Functional Analysis and Its Applications, Vol. 41, *No.* 4, *pp.* ??–??, 2007
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 41, *No.* 4, *pp.* 60–69, 2007

K-Finite Matrix Elements of Irreducible Harish-Chandra Modules are Hypergeometric Functions*

Yu. A. Neretin

Received March 31, 2006

To the memory of Dmitry Petrovich Zhelobenko, by whose classical book the author learned Lie groups and representations

Abstract. We show that each K-finite matrix element of an irreducible infinite-dimensional representation of a semisimple Lie group can be obtained from spherical functions by a finite collection of operations. In particular, each matrix element admits a finite expression via the Heckman–Opdam hypergeometric functions.

Key words: semisimple Lie group, Harish-Chandra module, infinite-dimensional representation, spherical function, matrix element, special function, Heckman–Opdam hypergeometric function.

1. Preliminaries and Notation

1.1. Semisimple groups. Notation. Let G be a linear real semisimple Lie group, and let K be the maximal compact subgroup in G. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{U}(\mathfrak{g})$ be its universal enveloping algebra. Let L_X and R_X, where $X \in \mathfrak{g}$, be the left and right Lie derivatives on G. Let $\mathfrak{U}_l(\mathfrak{g})$ (respectively, $\mathfrak{U}_r(\mathfrak{g})$) be the algebra of differential operators on G generated by the left (respectively, right) derivatives.

By P we denote the *minimal* parabolic subgroup** in G. Consider the decomposition $P = MAN$, where N is the nilpotent radical, MA is the reductive Levi factor of P, M is the compact subgroup, and $A \simeq (\mathbb{R}^*_+)^k$ is the vectorial subgroup.***

1.2. Example: $\mathbf{SL}(n,\mathbb{R})$. Let $G = \mathrm{SL}(n,\mathbb{R})$ be the group of real $n \times n$ matrices g with determinant 1. Then $K = \mathrm{SO}(n)$ is the subgroup of orthogonal matrices. The parabolic subgroup P consists of upper-triangular matrices

$$g = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad \prod_{j=1}^{n} a_{jj} = 1.$$

The nilpotent subgroup $N \subset P$ consists of matrices such that $a_{11} = a_{22} = \cdots = 1$.

The Levi subgroup $MA \subset P$ consists of diagonal matrices; i.e., $MA \simeq (\mathbb{R}^*)^{n-1}$.

The compact factor $M \simeq (\mathbb{Z}_2)^{n-1}$ consists of diagonal matrices with eigenvalues ± 1.

The vectorial subgroup $A \simeq (\mathbb{R}^*_+)^{n-1}$ consists of diagonal matrices with positive entries.

Recall that a complete *flag* $\mathscr{V}$ in $\mathbb{R}^n$ is a collection

$$\mathscr{V} \colon 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n, \qquad \dim V_j = j, \tag{1}$$

*Supported by grants FWF-P19064, NWO–047.017.015, and JSPS-RFBR 07.01.91209 and by the Russian Federal Agency for Nuclear Energy.

**For a definition, e.g., see [10] and [7]. For the groups $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{H})$, the subgroup P is the stabilizer of a complete flag; for the remaining classical series $\mathrm{SU}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}^*(2n)$, and $\mathrm{O}(n,\mathbb{C})$, this subgroup is the stabilizer of a complete isotropic flag.

***We denote by $\mathbb{R}^*$ (respectively, $\mathbb{R}^*_+$ or $\mathbb{C}^*$) the multiplicative group of real (respectively, positive real or complex) numbers.

of subspaces. The homogeneous space $\mathrm{SL}(n,\mathbb{R})/P$ is the space of all flags in $\mathbb{R}^n$. The subgroup $P \subset \mathrm{SL}(n,\mathbb{R})$ is the stabilizer of the standard flag

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n-1}$$

of coordinate subspaces.

