LIE DERIVATIVES OF SECTIONS OF NATURAL VECTOR BUNDLES

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Dedicated to the memory of Joseph A. Wolf

ABSTRACT. Time derivatives of pullbacks of push forwards along smooth curves of diffeomorphism of sections of natural vector bundles are computed in terms of Lie derivatives along adapted non-autonomous vector fields by extending a key lemma in [3].

1. INTRODUCTION

The following is an adaptation of the rather well known method of Lie derivation along mapping $N \to M$ as explained for differential forms in [5, 31.11] and more generally for purely covariant tensor fields in [4, 12.2–12.5]. It is used in newer proofs of the Poincaré lemma and the theorem of Darboux, see e.g. [6] and [8]. Namely, we prove the the following corollary 3.1 of [3, Lemma 6]; the need for this result arose during the preparation of [1].

Corollary. *Let φ_t be a smooth curve of local diffeomorphisms. Then we get two time dependent vector fields*

$$X_{t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0}\varphi_t \text{ and } Y_{t_0} = \partial_t|_{t_0}\varphi_t \circ \varphi_{t_0}^{-1}$$

Then for any natural vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have the first non-vanishing derivative

$$\partial_t \varphi(t)^* s = \varphi(t)^* \mathcal{L}_{Y(t)} s = \mathcal{L}_{X(t)}\, \varphi(t)^* s\,. \tag{1}$$

$$\partial_t \varphi(t)_* s = \partial_t (\varphi(t)^{-1})^* s = -\varphi(t)_* \mathcal{L}_{X(t)} s = -\mathcal{L}_{Y(t)}\, \varphi(t)_* s\,. \tag{2}$$

2. BACKGROUND FROM [3]

2.1. **Curves of local diffeomorphisms.** Let $\varphi : \mathbb{R} \times M \supset U_\varphi \to M$ be a smooth mapping where U_φ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that each φ_t is a diffeomorphism on its domain and $\varphi_0 = Id_M$. We

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say that φ_t is a *curve of local diffeomorphisms* though Id_M. From lemma 2.2 we see that if $\frac{\partial^j}{\partial t^j}|_0 \varphi_t = 0$ for all $1 \le j < k$, then $X := \frac{1}{k!}\frac{\partial^k}{\partial t^k}|_0 \varphi_t$ is a well defined vector field on M. We say that X is the *first non-vanishing derivative* at 0 of the curve φ_t of local diffeomorphisms. We may paraphrase this as $(\partial_t^k|_0 \varphi_t^*)f = k!\mathcal{L}_X f$.

2.2. **Lemma.** [3, Lemma 2] *Let $c : \mathbb{R} \to M$ be a smooth curve. If $c(0) = x \in M$, $c'(0) = 0, \dots, c^{(k-1)}(0) = 0$, then $c^{(k)}(0)$ is a well defined tangent vector in $T_x M$ which is given by the derivation $f \mapsto (f \circ c)^{(k)}(0)$ at x.*

2.3. **Natural vector bundles.** See [2, 6.14]. Let $\mathcal{M}f_m$ denote the category of all smooth m-dimensional manifolds and local diffeomorphisms between them. A *vector bundle functor* or *natural vector bundle* is a functor F which associates a vector bundle $(F(M), p_M, M)$ to each manifold M and a vector bundle homomorphism

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "p_M"] & F(N) \arrow[d, "p_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

to each $f : M \to N$ in $\mathcal{M}f_m$, which covers f and is fiber wise a linear isomorphism. If f is the embedding of an open subset of N then this diagram turns out to be a pullback diagram. We also point out that $f \mapsto F(f)$ maps smoothly parameterized families to smoothly parameterized families, see [2, 14.8]. Assuming this property all vector bundle functors were classified by [7]: They correspond to linear representations of higher jet groups, they are associated vector bundles to higher order frame bundles, see also [2, 14.8].

Examples of vector bundle functors are tangent and cotangent bundles, tensor bundles, densities, $M \mapsto L(TM, TM)$, and also the trivial bundle $M \times \mathbb{R}$.

2.4. **Pullback of sections.** Let F be a vector bundle functor on $\mathcal{M}f_m$ as described in 2.3. Let M be an m-manifold and let φ_t be a curve of local diffeomorphisms through Id_M on M. Then the flow φ_t, for fixed t, is a diffeomorphism defined on an open subset U_{φ_t} of M. The mapping

$$\begin{tikzcd} F(M) \arrow[d, "p_M"] & F(U_{\varphi_t}) \arrow[l] \arrow[r, "F(\varphi_t)"] \arrow[d] & F(M) \arrow[d, "p_M"] \\ M & U_{\varphi_t} \arrow[l] \arrow[r, "\varphi_t"] & M \end{tikzcd}$$

is then a vector bundle isomorphism.

