# RELATIVE DIFFERENTIAL CLOSURE IN HARDY FIELDS

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ABSTRACT. We study relative differential closure in the context of Hardy fields. Using our earlier work on algebraic differential equations over Hardy fields, this leads to a proof of a conjecture of Boshernitzan (1981): the intersection of all maximal analytic Hardy fields agrees with that of all maximal Hardy fields. We also generalize a key ingredient in the proof, and describe a cautionary example delineating the boundaries of its applicability.

## CONTENTS



## INTRODUCTION

Hardy's monograph *Orders of Infinity*[1] [30] founded an asymptotic calculus of non-oscillating real-valued functions, building on earlier ideas by du Bois-Reymond [13]. Hardy introduced the class of *logarithmic-exponential functions* (*LE-functions*, for short): functions constructed in finitely many steps from real constants and the identity function $x$ using arithmetic operations, exponentiation, and logarithm. He observed that this class allows one to describe the growth rates at infinity of many functions that naturally arise in mathematics. In his own words [30, p. 48]:

> *No function has yet presented itself in analysis the laws of whose increase, in so far they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.*

Typical examples of LE-functions defined on $(1, +\infty)$ are $x^r$ $(r \in \mathbb{R})$, $x^x$, $\mathrm{e}^{x^2}$, and $(\log x)(\log\log x)$. Much of the usefulness of the class of LE-functions stems from the fact that their germs at infinity form what Bourbaki [18] called a *Hardy field*: a field $H$ of germs at $+\infty$ of differentiable real-valued functions on intervals $(a, +\infty)$ $(a \in \mathbb{R})$ such that for any such function with germ in $H$, the germ of its derivative is also in $H$. The basic facts about Hardy fields are due to Bourbaki [18],

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Sjödin [52], Robinson [41], Boshernitzan [14]–[17], and Rosenlicht [45]–[49]. As background we shall mention such facts in this introduction.

A Hardy field $H$ such that each $f \in H$ has a smooth ($\mathcal{C}^\infty$) representative is called *smooth*; likewise we define when $H$ is *analytic*. Most Hardy fields from practice (like Hardy's field of LE-functions) are analytic; but not every Hardy field is analytic, or even smooth[2]. Every Hardy field is naturally a differential field, and an ordered field: the germ of a function $f$ is declared to be positive whenever $f(t)$ is eventually positive. In addition to the ordering we also use the asymptotic relations $\preccurlyeq$, $\prec$, $\asymp$ to compare germs $f$, $g$ in a Hardy field:

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad f = O(g) \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some real } c > 0 \\ f \prec g \quad &:\Longleftrightarrow \quad f = o(g) \quad :\Longleftrightarrow \quad |f| < c|g| \text{ for all real } c > 0 \\ f \asymp g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \text{ and } g \preccurlyeq f; \quad f \succ g \quad :\Longleftrightarrow \quad g \prec f. \end{aligned}$$

(The $\preccurlyeq$-notation of du Bois-Reymond predates the big $O$-notation of Bachmann and Landau, and is more convenient in dealing with Hardy fields.) The basic operations of calculus play well with the ordering and asymptotic relations on Hardy fields. For example, given any germs $f$, $g$ in a common Hardy field,

$$f > 0,\ f \succ 1 \Rightarrow f' > 0, \qquad f \prec g \not\asymp 1 \Rightarrow f' \prec g'.$$

The germ of a non-oscillating differentially algebraic function usually lies in a Hardy field. Besides the LE-functions, this is also the case for special functions like the error function erf, the exponential integral Ei, the Airy functions Ai and Bi, etc. Many differentially transcendental functions, like the Riemann $\zeta$-function and Euler's $\Gamma$-function, also have their germs in Hardy fields.