The Grassmannian Gr_α is the space of all α-dimensional subspaces in $\mathbb{R}^n$. The Grassmannian is the homogeneous space $\mathrm{Gr}_\alpha = \mathrm{SL}(n,\mathbb{R})/Q_\alpha$, where Q_α is the maximal parabolic subgroup consisting of the block $(\alpha + (n-\alpha)) \times (\alpha + (n-\alpha))$ matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.

1.3. Principal series. Any character of $A \simeq (\mathbb{R}_+^*)^k$ has the form $\chi_s(x_1,\dots,x_k) = \prod x_j^{s_j}$, where the s_j are complex numbers. Below we denote such characters by χ or χ_s. We denote by $\widehat{A}$ the group of all characters $A \to \mathbb{C}^*$.

Let $\chi \in \widehat{A}$. Let τ be an irreducible representation of M. Denote by $\chi \otimes \tau$ the representation of $P = MAN$ that is equal to χ on A, equal to τ on M, and trivial on N. By $\mathrm{Ind}_P^G(\chi \otimes \tau)$ we denote the representation of G induced from $\chi \otimes \tau$ (see [10] and [9]), i.e., a representation of the *principal nondegenerate series.*

Let us describe this construction in a self-contained way. Denote by Ξ the space of the representation $\chi \otimes \tau$. Consider the space $G \times \Xi$ and its quotient $G \times_P \Xi$ with respect to the equivalence relation

$$(g,\xi) \sim (gr, (\chi \otimes \tau)(r) \cdot \xi), \qquad g \in G,\ r \in P,\ \xi \in \Xi.$$

We have the natural map $G \times_P \Xi \to G/P$ given by $(g,\xi) \mapsto gP$; the fiber at each point is noncanonically isomorphic to Ξ. Thus, we obtain a bundle over G/P with fiber Ξ. The group G acts on the total space of the bundle by the transformations $h : (g,\xi) \mapsto (hg,\xi)$.

In particular, the group G acts on the space of smooth sections of the bundle. This action is just the representation $\mathrm{Ind}_P^G(\chi \otimes \tau)$.

1.4. Example: $\mathbf{SL}(n,\mathbb{R})$. Let us give an alternative description of the representations $\mathrm{Ind}_P^G(\chi \otimes 1)$, where 1 is the trivial one-dimensional representation of M.

Consider the flag space G/P and the corresponding Grassmannians $\mathrm{Gr}_\alpha = G/Q_\alpha$. We equip each space G/Q_α with a K-invariant volume form* Ω_α. For $\mathscr{V} \in G/P$, let V_α be its image under the map $G/P \to G/Q_\alpha$. (Thus, we forget all subspaces in (1) except for V_α.)

For $g \in G$, denote by $J_\alpha(g,\mathscr{V})$ the Jacobian

$$J_\alpha(g,\mathscr{V}) := [\Omega_\alpha(gV_\alpha)/\Omega_\alpha(V_\alpha)]$$

of the transformation $g\colon G/Q_\alpha \to G/Q_\alpha$ at the point V_α.

For $t_\alpha \in \mathbb{C}$, we define the representation

$$\rho_t(g) f(\mathscr{V}) := f(g\mathscr{V}) \prod_\alpha J_\alpha(g,\mathscr{V})^{t_\alpha}$$

of G in the space of functions on G/P.

One can readily verify that this family of representations coincides with the family $\mathrm{Ind}_P^G(\chi_s \otimes 1)$, where χ_s ranges in $\widehat{A}$ (and the dependence $s = s(t)$ is some linear transformation).

At the level of the Lie algebra, this representation is defined by

$$\rho(X) f(\mathscr{V}) = (Xf)(\mathscr{V}) + \sum_\alpha t_\alpha [\mathscr{L}_X \Omega_\alpha/\Omega_\alpha] f(\mathscr{V}),$$

where $X \in \mathfrak{g}$ and $\mathscr{L}_X$ is the Lie derivative (so that, in particular, $\mathscr{L}_X \Omega_\alpha/\Omega_\alpha$ is the divergence of the vector field X).

