit circle, densely and uniformly. However, if we extend the circle at the same time, then we obtain on the line a determinantal process which is given by the kernel

$$K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Later on, determinantal expressions were obtained for other distributions of the eigenvalues of matrices (for example, in the mid 1990s, C. Tracy and H. Widom obtained asymptotics of the type 'at the edge of the spectrum'). Kernels for a wide class of determinantal processes (and many expressions have accumulated over the past 25 years) have the form

$$\frac{f(x)g(y) - g(x)f(y)}{x-y},$$

where f and g are some functions. Expressions of similar structure occur also in the case of analysis on the spaces $\mathfrak{S}$ and $\mathfrak{U}$.

The work of Olshanski and Borodin from the late 1990s has been continued by many authors, and this has led to an explosive introduction of determinantal processes into mathematics (the term *determinantal point process* itself was introduced by the same authors). It turned out that these processes arise surprisingly often in a variety of problems in science. The formalism of determinantal processes was used by Borodin, Okounkov, and Olshanski to prove the well-known conjecture by J. Baik, P. Deift, and-K. Johansson concerning the asymptotic behaviour of the lengths of several largest increasing subsequences of a random permutation of large size [37]. In [49] Olshanski established the quasi-invariance of a determinantal process with gamma kernel under the action of the group of finitely supported permutations, and raised the question of the level of generality for which this kind of quasi-invariance holds. Together with his student V. E. Gorin, Olshanski constructed a q-analogue of the theory for $\mathfrak{U}$ (see [52]), and this direction of research is now being actively developed. Olshanski created an extremely elegant and effective method for asymptotic analysis of determinantal processes, based on the spectral theory of difference operators [44], [45]. His method has been used many times in the analysis of various measures on partitions and planar partitions. In a joint paper with Borodin he showed how determinantal processes lead to the asymptotic behaviour of a system of interacting particles that is known as ASEP (asymmetric simple exclusion process) [54], but was previously regarded as inaccessible for determinantal (or free-fermion) analysis. A theory of Markov processes preserving determinantal processes associated with classical special functions was also constructed [42], [50].

Another combinatorial-probabilistic aspect of Olshanski's work over the last 20 years is the **study of branching graphs and their boundaries.** This direction of research began with the work of Vershik and Kerov on the 'Young graph' and characters of the infinite-dimensional symmetric subgroup [65]. However, in Olshanski's hands these graphs began to take on their own lives, far away from the original motivations.

For definiteness, consider the *Gelfand–Tsetlin* graph. We recall that the irreducible representations $\rho_{\boldsymbol{\alpha}}$ of the group $\mathrm{U}(n)$ are indexed by *signatures* $\boldsymbol{\alpha}$, that is, sets of integers of the form $\alpha_1 \leqslant \cdots \leqslant \alpha_n$. The restriction of the representation $\rho_{\boldsymbol{\alpha}}$ to $\mathrm{U}(n-1)$ is the direct sum of all $\rho_{\boldsymbol{\beta}}$ such that

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{n-1} \leqslant \alpha_n$$

(in this case we write $\boldsymbol{\alpha} \downarrow \boldsymbol{\beta}$). The vertices of the Gelfand–Tsetlin graph are divided with respect to the levels $n = 0, 1, 2, \dots,$ the vertices on level n are indexed by signatures $\boldsymbol{\alpha}$, and edges connect only vertices of adjacent levels, under the condition $\boldsymbol{\alpha} \downarrow \boldsymbol{\beta}$. A function $\phi(n, \boldsymbol{\alpha})$ on the set of vertices is said to be *harmonic* if

$$\phi(n, \boldsymbol{\alpha}) = \sum_{\boldsymbol{\gamma}:\boldsymbol{\gamma}\downarrow\boldsymbol{\alpha}} \phi(n+1, \boldsymbol{\gamma}). \tag{9}$$