Characteristic of Hardy fields is that their elements are non-oscillating in a strong sense: if $f$ is a germ in a Hardy field $H$, then not only is the sign of $f(t)$ ultimately constant, but also each differential-polynomial expression in $f$ such as

$$g(t)f''(t)^3 - h(t)f'(t)f(t)^2 + 2 \qquad (g, h \in H).$$

This property is reflected in the existence of a field ordering on $H$ as well as the relations $\preccurlyeq$ and $\prec$ that are so useful in asymptotics. Functions that are non-oscillating in such a strong sense may be viewed as *tame*. In certain applications, establishing tameness is decisive: for example, it plays an important role in Écalle's work [23] on Dulac's conjecture (a weakened version of Hilbert's 16th Problem). An even stronger form of tameness is o-minimality [39], and the germs of definable univariate functions of an o-minimal expansion of the real field form a Hardy field. This leads to many further examples of Hardy fields [35, 42].

The class of LE-functions is rather small. For example, the antiderivatives of $\mathrm{e}^{x^2}$ have their germs in a Hardy field but are not LE-functions (Liouville, cf. [43]). Hardy's quote above notwithstanding, the functional inverse of $(\log x)(\log\log x)$ turned out to not even be asymptotic to an LE-function [22, 32] (and yet its germ also lies in a Hardy field). There are also (analytic) solutions of simple functional equations in Hardy fields ultimately outgrowing all LE-functions [16].

Any Hardy field has a unique algebraic Hardy field extension that is real closed; see [52] and [41]. For any germ $f$ in a Hardy field $H$, its exponential $\mathrm{e}^f$, its logarithm $\log f$ (if $f > 0$), and any primitive of $f$ lie in a Hardy field extending $H$; see [18]. More generally, if $H$ is a Hardy field and $P, Q \in H[Y] \setminus \{0\}$, then each germ of a $\mathcal{C}^1$-function $y$ satisfying $y'Q(y) = P(y)$ belongs to a Hardy field extending $H$;

see [45, Theorem 2] and [51]. In [9] we proved what is in some sense the ultimate result on solving algebraic differential equations in Hardy fields:

**Theorem.** *Given any Hardy field $H$, polynomial $P \in H[Y_0,\dots,Y_n]$ and $f,g \in H$ with $f<g$ and $P(f,f',\dots,f^{(n)})<0<P(g,g',\dots,g^{(n)})$, there is a $y$ in a Hardy field extension of $H$ such that $f<y<g$ and $P(y,y',\dots,y^{(n)})=0$.*

We convert this into an intermediate value property by means of the concept of a *maximal* Hardy field: a Hardy field that is not contained in any strictly larger one. Likewise we define maximal smooth Hardy fields and maximal analytic Hardy fields. Any Hardy field of either type is contained in a maximal one of the same type, by Zorn. Now the theorem can be rephrased as: maximal Hardy fields have DIVP (the *Differential Intermediate Value Property*), where a Hardy field $H$ is said to have DIVP if for all $P \in H[Y_0,\dots,Y_n]$ and $f,g \in H$ such that $f<g$ and $P(f,f',\dots,f^{(n)})<0<P(g,g',\dots,g^{(n)})$, there is a $y \in H$ such that $f<y<g$ and $P(y,y',\dots,y^{(n)})=0$. By the way, [9] also shows that maximal smooth and maximal analytic Hardy fields have DIVP.

DIVP essentially captures all properties of maximal Hardy fields that can be stated in the language of ordered differential fields, in analogy with the intermediate value property for ordinary polynomials capturing the property for an ordered field to be real closed. For a further explanation of this statement and numerous consequences of the above theorem[3] we refer to the introduction to [9].

