Mantles, Trains and Representations of Infinite Dimensional Groups

Yurii A. Neretin

The term "infinite dimensional group" is heuristic. It does not have a rigid definition. The most interesting types of groups which might be considered as infinite dimensional groups are:

groups of diffeomorphisms of manifolds;

the group of diffeomorphisms of a circle (it is the exceptional case among other groups of diffeomorphisms);

groups corresponding to Kac–Moody algebras;

infinite dimensional analogues of classical groups;

infinite analogues of symmetric groups;

current groups (i.e., groups of functions on some set X with values in fixed groups G);

groups of transformations of spaces with measure.

Looking at this list, it is easy to appreciate that the term "infinite dimensional group" is not quite suitable (symmetric groups have no dimension). It seems that a better term would be "large group" (as proposed by A.M. Vershik).

1. Phenomena

1.1. Let G be an infinite dimensional group. Then G is *not* a group, but only a *part* of some invisible to the naked eye semigroup $\Gamma = \Gamma(G)$ (the *mantle* of the group G). Strictly speaking, to any infinite dimensional group G, one can associate a semigroup $\Gamma \supset G$. The group G is dense in Γ and any representation ρ of G admits a unique extension to a representation $\bar{\rho}$ of the semigroup Γ.

Consider a representation ρ of G. Then the set $\rho(G)$ is dense (in the weak operator topology) in $\rho(\Gamma)$, and provides a method for a search of mantles. Unfortunately, the problem of describing the weak closure of the set $\rho(G)$ is often unexpectedly difficult (and the solution is also often unexpected).

1.2. The infinite dimensional group G is not a semigroup. It is only part of some invisible to the naked eye category $\mathcal{K} = \mathcal{K}(G)$. Strictly speaking, to any infinite dimensional group G, one can associate a category $\mathcal{K}$ (the *train* of G). The group G is the automorphism group $\mathrm{Aut}_{\mathcal{K}}(V)$ of some object V of the category $\mathcal{K}$. The semigroup $\Gamma = \Gamma(G)$ is the endomorphism semigroup $\mathrm{End}_{\mathcal{K}}(V)$ of the same object V. Any representation ρ of the group G admits a unique extension to a (projective) representation (R, ρ) of the category $\mathcal{K}$.

Consider a category $\mathcal{K}$. A projective representation (R, ρ) of the category $\mathcal{K}$ is a rule which associates to each object V of $\mathcal{K}$ a linear space $T(V)$ and to each morphism $P \in \mathrm{Mor}_{\mathcal{K}}(V, W)$ a linear operator $\rho(P) : R(V) \to R(W)$ such that for any three objects V, W and Y and for any two morphisms $P \in \mathrm{Mor}_{\mathcal{K}}(V, W)$, $Q \in \mathrm{Mor}_{\mathcal{K}}(W, Y)$, we have

$$\rho(Q)\rho(P) = c(Q, P)\rho(QP)$$

for some nonzero complex number $c(Q, P)$.

For a wide class of infinite dimensional groups (for (G, K)-pairs), there exists a universal construction of a train. In §5 we discuss one nontrivial example. This construction also gives us a method for the classification of representations of infinite dimensional groups.

The terms "mantle" and "trains" were introduced early by the author (cf. [23]).

1.3. Some classical areas of representation theory are in fact representation theories of categories. This is the case for the finite dimensional representations of classical groups, the highest weight representations of real classical groups $U(p, q)$, $Sp(2n, \mathbb{R})$, $SO^*(2n)$, representations of classical algebraic groups over finite characteristic fields and modular representations of Chevalley groups.

For some classical areas of representation theory, there exist representation theories of categories closely related (but not equivalent) to the representation theories of groups. This is the case for symmetric groups, cuspidal representations of p-adic groups, and complex representations of Chevalley groups.

1.4. There exist two universal objects for the representation theory of categories. The first is the "Weil" representation of the symplectic category $\overline{Sp}$ (see [21], [23]), and the second is the spinor representation of the orthogonal category $\overline{GD}$ (for the construction of this category, see [20]). These constructions are natural extensions of the classical constructions (K.O. Friedrichs, I. Segal, F. A. Berezin, D. Shale, W. F. Stinespring) of automorphisms of canonical commutation and anticommutation relations.

In trying to construct a representation of a category $\mathcal{K}$, it is useful to embed $\mathcal{K}$ in $\overline{Sp}$ or $\overline{GD}$ and to restrict the "Weil" representation or spinor representation to $\mathcal{K}$. In many cases such constructions allow us to obtain all representations of the category $\mathcal{K}$.

The purpose of this paper is to describe some details of this picture.

