Closed surfaces with different shapes that are indistinguishable by the square root normal form

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Abstract: The Square Root Normal Field (SRNF), introduced by Jermyn et al. in 2012, provides a way of representing immersed surfaces in $\mathbb{R}^3$, and equipping the set of these immersions with a "distance function" (to be precise, a pseudometric) that is easy to compute. Importantly, this distance function is invariant under reparametrizations (i.e., under self-diffeomorphisms of the domain surface) and under rigid motions of $\mathbb{R}^3$. Thus, it induces a distance function on the shape space of immersions, i.e., the space of immersions modulo reparametrizations and rigid motions of $\mathbb{R}^3$. In this paper, we give examples of the degeneracy of this distance function, i.e., examples of immersed surfaces (some closed and some open) that have the same SRNF, but are not the same up to reparametrization and rigid motions. We also prove that the SRNF does distinguish the shape of a standard sphere from the shape of any other immersed surface, and does distinguish between the shapes of any two embedded strictly convex surfaces.

Based on [Eric Klassen, PWM: Closed surfaces with different shapes that are indistinguishable by the SRNF. Achivum Mathematicum (Brno) 56 (2020), 107-114.]

and the lecture *Non-Injectivity of the SRNF Map and Measures on* S^2 by Eric Klassen at FSU in Tallahassee in 2020

Also of interest: [E. Hartman, M. Bauer, E. Klassen. Square Root Normal Fields for Lipschitz surfaces and the Wasserstein Fisher Rao metric. SIAM Journal on Mathematical Analysis 56.2 (2024): 2171-2190. arXiv:2301.00284]

Shape space

Problems in:

- Anthropology (space of hominid skulls)
- Computational Anatomy: (space of hearts, of brains, of parts of the brain, space of walking rythms, space of breathing lungs)
- Evolutionay biology (space of evolutionay trees; spaces of butterfly wings)

need meaningful distances to do statistical analysis of a point cloud in the shape space (means, principal component analysis, etc).

Infinite dimensional differential geometry offers a way to do this via geodesic distance of suitable Riemannian metrics.

Shape space II

M a template or model shape: a compact manifold, for simplicity's sake.

$$\begin{array}{ccc} \mathrm{Emb}(M,\mathbb{R}^d) & \xrightarrow{\subset} & \mathrm{Imm}(M,\mathbb{R}^d) \\ \downarrow \pi & & \downarrow \pi \\ B = \mathrm{Emb}(M,\mathbb{R}^d) & \xrightarrow{\subset} & B_i := \mathrm{Imm}(M,\mathbb{R}^2)/\,\mathrm{Diff}(M) \end{array}$$

Every $\mathrm{Diff}(M)$-invariant metric "above" induces a unique metric "below" on *shape space* such that π is a Riemannian submersion.

- The simplest Diff-invariant metric on Emb or Imm, *namely* $\mathrm{G}_f(h,h) = \int \langle h,h,\rangle\,\mathrm{vol}(f^*g_0)$, has vanishing geodesic distance: **not useful.**
- Higher order Sobolev metrics like $G_f^k(h,h) = \int \langle (1-\Delta^{f^*g_0})^k h, h\rangle\,\mathrm{vol}(f^*g_0)$: good choice, but geodesics costly to compute. Curvature complicates statistics.
- On $B(S^1,\mathbb{R}^2)$ the homogenuous Sobolev $\dot{H}^1$ metric allows for a (local) isometric mapping into flat space or a sphere: SQRT.

Motivation for SRNF

Square root transforms work nicely for plane curves; thus the wish to carry this over to surfaces. A precise presentation is in
[Bauer, Bruveris, Marsland, M: Constructing reparametrization invariant metrics on spaces of plane curves. Differential Geometry and its Applications 34 (2014), 139â165. arXiv:1207.5965]

Consider

$$G_q^{L^2}(h,h) = \int_{S^1} |h(\theta)|^2 d\theta\,, \text{ for } q \in C^\infty(S^1,\mathbb{R}^n), h \in T_q C^\infty(S^1,\mathbb{R}^n).$$

This is a flat weak Riemannian metric geodesic distance given by the L^2-norm

$$\operatorname{dist}^{L^2}(q_0,q_1)^2 = \int_{S^1} |q_0(\theta) - q_1(\theta)|^2 d\theta\,.$$

Consider the pullback metric $F^* G^{L2}$ on $\operatorname{Imm}(S^1,\mathbb{R}^2)$:

$$G_c^F(h,h) = (F^* G^{L^2})_c(h,h) = G_{F(c)}^{L^2}(D_{c,h}F, D_{c,h}F) = \int_{S^1} |D_{c,h}F|^2\, d\theta\,.$$

Theorem.

