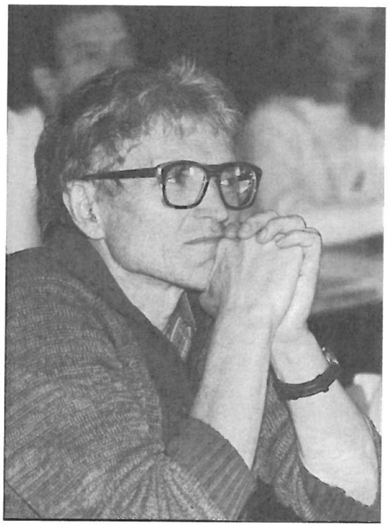
Russian Math. Surveys **68**:4 783–788 *Uspekhi Mat. Nauk* **68**:4 185–190

DOI 10.1070/RM2013v068n04ABEH004857

MATHEMATICAL LIFE

Rais Sal'manovich Ismagilov (on his 75th birthday)



On 10 July 2013, Doctor of the Physical and Mathematical Sciences and Professor Rais Sal'manovich Ismagilov turned 75. His main research interests lie in representation theory, infinite-dimensional groups, spectral theory, dynamical systems, and widths. He is the author of more than 90 research papers and the monograph *Representations of infinite-dimensional groups* [1]. In 1983 he was an invited speaker at the International Congress of Mathematicians in Warsaw. Here is a brief exposition of his best known results.

Spectral theory. Ismagilov started doing research under the guidance of Professor A. G. Kostyuchenko, whose chief focus was operator theory. In his first paper [2] Ismagilov introduced the notion of a *Carleman vector* for an unbounded operator A acting in a Hilbert space: a vector h such that $\|A^n h\| \leqslant m_n$ and $\sum m_{2n}^{-1/(2n)} < \infty$. The main result in that paper was that *if the Carleman vectors for a symmetric operator A are dense in the space, then A is essentially self-adjoint.* In fact, this was a reformulation in the language of operators of Carleman's famous test for the definiteness of the moment problem. However, Ismagilov's grasp of the relationship between these apparently quite different results was very unusual. This theorem has proved to be quite efficient in applications to concrete operators, and it led to several new tests for essential self-adjointness. We remark that five years later Nussbaum rediscovered Carleman vectors, calling them *quasi-analytic vectors.*

In the 1960s, after the papers by Rosenblum and Kato on preservation of the absolutely continuous component of a self-adjoint operator perturbed by a trace-class operator, the question of the behaviour under perturbations of the singular components of the spectrum arose in a natural way. One of the central results in Ismagilov's paper "The spectrum of Toeplitz matrices" [3] was a description of the

AMS 2010 *Mathematics Subject Classification.* Primary 01A70.

connections between the spectral types of operators A and PAP, where A is selfadjoint and P is the projection onto a codimension-1 subspace. This result said that the singular components of these operators are mutually singular (of course, their absolutely continuous components are unitarily equivalent by the Kato–Rosenblum theorem on trace-class perturbations). It was easily deduced from this result that the spectrum of a Toeplitz operator is absolutely continuous (an analytic proof of this was obtained earlier by Rosenblum). Finally, the multiplicity function was indicated, and thus a complete description of the spectrum of a Toeplitz operator was given. The same paper contained without proof another beautiful result (its proof was given in Ismagilov's Ph.D. dissertation): *if A and B are bounded selfadjoint operators and AB is of trace class, then $(A+B)_{\mathrm{a}} \simeq A_{\mathrm{a}} \oplus B_{\mathrm{a}}$ in the sense of unitary equivalence* (here A_{a} is the absolutely continuous component of the operator A). Subsequently Howland and Kato, who understood the importance of this result, published an independent proof of it (close to Ismagilov's) in [4].