2. Categories of Linear Relations

2.1. Linear relations. Consider linear spaces V, W. By definition a linear relation $P : V \rightrightarrows W$ is a subspace in $V \oplus W$. Consider linear relations $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$. The product QP consists of all pairs $(v, y) \in V \oplus Y$ such that there exists $w \in W$ satisfying conditions $(v, w) \in P$, $(w, y) \in Q$.

For any linear relation $P : V \rightrightarrows W$ we define

(a) the kernel $\mathrm{Ker}P = P \cap V$;

(b) the image $\mathrm{Im}P$, as the the projection of P to W;

(c) the domain $\mathrm{Dom}P$, as the projection of P to V;

(d) The indefinity Indef $(P) = P \cap W$.

2.2. The category GA. Objects of GA are finite dimensional complex linear spaces. The set $\mathrm{Mor}_{GA}(V, W)$ consists of all linear relations $P : V \rightrightarrows W$ and of the formal morphism null $= \mathrm{null}_{V,W}$ (it is not a linear relation). The product of null and any morphism equals null. Consider linear relations $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$. If

$$\mathrm{Indef}(P) \cap \mathrm{Ker}Q = 0 \qquad \mathrm{Dom}Q + \mathrm{Im}P = W,$$

then the product QP in category GA is the usual product of linear relations. If one of the conditions (1) does not hold, then $QP =$null.

2.3. The Category C. An object V of category C is a finite dimensional complex linear space provided by a skew-symmetric bilinear form $L_V(\cdot, \cdot)$. Let V, W be objects of C. Consider, in $V \oplus W$, the bilinear form

$$L_{V \oplus W}((v, w)(v', w')) = L_V(v, v') - L_W(w, w').$$

The set $\mathrm{Mor}_C(V, W)$ consists of maximal isotropic subspaces in $V \oplus W$ and $\mathrm{null}_{V,W}$. The composition of morphisms is defined as in a category GA.

Let V_{2n} be a $2n$-dimensional object of C. It is easy to see that the automorphism group $\mathrm{Aut}_C(V_{2n})$ is the complex symplectic group $C_n \simeq Sp(2n, \mathbb{C})$. Let (T, τ) be a representation of category $\mathcal{C}$. Then the group $Sp(2n, \mathbb{C})$ acts in the space $T(V_{2n})$.

By definition, a projective representation of category C is holomorphic if the corresponding representations of the groups C_n are holomorphic.

Theorem 1. [22] (a) *The projective holomorphic representations of the category* C *are completely reducible.*

(b) *The holomorphic irreducible representations of category* C *are enumerated by diagrams of the form*

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 \Leftarrow & 0 - & 0 - & 0 & \cdots \end{array}$$

where a_j *are nonnegative integers, and only a finite number of them are nonzero. Let* a_α *be the extreme right nonzero label. If* $n < \alpha - 1$, *then the corresponding representation of* $C_n \simeq Sp(2n, \mathbb{C})$ *is a zero-dimensional representation. If* $n \geq \alpha - 1$, *then the corresponding representation of* C_n *has the following labels in the Dynkin diagram of type* C_n

$$\begin{array}{ccccc} a_1 & a_2 & & & a_n \\ 0 \Leftarrow & 0 - & 0 - & \cdots - & 0 \end{array}$$

For analogous theorems for the series of groups $A_n \simeq SL(n+1, C)$, $B_n = SO(2n+1, \mathbb{C}), D_n = SO(2n, \mathbb{C})$, see [22].

2.4. Shmul'yan's category $\mathcal{U}$. An object V of the category $\mathcal{U}$ is a complex finite dimensional linear space V equipped with a *nondegenerate* Hermitian form $M_V(\cdot,\cdot)$. Let p_V and q_V be the inertia indexes of M_V. A morphism $P : V \to W$ is a linear relation $P : V \rightrightarrows W$ such that

(a) If $(v, w) \in P$, then $M_V(v, v) \geq M_W(w, w)$; in other words, P "contracts" the form M.

(b) $\dim P = p_V + q_W$ (i.e., P has the maximal possible dimension among all subspaces satisfying condition (a)).

(c) If $(v, 0) \in P$, $v \neq 0$, then $M_V(v, v) > 0$. If $(0, w) \in P$, $w \neq 0$, then $M_W(w, w) < 0$ (this is a technical condition, inequalities $M_V(v, v) \geq 0$, $M_W(w, w) \leq 0$ follow from condition (a).)

The product of morphisms is the usual product of linear relations.

