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New Math in the Classroom

In the 1970s the wave of New Math surged into secondary education. Structures inspired by Bourbaki were certainly included, but how much was the group itself involved?

"Definition 4: A subset I of an ordered set E is called an *interval* of E if it satisfies the implication $(x \in I \text{ and } y \in I \text{ and } x \leq z \leq y) \Rightarrow (z \in I)$."

"Theorem 6: The identity is the only automorphism of the field R."

"Theorem: Addition and composition of functions equips the set $L(V)$ of endomorphisms of a vector space B with the structure of a unitary ring."

"Every involutive endomorphism is a symmetry."

"Definition: We say that an orthogonal endomorphism of ϕ of E_3 is a vector rotation to express the fact that the subspace of variables invariant under ϕ has dimension 1 or 3."

"Theorem 2: Let F be a vector function and x_0 be an accumulation point in its domain of definition. Then $\lim_{X \to X_0} \overrightarrow{F(x)} = \vec{0} \Leftrightarrow \lim_{X \to X_0} \|\overrightarrow{F(x)}\| = 0$."

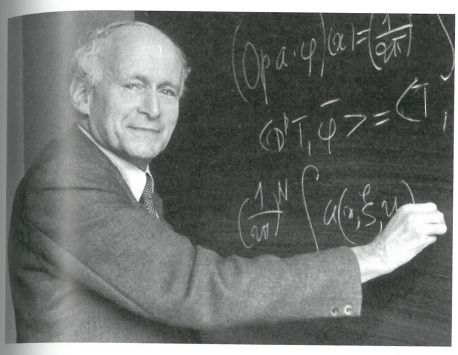
THEOREM AND DEFINITION

Given: a graduated line (Δ, g).

1. For each pair (a', b') of real numbers such that $a' \neq 0$, the function g' from Δ to R defined for any element M of Δ by $g'(M) = a' \cdot g(M) + b'$ is bijective.
2. The family of all bijections thus defined has the following property:
   For any two bijections g' and g'' of this family, there exists a pair (a, b) of real numbers such that $a \neq 0$ and $g''(M) = a' \cdot g'(M) + b$ for every element M of Δ.

Any bijection belonging to this family is called a graduation of Δ. The number $g'(M)$ is called the abscissa of M in the graduation g'.

A property of a graduated line and the definition of a graduation...in eighth grade!



Laurent Schwartz teaching about the theory of distributions. Laurent Schwartz's course revolutionized mathematics teaching at l'École Polytechnique.

Are these statements from Bourbaki's *Éléments de mathématique*? No. Or maybe from mathematical courses for university students or for students preparing to enter the presitigious Grandes Écoles? Once again, no. In fact, they are simply extracts of a French mathematics textbook for high school seniors, published in the 1970s by Hachette in their Aleph$_0$ series. And this textbook consists of six volumes—two for algebra, two for geometry, two for analysis—for a total of over 1500 pages! Its content would leave today's math student stunned.

In France and around the world, this was the era of New Math, an ambitious reform in elementary and secondary math education. However, it was a failed reform whose biggest achievements were wasting gallons of ink and creating heated controversy among parents, teachers, mathematicians, educational psychologists, and others. It was a setback whose results remain today as scars in the educational system. Some people connect Bourbaki with this reform. Why? Are they right?

Bourbaki Attacks Higher Education

Before tackling the subject of seconday education, let's discuss the reform of higher education. At least in France, Bourbaki clearly played an active role in this reform. Indeed, this was the very nature of the