Some (germs of) functions are "absolutely tame" in the sense that they belong to *every* maximal Hardy field. This holds in particular for all LE-functions, and Boshernitzan [14, 15, 16] promoted the study of these germs as natural generalizations of LE-functions. The Hardy field $\mathrm{E}$ of absolutely tame germs is rather extensive: for example, it is closed under exponentiation, logarithm, and taking antiderivatives; more generally, any solution $y \in \mathcal{C}^1$ of an equation $y'Q(y)=P(y)$ where $P,Q \in \mathrm{E}[Y]\setminus\{0\}$ also has its germ in $\mathrm{E}$ (by a result mentioned earlier). Thus, for instance, arctan and the Gaussian integrals $\int^x \mathrm{e}^{-t^2}\,dt$ are in $\mathrm{E}$. Moreover, if $f \in \mathrm{E}$ and $f \preccurlyeq 1$, then $\cos f, \sin f \in \mathrm{E}$. But Boshernitzan [16, Proposition 3.7] also exhibited germs in Hardy fields not belonging to $\mathrm{E}$: the germ of *any* $\mathcal{C}^2$-function satisfying $y''+y=\mathrm{e}^{x^2}$ lies in a Hardy field[4], but no maximal Hardy field contains more than one[5] and hence none of them lies in $\mathrm{E}$. As a consequence, $\mathrm{E}$ does not have DIVP[6]. Boshernitzan also considered the intersection $\mathrm{E}^\infty$ of all maximal smooth Hardy fields and the intersection $\mathrm{E}^\omega$ of all maximal analytic Hardy fields, and conjectured that $\mathrm{E}=\mathrm{E}^\infty=\mathrm{E}^\omega$ [14, §10, Conjecture 1]. As positive evidence, in [15, (20.1)] he obtained $\mathrm{E}\subseteq\mathrm{E}^\infty\subseteq\mathrm{E}^\omega$. In this paper we prove this conjecture:

**Theorem A.** *Any germ lying in all maximal analytic Hardy fields also lies in all maximal Hardy fields.*

By [15, Theorem 14.3], each $f \in \mathrm{E}^\omega$ is differentially algebraic. So it may not be surprising that the proof of Theorem A centers on a study of the notion of relative differential closure in the context of Hardy fields: if $E \supseteq H$ are Hardy fields, the *differential closure* of $H$ in $E$ is the set of all $y \in E$ which are differentially algebraic over $H$, that is, satisfy an equation $P(y,y',\dots,y^{(n)})=0$ for some $n$ and nonzero $P \in H[Y_0,\dots,Y_n]$, and we say that $H$ is *differentially closed* in $E$ if it equals its differential closure in $E$. A crucial ingredient for this study is a differential transcendence result [ADH, 16.0.3], of which we prove here a variant:

**Theorem B.** *Let $E \supseteq H$ be Hardy fields where $H$ properly extends $\mathbb{R}$ and has* DIVP. *Then $H$ is differentially closed in $E$ iff $E \cap \exp(H) \subseteq H$.*

We include an example showing that the hypothesis $H \supset \mathbb{R}$ in Theorem B cannot be replaced by the condition $E \cap \mathbb{R} = H \cap \mathbb{R} \neq H$. This uses a result of Rosenlicht [44].

**From du Bois-Reymond and Hardy to Écalle, Conway, and Gödel.** An attractive feature of Hardy fields is that their elements are actual functions (more precisely, germs of such). To conclude this introduction we recall two alternative universal frameworks for tame asymptotics in which growth rates are explicitly represented, namely *transseries* and *surreal numbers.* Transseries (à la Écalle [23]) are constructed from a formal indeterminate $x$ and the real numbers using the field operations, exponentiation, logarithms, and certain kinds of *infinite* summation[7]. They are generalized series (as in Hahn [27]) with real coefficients and monomials that themselves are exponentials of "simpler" transseries, such as the third term of

$$f = \mathrm{e}^{\frac{1}{2}\mathrm{e}^{x}} - 5\,\mathrm{e}^{x^2} + \mathrm{e}^{x+x^{1/2}+x^{1/3}+\cdots} + \sqrt[3]{2}\log x - x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \cdots + 5\,\mathrm{e}^{-x^{3/2}}.$$

The transseries form a field which, like each Hardy field, naturally comes equipped with a derivation $\frac{d}{dx}$ and an ordering making it an ordered differential field $\mathbb{T}$.

Surreal numbers, invented by J. H. Conway [20] in connection with game theory, have a more combinatorial flavor. They include both real numbers and Cantor's ordinal numbers, forming a proper class naturally equipped with an ordering and arithmetic operations making it a real closed ordered field extension $\mathbf{No}$ of $\mathbb{R}$. For example, with $\omega$ the first infinite ordinal, $\omega - \pi$, $1/\omega$, $\sqrt{\omega}$, make sense as surreal numbers[8]. Recently, Berarducci and Mantova [12] constructed a derivation $\partial_{\mathrm{BM}}$ on $\mathbf{No}$ making it an ordered differential field with field of constants $\mathbb{R}$ and $\partial_{\mathrm{BM}}(\omega) = 1$.