If $F : \mathrm{Imm}(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^n)$ *satisfies*

$$F(c \circ \varphi) = \sqrt{\varphi'} F(c) \circ \varphi,$$

with $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ *and* $\varphi \in \mathrm{Diff}(S^1)$ *and if* F *is infinitesimally injective, i.e.,* $D_{c,\cdot}F$ *is injective for all* c*, then* G^F *is a Riemannian metric on* $\mathrm{Imm}(S^1, \mathbb{R}^2)$ *that is invariant under the reparameterization group* $\mathrm{Diff}(S^1)$*.*

Arc length derivative and measure: $D_s c = \frac{1}{|c'|\partial_\theta}$, $ds = |c'| d\theta$.
Example: Take a smooth function $f \in C^\infty(\mathbb{R}^{2m}, \mathbb{R}^n)$ and define the transform F as $F(c) = \sqrt{|c'|}\, f \circ (c, D_s c, \ldots, D_s^{m-1} c)$. The image of F is a submanifold of the flat pre-Hilbert space.

Example: (SRVT), [Srivastava2011]:

$$R: \operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Transl} \to C^\infty(S^1, \mathbb{R}^2)$$

$$R(c) = |c'|^{1/2} v \quad . \qquad \text{Image is open subset}$$

$$R^* G^{L^2} : G_c^{1,1/2}(h,h) = \int_{S^1} \langle D_s h, n\rangle^2 + \tfrac{1}{4} \langle D_s h, v\rangle^2 \, ds \,,$$

Example:

$$R^{a,b}: \operatorname{Imm}([0,2\pi], \mathbb{R}^2)/\operatorname{Transl} \to C^\infty([0,2\pi], \mathbb{R}^3)$$

$$R^{a,b}(c) = |c'|^{1/2} \left(a \begin{pmatrix} v \\ 0 \end{pmatrix} + \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad .$$

Here $a, b \in \mathbb{R}^+$ are positive numbers with $4b^2 \geq a^2$. Image is: curves in a cone. Pulls back elastic metrics:

$$R^* G_c^{L^2\, a,b}(h,h) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle^2 + b^2 \langle D_s h, v\rangle^2 \, ds \,.$$

Movie using soliton solution for the $\dot{H}^1$-metric on space of plane curves

$$G_c^{\dot{H}^1}(h,k) = \int_{S^1} \langle D_s h, n\rangle . \langle D_s k, n\rangle \, ds \text{ on } B_i(S^1, \mathbb{R}^2)/\text{Translations}$$

extends to the boundary consisting of Lipschitz curves. This boundary contains many finite dimensional submanifolds of polygonal curves (for each fixed number of nodes) which are geodesically convex. Their geodesics are **soliton-solutions** of the original equation in the sense that their momenta are finite sums of delta distributions which are transported by the geodesic flow. Such solitons form the the final movie. The translations are put into the movie by hand. A suitable SQRT is compatible with this extension.

[Bauer, Bruveris, Harms, M: Soliton solutions for the elastic metric on spaces of curves. Discrete and Continuous Dynamical Systems 38, 3 (March 2018), 1161-1185. https://doi.org/10.3934/dcds.2018049 arxiv:1702.04344]

Review of the SRNF Map
(From here on, I follow Eric Klassen's talk)

Let M be an oriented surface (with or without boundary), and assume it has a Riemannian metric.

$$\mathrm{Imm}(M, \mathbb{R}^3) = \{\text{the set of immersions } M \to \mathbb{R}^3\}$$

Given $f \in \mathrm{Imm}(M, \mathbb{R}^3)$, define the area-multiplication factor of f to be a function

$$a : M \to \mathbb{R}_+.$$

Precise formula for $a(x)$: let $\{v, w\}$ be an orthonormal basis of $T_x M$. Then

$$a(x) = |df_x(v) \times df_x(w)|.$$

For $f \in \mathrm{Imm}(M, \mathbb{R}^3)$, we define the oriented unit normal function

$$n : M \to S^2$$

$$n(x) = \frac{df_x(v) \times df_x(w)}{|df_x(v) \times df_x(w)|},$$

where $\{v, w\}$ is an oriented orthonormal basis of $T_x M$.