group's initial project, the project to write a new treatise on analysis that would replace the obsolete and unsuitable texts used at the time in higher math education. The Bourbaki pioneers, who for the most part were professors at universities outside Paris, started to modernize math education in their respective institutions as soon as the group had been formed. For example, Henri Cartan said that "at Clermont-Ferrand, where the University of Strasbourg had moved in 1939, I taught differential calculus on Banach spaces. This was a real revolution." In this way, students at various universities outside the capital were introduced to slightly more modern mathematics than what had been taught to students just a couple years before. This revolution arrived in Paris somewhat later. While the students of l'École Normale Supérieure didn't have to wait long to benefit from Henri Cartan, who arrived in 1940, students at other Parisian institutions had to wait until the arrival of Gustave Choquet in 1954. According to Henri Cartan, at the end of each academic year the dean would hold a meeting with the math faculty to distribute the teaching of the math classes for the following year. The class on differential and integral calculus had been taught for many years by Georges Valiron, who gave traditional and boring lectures. One year, though, illness prevented Valiron from continuing to teach. At this point, Henri Cartan recalled, "I recommended Choquet, who shared our point of view on teaching." So, Gustave Choquet, who was never a member of Bourbaki, took on the job. As Choquet described to Marian Schmidt, "Until then, the course mostly followed the elementary sections of Goursat's treatise. I modified the focus and content of the course. I thus had the good fortune of being behind the revolution in French mathematical teaching, at first in the classes for advanced university students, and later, as the reforms spread contagiously, for the younger students as well." However, this revolution was not accepted by all, and conservative mathematicians protested bitterly. "For example, I remember that Henri Villat demanded, 'How can you expect students to understand when I don't understand it myself?'" Henri Cartan recalled.

In his autobiographical book *Mathématique: (récit)* (which mentions Bourbaki several times), the author-mathematician Jacques Roubaud describes how students received Choquet's revolution: "Faced with the sudden metamorphosis of mathematics occurring as they watched (and especially as they listened), even the most hardened students, those who had taken classes preparing them for the Grandes Écoles entrance exams and who had survived the slaughter of the exams to earn their certificate in general mathematics, felt their most established convictions waver. Over the course of their studies, these students had developed a stable and familiar view of mathematics, and now this view

Henri Cartan in Bombay, 1960.

was changing so much that it was becoming locked away, sealed before their eyes. And they didn't find the new face of mathematics to be especially pretty. Those students who had had to repeat a year were the most noticeably distraught, as they found nothing in common between Choquet's classes and Valiron's of the year before. It was as if this science had been replaced by a new one over the summer vacation, a new science that had kept the name of the old one merely out of convenience." Choquet quickly gained reinforcements. By the following year, several professor and lecturer positions had been vacated. According to Martin Andler, a "fierce battle" ended with victory on the part of the revolutionaries. The new faculty members for the 1955 academic year were Chevalley, Ehresmann, Pisot, Zamansky, Godement. All of these, except Zamansky, were at some point members of Bourbaki.

Bourbaki Reaches l'École Polytechnique

The situation at l'École Polytechnique also deserves to be discussed. From World War I until the 1950s, teaching at this elite school—including the teaching of mathematics—was mediocre, despite the presence of Paul Lévy (a great probabilist whose work was long unrecognized) and of Gaston Julia, who at the time was nearing the end of his career. The hiring of Laurent Schwartz (a member of Bourbaki) in 1959, following the retirement of Paul Lévy (Schwartz's father-in-law) gave new life to this institution, which had produced almost no new mathematicians for many years. Schwartz's course on analysis was very succesful, and the changes he introduced—despite resitstance—led to a profound revival of teaching at l'École Polytechnique, especially after 1968. Jean-Pierre Bourguignon described how "there was such a contrast between Schwartz's teaching and the teaching of other subjects that the need to completely renovate the other courses became obvious." According to Bourguignon, the success of this renovation was so great that l'École Polytechnique today produces mathematicians of the highest quality: "At the most recent international congress of mathematicians, the one in Berlin, four of the ten plenary addresses were given by French researchers, and three of those four had graduated from l'École Polytechnique."