1.5. Irreducible representations. For fixed τ, the representation $\mathrm{Ind}_P^G(\chi_s \otimes \tau)$ is irreducible for generic χ_s. The singular set of s is contained in some locally finite family of complex hyperplanes.

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*Or density if G/Q_α is nonorientable.

By the subquotient theorem,* each irreducible representation of G is *equivalent* to a subquotient (and even to a subrepresentation) of some representation $\mathrm{Ind}_P^G(\chi \otimes \tau)$; e.g., see [3] and [10].

Recall the meaning of the term *equivalent* in the previous statement. Let ρ be a representation of G in a complete locally convex linear topological space V. A vector $v \in V$ is K-*finite* if the set Kv spans a finite-dimensional subspace. The space of K-finite vectors is dense in V. If ρ is irreducible, then the Lie algebra $\mathfrak{g}$ preserves the subspace of K-finite vectors. A natural equivalence of irreducible representations is an equivalence of the corresponding $\mathfrak{g}$-modules of K-finite vectors.**

Let V° be the dual space. By $\{\,\cdot\,,\,\cdot\,\}$ we denote the pairing $V \times V^\circ \to \mathbb{C}$. A *matrix element* of the representation ρ is a function on G of the form

$$f(g) = \{\rho(g)h, h^\circ\}, \quad \text{where } h \in V \text{ and } h^\circ \in V^\circ \text{ are fixed.}$$

A matrix element is said to be K-*finite* if the vectors h and h° are K-finite.

1.6. Spherical representations (see [10] **and** [8]**).** Recall that an irreducible representation ρ of the group G in a space V is said to be *spherical* if there exists a K-fixed vector $v \in V$. By Gelfand's theorem, such a vector is unique, and hence we have the canonical matrix element

$$\Psi(g) = \{\rho(g)v, v^\circ\}, \quad \text{where } v \in V \text{ and } v^\circ \in V^\circ \text{ are the spherical vectors and } \{v, v^\circ\} = 1.$$

This element is called a *spherical function.* By construction, $\Psi(g)$ is biinvariant with respect to K,

$$\Psi(k_1 g k_2) = \Psi(g), \qquad k_1, k_2 \in K.$$

Hence spherical functions can be viewed as functions on the double cosets $K \setminus G/K$.

In particular, Ψ is uniquely determined by its values on the subgroup A. Recall that $G = KAK$.

On the other hand, spherical functions can be treated as functions on the (symmetric) space G/K.

Spherical functions are special cases of the Heckman–Opdam multivariate hypergeometric functions; see [6].

Each representation $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ of the principal series has a unique K-invariant vector.*** For generic s, the representation $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ is irreducible and hence is a spherical representation itself. Each reducible representation $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ contains a unique spherical irreducible subquotient, and Q1
each spherical representation of G can be obtained in this way.

Spherical representations are numbered by orbits of the Weyl group W on $\widehat{A} \simeq \mathbb{C}^k$. For a given $s \in \mathbb{C}^k$, we denote by $\Psi_s(g)$ the corresponding spherical function on G. This expression is holomorphic in s.

2. Statement of the Result

2.1. Formulation. For a finite-dimensional representation ξ of G, let $\mathfrak{M}(\xi)$ be the space of finite linear combinations of matrix elements of ξ.

Let V be a Harish-Chandra module (e.g., see [10]) over G. We denote by $\pi(g)$ the operators of representation of G in some completion of V. Let σ range in the set $\widehat{K}$ of all irreducible representations of K. Let $V = \bigoplus_\sigma V_\sigma$ be the decomposition of V into a direct sum of K-isotypical components (e.g., see [9]).

Proposition. (a) *Let* $(\pi, V) = \mathrm{Ind}_P^G(\chi \otimes \tau)$ *be irreducible. Let* V° *be the dual module. Let* $v \in V_\sigma$ *and* $w \in V_\theta^\circ$. *There exists an irreducible finite-dimensional representation* ξ *of* G *and an* $s \in \mathbb{C}^k$ *such that the matrix element* $\{\pi(g)v, w\}$ *is a finite sum of the form*

$$\{\pi(g)v, w\} = \sum_j h_j(g) \cdot p_j q_j \Psi_s(g),$$

*This theorem is due to Harish-Chandra and Casselman and has strong versions due to Zhelobenko–Naimark (complex groups) and Langlands (real groups) with explicit indication of the subquotients.