The group $\mathrm{Aut}_{\mathcal{U}}(V)$ is the pseudounitary group $\mathcal{U}(p_V, q_V)$. It is easy to see that $\dim \mathrm{Aut}_{\mathcal{U}}(V) = \frac{1}{2}\dim \mathrm{End}_{\mathcal{U}}(V)$. For an analogue of Theorem 1, see [22].

2.5. The Category Sp. An object $V = V_{2n}$ of Sp is a direct sum $V_+ \oplus V_- = \mathbb{C}^n \oplus \mathbb{C}^n$ $(n = 0, 1, 2, \ldots)$ equipped with two forms

$$L_V((v^+, v^-), (w^+, w^-)) = \Sigma(v_j^+ w_j^- - v_j^- w_j^+)$$
$$M_V((v^+, v^-), (w^+, w^-)) = \frac{1}{i}\Sigma(v_j^+ \overline{w}_j^- - v_j^- \overline{w}_j^+)\,.$$

So V is an object of both categories, C and D. Morphisms $V \to W$ are subspaces $P \subset V \oplus W$ such that P is a morphism of both categories. A product is the usual product of linear relations. It is easy to see that $\mathrm{Aut}_{Sp}(V_{2n})$ is the real symplectic group $Sp(2n, \mathbb{R})$.

2.6. The Potapov transformation

Proposition. *Fix the objects V, W of Sp. Let P be a subspace $P \subset V \oplus W$. The following conditions are equivalent.*

(A) $P \in \mathrm{Mor}_{Sp}(V, W)$;

(B) *P is the graph of an operator*

$$\Pi(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : V_+ \oplus W_- \to V_- \oplus W_+$$

satisfying three conditions:

(a) $\|\Pi(P)\| \leq 1$ *where* $\|\cdot\|$ *denotes the usual Euclidean norm*

(b) *the matrix* $\Pi(P)$ *is symmetric*

(c) $\|K\| < 1$, $\|N\| < 1$.

2.7. The "Weil" representation (We, we) of the category Sp. Let V be an object of Sp. The space $We(V)$ is the bosonic Fock space, i.e., the space of holomorphic functions on V_+ with the scalar product

$$\langle f, g\rangle = \int\int f(v)\overline{g(v)}\, d\mu(v)$$

where $\mu(v)$ denotes the Gaussian measure with density $\exp(-\Sigma|v_j^+|^2)/(2\pi)$.

Let $P \in \mathrm{Mor}_{Sp}(V, W)$. Let $\Pi(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ and let $k_{ij}, \ell_{i\alpha}, \ldots$ be the matrix elements of $K, L, \ldots$. Put

$$we(P)f(z) = \int\int \exp\left\{\frac{1}{2}\Sigma k_{\alpha\beta} z_\alpha z_\beta + \Sigma \ell_{\alpha\gamma} z_\alpha \bar{u}_\gamma + \frac{1}{2}\Sigma m_{\delta\gamma}\bar{u}_\delta \bar{u}_\gamma\right\} f(u) d\mu(u).$$

Theorem 2. ([15], [21]) (We, we) *is a projective representation of the category* Sp. *The corresponding representations of the groups* $Sp(2n, \mathbb{R}$ *arethe usual Weil representations.*

Remark. The above-mentioned "universal" category $\overline{Sp}$ consists of the category Sp and some infinite dimensional objects, see [21], [23].

3. The category *Shtan*

3.1. The group Diff **and the semigroup** Γ**.** Let Diff be the group of orientation-preserving (analytic) diffeomorphisms of the circle. An element of the semigroup Γ is a triple (R, r^+, r^-) where

(a) R is a Riemann surface (one dimensional complex manifold) which is homeomorphic to some annulus $a \le |z| \le b$;

(b) r^+, r^-: $e^{i\varphi} \mapsto R$ are analytic parametrizations of the components of the boundary of the surface R. The surface R lies on the right of the path $r^+(e^{i\varphi})$ and on the left of the path $r^-(e^{i\varphi})$;

(c) Two triples (R, r^+, r^-) and (Q, q^+, q^-) are equivalent if there exists a biholomorphic map $\pi : R \to Q$ such that $q^\pm(e^{i\varphi}) = \pi \circ r^\pm(e^{i\varphi})$.

The product of (R, r^+, r^-) and (Q, q^+, q^-) is the triple (P, p^+, p^-) where P is the Riemann surface obtained from $R \cup Q$ by glueing together the pairs $q^+(e^{i\varphi})$, $r^-(e^{i\varphi})$, and $p^+(e^{i\varphi}) = r^+(e^{i\varphi})$, $p^-(e^{i\varphi}) = q^-(e^{i\varphi})$.