Although Bourbaki actively contributed to the renovation of higher mathematics education, this renovation was not entirely the work of Bourbaki. For one thing, this contribution is not the work of Bourbaki (excepting the indirect influence of *Éléments de mathématique*) but rather the work of individual members. It is highly unlikely that

Jean Piaget in 1977.

Bourbaki developed a collective strategy to obtain key positions at universities or the Grandes Écoles (including l'École Normale Supérieure and l'École Polytechnique) or to redefine the content of mathematics courses at these institutions. In addition, the members of Bourbaki were not the only mathematicians to renovate higher mathematics education. Gustave Choquet is the most obvious non-Bourbaki to contribute to this revolution, and others include Jean Leray and André Lichnerowicz, who were appointed as professors at the prestigious Collége de France in 1947 and 1952 respectively.

It is much harder to assess the role of Bourbaki in the New Math movement in secondary education. This famous reform started in the 1950s, when it became apparent that the mathematics taught in secondary schools was worn out and poorly adapted to the modern demands of economics, technology, science and culture. This realization affected many countries, both in France and around the world. But what exactly triggered it?

In the 1950s and 1960s, the West was undergoing a profound transformation that involved economic and industrial growth, scientific and technological progress, and cultural change. Training excellent scientists and engineers was considered necessary for economic development. Furthermore, the competition between the Soviet Union and the West, intensified by the launch of the first Sputnik in 1957, made the West worried about falling behind in technology. A solid knowledge of mathematics was considered necessary for all scientists and engineers.

The evolution of mathematics itself added to this situation. Partly due to the influence of Bourbaki, mathematics had become seen throughout the world as a unified subject based on set theory and built out of general structures (groups, rings, fields, etc.) defined by axioms. This sort of mathematics was only vaguely related to that being taught in elementary and secondary schools, despite the mathematical revolution taking place in universities. Thus mathematicians, as well as young teachers who had taken Bourbaki-influenced classes in college, wished to modernize the teaching of their subject.

André Lichnerowicz (1915–1998) in 1988.

Mathematics Was Everywhere

Linked to this evolution was the idea that mathematics (or the singular noun *mathématique*, as Bourbaki called the subject) was a universal language intended to be used in all disciplines, and in particular in all sciences, including even humanities and social sciences. This went with the modern trend in mathematics of emphasizing the relations

At the 1954 Bourbaki conference in Murols: Roger Godement, Jean Dieudonné, André Weil, Saunders Mac Lane, and Jean-Pierre Serre (from left to right). Roger Godement ardently supported the implementation of Bourbaki's philosophy in university-level math education.

between objects (numbers, functions, geometric figures, and so forth) rather than the objects themselves, whose characteristics were considered to be of little importance. Many declared that "mathematics is everywhere," meaning that it is an essential part of everyone's academic and cultural knowledge. It is likely that this point of view was based partially on the fashion of structuralism affecting philosophy, literature, ethnology, linguistics, and psychology.

A final factor in sparking the New Math revolution was the new trends in pedagogy. Jean Piaget saw an analogy between the mental structures created when a child learns mathematics and the mother-structures (algebraic structures, ordered structures, and topological structures) that Nicolas Bourbaki discussed in its article *L'architecture des mathématiques* ("The Structure of Mathematics"). Furthermore, Piaget and many other educational psychologists emphasized the importance of active learning in a child's intellectual development, which created a movement for teaching based on observations, experiments, and inferences made by the students with the help of their teachers rather than on knowledge obtained directly from the teacher. The Bourbaki style of mathematics seemed better suited to this new sort of teaching than the traditional style did. In addition, mathematics seemed to be more democratic than other subjects, since its emphasis on

concepts required no cultural prerequisites. Many argued, erroneously or otherwise, that students would find math more accessible than Latin and Greek, which the schools traditionally used to choose their top students. Such an argument was by no means insignificant in a time when more students were continuing their education through high school and in the social and political atmosphere that led to the May 1968 student uprisings in France.

"Down With Euclid!"