Thus maximal Hardy fields, the field of transseries $\mathbb{T}$, and $\mathbf{No}$ are all extensions of $\mathbb{R}$ to ordered differential fields containing both infinite and infinitesimal elements. It is natural to ask about the canonicity of such extensions of the continuum. By [ADH] and [4, 9], they share the same first-order theory, as ordered differential fields. In [10] we show that under Cantor's Continuum Hypothesis (CH), any maximal Hardy field is in fact *isomorphic* to the ordered differential subfield $\mathbf{No}(\omega_1)$ of $\mathbf{No}$ (consisting of the surreal numbers of countable length). In [2] this is also shown for maximal smooth and maximal analytic Hardy fields[9] Without even assuming CH, one can embed $\mathbb{T}$ into each maximal analytic Hardy field and also into $\mathbf{No}(\omega_1)$, cf. [2, 4]. One is left to wonder what Gödel would have made of this in light of his long standing interest in CH, his appreciation of A. Robinson's nonstandard analysis, and his musings, reported by Conway, whether or not *a solution to the Continuum Hypothesis might yet be possible, but only once the correct theory of infinitesimals had been found* [40, pp. 209–213].

**Organization of the paper.** To keep the length of this paper at bay we assume familiarity with the basic setup of [ADH]. (For a brief synopsis which should suffice for reading the present paper see the section *Concepts and Results from* [ADH] in the introduction to [8][10].) Section 1 has additional definitions and results from the papers [7, 8, 9] used in this note. Section 2 contains a Hensel type Lemma for analytic functions on Hahn fields. This is applied to solve certain equations involving power functions in $\mathbb{T}$, for use in connection with Theorem B. We then study differential closure, first in the general setting of differential fields (Section 3), then in $H$-fields (Sections 4 and 5), before proving Theorems A and B in Section 6.

**Notations and conventions.** For this we follow [ADH]. In particular, $m$, $n$ range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given an ordered abelian group $\Gamma$, additively written, we put $\Gamma^{>} := \{\gamma \in \Gamma : \gamma > 0\}$. For an additively written abelian group $A$ set $A^{\neq} := A \setminus \{0\}$. Given a commutative ring $R$ (always with identity 1), $R^{\times}$ denotes the multiplicative group of units of $R$. (So if $K$ is a field, then $K^{\neq} = K^{\times}$.) If $R$ is a differential ring (by convention containing $\mathbb{Q}$ as a subring) and $y \in R^{\times}$, then $y^{\dagger} = y'/y$ denotes the logarithmic derivative of $y$, so $(yz)^{\dagger} = y^{\dagger} + z^{\dagger}$ for $y, z \in R^{\times}$, and thus $R^{\dagger} := \{y^{\dagger} : y \in R^{\times}\}$ is an additive subgroup of $R$. The prefix "d" abbreviates "differentially"; for example, "d-algebraic" means "differentially algebraic".

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## 1. Preliminaries

We begin by recalling definitions, notations, and facts around germs, Hausdorff fields, and Hardy fields as needed later. Next we briefly discuss the main result of our paper [9] on the first-order theory of maximal Hardy fields. Finally, we include some material on general asymptotic differential algebra from [8], before discussing polar coordinates for germs in complexifications of Hardy fields.

**Germs.** Let $a$ range over $\mathbb{R}$ and $r$ over $\mathbb{N} \cup \{\omega, \infty\}$. We let $\mathcal{C}^r$ be the $\mathbb{R}$-algebra of germs at $+\infty$ of $\mathbb{R}$-valued $\mathcal{C}^r$-functions on half-lines $(a, +\infty)$, for varying $a$, where $\mathcal{C}^{\omega}$ means "analytic". Thus $\mathcal{C} := \mathcal{C}^0$ consists of the germs at $+\infty$ of continuous functions $(a, +\infty) \to \mathbb{R}$, and

$$\mathcal{C} \ = \ \mathcal{C}^0 \ \supseteq \mathcal{C}^1 \ \supseteq \ \mathcal{C}^2 \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^{\infty} \ \supseteq \ \mathcal{C}^{\omega}.$$