Consider linear combinations of characters $F_n := \sum_{\boldsymbol{\alpha}} \phi(n, \boldsymbol{\alpha})\chi_{\boldsymbol{\alpha}}$; then the harmonic condition means that the restriction of F_n to $\mathrm{U}(n-1)$ is equal to F_{n-1}. In this language, characters of type II_1 of the group $\mathrm{U}(\infty)$ (or, in other words, the spherical functions of the pair $\mathrm{U}(\infty) \times \mathrm{U}(\infty) \supset \operatorname{diag} \mathrm{U}(\infty)$) are the extreme points Φ_ω of the space of non-negative harmonic functions (using the formal language, these are points in the minimal boundary of the Markov process). The problem of writing out the spectral measure μ for $\mathfrak{U}$ becomes the problem of decomposing some specific positive harmonic function Ψ with respect to the extreme points,

$$\Psi(n,\boldsymbol{\alpha}) = \int \Phi_\omega(n,\boldsymbol{\alpha})\, d\mu(\omega).$$

Olshanski understood that here, in analogy with the theory of spherical functions, one can go beyond representation theory. In the definition of harmonic function we can correctly introduce coefficients of the right-hand side of (9), and then new problems on extreme points arise, as well as problems of decomposing some natural harmonic functions with respect to extreme points.

Olshanski developed a wide range of methods applicable to problems of distinguishing boundaries: in joint papers with Kerov and Okounkov ([35], [34], [44]) they used the combinatorics of symmetric and shifted symmetric functions and generalised results on boundaries to the general values of the Jack parameter θ (this parameter generalises the dimension of the ground field in the theory of spherical functions and is associated with a random matrix parameter which is usually denoted by β; $\theta = 1$ in the above example with $\mathrm{U}(n)$). Olshanski and Borodin [51] found beautiful determinantal formulae for multiplicities in the problem of the restriction of irreducible representations of $\mathrm{U}(n)$ to $\mathrm{U}(m)$.

A fresh turn in the development of this topic is related to the study of q-deformations of classical objects. The first result of this kind was obtained by Olshanski and A. V. Gnedin [47], [48]: they solved the q-version of the classical de Finetti problem on the characterisation of all sequences of random variables that are invariant under permutations. Recently, Olshanski and Gorin [52] managed to find a new and still not fully understood q-deformation in a much more complicated problem connected with the unitary group $\mathrm{U}(\infty)$. An unexpected consequence and an offshoot of this activity was the discovery of a new class of non-homogeneous symmetric orthogonal polynomials in infinitely many variables by Olshanski and C. Cuenca [55], [56].

Another application of the theory of boundaries was L. A. Petrov's construction of infinite-dimensional Markov diffusions closely connected with population genetics. Borodin and Olshanski [46], [50] showed how such processes can be constructed from branching graphs, by translating into the probabilistic language and further developing the idea of approximation of infinite-dimensional groups by finite-dimensional subgroups. Surprisingly, this approach enabled the authors to prove properties and theorems on infinite-dimensional diffusions that had not yielded to the efforts of experts in classical probability theory.

Now we turn to Olshanski's works on **Yangians**. In working with representations of infinite-dimensional groups in the 1980s he sought to apply the technique of universal enveloping algebras to these representations, and he proposed an algebraic parallel to the semigroup construction. Fix a number m and consider the Lie algebra $\mathfrak{gl}(n)$ of all matrices of order n and its subalgebra $\mathfrak{gl}(n-m)$ consisting of block $(m+(n-m))$ matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$. We denote the standard basis in $\mathfrak{gl}(n)$ by $e_{\alpha\beta}$. Consider the universal enveloping algebra $\mathscr{U}(\mathfrak{gl}(n))$ and in it the centraliser $\mathscr{A}_m(n)$ of the subalgebra $\mathfrak{gl}(n-m)$. Referring to classical invariant theory, one can show that $\mathscr{A}_m(n)$ is generated by the elements of the form p_M and $p_{ij|M}$:

$$p_M := \sum_{\alpha_1,\dots,\alpha_M=1}^{n} e_{\alpha_1\alpha_2}e_{\alpha_2\alpha_3}\cdots e_{\alpha_M\alpha_1}, \quad p_{ij|M} := \sum_{\alpha_1,\dots,\alpha_{M-1}=1}^{n} e_{i\alpha_1}e_{\alpha_1\alpha_2}\cdots e_{\alpha_{M-1}j},$$