For most countries, the New Math revolution consisted of four phases. The first was a phase of realization and reflection. An important event of this stage was the November 1959 Colloque de Royaumont, a ten-day conference organized by the Organisation for European Economic Cooperation (now called the Organisation for Economic Cooperation and Development) to promote a reform in the content and methodology of secondary math education. It was during this conference that Jean Dieudonné, one of the participants, threw out a provocative declaration in reference to the teaching of geometry: "Down with Euclid!"

The years from 1964 through 1967 marked the second phase, a preparatory one in which committees formed. The third phase saw educational experiments carried out and new curriculums implemented. The fourth and final stage, which occurred during the early 1970s, was the general implementation of the new curriculums. In France, one important step in the New Math reform was the government's creation of the Lichnerowicz Commission in 1967. Chaired by André Lichnerowicz, this committee comprised a total of eighteen university and high school teachers, including Gustave Choquet, the physicist Louis

2.1 CORPS ℂ DES MATRICES $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

2.1.1 Définition.

On appelle 𝔐 l'ensemble des matrices carrées d'ordre 2 à termes réels. Sur le corps ℝ des nombres réels, on appelle ℂ le sous-ensemble des matrices M(*a*, *b*) de la forme :

$$\mathbf{M}(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

pour tout *a* et pour tout *b*.

The definition of complex numbers in terms of matrices, as taught to twelfth-grade students.
(For English translation, see appendix.)

Néel, Pierre Samuel (a member of Bourbaki at the time), Charles Pisot (a former member of Bourbaki), and André Revuz (one of the main proponents of the reform). As the job of the Lichnerowicz Commission was to propose new curriculums, this committee played a key part in the New Math reform.

It would be tedious to describe the new curricula in detail, but their general principles are simple to explain. The new curricula introduced the principles of mathematical and formal logic and of naive set theory earlier than the old curricula, including an introduction to groups, rings, fields, and vector spaces presented axiomatically; introduced complex numbers (in senior year) and principles of probability theory; and replaced traditional geometry with linear algebra (which deals with linear equations and vector spaces). The new curricula emphasized rigor in definitions, theorems, and proof-writing while deemphasizing all numerical, algebraic, and trigonometric calculations. The reforms in other countries were of the same nature as those in France.

A Revolution and a Counterrevolution

The reform certainly went too far, especially in the implementation of the curricula in classrooms and textbooks. The results of the reform quickly aroused harsh criticism, which the media reported to the general public. The abandonment of geometry, or more precisely its algebrization, was one of the main targets of the complaints. Another was the excessive formalism and abstraction of New Math. "[...] New Math is not in touch with reality," and "[...] Set theoretical algebra does not teach students how to reason" are just two of the many criticisms published by the magazine *Sciences et Vie* in a series of articles (1971-1972) attacking the reform. Eminent mathematicians like Jean Leray and René Thom also charged against New Math. As Leray wrote

THEOREM AND DEFINITION

For any two pairs (D_1, D_1') and (D_2, D_2') of vector rays in E_2 the relation "There exists a vector rotation f of E_2 such that

$$f(D_1) = D_1' \quad \text{and} \quad f(D_2) = D_2'"$$

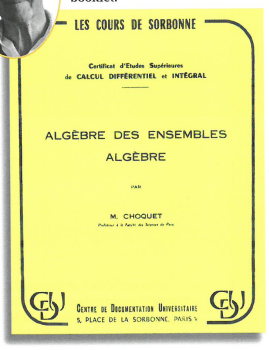
is an equivalence relation in DxD, where D is the set of vector rays in E_2.

An equivalence class for this relation is called **an angle of two vector rays in E_2.**

The definition of the angle between two rays, as taught in 1971 to eleventh-grade students...with no diagram!

in the October 1971 issue of *Gazette des mathématiens*, "'New Math' is a series of concepts that are defined without reference to their characteristic properties (axioms) and that fail to lead to statements of any remarkable properties. It's impossible to reason with these concepts or to find interest in them. Learning them is a test of memory that poisons intelligence." Jean Dieudonné denounced it as "a new method of education, a more agressive and stupid method that flies the banner of modernism."