The group Diff is contained in the "boundary" of the semigroup Γ; elements of Diff correspond to "infinitely short tubes." There exists a natural complex structure on Γ (it is not evident).

3.2. Representations of Γ**.** The group Diff has one well-known and interesting class of representations, the so-called highest weight representations.

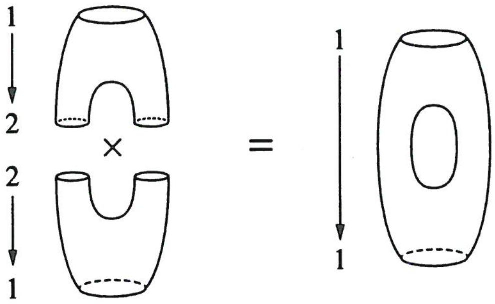
Theorem 3. ([18], [19]) *Any irreducible highest weight representation of Diff has a unique extension to a holomorphic representation of the semigroup* Γ.

A proof of the theorem is based on the embedding of the semigroup Γ in semigroups of linear relations (see §2).

3.3. The category *Shtan* **of Kontsevich–Segal.** Objects of the category *Shtan* are nonnegative integers. A morphism $m \to n$ is a family (R, r_i^+, r_j^-), $1 \le i \le m$, $1 \le j \le n$ where

(a) R is a compact Riemann surface with a boundary.

(b) $r_i^+, r_j^-\colon e^{i\varphi} \mapsto R$ are analytic parametrizations of components of the boundary. The surface R lies on the right side of paths $r_i^+(e^{i\varphi})$ and on the left side of paths $r_j^-(e^{i\varphi})$. The definition of the product of morphisms is the same as for the semigroup Γ.



Some of the highest weight representations of Diff extend to representations of the category *Shtan* ([19], [20]). Kontsevich and Segal considered conformal quantum field theory in terms of the representation theory of the category *Shtan*, see [37].

4. Heavy groups

4.1. The group $O(\infty)$. Denote by $O(\infty)$ the group of all orthogonal operators in the real Hilbert space ℓ_2. Denote by $\Gamma O(\infty)$ the semigroup of all operators A in the real Hilbert space ℓ_2 such that $\|A\| \leq 1$. We equip $O(\infty)$ and $\Gamma O(\infty)$ with the weak operator topology.

Theorem 4. *Any representation of $O(\infty)$ admits a unique extension to a representation of $\Gamma O(\infty)$.*

4.2. The Category $\bar{O}$. The objects of $\bar{O}$ are the real Euclidean spaces $\mathbb{R}^1, \mathbb{R}^2, \dots, \ell_2$. Morphisms $V \to W$ are the linear operators $A : V \rightrightarrows W$ such that $\|A\| \leq 1$.

Theorem 4′. *Any representation ρ of the group $O(\infty)$ admits a unique extension to a representation (R, ρ) of the category $\bar{O}$.*

Now we briefly describe a construction of an extension (R, ρ). Consider the space $H_n \subset \ell_2$ consisting of all vectors of type $(x_1, x_2, \dots, x_n, 0, 0, \dots)$. Let P_n be the orthogonal projector onto H_n. Put $R(\mathbb{R}^n) = \mathrm{Im}\rho(P_n)$.

Consider the morphism $A : \mathbb{R}^n \to \mathbb{R}^m$. Consider an infinite matrix $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma O(\infty)$.

Set

$$\rho(A) = \rho(\tilde{A})\,|_{R(\mathbb{R}^n)} : R(\mathbb{R}^n) \to R(\mathbb{R}^m).$$

Then (R, ρ) is a representation of the category $\bar{0}$. The classification of representations of category $\bar{0}$ is a simple problem which allows us to classify the representations of $O(\infty)$.

4.3. Heavy groups. The term "heavy groups" is also heuristic. There are three types of heavy groups:

1. $O(\infty), \mathcal{U}(\infty), Sp(\infty)$
2. The complete infinite symmetric group S_∞.
3. The group Ams (respectively Ams_∞) of automorphisms of the Lebesgue space with finite (respectively σ-finite) continuous measure. This group is equipped with a weak topology (see [24], [42]).

It seems that these groups are quite different, but all heavy groups have strangely similar properties. The role of the heavy groups in the representation theory of infinite dimensional groups is like the role of compact groups in the representation theory of Lie groups.

4.4. The train of S_∞. The objects of category PB are sets $\emptyset, \{1\}, \{1,2\}, \dots, \{1,2,\dots\}$. Morphisms $M \to N$ are partially defined injective maps $M \to N$. Any representation of S_∞ admits a unique extension to a representation of PB.