**An "irreducible Harish-Chandra module" mentioned in the title is the same as the $\mathfrak{g}$-module of K-finite vectors of an irreducible representation.

***In the model of Sec. 1.4, it is a constant function.

where $h_j \in \mathfrak{M}(\xi)$, $p_j \in \mathfrak{U}_l(\mathfrak{g})$, *and* $q_j \in \mathfrak{U}_r(\mathfrak{g})$.

(b) *Let* $(\pi, V) = \operatorname{Ind}_P^G(\chi \otimes \tau)$ *be reducible. Then each* K*-finite matrix element admits a representation*

$$\{\pi(g)v, w\} = \lim_{\varepsilon \to 0} \sum_j h_j(\varepsilon; g) \cdot p_j(\varepsilon) q_j(\varepsilon) \Psi_{s+\varepsilon t}(g), \tag{2}$$

where s *and* t *are some elements of* $\mathbb{C}^k$, *the sum is finite, and* $h_j(\varepsilon; g) \in \mathfrak{M}(\xi)$, $p_j(\varepsilon) \in \mathfrak{U}_l(\mathfrak{g})$, *and* $q_j(\varepsilon) \in \mathfrak{U}_r(\mathfrak{g})$ *rationally depend on the parameter* ε.

(c) *For an arbitrary* K*-finite matrix element of an irreducible Harish-Chandra module, there exists a representation of the form* (2).

2.2. Application. Domains of holomorphy of matrix elements. Let $G_{\mathbb{C}}$ be the complexification of the group G. Let $K_{\mathbb{C}}$ and $A_{\mathbb{C}} \subset G_{\mathbb{C}}$ be the complex subgroups corresponding to K and A. In particular, $A_{\mathbb{C}}$ is the complex torus $(\mathbb{C}^*)^k$.

Corollary. *Each domain* $\Omega \subset G_{\mathbb{C}}$, $\Omega \supset G$, *of holomorphy of all spherical functions is a domain of holomorphy of all* K*-finite matrix elements of all irreducible Harish-Chandra modules over* G.

Proof. A priori, the sum on the right-hand side in (2) is holomorphic for $g \in \Omega$ and $0 < |\varepsilon| < \delta$ and may have a pole at $\varepsilon = 0$. In the latter case, the function has an infinite limit at all points of the hypersurface $\varepsilon = 0$ except for a complex submanifold of codimension 1.

On the other hand, it has a finite limit as $\varepsilon \to 0$ for each $g \in G$. Hence our expression has in fact no pole on the surface $\varepsilon = 0$. □

Let us discuss some versions of this corollary. Recall that $A_{\mathbb{C}} \simeq (\mathbb{C}^*)^k$. We denote elements of this group by $a = (a_1, \dots, a_k)$.

1°. *The Akhiezer–Gindikin domain.* Let $\mathscr{A} \subset A_{\mathbb{C}}$ be the subset consisting of a such that $|\arg a_j| < \pi/4$. The Akhiezer–Gindikin domain $A\Gamma$ (see [1] and also [19]) is defined to be $G \cdot \mathscr{A} \cdot K_{\mathbb{C}}$. Since all spherical functions are holomorphic in $A\Gamma$, we see that all K-finite matrix elements are holomorphic in $A\Gamma$ as well. (This was established in [12].)

2°. *The Casselman phenomenon.*

Each K*-finite matrix element of the group* G *admits an extension to a multivalued holomorphic function* $G_{\mathbb{C}}$ *having singularities* (*branchings*) *on some fixed* (*nonsmooth*) *submanifold of codimension* 1 *in* $G_{\mathbb{C}}$.