The counterrevolution was in vain, and the New Math revolution continued to be criticized around the world as a global failure, even while this failure was never measured quantitatively by statistics. "There's no doubt about the failure. The teachers complained of being unable to teach according to the curricula and textbooks prescribed by the ministers, the students didn't seem to understand these 'unifying concepts,' and the parents realized that their children could no longer count or solve problems," explained Anna Sierpinska, a Polish woman living in Canada who has served as the vice-president of the International Committe on Mathematical Education. But did New Math fail? To what extent was the new material understood less than the old material? The new curricula did have a certain coherence, so why did their implementation create such extreme problems? An objective and detailed analysis of these questions is lacking even to this day. Some hints at answers do exist in criticisms expressed here and there, but they don't create a clear and global view of the reform's problems. The reasons for the failure remain poorly understood, concealing any lessons that might be learned for the future.



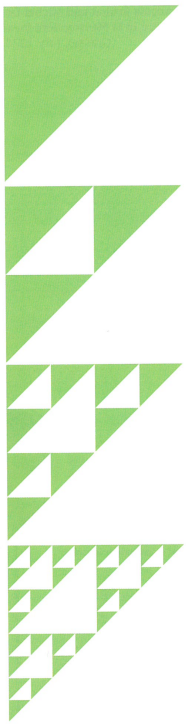
Gustave Choquet and his revolutionary booklet.

On the other hand, we do have the retrospective comments of Laurent Schwartz and Gustave Choquet, two mathematicians who were also renowned as teachers. As Choquet told Marian Schmidt in 1990, "in an abstract and sterilized world, the progress of a hundred years would allow us to elegantly and rigorously present fundamental concepts and theorems *ab ovo*, without referring to experience or geometric intuition. This is what happened, both in France and in other countries, with the drama of the mathematics reform at the end of the 1960s. Jean Dieudonné's famous proclamation 'Down with Euclid!' represents the focus of the ministerial committee in charge of developing new curricula for mathematics education in middle schools and high schools. The motivating idea of the movement was that, since fundamental concepts are necessary for all logical constructions, these concepts (including logic, set theory, algebra, and linear algebra) should be taught first. The result was bound to be catastrophic,

François Bruhat and Michel Demazure in la Messuguière (near Grasse), July 1975.

since all pedagogical considerations, such as the students' motivation and previous knowledge, the teachers' education, and the writing of reasonable textbooks, were swept aside. Also, little effort was made to meet the demands of physicists and engineers." In the official report *La France en mai 1981: l'enseignement et le développement scientifique* ("France in May 1981: Scientific education and development") published in December 1981, Schwartz writes that "[...] the teachers, parents, and children didn't learn modern mathematics but rather just the basic language of an extraordinarily vast and modern subject [...] of which the definitions given in classrooms (around the world!) were only the ABCs of the subject. [...] Little by little, all the richness of the mathematics previously taught in high schools, all the theorems, geometric figures, and connections with other sciences, has been replaced by a plethora of axioms and definitions. Most students find these incomprehensible, and the effect is that the results are *very poor*. Mathematics is rich when it introduces few concepts and structures and many theorems, but the New Math taught in schools introduces an enormous number of concepts and definitions and almost no theorems. This is *very poor* mathematics. [...] The goal of mathematics is *not* to prove rigorously things that everyone knows. Instead, the goal is to find rich results and then, in order to make sure they are true, to prove them."