The Square Root Normal Field

Define the square root normal field (SRNF) of f[1] to be

$$q_f : M \to \mathbb{R}^3$$

where

$$q_f(x) = \sqrt{a(x)}n(x).$$

If $f \in \mathrm{Imm}(M, \mathbb{R}^3)$, then $q_f \in C^\infty(M, \mathbb{R}^3)$

[1] Jermyn, Kurtek, Klassen, and Srivastava, *Elastic Matching of Parametrized Surfaces Using Square Root Normal Fields*, European Conference on Computer Vision, Florence, Italy, Nov. 2012.

Action of $\mathrm{Diff}_+(M)$

$\mathrm{Diff}_+(M)$ = the group of orientation-preserving diffeomorphisms $M \to M$. $\mathrm{Diff}_+(M)$ acts on $\mathrm{Imm}(M, \mathbb{R}^3)$ from the right by composition.

Let $\mathrm{Diff}_+(M)$ act on $C^\infty(M, \mathbb{R}^3)$ from the right by

$$(q * \gamma)(x) = \sqrt{b(x)}\, q(\gamma(x)),$$

where $\gamma \in \mathrm{Diff}_+(M)$, and $b : M \to \mathbb{R}_+$ denotes the area multiplication factor $\det(T\gamma)$ of γ.

This action is defined so that $q_{f \circ \gamma} = q_f * \gamma$. It is the action of $\mathrm{Diff}_+(M)$ on the space of $\mathbb{R}^3$-valued half-densities.

Each element of $\mathrm{Diff}_+(M)$ acts on $C^\infty(M, \mathbb{R}^3)$ by a linear isometry, if you put the L^2 metric on $C^\infty(M, \mathbb{R}^3)$.

Metric on Shape Space

Because of the isometric action, it makes sense to define a distance function on the shape space

$$\mathcal{S}(M,\mathbb{R}^3) := \mathrm{Imm}(M,\mathbb{R}^3)/\mathrm{Diff}_+(M)$$

by

$$d([f],[g]) = \inf_{\gamma\in\mathrm{Diff}_+(M)} \|q_f - q_{g\circ\gamma}\|_2.$$

(One might also want to mod out by rigid motions and/or rescaling, but for simplicity I'll ignore that here.)

Injectivity of SRNF

Question: Does

$$q_f = q_g \implies [f] = [g]?$$

Note: $[f]$ denotes the $\mathrm{Diff}_+(M)$-orbit of f in $\mathrm{Imm}(M, \mathbb{R}^3)$.

In other words, if two immersed surfaces have the same SRNF, must they have the same shape?

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Answer: NO.

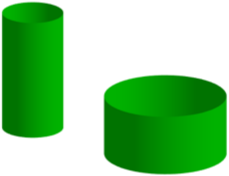
Examples of non-injectivity: Cylinders.

Let $M = S^1 \times [0,1]$, a standard cylinder (with two circles as boundary). Define two embeddings f and g from $M \to \mathbb{R}^3$ by

$$f((x,y),z) = (x,y,z)$$

$$g((x,y),z) = (rx, ry, z/r)$$

where $r > 1$. Then $q_f = q_g$. But, clearly $f(M)$ and $g(M)$ don't have the same shape: one is tall and thin, the other is short and fat.



Examples of non-injectivity: Paraboloids

Let a and b be non-zero real numbers and let $S_{a,b}$ denote the graph of the function $z = ax^2 + by^2$ in $\mathbb{R}^3$.

Claim: *If $ab = cd$, then we can parametrize $S_{a,b}$ and $S_{c,d}$ in such a way that they have the same SRNF.*

Proof: For a given (a, b), parametrize $S_{a,b}$ by $B : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$B(x,y) = \left(\frac{x}{a}, \frac{y}{b}, \frac{x^2}{a} + \frac{y^2}{b} \right).$$

An easy computation then yields

$$B_x \times B_y = \left(-\frac{2x}{ab}, -\frac{2y}{ab}, \frac{1}{ab} \right).$$

Since $B_x \times B_y$ depends only on ab (not on a and b individually), and the SRNF can be expressed as $\frac{B_x \times B_y}{\sqrt{|B_x \times B_y|}}$, the theorem follows. □

It's clear that $S_{1,1}$ and $S_{\frac{1}{2},2}$ don't have the same shape, so this is another example of non-injectivity.