First, consider the case $G = \mathrm{SL}(n, \mathbb{R})$. Let $\mathscr{X}_n$ be the space of complex symmetric matrices with determinant 1. The group $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ acts on $\mathscr{X}_n$ by the formula $g\colon X \mapsto gXg^t$. The subgroup $K_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$ is the stabilizer of the point $X = E$; obviously, $\mathscr{X}_n \simeq G_{\mathbb{C}}/K_{\mathbb{C}}$.

Let $\mathscr{X}_n^\circ \subset \mathscr{X}_n$ be the set of matrices with pairwise distinct eigenvalues. Each matrix $\in \mathscr{X}_n^\circ$ can be reduced to diagonal form by an orthogonal transformation. According to [6], each spherical function can be extended as a branching analytic function to the submanifold of diagonal matrices in $\mathscr{X}_n$; the functions thus obtained have branching on the set of matrices with multiple eigenvalues. Next, we extend these functions to $\mathscr{X}_n$ by $K_{\mathbb{C}}$-invariance and lift them from the homogeneous space $G_{\mathbb{C}}/K_{\mathbb{C}}$ to the group $G_{\mathbb{C}}$. Let $G_{\mathbb{C}}^\circ$ be the preimage of $\mathscr{X}_n^\circ$ in $G_{\mathbb{C}}$. A shorter description of $G_{\mathbb{C}}^\circ$ is as follows: *a matrix* $g \in G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ *is contained in* $G_{\mathbb{C}}^\circ$ *if the eigenvalues of* $g^t g$ *are pairwise distinct.*

By construction, the spherical functions of the group $G = \mathrm{SL}(n, \mathbb{R})$ admit multivalued holomorphic extension to the domain $G_{\mathbb{C}}^\circ \subset G_{\mathbb{C}}$. Hence all K-finite matrix elements of $\mathrm{SL}(n, \mathbb{R})$ can be extended to this domain.

Now let us describe the domain $G_{\mathbb{C}}^\circ$ for an arbitrary linear semisimple group G. Consider the symmetric space $\mathscr{X} = G_{\mathbb{C}}/K_{\mathbb{C}}$ and the $A_{\mathbb{C}}$-orbit D of the initial point in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Let D° be the set obtained from D by removing all points having a nontrivial stabilizer in the Weyl group. We define $\mathscr{X}^\circ$ as the union of all $K_{\mathbb{C}}$-orbits that meet D°. Now $G_{\mathbb{C}}^\circ$ is the preimage of $\mathscr{X}^\circ$ in $G_{\mathbb{C}}$.

Each K-finite matrix element of G extends to be a multivalued holomorphic function on $G_{\mathbb{C}}^\circ$.

This fact is known; it was discovered in Casselman's famous unpublished paper, which is unavailable to the author; however, see [3].

3°. Thus, we see that a representation ρ of G can be extended to a branching holomorphic function on $G_{\mathbb{C}}^{\circ}$ ranging in the space of infinite matrices. A product $\rho(g_1)\cdot\rho(g_2)$ of such matrices generally diverges but is sometimes well defined (see [14] and [12]).

For any $X\in\mathfrak{g}_{\mathbb{C}}$ and $g\in G_{\mathbb{C}}^{\circ}$, we have

$$\frac{d}{d\varepsilon}\rho(\exp(\varepsilon X)g)=\rho(X)\rho(g).$$

The matrix $\rho(X)$ has only finitely many nonzero matrix elements in each row and each column, and hence the product of matrices is well defined.

4°. *Groups of complex type.* For the groups $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$, and $\mathrm{SU}(p,q)$, closed-form expressions for the spherical functions are known (see [4] and [2]). Hence the expressions for the analytic continuations can be written out as well.