We also mention two of Ismagilov's papers on Sturm–Liouville operators, containing results now regarded as classical. In the first paper [5] he found a remarkable localization property of the Sturm–Liouville operator $L_q = -d^2/dx^2 + q$ on the whole of $\mathbb{R}$. For an arbitrary interval $\Delta = (a, b)$ he introduced the quantity

$$\lambda(\Delta) = \inf(L_q y, y), \quad \text{where } y \in C_0^\infty(\Delta), \ \|y\|_{L^2(a,b)} = 1,$$

and proved the inequality $\lambda(-\infty, \infty) > \inf_{|\Delta|=1} \lambda(\Delta) - 32$. This means that if an operator is uniformly semibounded on each interval of a fixed length with Dirichlet conditions at the endpoints, then it is semibounded on the whole axis. This result was not duly appreciated when it was published and was rediscovered 18 years later (see [6] for the details). Subsequently Simon introduced the term *IMS-localization formula* (for Ismagilov, Morgan, and Sigal, the first of whom discovered the formula, while the other two extended it to more general operators).

In the other paper [7] Ismagilov discussed the deficiency indices of a Sturm–Liouville operator $L_q = -d^2/dx^2 + q$, $y(0) = 0$ on the half-axis $\mathbb{R}^+$. It turned out that one can determine whether L_q is essentially self-adjoint (that is, has zero deficiency indices) when one has information about the behaviour of q not on the whole of $\mathbb{R}^+$ but only on a certain system of disjoint closed intervals Δ_k tending to infinity. In particular, *the deficiency indices are equal to zero if $\sum_k |\Delta_k|^2 = \infty$ and $q(x) > -C|\Delta_k|^{-2}$ for $x \in \Delta_k$* (there are also other sufficient conditions). However, there is no such result when the system of intervals is replaced by a closed nowhere dense unbounded set.

Ismagilov's interest in operator theory has never faded. In the past decade he wrote several papers in this area (some of them together with Kostyuchenko). Here we point out new and fundamental results on the theory of Sturm–Liouville operators with a matrix-valued potential or with a potential which does not ensure semiboundedness of the operator.

Representation theory and infinite-dimensional groups. In 1964–1968, under the influence of Naimark's work, Ismagilov studied group representations on infinite-dimensional spaces with an indefinite inner product. New phenomena appear in this situation in comparison with unitary representations (the spectra get more complicated, and complete reducibility fails).

In the same period he began a search for new methods in representation theory. In the classical theory a prominent part is played by convolution algebras of functions which are constant on double cosets $K\backslash G/K$, where K is a compact subgroup of the group G (for example, Iwahori–Hecke algebras). Now let $G = \mathrm{SL}(2,\mathbb{F})$, where $\mathbb{F}$ is a non-locally compact non-Archimedean field, and let $K \subset \mathrm{SL}(2,\mathbb{F})$ be a subgroup of matrices with integer entries. Ismagilov discovered a surprising fact which has no finite-dimensional analogue: in this case the double coset space has the natural structure of a semigroup acting in the space of K-fixed vectors of unitary representations of G. Later it was understood that such semigroup operations are quite common for infinite-dimensional groups (including classical and symmetric groups) and can be a powerful tool in their investigation. This cycle of Ismagilov's papers was one of the starting points for the theory of infinite-dimensional groups.

In the 1970s to early 80s he was interested in representations of diffeomorphism groups and current groups, an area new at that time and now understood to be complicated. He looked at various approaches to the subject: representations on spaces of functions on point configurations, representations induced from representations of the fundamental group, constructions of central extensions with the use of amalgams, highest-weight representations of a diffeomorphism group (the subject of a talk at Kirillov's seminar in 1980, not published), representations of the group of diffeomorphisms preserving a foliation, use of the Araki scheme, and constructions of actions of diffeomorphism groups on infinite-dimensional symplectic manifolds. We discuss two series of papers.