So what role did Bourbaki play in the introduction of New Math? Dieudonné expressed his personal opinions on education with provocative exclamations, and while he didn't play an official role in developing the new curricula, he did greatly influence them (which is apparent, for example, in the algebraic presentation of geometry). Pierre Samuel



Jean-Pierre Kahane believes that the Sierpinski triangle, a fractal structure, is an analogy for the way mathematics curriculums are being drained of their content more and more each year.

participated in the Lichnerowicz Commision, but so did seventeen other people. In addition, Samuel was not among the most radical. "Lichnerowicz and I were quite moderate, but some of the committee members took things too far," he explained. Cartan and Schwartz gave addresses on contemporary mathematics to the *Association des professeurs de mathématiques*, an association whose members included many high school teachers enthusiastic about the reform. This was basically the extent of the individual contributions by Bourbaki members. As a group, Bourbaki took part neither in the reform nor in the debates surrounding it. According to Pierre Samuel, "Bourbaki held no opinions on high school education. The group didn't even hold any opinions about education in the first years of university, although some members wanted to write a mini-Bourbaki for these earlier years. This idea was abandoned, as we already had a lot to do and there were already good books for this level."

Supicious but Silent

Michel Demazure recalls that the group viewed this reform with much suspicion, and that certain members were strongly against it: "What we all shared was contempt for the pedagogical movement. We were preoccupied with the content of courses, not how to teach them." But although Bourbaki did not participate in the reform, the group did influence it indirectly. According to Anna Sierpinska, "Bourbaki's influence on this movement was particularly evident in the underlying mathematical philosophy of the choices and in the organization of the mathematical content of the new curricula. The goal was to build mathematical knowledge from the earliest classes, starting even in preschool, as an enormous, unified structure built on general concepts like sets, orders, relations, and groups." Thus Bourbaki's vision of mathematics had spread to the world of mathematicians and then to that of higher education; from there, it spread to high school teachers who proposed using it to revolutionize secondary math education, sometimes even explicitly mentioning Bourbaki in their arguments. But Bourbaki, who never claimed that the methods used in the group's treatise could be displaced to secondary education, didn't hold itself responsible for the misinterpretations of its philosophy and didn't try to perfect the extension of its philosophy to high school teaching. This attitude may deserve to be criticized, and indeed Pierre Cartier did: "Bourbaki acted very hypocritically in terms of the reforms in secondary education. The group influenced the movement quite a bit, but it denied any responsibility for the results." Meanwhile, Michel Demazure saw an element of Jansenism in the attitude held by Bourbaki, whose members were mostly Protestant: "Bourbaki didn't try to justify itself, and the group didn't feel responsible for the misinterpretations others made of its work. The Bourbakis were all against the use of the treatise as a model for teaching, but they held a certain Jansenist philosophy that said, 'Let people believe what they will.'"

Whatever the cause and extent of their failure, the various reforms of the New Math revolution were abandoned towards the end of the 1970s. Following New Math was a counterreform, leading to much less ambitious curricula. Traditional geometry was reintroduced, superfluous formalism was abandoned, calculations were reemphasized. But today's curriculums are also questioned. They seem less coherent, not as well constructed. Each year, more and more topics are removed to make the curricula easier, to the point where Jean-Pierre Kahane, a mathematician who chairs a committee on mathematical education, said that "today's curricula are like Sierpinski's triangle." Vector spaces are no longer included, even though they don't seem to pose any exceptional difficulties and play a considerable role in mathematics and its applications. In addition, number theory has almost completely disappeared. Today's curricula demands little from students' imagination and reasoning abilities and fails to prepare students to create and write proofs. "Students are not really taught to reason," Pierre Samuel complains, while Michel Demazure laments that "now, most of the exercises almost contain their answers in their statements." It seems that abandoning the New Math revolution was throwing the baby out with the bathwater. Today, mathematics inspires in many students as much fear, lack of interest, and even hatred as ever. And what Dieudonné wrote in his 1987 book *Pour l'honneur de l'esprit humain* unfortunately remains true today. "Nothing taught in high school mathematics was discovered after 1800," he writes, although it would be more accurate to replace "nothing" by "almost nothing."