4.5. The train of Ams. An object (M, μ) of the category Pol is a Lebesgue spaces M with a probability measure μ. A morphism $(M, \mu) \to (N, \nu)$ of the category Pol (polimorphism, see [42]) is a probability measure κ on $M \times N$ such that

1. The projection of κ onto M is μ.
2. The projection of κ onto N is ν.

Let $\kappa : (M, \mu) \to (N, \nu)$, $\sigma : (N, \nu) \to (L, \lambda)$ be morphisms of the category Pol. Let $\kappa_m(n)$ be conditional measures on the sets $m \times N$ ($m \in M$) and let $\sigma_n(\ell)$ be conditional measures on the sets $n \times L$. Then the measure $\sigma\kappa$ on $M \times L$ is defined by the equality

$$(\sigma\kappa)_m(\ell) = \int \sigma_n(\ell) d\kappa_m(n)$$

for conditional measures.

Remark. The construction becomes clear when the sets M, N, and L are finite.

Remark. Speaking informally, a polymorphism $\kappa : (M, \mu) \to (N, \nu)$ is a "map" which "spreads" any point $m \in M$ to a probability measure $\kappa_m(n)$ on N.

Representations of *Ams* extend to representations of the category *Pol*.

4.6. Representations. The classification of representations for $S(\infty)$ was obtained by A. Lieberman in 1972, (see [14] and [28]); for $O(\infty), \mathcal{U}(\infty), Sp(\infty)$ classification was obtained by A.A. Kirillov in 1973 (see [12] and [27]); for *Ams* and Ams_∞ see [24]. In all cases any representation can be realized in the tensor products of the simplest representations.

5. (G, K) – pairs and Ismagilov–Olshanskii multiplicativity for double cosets

The "term" (G, K)-pair is also heuristic. Roughly speaking, a (G, K)-pairs is a group G together with a heavy subgroup K.

5.1. The pair (GL, O) and its train. Denote by (GL, O), the group of those bounded invertible operators in the real Hilbert space ℓ_2 which can be represented in the form $A(1+T)$ where $A \in O(\infty)$ and T is a Hilbert–Schmidt operator (the operator T is Hilbert–Schmidt if $\Sigma|t_{ij}|^2 < \infty$ where t_{ij} are matrix elements of T).

Let $K_0 = K = O(\infty)$. Let K_n consist of all matrices of the form

$$\begin{pmatrix} E_n & 0 \\ 0 & Q \end{pmatrix} \in O(\infty)$$

where E_n is the unit $n \times n$-matrix and $Q \in O(\infty)$.

The objects of the category GLO are nonnegative integers $0, 1, 2, \dots$. The set $\mathrm{Mor}_{GLO}(n, m)$ consists of double cosets $\gamma \in K_m \backslash G / K_n$. Double cosets are matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ defined up to equivalence

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} E_m & \\ & \mathcal{U}_1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_n & \\ & \mathcal{U}_2 \end{pmatrix} = \begin{pmatrix} A & B\mathcal{U}_2 \\ \mathcal{U}_1 C & \mathcal{U}_1 D \mathcal{U}_2 \end{pmatrix}$$

where $\mathcal{U}_1, \mathcal{U}_2 \in O(\infty)$.

Consider the double cosets $\gamma \in K_m \backslash G / K_n$ and $\delta \in K_m \backslash G / K_\ell$. Let

$$P_m = \begin{pmatrix} E_m & \\ & 0 \end{pmatrix} .$$

Consider a sequence $\pi_1, \pi_2, \dots, \in K_m$ such that $\pi_j \to P_m$ weakly. Let $g \in \gamma$ and $h \in \delta$. Consider double cosets $\sigma_j \in K_\ell \backslash G/K_n$ containing the elements $h\pi_j g$. It is easy to see that the sequences $\{\sigma_j\}$ has a limit σ in the natural topology on $\mathrm{Mor}(n, \ell)$. By definition, σ is the product of γ and δ. Let

$$g = \begin{pmatrix} A & B \\ \underbrace{C}_{n} & \underbrace{D}_{\infty} \end{pmatrix} \begin{matrix} \}m \\ \}\infty \end{matrix} \qquad h = \begin{pmatrix} K & L \\ \underbrace{M}_{m} & \underbrace{N}_{\infty} \end{pmatrix} \begin{matrix} \}\ell \\ \}\infty \end{matrix}$$

Then $\sigma \in \mathrm{Mor}(n, \ell)$ is the double coset containing

$$h * g = \begin{pmatrix} AP & B & AQ \\ CP & D & CQ \\ R & O & T \end{pmatrix}.$$

The multiplication $(g, h) \mapsto h * g$ is a known multiplication of so-called operator nodes (= colligations) introduced by M.G. Krein.