We consider a section $s \in \Gamma(F(M))$ of the vector bundle $(F(M), p_M, M)$ and we define for $t \in \mathbb{R}$ *pullback* and *push forward* as

$$\varphi_t^* s := F(\varphi_t^{-1}) \circ s \circ \varphi_t\,, \quad (\varphi_t)_* s = (\varphi_t^{-1})^* s = F(\varphi_t) \circ s \circ \varphi_t^{-1}\,.$$

These are local sections of the bundle $F(M)$. If φ_t is smooth curve of diffeomorphisms these are global sections. For each $x \in M$ the value $(\varphi_t^* s)(x) \in F(M)_x := p_M^{-1}(x)$ is defined, if t is small enough. So in the vector space $F(M)_x$ the expression $\frac{d}{dt}|_0(\varphi_t^* s)(x)$ makes sense and therefore the section $\frac{d}{dt}|_0(\varphi_t)^* s$ is globally defined and is an element of $\Gamma(F(M))$. If $\varphi_t = \mathrm{Fl}_t^X$ is the flow of a vector field X on M this section

$$\mathcal{L}_X s := \tfrac{d}{dt}|_0(\mathrm{Fl}_t^X)^* s$$

is called the *Lie derivative* of s along X. It satisfies $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$, see [2, 6.20].

2.5. **Lemma.** [3, Lemma 6] *Let φ_t be a smooth curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X = \partial_t^k|_0\varphi_t$. Then for any vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have the first non-vanishing derivative*

$$k!\mathcal{L}_X s = \partial_t^k|_0 \varphi_t^* s.$$

3. The result

For the following we consider only first derivatives instead of first vanishing ones.

3.1. **Corollary.** *Let φ_t be a smooth curve of (local) diffeomorphisms. Consider the two time dependent vector fields*

$$X_{t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0}\varphi_t \text{ and } Y_{t_0} = \partial_t|_{t_0}\varphi_t \circ \varphi_{t_0}^{-1}$$

Then for any vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have

(1) $$\partial_t \varphi(t)^* s = \varphi(t)^* \mathcal{L}_{Y(t)} s = \mathcal{L}_{X(t)}\, \varphi(t)^* s\,.$$

(2) $$\partial_t \varphi(t)_* s = \partial_t(\varphi(t)^{-1})^* s = -\varphi(t)_* \mathcal{L}_{X(t)} s = -\mathcal{L}_{Y(t)}\, \varphi(t)_* s\,.$$

Proof. Let $\tilde\varphi_t = \varphi_{t_0}^{-1} \circ \varphi_{t+t_0}$, a smooth curve of (local) diffeomorphisms through Id_M. We have

$$\partial_t|_0\tilde\varphi_t = \partial_t|_0 \varphi_{t_0}^{-1} \circ \varphi_{t+t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_0\varphi_{t+t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0}\varphi_t = X_{t_0}\,.$$

By Lemma 2.5 we we get that

$$\begin{aligned} &\mathcal{L}_{X_{t_0}} s = \partial_t|_{t_0}\tilde\varphi_t^* s = \partial_t|_{t_0}(\varphi_{t_0}^{-1} \circ \varphi_t)^* s = \partial_t|_{t_0}\varphi_t^*(\varphi_{t_0}^{-1})^* s \\ \Longrightarrow \quad &\mathcal{L}_{X_{t_0}} \varphi_{t_0}^* s = \partial_t|_{t_0}\varphi_t^*(\varphi_{t_0}^{-1})^* s \quad \text{which is part of (1).} \end{aligned}$$

For the second part of (1) we consider $\bar\varphi_t = \varphi_{t+t_0}\circ\varphi_{t_0}^{-1}$, another smooth curve of local diffeomorphisms through Id_M. Here we have, again by Lemma 2.5,

$$\begin{aligned}
&\partial_t|_0\bar\varphi_t = \partial_t|_0\varphi_{t+t_0}\circ\varphi_{t_0}^{-1} = \partial_t|_{t_0}\varphi_t\circ\varphi_{t_0}^{-1} = Y_{t_0}\,.\\
&\mathcal{L}_{Y_{t_0}}s = \partial_t|_0\bar\varphi_t^*s = \partial_t|_0(\varphi_{t+t_0}\circ\varphi_{t_0}^{-1})^*s = \partial_t|_0(\varphi_{t_0}^{-1})^*\varphi_{t+t_0}^*s\\
&\qquad = (\varphi_{t_0}^{-1})^*\partial_t|_0\varphi_{t+t_0}^*s \;\text{ since } (\varphi_{t_0}^{-1})^* : \Gamma(F(M))\to\Gamma(F(M)) \text{ is bounded linear}\\
&\qquad = (\varphi_{t_0}^{-1})^*\partial_t|_{t_0}\varphi_t^*s \text{ which implies the second part of (1).}
\end{aligned}$$

To show (2) note first that

$$\begin{aligned}
&0 = \partial_t(\mathrm{Id}) = \partial_t(\varphi_t^{-1}\circ\varphi_t) = (\partial_t\varphi_t^{-1})\circ\varphi_t + T\varphi_t^{-1}\circ\partial_t\varphi_t\\
&\partial_t(\varphi_t^{-1}) = -T\varphi_t^{-1}\circ(\partial_t\varphi_t)\circ\varphi_t\\
&T\varphi_t\circ\partial_t(\varphi_t^{-1}) = -(\partial_t\varphi_t)\circ\varphi_t^{-1} = -Y_t\\
&(\partial_t\varphi_t^{-1})\circ\varphi_t = -T\varphi_t^{-1}\circ(\partial_t\varphi_t) = -X_t
\end{aligned}$$

Hence, replacing φ_t by φ_t^{-1} in (1) replaces X_t by $-Y_t$ and Y_t by $-X_t$ and noting that $(\varphi_t)_*s = (\varphi_t^{-1})^*s$ transforms (1) into (2). $\square$

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