where $1 \leqslant i,j \leqslant m$, $M \in \mathbb{N}$, and the elements p_M lie in the centre of $\mathscr{A}_m(n)$. There is a natural homomorphism $o_m\colon \mathscr{A}_m(n) \to \mathscr{A}_m(n-1)$ defined by analogy with the Harish-Chandra homomorphism (it turns out that it takes the generators p_M and $p_{ij|M}$ of $\mathscr{A}_m(n)$ to the corresponding generators of the algebra $\mathscr{A}_m(n-1)$). This enables us to define the chain

$$\mathscr{A}_m(m) \xleftarrow{o_{m+1}} \mathscr{A}_m(m+1) \xleftarrow{o_{m+2}} \cdots \xleftarrow{o_n} \mathscr{A}_m(n) \xleftarrow{o_{n+1}} \cdots$$

and its projective limit $\mathscr{A}_m$ (the *Olshanski algebra*). The same elements p_M and $p_{ij|M}$ can be regarded as the generators of this algebra. Olshanski showed that one obtains an algebra with quadratic relations that can be explicitly written out and have a simple form. Then he introduced the algebra $\mathscr{A}$ as the inductive limit $\lim_{m\to\infty}\mathscr{A}_m$ and proposed treating this algebra as an analogue of an enveloping algebra for $\mathfrak{gl}(\infty)$.

This construction was announced in 1987 [13], and the details appeared in [14] and [20] (for a thorough exposition see the survey [27] and, in greater detail, A. I. Molev's book [61]). B. L. Feigin suggested that the algebras $\mathscr{A}_m$ and $\mathscr{A}$ are connected with the Yangian.[24] It turned out that this is indeed the case, and the algebra $\mathscr{A}_m$ can be decomposed into the tensor product

$$\mathscr{A}_m = \mathscr{A}_0 \otimes \mathscr{Y}(\mathfrak{gl}(m))$$

of the algebra $\mathscr{A}_0$ of virtual Laplace operators that is generated by the elements p_M and the Yangian $\mathscr{Y}(\mathfrak{gl}(m))$ associated with the Lie algebra $\mathfrak{gl}(m)$. It is interesting that Yangians naturally 'grow up' as objects of representation theory.[25]

At that time Olshanski expected that application of a similar centraliser construction to other series of classical Lie algebras would lead to the corresponding Yangians introduced by V. G. Drinfeld in 1985. However, in the cases of the orthogonal Lie algebra $\mathfrak{o}(N)$ and the symplectic Lie algebra $\mathfrak{sp}(2n)$, the projective limit of the centralisers turned out to be connected with new algebras $\mathscr{Y}^+(N)$ and $\mathscr{Y}^-(2n)$, respectively, which he called *twisted Yangians*. No direct connection of these algebras with Drinfeld's Yangians for $\mathfrak{o}(N)$ and $\mathfrak{sp}(2n)$ was discovered. The twisted Yangians $\mathscr{Y}^+(N)$ and $\mathscr{Y}^-(2n)$ are identified with subalgebras of the corresponding Yangians $\mathscr{Y}(\mathfrak{gl}(N))$ or $\mathscr{Y}(\mathfrak{gl}(2n))$, and these subalgebras are co-ideals with respect to the Hopf algebra structures. These results were first published in [22], and a detailed exposition was given later in [27] and [36].

[24] Feigin means the Yangian $\mathscr{Y}(\mathfrak{gl}(m))$ of the algebra $\mathfrak{gl}(m)$; it was known as the 'Leningrad algebra' and arose in the early 1980s, in the school of L. D. Faddeev in connection with the 'quantum method of the inverse problem' (for example, see [60]).

[25] By the way, this gives a possibility for constructing representations of the Yangians themselves, based on representations of the centralisers. In this way, a natural class of the so-called 'tame' or 'skew' representations of Yangians arises, which were studied in detail by M. L. Nazarov and V. O. Tarasov; their superversions were considered by Molev.