For example, let $G=\mathrm{SL}(n,\mathbb{C})$. Then $G_{\mathbb{C}}\simeq\mathrm{SL}(n,\mathbb{C})\times\mathrm{SL}(n,\mathbb{C})$, and the subgroup $G\subset G_{\mathbb{C}}$ consists of elements of the form $(g,\overline{g})$. Spherical functions on $G_{\mathbb{C}}$ are given by*

$$\Psi_s(g,h)=\frac{\det_{1\leqslant j,k\leqslant n}\{\lambda_j^{s_k/2}\}}{\det_{1\leqslant j,k\leqslant n}\{\lambda_j^k\}\cdot\det_{1\leqslant j,k\leqslant n}\{s_j^k\}},\tag{3}$$

where the λ_j are the eigenvalues of gh^t. The function $\Psi(g)$ is multivalued, since so is a power of a complex number. Our expression is well defined if all λ_j are pairwise distinct.

Further, the powers are single-valued in the domain $\arg\lambda_j<\pi$, and the zeros of the denominator coincide with those of the numerator. Thus, we obtain a holomorphic function on $G_{\mathbb{C}}$ minus the real submanifold

$$\{\arg\lambda_j=\pi\ \text{ for some } j\}$$

of codimension 1. However, bypassing the remote singularity and returning to the group G, we obtain another branch. This branch, generally speaking, has a singularity on G supported by matrices with multiple singular numbers. Q2

For all complex semisimple groups, spherical functions can be written out explicitly in the spirit of (3); see [4]. In particular, all their matrix elements are elementary** functions.

For the groups $\mathrm{U}(p,q)$, the analog of (3) contains a determinant of one-dimensional hypergeometric functions, see [2]. Hence all their matrix elements can be expressed in the terms of usual Gauss hypergeometric functions (including derivatives of hypergeometric functions with respect to indices).

5°. *Extended braid groups and monodromy.* Let $\mathscr{Z}(\mathfrak{g})$ be the center of the enveloping algebra. Fix a character λ of $\mathscr{Z}(\mathfrak{g})$. Let σ and θ be irreducible representations of K acting in spaces W_σ and W_θ, respectively. Consider (branching) functions $f\colon G^{\circ}\to\mathrm{Hom}(W_\sigma,W_\theta)$ satisfying the conditions

- $f(k_1gk_2)=\theta(k_1)f(g)\sigma(k_2^{-1})$ for $g\in G_{\mathbb{C}}^{\circ}$ and $k_1,k_2\in K_{\mathbb{C}}$.
- $pf=\lambda(p)f$ for each $p\in\mathscr{Z}(\mathfrak{g})$.

The monodromy group for this problem is the so-called extended Artin braid group, see [6]. As far as I know, these monodromy representations have never been studied.

6°. *Large domains of holomorphy.* There are exceptional situations in which a unitary representation admits a holomorphic continuation to the entire complex group or into a subsemigroup; apparently, such cases are well understood.

For infinite-dimensional representations of semisimple Lie groups, the only possible case is highest weight representations, which admit an extension to Olshanski semigroups (see [18]). Some other situations are discussed in [15], [16, Secs. 1.1, 4.4, 5.4, 7.4–7.6, 9.7], [13], and [5]. It may happen that there are other cases of unexpectedly large (nonsemigroup) domains of holomorphy. As far as I know, this question has never been considered.

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*This formula is the analytic continuation of the Weyl character formula.

**but very complicated . . .

2.3. Nonlinear semisimple Lie groups. For universal coverings of the groups $\mathrm{SU}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, and $\mathrm{SO}^*(2n)$, our construction survives; we only have to replace spherical functions by appropriate Heckman–Opdam hypergeometric functions (see [7, Chap. 1]).

I do not know whether it is possible to express the matrix elements of universal covering groups of $\mathrm{SO}(p,q)$ and $\mathrm{SL}(n,\mathbb{R})$ via the Heckman–Opdam hypergeometric functions.

3. Proof of the Proposition

First, we prove our assertion for the modules $\mathrm{Ind}_P^G(\chi_s \otimes 1)$; this is more or less obvious. Therefore, the assertion is valid for tensor products of such modules by finite-dimensional representations. The class of representations thus obtained contains whatever one likes as subfactors.