The complexification $\mathcal{C}[i] = \mathcal{C} + \mathcal{C}i$ of $\mathcal{C}$ is the $\mathbb{C}$-algebra consisting of the germs of continuous functions $(a, +\infty) \to \mathbb{C}$, for varying $a$. For $f, g \in \mathcal{C}$ we set

$$|f + ig| \ := \ \sqrt{f^2 + g^2} \in \mathcal{C}.$$

We have the $\mathbb{C}$-subalgebra $\mathcal{C}^r[i] = \mathcal{C}^r + \mathcal{C}^r i$ of $\mathcal{C}[i]$. For $n \geqslant 1$ we have the derivation $g \mapsto g' \colon \mathcal{C}^n[i] \to \mathcal{C}^{n-1}[i]$ such that (germ of $f$)$'$ = (germ of $f'$) for $\mathcal{C}^n$-functions $f \colon (a, +\infty) \to \mathbb{R}$, and $i' = 0$. Therefore $\mathcal{C}^{<\infty}[i] := \bigcap_n \mathcal{C}^n[i]$ is naturally a differential ring with ring of constants $\mathbb{C}$, and $\mathcal{C}^{<\infty} := \bigcap_n \mathcal{C}^n$ is a differential subring of $\mathcal{C}^{<\infty}[i]$ with ring of constants $\mathbb{R}$. Note that $\mathcal{C}^{<\infty}[i]$ has $\mathcal{C}^{\infty}[i]$ as a differential subring, $\mathcal{C}^{<\infty}$ has $\mathcal{C}^{\infty}$ as a differential subring, and $\mathcal{C}^{\infty}$ has in turn the differential subring $\mathcal{C}^{\omega}$.

**Asymptotic relations.** We often use the same notation for a $\mathbb{C}$-valued function on a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ as for its germ if the resulting ambiguity is harmless. We equip $\mathcal{C}$ with the partial ordering given by $f \leqslant g :\Leftrightarrow f(t) \leqslant g(t)$

for all sufficiently large real $t$, and equip $\mathcal{C}[\mathrm{i}]$ with the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ defined as follows: for $f, g \in \mathcal{C}[\mathrm{i}]$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in \mathbb{R}^{>}, \\ f \prec g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[\mathrm{i}]^{\times} \text{ and } |f| \leqslant c|g| \text{ for all } c \in \mathbb{R}^{>}, \\ f \sim g \quad &:\Longleftrightarrow \quad f - g \prec g. \end{aligned}$$

**Hausdorff fields.** Let $H$ be a *Hausdorff field*: a subfield of $\mathcal{C}$. Then the partial ordering of $\mathcal{C}$ restricts to a total ordering on $H$ which makes $H$ into an ordered field. The ordered field $H$ has a convex subring $\mathcal{O} := \{f \in H : f \preccurlyeq 1\}$, which is a valuation ring of $H$, and we consider $H$ accordingly as a valued ordered field. Moreover, $H[\mathrm{i}]$ is a subfield of $\mathcal{C}[\mathrm{i}]$, and $\mathcal{O} + \mathcal{O}\mathrm{i} = \big\{f \in H[\mathrm{i}] : f \preccurlyeq 1\big\}$ is the unique valuation ring of $H[\mathrm{i}]$ whose intersection with $H$ is $\mathcal{O}$. In this way we consider $H[\mathrm{i}]$ as a valued field extension of $H$. The asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $\mathcal{C}[\mathrm{i}]$ restrict to the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $H[\mathrm{i}]$ that $H[\mathrm{i}]$ has as a valued field (cf. [ADH, (3.1.1)]; likewise with $H$ in place of $H[\mathrm{i}]$.

**Hardy fields.** Let $H$ be a *Hardy field*: a differential subfield of $\mathcal{C}^{<\infty}$. Then $H$ is a Hausdorff field, and we consider $H$ as an ordered valued differential field with ordering and valuation as above. Any Hardy field is a pre-$H$-field; if it contains $\mathbb{R}$, it is is an $H$-field. We equip the differential subfield $H[\mathrm{i}]$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ with the unique valuation ring lying over that of $H$. Then $H[\mathrm{i}]$ is a pre-d-valued field of $H$-type with small derivation, and if $H \supseteq \mathbb{R}$, then $H[\mathrm{i}]$ is d-valued with constant field $\mathbb{C}$.