Theorem 5. *There exists a canonical one-to-one correspondence between unitary representations of (GL, O) and representations of the category GLO.*

5.2. Characteristic functions. Consider $\gamma \in \mathrm{Mor}_{GLO}(k, n)$; let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \gamma.$$

Let λ be a point of Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$. We want to construct a linear relation $P_\gamma(\lambda) : V_{2k} \rightrightarrows V_{2n}$ (where V_{2k} is the same as in 2.5). Let $V_{2n} = V_{2n}^+ \oplus V_{2n}^-$ where $V_{2n}^\pm$ are isotropic subspaces relative to both forms L and M. Let $p^\pm \in V_{2n}^\pm$, $q^\pm \in V_{2k}^\pm$. Let $x, y \in \ell_2$. The element $(p; q) = (p^+, p^-; q^+, q^-) \in V_{2n} \oplus V_{2k}$ is contained in $P_\gamma(\lambda)$ if there exists $x, y \in \ell_2$ such that

$$\begin{pmatrix} p^+ \\ x \\ q^- \\ -\lambda^{-1} y \end{pmatrix} = \begin{pmatrix} A\,B & \\ C\,D & \\ & A^t\,C^t \\ & B^t\,D^t \end{pmatrix} \begin{pmatrix} q_+ \\ y \\ p^- \\ \lambda x \end{pmatrix}$$

(by A^t we denote matrix transposed to A).

Theorem 6. (a) *Let $Re\lambda > 0$. Then*

$$P_\gamma(\lambda) \in \mathrm{Mor}_{sp}(V_{2k}, V_{2n}).$$

(b) *Let* $Re\lambda = 0$. *Let* $(p,q) \in P_\gamma(\lambda)$. *Then* $M(p,p) = M(q,q)$ *(so, in the language of function theory, the function* $P_\gamma(\lambda)$ *is the interior function*

(c) $P_\gamma(-\lambda) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} P_\gamma(\lambda) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

(d) $\overline{P_\gamma(\lambda)} = P_\gamma(\bar{\lambda})$

(e) $P_{\gamma_1\gamma_2}(\lambda) = P_{\gamma_1}(\lambda)P_{\gamma_2}(\lambda)$ (we consider the usual product of linear relations).

5.3. The construction of 5.1 is a particular case of Ismagilov–Olshanskii multiplicativity. (We follow [31]). There exists many examples of (G,K)-pairs. For the case $K = \mathcal{U}(\infty), O(\infty), Sp(\infty)$ see [31]; for the case $K = S(\infty)$, see [29] and for the case $K = Ams$, Ams_∞, see [24].

6. Weak closure

6.1. Let ρ be a unitary representation of a group G. Consider the set $\rho(G)$ which consists of all operators $\rho(g), g \in G$. Let $\Gamma(\rho, G) = \overline{\rho(G)}$ be the closure of $\rho(G)$ in weak operator topology. It is easy to see that $\Gamma(\rho, G)$ is a compact (semitopological) semigroup.

This simple construction was proposed and used by Olshanskii. It can be applied, in particular, to studying the mantle of G. Indeed let Γ be the mantle of G. Then (see (1.1)).

$$\rho(\Gamma) \subset \overline{\rho(G)} = \Gamma(\rho, G)\,.$$

So we have some information about Γ.

Example. Let the group Diff act in $L^2(S^1)$ by formula

$$\rho(g)f(\varphi) = f(g(\varphi))g'(\varphi)^{\frac{1}{2}}\,.$$

Then the semigroup $\Gamma(\rho, G)$ contains all operators of type

$$f(\varphi) \mapsto \alpha(\varphi)f(\varphi)$$

where $0 \le \alpha(\varphi) \le 1$.

Example 2. Let M^n be an n-dimensional manifold, $n \ge 2$ with a volume form ω. Let D be the group of all diffeomorphisms of M^n preserving ω. Let D act in $L^2(M^n)$ by the formula

$$\rho(g)f(x) \;=\; f(g(x)).$$

Then $\Gamma(\rho, G) = \mathrm{End}_{Pol}(M^n)$.

6.2. How does the semigroup $\Gamma(\rho, G)$ depend on G? I don't know the answer to this question. Let R be some series of unitary representation of G. It is easy to show (see [31]) that there exists a universal compact semigroup $\Gamma(R; G)$ such that

(a) G is dense in $\Gamma(R; G)$;

(b) Any representation $\rho \in R$ of G extends to $\Gamma(R; G)$;

(c) $\rho(\Gamma(R; G)) \simeq \Gamma(\rho, G)$ for any $\rho \in R$.