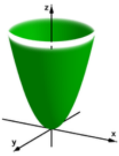


Figure: $z = x^2 + y^2$

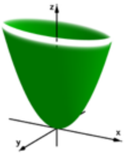


Figure: $z = \frac{1}{2}x^2 + 2y^2$

The examples given so far have been known for quite awhile, but this phenomenon can occur for closed surfaces as well.

Useful Theorem

Suppose S_1 and S_2 are oriented surfaces in $\mathbb{R}^3$, and suppose there is an orientation preserving diffeomorphism $\phi : S_1 \to S_2$ that preserves area and normal direction: $n(\phi(x)) = n(x) \in \mathbb{R}^3$ for all $x \in S_1$. Let $f : M \to S_1$ be a parametrization of S_1, so $\phi \circ f$ is a parametrization of S_2. Then

$$q_f = q_{\phi \circ f}.$$

Follows immediately from the definition of SRNF.

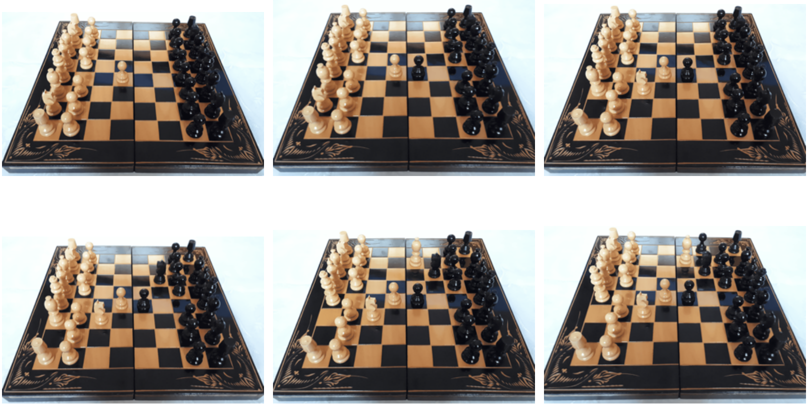


Figure: The SRNF cannot distinguish between these chess positions

Chessboard Example

Let S_1 and S_2 be two chessboards, each with all the usual pieces on it, but occupying different squares. Think of each of these as a surface that is topologically a sphere (imagine that the board has some thickness, and that the corners are rounded to make it smooth). Define a diffeomorphism $\phi : S_1 \to S_2$ as follows:

1. ϕ maps each piece on S_1 to the corresponding piece on S_2 by a translation.
2. ϕ maps the bottom and sides of the board in S_1 to the bottom and sides of the board in S_2 by the identity map.
3. ϕ maps the top of the board in S_1 (minus a round hole where each piece stands) to the top of the board in S_2 (also minus a hole for each piece) by an area-preserving map that takes the boundary of each piece in S_1 to the boundary of the corresponding piece in S_2.

Since ϕ is area-preserving and preserves normal vectors, it follows from Theorem 1 that S_1 and S_2 can be parametrized to have precisely the same SRNF. Therefore, the SRNF cannot distinguish between any two positions on a chessboard!

To construct this example it was crucial that the board had a "flat area" around the pieces, so that we were free to move around points (in an area-preserving manner) without changing the normal vectors.
Without such a flat place, I don't know if an example like this can be constructed, of two non-equivalent smooth immersions f and g of a closed surface such that $q_f = q_g$.

However, I can give an example of two non-equivalent surfaces with no flat places such that $[q_f] = [q_g]$, where the brackets denote the L^2 closures of the Diff_+ orbits of q_f and q_g. Thus, even without flat places, the distance between two non-equivalent surfaces can be 0.

Two Questions:

In the first part of this talk, we gave examples of the non-injectivity of the SRNF map. We now consider two related questions:

1. Given $f, g \in \mathrm{Imm}(S^2, \mathbb{R}^3)$, can we give a geometric criterion for when $d([f],[g]) = 0$?
2. What is the L^2-closure of the image of the SRNF map $\mathrm{Imm}(S^2, \mathbb{R}^3) \to L^2(S^2, \mathbb{R}^3)$?