Recall that $H$ is said to be *maximal* if it has no proper Hardy field extension, and that every Hardy field has a maximal Hardy field extension. The intersection $\mathrm{E}(H)$ of all maximal Hardy field extensions of $H$ is a Hardy field extension of $H$, called the *perfect hull* of $H$, and if $\mathrm{E}(H) = H$, then $H$ is said to be *perfect.* We also say that $H$ is d-*maximal* if it has no proper d-algebraic Hardy field extension. Zorn yields a d-maximal d-algebraic Hardy field extension of $H$, hence the intersection $\mathrm{D}(H)$ of all d-maximal Hardy fields containing $H$ is a d-algebraic Hardy field extension of $H$, called the d-*perfect hull* of $H$. We call $H$ d-*perfect* if $\mathrm{D}(H) = H$. If $H$ is d-perfect, then $H \supseteq \mathbb{R}$ and $H$ is a Liouville closed $H$-field, by [7, remarks after Proposition 4.2]. We have $\mathrm{D}(H) \subseteq \mathrm{E}(H)$; indeed, by [7, Lemma 4.1]:

**Lemma 1.1.** $\mathrm{D}(H) = \big\{f \in \mathrm{E}(H) : f$ *is* d-*algebraic over* $H\big\}$.

A *smooth Hardy field* is a Hardy field $H \subseteq \mathcal{C}^{\infty}$, and an *analytic Hardy field* is a Hardy field $H \subseteq \mathcal{C}^{\omega}$. Instead of smooth and analytic Hardy fields we also speak of $\mathcal{C}^{\infty}$- and $\mathcal{C}^{\omega}$-Hardy fields. Let $r \in \{\infty, \omega\}$. A $\mathcal{C}^{r}$-*maximal Hardy field* is a $\mathcal{C}^{r}$-Hardy field which has no proper $\mathcal{C}^{r}$-Hardy field extensions. If $H \subseteq \mathcal{C}^{r}$, then we let $\mathrm{E}^{r}(H)$ be the intersection of all $\mathcal{C}^{r}$-maximal Hardy fields containing $H$. Thus using the notation from the introduction, our main objects of interest in this paper are $\mathrm{E} = \mathrm{E}(\mathbb{Q})$, $\mathrm{E}^{\infty} = \mathrm{E}^{\infty}(\mathbb{Q})$, and $\mathrm{E}^{\omega} = \mathrm{E}^{\omega}(\mathbb{Q})$. Instead of "$\mathcal{C}^{\infty}$-maximal" and "$\mathcal{C}^{r}$-maximal" we also write "maximal smooth" and "maximal analytic", respectively. The following is [9, Corollary 7.8]:

**Proposition 1.2.** *Suppose $H$ is smooth. Then every* d-*algebraic Hardy field extension of $H$ is also smooth; in particular,* $\mathrm{D}(H)$ *is smooth. Likewise with "smooth" replaced by "analytic".*

Let now $H$ be a $\mathcal{C}^{\infty}$-Hardy field. Then by Proposition 1.2, $H$ is d-maximal iff $H$ has no proper d-algebraic $\mathcal{C}^{\infty}$-Hardy field extension; thus every $\mathcal{C}^{\infty}$-maximal Hardy field is d-maximal, and $H$ has a d-maximal d-algebraic $\mathcal{C}^\infty$-Hardy field extension. The same remarks apply with $\omega$ in place of $\infty$.

**The main result of** [9]**.** A *closed $H$-field* (or *$H$-closed field*) is a Liouville closed, $\omega$-free, newtonian $H$-field. By [ADH], the closed $H$-fields with small derivation are precisely the models of the elementary theory of $\mathbb{T}$ as an ordered valued differential field. Hence by the next theorem, every d-maximal Hardy field as an ordered valued differential field is elementarily equivalent to $\mathbb{T}$.