For Lie algebras of all the four classical series, the corresponding (twisted) Yangians have an important property: the point (evaluation) epimorphisms into the universal enveloping algebras are well defined:

$$\mathscr{Y}(\mathfrak{gl}(m)) \to \mathscr{U}(\mathfrak{gl}(m)), \qquad \mathscr{Y}^+(N) \to \mathscr{U}(\mathfrak{o}(N)), \qquad \mathscr{Y}^-(2n) \to \mathscr{U}(\mathfrak{sp}(2n)).$$

This is important for applications to the classical representation theory. I. V. Cherednik discovered that the twisted Yangians can be used in classical representation theory as a tool to separate the multiplicities. Since the defining relations of Yangians and twisted Yangians admit a special 'R-matrix' form, their structure can be investigated with the help of the corresponding techniques originating from work in the Leningrad school of Faddeev. In particular, these techniques can be used to construct central elements in Yangians, and thus the use of the point homomorphisms leads to new constructions of Casimir elements for classical Lie algebras and to the corresponding 'Capelli identities' (see the aforementioned book [61] by Molev). Using Olshanski's centraliser construction, in 1998 Molev was able to construct bases of Gelfand–Tsetlin type for representations of symplectic Lie algebras (a question concerning such bases was posed by D. P. Zhelobenko in the early 1960s) and also *weight* bases for orthogonal algebras.

The joint paper [28] with Nazarov was devoted to applications of the R-matrix technique to the construction of families of commutative subalgebras of Yangians. Using point homomorphisms, the authors were able to give a positive solution of É. B. Vinberg's problem of quantization: they constructed maximal commutative subalgebras of classical universal enveloping algebras, so that the corresponding adjoint graded algebras are the commutative Poisson 'shift of argument' subalgebras constructed by A. S. Mishchenko and A. T. Fomenko in the symmetric algebras.

Olshanski's papers on Yangians stimulated further research in this area, both by experts in representation theory and by experts in mathematical physics (N. Guai, E. Ragusi, Molev, Nazarov and S. M. Khoroshkin, V. Regelskis, C. Wendlandt, J. Brown, H. Chen, X. Ma, L. O. Chekhov, M. Mazzocco, H. Bao, G. Wang, S. Kolb, B. Vlaar, G. Letzter, M. Balagović, and others). In particular, various generalisations of the centraliser construction for other families of algebras were obtained. and the twisted Yangians were investigated as symmetry algebras in models of statistical mechanics.

The centraliser construction has an unexpected analogue. Namely, it can be applied to the chain of group algebras of symmetric groups,

$$\mathbb{C}[\mathrm{S}(1)] \subset \mathbb{C}[\mathrm{S}(2)] \subset \cdots \subset \mathbb{C}[\mathrm{S}(n)] \subset \mathbb{C}[\mathrm{S}(n+1)] \subset \cdots$$

which leads to *degenerate affine Hecke algebras* [13], [11], [38].

The **special functions of representation theory** occupy a very important place in Olshanski's work. Of course, the specific form of irreducible representations, their characters, and so on, are connected with the richest theory of special functions, as was already clear to the founders of representation theory. The innovation and importance of Olshanski's works come in large part from the new point of view on classical representation theory which is opened from the vantage point of the infinite-dimensional theory and, in particular, from the point of view of the theory he developed for representations of infinite-dimensional classical groups.

As the simplest (but very striking) example we can take the representations of the symmetric group $S(n)$ and the corresponding irreducible character $\chi^\lambda(\sigma) = \chi^\lambda_\mu$ indexed by two partitions λ and μ of the integer n. The partition μ encodes the lengths of the cycles of the permutation $\sigma \in S(n)$, and the traditional theory of representations regards μ as an argument of the function χ^λ_μ and λ as a parameter. On the contrary, from the point of view of representations of $S(\infty) = \bigcup_n S(n)$, it is important to fix non-trivial cycles of σ and regard λ as an argument. In their 1994 paper [24], Kerov and Olshanski first proved the remarkable result that the normalised characters $f_\sigma(\lambda) = \chi^\lambda(\sigma)/\chi^\lambda(1)$ form a basis in a certain algebra Λ^* of polynomials in the discrete variable λ, and then explained the importance and naturalness of this algebra and its connection with the classical algebra Λ of symmetric functions. In particular, identification of the highest-degree component of f_σ is a fundamental step in the modern understanding of the asymptotic theory of characters which has its origin in works of Vershik and Kerov.