Examples. Let R be a set of representatives of equivalence classes of all unitary representations. Then

$$\Gamma(R; O(\infty)) \simeq \mathrm{End}_{\bar{O}}(\ell_2) \qquad \Gamma(R; S(\infty)) = \mathrm{End}_{PB}(\mathbb{N})$$
$$\Gamma(R; Ams) \simeq \mathrm{End}_{Pol}(M)$$

where M is a Lebesgue space with continuous measure.

However in some cases semigroups $\Gamma(R, G)$ turn out to be "pathological" objects.

6.3. ***H*-polymorphisms.** Let us now describe some categories which appear as a result of the "universalization" of the semigroups $\Gamma(\rho, G)$ ("universalization" was applied to groups of diffeomorphisms (see [24])).

Let H be a fixed group. Objects of the category H-*Pol* are finite spaces with probability measure. Morphisms $(M, \mu) \to (N, \nu)$ are measures κ on $H \times M \times N$ such that

1. The projection of κ to M is μ,
2. The projection of κ to N is ν.

Let $m_1, \dots, m_\alpha$ be the points of M and let $n_1, \dots, n_\beta$ be the points of N. Then it is possible to consider κ as an $\alpha \times \beta$-matrix, the elements λ_{ij} of this matrix are measures on H and

$$\Sigma_i \lambda_{ij}(H) = \mu(m_j) \qquad \Sigma_j \lambda_{ij}(H) = \nu(n_i).$$

Let $\{\rho_{jk}\}$ be another of the same type matrix. The product of the morphisms $\{\lambda_{ij}\}$ and $\{\rho_{jk}\}$ is the matrix

$$\sigma_{ik} = \sum_j \mu(m_j)^{-1} \lambda_{ij} * \rho_{jk}$$

where $*$ denotes ususal convolution of measures on the group H.

For the case of infinite measure spaces, see [24]. Let M^n be an n-dimensional manifold, where $n \geq 2$; the "universal" semigroups corresponding to the known series of representations of groups of diffeomorphisms of M^n, are different semigroups of type $\mathrm{End}_{H-Pol}(M^n)$; the group H (in different cases) is $\mathbb{R}$ or some jet group or fundamental group $\pi_1(M^n)$.

7. Universal completions of classical groups

Let us consider the result of the application of some infinite dimensional constructions (see §6) to the classical case. Let us consider "the mantle of the group $GL(n, \mathbb{C})$"; for other classical groups see [25].

7.1. Resolving sequences. Resolving sequences in $\mathbb{C}^n$ are sequences of linear relations

$$P_1, P_2, \ldots, P_k : \quad \mathbb{C}^n \rightrightarrows \mathbb{C}^n$$

defined up to a multiplier ($k \geq 0$), such that

$$\mathrm{Dom}\, P_{j+1} \subset \mathrm{Ker}\, P_j \qquad \mathrm{Indef} P_{j+1} \subset \mathrm{Im}\, P_j$$

Remark. Let us consider the space S^n of all resolving sequences satisfying

$$\begin{aligned} \mathrm{Indef}\, P_1 &= 0 & \mathrm{Ker}\, P_k &= 0 \\ \mathrm{Dom}\, P_{j+1} &= \mathrm{Ker}\, P_j & \mathrm{Indef}\, P_{j+1} &= \mathrm{Im}\, P_j. \end{aligned}$$

Then the space S^n coincides with the variety of complete collineations constructed by J. G. Semple in 1948 (see [5]).

7.2. A multiplication. Let $(P_1, \ldots, P_k)$, $(Q_1, \ldots, Q_s)$ be resolving sequences in $\mathbb{C}^n$. Consider all products $\alpha_{ij} = P_i Q_j$ such that $P_i Q_j \neq$ null (see 2.1). Then α_{ij} is a resolving sequence (of course we have to do some permutation of α_{ij}).

Denote by $\overline{GL_n}$ the semigroup of all resolving sequences in $\mathbb{C}$. There exists a natural (non-Hausdorff) topology on $\overline{GL_n}$ and $GL(n, \mathbb{C})$ is dense in $\overline{GL_n}$; see [25].