**Theorem 1.3.** *For a Hardy field $H$, the following are equivalent:*

(i) *$H$ is a* d-*maximal Hardy field;*
(ii) *$H \supseteq \mathbb{R}$ and $H$ is a closed $H$-field;*
(iii) *$H \supseteq \mathbb{R}$ and $H$ is a Liouville closed $H$-field having* DIVP*.*

Here [9, Theorem 11.19] is the equivalence (i) $\Leftrightarrow$ (ii), and (ii) $\Leftrightarrow$ (iii) is [5, Corollary 1.7]. The preceding theorem and Proposition 1.2 yield [9, Corollary 11.20]:

**Corollary 1.4.** *Each Hardy field $H$ has a* d-*algebraic $H$-closed Hardy field extension. If $H$ is a $\mathcal{C}^\infty$-Hardy field, then so is any such extension, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

**Logarithmic derivatives.** Let $K$ be a differential field. The *group of logarithmic derivatives* of $K$ is the additive subgroup $K^\dagger = \{f^\dagger : f \in K^\times\}$ of $K$. If $K$ is algebraically closed or real closed, then $K^\dagger$ is divisible. Here is [8, Lemma 1.2.1]:

**Lemma 1.5.** *Suppose $K^\dagger$ is divisible, $L$ is a differential field extension of $K$ such that $L^\dagger \cap K = K^\dagger$, and $M$ is a differential field extension of $L$ and algebraic over $L$. Then $M^\dagger \cap K = K^\dagger$.*

Suppose that $H$ is a real closed asymptotic field whose valuation ring $\mathcal{O}$ is convex with respect to the ordering of $H$, and $K := H[i]$. Then $\mathcal{O}_K = \mathcal{O} + \mathcal{O}i$ is the unique valuation ring of $K$ with $\mathcal{O}_K \cap H = \mathcal{O}$ [ADH, 3.5.15]. Equipped with this valuation ring, $K$ is an asymptotic field extension of $H$ [ADH, 9.5.3], and if $H$ is $H$-asymptotic, then so is $K$. With $\operatorname{wr}(a,b) := ab' - a'b$ (the wronskian of $a, b$), set

$$S := \{y \in K : |y| = 1\}, \qquad W := \{\operatorname{wr}(a,b) : a, b \in H,\ a^2 + b^2 = 1\}.$$

Then $S$ is a subgroup of $\mathcal{O}_K^\times$ with $S^\dagger = Wi$ and $K^\dagger = H^\dagger \oplus Wi$ by [8, Lemma 1.2.4]. Recall from [ADH, 14.2] that for asymptotic fields $E$ (such as $H$, $K$) we defined

$$\mathrm{I}(E) := \{f \in E : f \preccurlyeq g' \text{ for some } g \preccurlyeq 1 \text{ in } E\}.$$

Since $\partial\mathcal{O} \subseteq \mathrm{I}(H)$, we also have $W \subseteq \mathrm{I}(H)$, and thus: $W = \mathrm{I}(H) \iff \mathrm{I}(H)i \subseteq K^\dagger$. Moreover, by [8, Lemma 1.2.13] we have $W = \mathrm{I}(H) \subseteq H^\dagger \iff \mathrm{I}(K) \subseteq K^\dagger$.

**Lemma 1.6.** *Suppose $H$ is $H$-asymptotic with asymptotic integration, and $K$ is* 1-*linearly newtonian. Then $K^\dagger = H^\dagger \oplus \mathrm{I}(H)i$. Moreover, if $F$ is a real closed asymptotic extension of $H$ whose valuation ring is convex, then*

$$F[i]^\dagger \cap K = (F^\dagger \cap H) \oplus \mathrm{I}(H)i.$$

*Proof.* By [8, Corollary 1.2.14] we have $\mathrm{I}(K) \subseteq K^\dagger$ and thus $W = \mathrm{I}(H)$ and $K^\dagger = H^\dagger \oplus \mathrm{I}(H)i$ by the remarks before the lemma. The second part of the lemma follows from this and [8, Corollary 1.2.15]. □