One might think that the function $f_\mu(\lambda)$ is not defined for $|\lambda| < |\mu|$, but in fact it is defined and *vanishes* in this case. This remarkable property is the basis of a characterisation of the *Schur interpolation functions* s^*_μ studied by Olshanski and Okounkov [30]–[32]. These functions can be defined as a kind of Newton interpolation polynomials in the algebra Λ^*, and the fact that their components of highest degree $|\mu|$ coincide with the ordinary Schur functions, that is, with the characters of irreducible representations of the group GL, reflects the deep symmetries in representation theory. For general classical groups G, interpolation functions of this kind are inseparably connected with binomial formulae (that is, with the expansion of irreducible characters into series in a neighbourhood of $1 \in G$), important bases in $\mathscr{U}(\mathfrak{g})$, the asymptotic theory of characters of G, and many other questions.

Olshanski's idea that the theory of interpolation functions can be extended naturally from algebras of type Λ^* to their deformations associated with MacDonald polynomials and related multidimensional special functions turned out to be extraordinary in its importance and insight. Although the group G itself recedes to the background in this theory, many other important structures and applications arise that play a central role in modern mathematics. The corresponding theory of interpolation functions developed by Olshanski, Okounkov, F. Knop, S. Sahi, E. M. Rains, and others plays a key conceptual and technical role in the current state of the theory of special functions.

A few words about Grigori Olshanski's **biography**. He was born in Moscow on January 8, 1949, in the family of the movie scriptwriter and author Iosif Grigor'evich Olshanski. After completing his postgraduate studies, Grigori worked from 1972 to 1975 as a junior researcher in the Research Institute for Preschool Education of the Academy of Pedagogical Sciences of the USSR. In 1975–1987 (when his main papers on representations of infinite-dimensional classical groups were written) he was a senior researcher at VNIIPIStromsyr'ë (the All-Union[26] Research and Design Institute for Problems of Extraction, Transporting, and Processing of Raw Materials in the Construction Materials Industry). In 1987–1990 he was a senior researcher in the Department of Glaciology of the Institute of Geography of the

[26] Now this word is deleted from the name of the institute (which deals with quarrying), but the abbreviation FGUP (Federal State Unitary Enterprise) is added in front.

USSR Academy of Sciences. Since 1991 Olshanski has worked in the Institute for Information Transmission Problems of the Russian Academy of Sciences (as a leading and then a principal researcher). Since 2013, he has also been a professor in the Higher School of Economics, and since 2017, a Skoltech professor.

In the 1970s and 80s Olshanski was one of the most active participants in Kirillov's seminar on representation theory, at that time one of the most important mathematical seminars in Moscow. Moreover, Olshanski put up a blackboard at home and held regular meetings and small seminars with young people (one of these seminars was mentioned above). In the 1980s and early 90s he was an actual research supervisor of several participants of Kirillov's seminar, and some among his students of that time later became well-known authors, including Borodin, Molev, Nazarov, and Okounkov. Among later students were Alexei I. Bufetov, Gorin, V. N. Ivanov, A. A. Osinenko, and Petrov. Since 2009, Olshanski has supervised a seminar (in conjunction with the Steklov Mathematical Institute) at the Independent University of Moscow (now together with Aleksandr I. Bufetov, A. V. Dymov, and A. V. Klimenko as co-supervisors).

Olshanski is an author of 110 research papers and the 2017 book (with Borodin) *Representations of the infinite symmetric groups* [53]. He was also the editor of the collection of papers *Kirillov's seminar on representation theory* published by the American Mathematical Society in 1998.

Olshanski was an invited speaker at the 4th European Congress of Mathematics in 2004 (joint talk with Borodin, "Representation theory and random point processes") and at the International Congress of Mathematicians in 2014 ("The Gelfand–Tsetlin graph and Markov processes"). He is a member of the board of the Moscow Mathematical Society and the editorial boards of the journals *Funktsional'nyi Analiz in ego Prilozheniya*,[27] *Transformation Groups*, *Journal of Lie Theory*, and *SIGMA*.

Grigori Iosifovich Olshanski has had a very strong mathematical influence on each of us. We cordially congratulate him on his seventieth birthday and sincerely wish him good health and further successes in his research and in teaching young mathematicians.

A. M. Borodin, Aleksandr I. Bufetov, Aleksei I. Bufetov, A. M. Vershik, V. E. Gorin, A. I. Molev, V. F. Molchanov, R. S. Ismagilov, A. A. Kirillov, M. L. Nazarov, Yu. A. Neretin, N. I. Nessonov, A. Yu. Okounkov, L. A. Petrov, and S. M. Khoroshkin

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