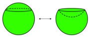
In the second half of the talk, I will conjecture answers to both of these questions.

Immersions and Measures

If $f : M \to \mathbb{R}^3$ is an immersion, then

$$\mu_{q_f}(U) = \text{ the area of the portion of } f(M) \text{ where } n(x) \in U.$$

It's easy to see that if f and g differ only by reparametrization and translation, then $\mu_{q_f} = \mu_{q_g}$. But the converse is not true: There are plenty of examples where $\mu_{q_f} = \mu_{q_g}$, but f and g have completely different shapes!



These two embeddings of the sphere, one convex and one non-convex, induce the same measure on S^2. Also, they are not distinguishable by the SRNF, because $[q_f] = [q_g]$. Another (smoother) example would be the chessboard example given earlier in the talk.

Anwer to Question 1. If $q \in L^2(M, \mathbb{R}^3)$, let $[q]$ denote the L^2-closure of the $\text{Diff}_+(M)$-orbit of q.
Let $q_1, q_2 \in L^2(M, \mathbb{R}^3)$. Then

$$[q_1] = [q_2] \iff \mu_{q_1} = \mu_{q_2}$$

This means the SRNF can only distinguish two immersions if they induce different measures on S^2. Or: the SRNF only sees on how much area each normal vector is attained; it ignores the location of these normal vectors.

Answer to Question 2 Define

$$\Psi : \text{Imm}(S^2, \mathbb{R}^3) \to L^2(S^2, \mathbb{R}^3)$$

by $\Psi(f) = q_f$. The L^2-closure of $\Psi(\text{Imm}(S^2, \mathbb{R}^3))$ is

$$\left\{ q \in L^2(S^2, \mathbb{R}^3) : \int_{S^2} |q| q \, dA = 0 \right\}$$

where dA denotes the usual area form on S^2.
This is analogous to a familiar fact about SRVFs for curves: An L^2 function $q : I \to \mathbb{R}^n$ is the SRVF of a *closed* curve if and only if $\int_0^1 |q| q \, dx = 0$.

All conjectures have been proved in:
[E. Hartman, M. Bauer, E. Klassen. Square Root Normal Fields for Lipschitz surfaces and the Wasserstein Fisher Rao metric. SIAM Journal on Mathematical Analysis 56.2 (2024): 2171-2190. arXiv:2301.00284]
Core of proof.
Theorem.[Minkowski, Fenchel, Jessen and Alexandrov, circa 1903]
The correspondence

$$f \longleftrightarrow \mu_{q_f}$$

gives a bijection between convex embeddings of S^2 *in* $\mathbb{R}^3$ *(up to translation and reparametrization) and measures* μ *on* S^2 *satisfying* $\int_{S^2} x \, d\mu(x) = 0$.

Indistinguishable Surfaces

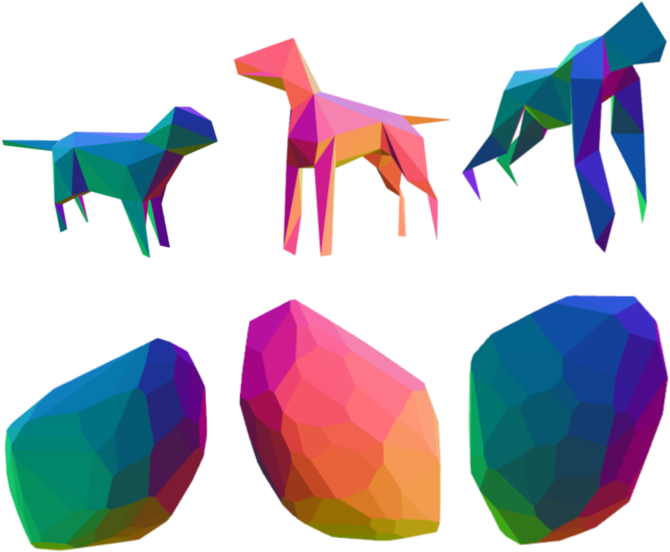


Figure: The SRNF cannot distinguish between these surfaces

Figure: neither between these surfaces

Thank you, audience, for your attention!