3.1. Proof for spherical representations. Let ρ be a spherical representation with parameter s in a space V. Let $h \in V$ be the spherical vector, and let h° be the spherical vector in V°. Since ρ is irreducible, it follows that vectors $v \in V_\sigma$ and $w \in V_\theta^\circ$ can be represented in the form $v = p(X) \cdot h$ and $w = q(X) \cdot h^\circ$, where $p(X)$ and $q(X)$ are appropriate elements of the enveloping algebra. Let $q(-X)$ be the element obtained from $q(X)$ by the standard antiinvolution on $\mathfrak{U}(\mathfrak{g})$. (It is defined as $X \mapsto -X$ for $X \in \mathfrak{g}$.) Then

$$\begin{aligned}\{\rho(g)v, w\} &= \{\rho(g) \cdot p(X) \cdot h, q(X) \cdot h^\circ\} \\ &= \{q(-X) \cdot \rho(g) \cdot p(X) \cdot h, h^\circ\} = q(-L_X)\Psi_s p(R_X).\end{aligned}$$

3.2. Representations $\mathrm{Ind}_P^G(\chi \otimes 1)$. The standard realization. Consider the flag space G/P and the corresponding Grassmannians, i.e., the quotient spaces G/Q_α, where the $Q_\alpha \supset P$ are maximal parabolic subgroups in G. Starting from this point, we can reproduce the construction in Sec. 1.4 word for word.

Thus, we have obtained a realization of the family $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ such that

1. The action of K is independent of χ_s.

2. The operators of representation of the group G are continuous functions of s. More precisely, each K-finite matrix element is a continuous function of s.

3. The operators of representation of the Lie algebra $\mathfrak{g}$ are linear expressions in the parameters s.

3.3. Proof for the representations $\mathrm{Ind}_P^G(\chi \otimes 1)$. If $\chi = \chi_s$ is in general position (in fact, $s \in \mathbb{C}^k$ lies outside a locally finite family of complex hyperplanes), then π is an irreducible spherical representation. This situation was considered in Sec. 3.1.

Now let us examine the case of reducible π. The continuity of matrix elements as functions of the parameters s follows from Sec. 3.2.

Let V be the space of K-finite functions on the flag space G/P (see Secs. 1.4 and 3.2). Let $1 \in V$ be the function $f(\cdot) = 1$. Fix $\sigma, \theta \in \widehat{K}$, $v \in V_\sigma$, and $w \in V_\theta$.

Let $\mathfrak{U}^N(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ be the subspace consisting of all elements of degree $\leqslant N$. Let N be large enough that $\mathfrak{U}^N(\mathfrak{g}) \cdot 1$ contains the entire subspace V_σ for all generic characters* χ_s. Consider a sequence $r_1, r_2, \dots \in \mathfrak{U}^N$ such that, for generic χ_s,

1. $r_j \cdot 1$ are linearly independent in $\mathrm{Ind}_P^G(\chi_s \otimes 1)$.

2. Their linear span contains V_σ.

These properties remain valid for all s outside a certain algebraic submanifold $\mathscr{M}$ in the parameter space. Now we express the vector v as a linear combination of $r_j \cdot 1$, $v = \sum c_j(s) r_j \cdot 1$, where c_j are certain rational functions.

We reproduce the same arguments for V_θ°.

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*Decompose $\mathfrak{U}(\mathfrak{g}) = \bigoplus \mathfrak{U}(\mathfrak{g})_\sigma$ according to the adjoint action of K. The space $\mathfrak{U}(\mathfrak{g})_\sigma$ is a finitely generated module over the center $\mathscr{Z}(\mathfrak{g})$ of the enveloping algebra. For sufficiently large N, the subspace $\mathfrak{U}^N(\mathfrak{g})$ contains all generators of $\mathfrak{U}(\mathfrak{g})_\sigma$.

Let Y be an element of the Lie algebra of the group K, and let $p(X) \in \mathfrak{U}(\mathfrak{g})$. Then $(Yp(X) - p(X)Y) \cdot 1 = Yp(X) \cdot 1$. Hence $V_\sigma \subset \mathfrak{U}(\mathfrak{g})_\sigma \cdot 1 \subset \mathfrak{U}^N(\mathfrak{g}) \mathscr{Z}(\mathfrak{g}) \cdot 1 = \mathfrak{U}^N(\mathfrak{g}) \cdot 1$.