Consider the *current group* $\mathscr{M} = C^\infty(X,K)$ of smooth maps r from a manifold X to a compact Lie group K (for instance, $\mathrm{SU}(2)$) and the L^2 space of functions from X to the Lie algebra $\mathfrak{k}$. We look at the natural unitary action $r\colon f(x) \mapsto r(x)^{-1}f(x)r(x)$ of the group $\mathscr{M}$ on this L^2 space and modify it as follows: $r\colon f(x) \mapsto r(x)^{-1}f(x)r(x) + r(x)^{-1}dr(x)$. This gives us an action of $\mathscr{M}$ by isometric affine transformations. On the other hand, the group of isometric affine transformations of a Hilbert space has a standard representation on the Fock space. The resulting action of the current group on the Fock space subsequently became an object of interest for many authors ('energy representations' was the term used by Høegh-Krohn and Albeverio, who rediscovered them later).

Another important result was the construction of an action of a diffeomorphism group on the L^2-space on a set of Poisson configurations (this action was also constructed by Goldin, Grodnik, Powers, and Sharp and by Vershik, Gelfand, and Graev). We remark that the resulting representation can be realized in the L^2-space with respect to a Gaussian measure, which gives us a canonical correspondence between L^2-spaces with respect to Gaussian and Poisson measures.

Several of Ismagilov's papers from 2006–2008 were concerned with analogues of the Racah coefficients for infinite-dimensional representations. Namely, for a given group G consider the tensor product $(\rho_1 \otimes \rho_2) \otimes \rho_3 = \rho_1 \otimes (\rho_2 \otimes \rho_3)$ of three representations of G. Decomposing it into irreducible representations in different orders, we obtain ostensibly different answers, which are related by a non-trivial transition matrix (operator). For finite-dimensional representations of SL_2 this produces Racah's ${}_4F_3$-polynomials well known in mathematical physics. Ismagilov was the first to look at this problem in the infinite-dimensional case, for $\mathrm{SL}_2(\mathbb{C})$ and the isometry group of $\mathbb{R}^3$. He obtained answers that turned out to be connected with interesting geometry (moduli spaces of flat connections on surfaces and Klyachko's varieties of hinged polygonal lines).

Dynamical systems. Here we mention papers from 1973–2002 devoted to sparse random walks on locally compact groups G and matrix Riesz products. We shall discuss the simplest case. Let $G = \mathbb{R}$, and fix two sequences of positive numbers p_k and q_k with $p_k + q_k = 1$ and a sequence $h_k \in \mathbb{R}$. Assume that a point wanders in $\mathbb{R}$ in such a way that at the kth step it stays in place with probability p_k or moves by h_k with probability q_k. We consider the tail sigma-algebra of this random walk and the induced measure ν on it (we view two trajectories of the walk as equivalent if they coincide starting from some point, and we consider measurable subsets consisting of equivalence classes). If the sequences p_k, q_k, and h_k are constant, then the resulting measure space is trivial (by the zero-one law). More general cases (*Mackey actions*, a *special flow*) have been extensively studied by various authors (Golodets, Sinel'shchikov, Zimmer). In 1986 Ismagilov proposed a criterion for the triviality of the tail algebra (in terms of the Wiener algebra and the behaviour of the products $\prod_{k=m}^{n}(1+e^{ish_k})$). After that he considered the case of 'sparse walks', when the sequence h_k grows rapidly: $h_k > h_1 + \dots + h_{k-1}$. He showed that the spectral measure for the action of $\mathbb{R}$ on $L^2(\nu)$ was given by the 'Riesz product'

$$\sigma(s) = \prod_{k=1}^{\infty}\bigl(1 + 2\sqrt{p_k q_k}\cos(h_k s)\bigr) = \prod_{k=1}^{\infty}\bigl|\sqrt{p_k} + \sqrt{q_k}e^{ish_k}\bigr|^2,$$

which converges in the sense of distributions to a positive measure on $\mathbb{R}$ (according to F. Riesz). If $h_{k+1}/h_k > 3 + \varepsilon$ and $\sum p_k q_k = \infty$, then the measure obtained is purely singular.

Widths. Ismagilov's papers on widths made him widely known. The value $d_n(C, X)$ of the Kolmogorov n-width characterizes the best approximation of a subset C of a normed space X by n-dimensional subspaces.