Theorem 6. (a) *Any irreducible representation ρ of $GL(n, \mathbb{C})$ admits a canonical extension to a projective representation $\bar{\rho}$ of the semigroup $\overline{GL_n}$.*

(b) *Consider the closure $\overline{\mathbb{C} \cdot \rho(GL_n)}$ of the set of all operators of type $\lambda \cdot \rho(g)$ where $\lambda \in \mathbb{C}, g \in GL(n, \mathbb{C})$. Then $\overline{\mathbb{C} \cdot \rho(GL(n, \mathbb{C}))} = \mathbb{C} \cdot \bar{\rho}(\overline{GL_n})$.*

(c) *The action of $GL(n, \mathbb{C})$ on $GL(n, \mathbb{C})/O(n, \mathbb{C})$ extends to an action of $\overline{GL_n}$ in some completion of $GL(n, \mathbb{C})/O(n, \mathbb{C})$.*

8. Historical remarks

For a long time the representation theory of infinite dimensional groups was extremely disconnected. One of the results of the last years is the bringing together of the different theories. Unfortunately, up until now, while there have been some attempts to present the general picture, there is still no text which gives the relations between the different theories, see [23], [24], [26], [31].

The theory of the highest weight representation has been developed in particular by V.G. Kac, G. Segal, J. Lepowsky, R.L. Wilson, I.B. Frenkel, R.L. Goodman, N. Wallach ([11], [17], [34]).

For the infinite dimensional classical groups, one may refer to I. Segal, F.A. Berezin, D. Shale, W.F. Stinespring, A.A. Kirillov, S. Stratila, D. Voiculescu, R. Boyer, G.I. Olshanskii, A.M. Vershik, S.V. Kerov, D. Pickrell [4], [30], [33].

For the infinite symmetric group, see the works of E. Thoma, A. Lieberman, A.M. Vershik, S.V. Kerov, G.I. Olshanskii, see [28].

There are two quite different representation theories for current groups: Araki multiplicative integral and "energy representations." For the Araki multiplicative integral, see the works of H. Araki, R.F. Streater, A.M. Vershik–I.M. Gelfand– M.I. Graev, A. Guichardet, K.R. Parthasarthy - K. Schmidt (see [24], [40]). For the "energy representations," see the works of R.S. Ismagilov, A.M. Vershik–I.M. Gelfand–M.I. Graev, I.B. Frenkel., S. Albeverio, R. Hoegh-Krohn, D. Testard, M.P. and P. Malliavin, L. Gross [1].

The theory of mantles and trains is closely related to the theory of completions of symmetric spaces. For complex symmetric spaces, see the works of E. Study, J.G. Semple, J.A. Tyrell, C. De Concini, C. Procesi ([5]); for real symmetric spaces, see the works of H. Furstenberg, I.I. Piatetskii-Shapiro, I. Satake, F.I. Karpelevich, G.F. Kushner, A. Borel, T. Oshima, see [36]. One may consult M. Putcha and L.E. Renner on algebraic semigroups [35]; for analogues of such phenomena for the affine algebra, see [2].

Polymorphisms (or stochastic kernels) were introduced by E. Hopf (1953), see [13], [42].

For linear relations in operator theory, see the works of M.G. Krein and Shmul'yan [38]. The characteristic function of an operator was introduced by M. Livshic, see [3].

The categories of linear relations are closely related to Lie semigroup theory (E.B. Vinberg, S.M. Paneitz, G.I. Olshanskii, see [6]). For representations of groups of diffeomorphisms, see the works of R.S. Ismagilov, A.A. Kirillov, A.M. Vershik – I.M. Gelfand – M.I. Graev [10], [41].

For the first time, multiplicativity theorems were obtained by E. Thoma (1964), see [39], and R.S. Ismagilov (1968), see [8] and [9]. From 1978–1980, G.I. Olshanskii formulated the principle of the semigroup extension and obtained more general multiplicativity theorems. The semigroup Γ was constructed by the author in 1986 (and later by G. Segal). For some years (approximately 1981–1987) a group of Moscow mathematicians, discussed (proposed by G.I. Olshanskii) the problem of describing of the mantle of the "Weil" representation. In 1987 (see [15]), the problem was solved and after the autumn of 1987 the picture described in papers (see [19]–[25], [29], [31]). step by step became clear .

9. Some problems

I want only to formulate some directions which seem interesting.

1. A description of the whole mantle and the whole train of Diff. The semigroup Γ and the category *Shtan* are only parts of the mantle and of the train of Diff (for some highest weight representations ρ of Diff the tensor product $\rho \otimes \rho^*$ have nontrivial deformations. Those "new" representations don't admit extensions to Γ, see [17]).
2. Harmonic analysis on $\mathrm{Mor}_{Shtan}(0, n)$.
3. A classification of irreducible representations of (G, K)-pairs.
4. Extension of the theory to nonlinear actions of groups.

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Novozavodskaya 25/11-6-27
Moscow 121309, Russia

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