Applying Sec. 3.1, we find that our matrix element has the form $\Xi(s)\Psi_s(g)$, where $\Xi(s)$ is an element of $\mathfrak{U}_l \otimes \mathfrak{U}_r$ depending rationally on s.

Now let s_0 be a singular value of s. Consider a complex line $\gamma(\varepsilon) = s_0 + \varepsilon t$ avoiding the submanifold $\mathscr{M}$ and the singular values of the parameter s for small $|\varepsilon| > 0$. Then $\Xi(s_0+\varepsilon t)\Psi_s(s_0+\varepsilon t)$ is the desired approximation to our matrix element.

3.4. Main trick. Let ξ be an irreducible finite-dimensional representation of G in a space H. Following [11], consider the tensor product

$$\pi \otimes \xi = \mathrm{Ind}_P^G(\chi \otimes 1) \otimes \xi = \mathrm{Ind}_P^G(\chi \otimes \xi|_P). \tag{4}$$

The representation $\xi|_P$ is in general not irreducible and admits a finite filtration

$$H_1 \supset H_2 \supset H_3 \supset \dots$$

with irreducible subquotients. The nilpotent subgroup $N \subset P$ acts on the subquotients H_j/H_{j+1} trivially. The representations of the subgroup $MA \subset P$ on H_j/H_{j+1} have the form $\mu_j \otimes \tau_j$ for some characters μ_j of A and some irreducible representations τ_j of M.

Thus, the representation $\pi \otimes \xi$ has a filtration whose subquotients are representations of the principal series of the form $\mathrm{Ind}_P^G([\chi \cdot \mu_j] \otimes \tau_j)$.

3.5. Fix a representation $\widetilde{\tau}$ of M and a character $\widetilde{\chi}$ of A. We intend to realize $\mathrm{Ind}_P^G(\widetilde{\chi} \otimes \widetilde{\tau})$ as Q3
a subquotient in an appropriate tensor product (4).

We can choose a representation ξ of G such that the restriction of ξ to M contains $\widetilde{\tau}$.* Then the restriction of ξ to $P = MAN$ contains a subquotient of the form $\widetilde{\mu} \otimes \widetilde{\tau}$ with some character $\widetilde{\mu}$.

Next, we choose a character χ of A such that $\chi \cdot \widetilde{\mu} = \widetilde{\chi}$. Thus, we find that $\mathrm{Ind}_P^G(\chi \otimes 1) \otimes \xi$ contains a given representation $\mathrm{Ind}_P^G(\widetilde{\chi} \otimes \widetilde{\tau})$ as a subquotient.

3.6. K-finite matrix elements of $\mathrm{Ind}_P^G(\widetilde{\chi} \otimes \widetilde{\tau})$ are contained in the set of K-finite matrix elements of $\mathrm{Ind}_P^G(\chi \otimes 1) \otimes \xi$. The latter matrix elements are linear combinations of products of K-finite matrix elements of $\mathrm{Ind}_P^G(\chi \otimes 1)$ by matrix elements of ξ. This completes the proof of (a) and (b).

3.7. End of proof. Assertion (c) follows from (a), (b), and the subquotient theorem.

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*Proof. Let G_c be the compact form of G. Consider the induced representation $\mathrm{Ind}_M^{G_c}(\widetilde{\tau})$. Let ξ be an irreducible subrepresentation. We can treat ξ as a representation of G. By the Frobenius reciprocity (see [9]), its restriction to M contains $\widetilde{\tau}$.

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Institute of Theoretical and Experimental Physics, Math. Phys. Group
University of Vienna
e-mail: neretin@mccme.ru

Translated by Yu. A. Neretin

Questions

Q1. Ср. русский текст.
Q2. Имеются в виду собственные значения?
Q3. Почему крышки заменились на волны? Это нежелательно.