In his first paper on widths [8] Ismagilov showed that the widths of a subset of a Hilbert space X have a two-sided bound in terms of the eigenvalues and eigenfunctions of a certain integral operator whose kernel can be expressed in terms of the inner products of vectors in the subset. This enabled him to prove new results on the asymptotic behaviour of widths, and it drew the attention of experts in operator theory to this area.

Another of Ismagilov's papers [9] marked a crucial step in the study of the asymptotic behaviour of the widths in $L_q(\mathbb{T})$ of the Sobolev classes $W_p^r(\mathbb{T})$ for $p \leqslant q$ and $q \geqslant 2$. It is easy to show that the deviation $d(W_p^r(\mathbb{T}), \mathscr{T}_n, L_q(\mathbb{T}))$ of the Sobolev class from the spaces $\mathscr{T}_n$ spanned by the trigonometric polynomials of degree $\leqslant n$ is weakly (with respect to n) equivalent to $n^{-(r-(1/p-1/q)_+)}$. It was also fairly quickly proved that approximation by the spaces of trigonometric polynomials yields the correct order of decrease of the Kolmogorov widths for $p > q$. Experts were fairly sure that this should also hold for $1 \leqslant p \leqslant q$ and $q \geqslant 2$. However, Ismagilov disproved this: for $1 \leqslant p < \infty$ and $q = \infty$ the asymptotic behaviour is different. To this end he used methods from other areas, methods completely unexpected to those interested in these problems. In particular, it turned out that the linear span of n non-consecutive trigonometric polynomials can produce a better approximation than the space $\mathscr{T}_n$ of trigonometric polynomials of degree n. This led to the new notions of *trigonometric* and *absolute widths*, which later proved to be very important. This was a breakthrough in a fortress wall, after which many authors became interested in the problem. A fundamentally new approach to this circle of problems was found by Kashin, and further developments were associated with the names of Belinskii, Galeev, Gluskin, Kulanin, Maiorov, Temlyakov, and others.

On finishing his postgraduate studies, Ismagilov was assigned to the newly organized Moscow Institute for Electronic Machine Building, which soon became a concentration point for experts in mathematics and a centre of mathematical education in Moscow. He worked there for 25 years, as a professor since 1978 (he defended his D.Sc. dissertation in 1973), and from 1982 to 1988 he was the head of the Department of Algebra and Analysis. For the last 20 years he has been at Bauman University (Moscow State Technical University). Ismagilov's lectures have always been distinguished by their clarity and by his ability to make complicated things sound simple, even to listeners far from mathematics.

We end by a quotation from his note "The Steiner chain or a passion for computations", on Steiner's porism, which was published in the journal *Kvant* in 2003.

The ideas presented in the previous section were developed by me in 1954, *during the summer vacation after my 9th school year. Of course, I did not know at that time that the problem had been stated and solved by J. Steiner as long ago as the* 19*th century. Neither did I know about inversion or about solving recursion relations. However, this was precisely a case when ignorance turned into a blessing. Because what would have happened if someone had quickly provided me with all this information? I would have been deprived of a couple of exciting weeks, full of everything that constitutes creative work: intense curiosity (about whether the chain of circles will close or not), the fortuitous idea of using Vièta's formula, and the final formula which gave a solution to the puzzle. I was lucky: in a village school of* 1954 *no one could imagine today's kaleidoscope of extracurricular groups, olympiads, specialized schools and lyceums, "Steps to the Future" (there are now such events), and so forth and so on. A well-known maxim says that creative work on one's own has greater value than a bulk of knowledge, and it is intellectual hunger rather than satiety that drives one to creativity.*

I will finish with this maxim, but let the reader who wants to follow it maintain a sense of proportion and common sense.

In congratulating Rais Sal'manovich Ismagilov on his birthday, we wish him every kind of well-being and further successful work for the sake of mathematics and mathematical education.

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Translated by N. KRUZHILIN