**The universal exponential extension.** *In this subsection $K$ is a differential field with algebraically closed constant field $C$ and divisible group $K^\dagger$ of logarithmic derivatives.* An *exponential extension* of $K$ is a differential ring extension $R$ of $K$ such that $R = K[E]$ for some $E \subseteq R^\times$ with $E^\dagger \subseteq K$. By [8, Section 2.2, especially Corollary 2.2.11 and remarks preceding it], there is an exponential extension $\mathrm{U}$ of $K$ with $C_{\mathrm{U}} = C$ such that every exponential extension $R$ of $K$ with $C_R = C$ embeds into $\mathrm{U}$ over $K$; any two such exponential extensions of $K$ are isomorphic over $K$. We call $\mathrm{U}$ the *universal exponential extension* of $K$, denoted by $\mathrm{U}_K$ if we want to stress the dependence on $K$. By its construction in [8, Section 2.2], $\mathrm{U}$ is an integral domain, and $(\mathrm{U}^\times)^\dagger = K$ by [8, remarks before Example 2.2.4]. We denote the differential fraction field of $\mathrm{U}$ by $\Omega$ (or $\Omega_K$); then $C_\Omega = C$ by [8, remark before Lemma 2.2.7]. If $L$ is a differential field extension of $K$ such that $C_L$ is algebraically closed and $L^\dagger$ is divisible, then by [8, Lemma 2.2.12] the natural inclusion $K \to L$ extends to an embedding $\mathrm{U}_K \to \mathrm{U}_L$ of differential rings.

Suppose that $K$ is d-valued of $H$-type with $\Gamma \neq \{0\}$ and with small derivation. By [8, Lemma 2.5.1], the valuation of $K$ extends to a valuation on $\Omega$ that makes $\Omega$ a d-valued extension of $K$ of $H$-type with small derivation, called a *spectral extension* of the valuation of $K$ to $\Omega$. By [8, Lemma 2.5.3 and Corollary 2.5.5] we have:

**Lemma 1.7.** *If $K$ is $\lambda$-free and $\mathrm{I}(K) \subseteq K^\dagger$, then the $H$-asymptotic couple $(\Gamma_\Omega, \psi_\Omega)$ of $\Omega$ equipped with a spectral extension of the valuation of $K$ is closed with $\Psi_\Omega := \{\psi_\Omega(\gamma) : \gamma \in \Gamma_\Omega^{\neq}\} \subseteq \Gamma$.*

**Some facts about complexified Hardy fields.** *In this subsection, $H$ is a Hardy field.* The $H$-asymptotic field extension $K := H[\mathrm{i}]$ of $H$ is a differential subring of $\mathcal{C}^{<\infty}[\mathrm{i}]$. The next proposition, [7, Proposition 6.11], considers the condition $\mathrm{I}(K) \subseteq K^\dagger$ in this setting:

**Proposition 1.8.** *Suppose $H \supseteq \mathbb{R}$ is closed under integration. Then the following are equivalent:*

(i) $\mathrm{I}(K) \subseteq K^\dagger$*;*
(ii) $\mathrm{e}^f \in K$ *for all* $f \in K$ *with* $f \prec 1$*;*
(iii) $\mathrm{e}^\phi, \cos\phi, \sin\phi \in H$ *for all* $\phi \in H$ *with* $\phi \prec 1$.

Next we discuss "polar coordinates" of nonzero elements of $K$:

**Lemma 1.9.** *Let $f \in \mathcal{C}[\mathrm{i}]^\times$. Then $|f| \in \mathcal{C}^\times$, and $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ for some $\phi \in \mathcal{C}$. Such $\phi$ is unique up to addition of an element of $2\pi\mathbb{Z}$. If also $f \in \mathcal{C}^r[\mathrm{i}]^\times$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, then $|f| \in \mathcal{C}^r$ and $\phi \in \mathcal{C}^r$ for such $\phi$.*

*Proof.* It is clear that $|f| \in \mathcal{C}^r$ if $f \in \mathcal{C}^r[\mathrm{i}]^\times$. To show existence of $\phi$ we replace $f$ by $f/|f|$ to arrange $|f| = 1$. Take $a \in \mathbb{R}$ and a continuous representative $[a, +\infty) \to \mathbb{C}$ of $f$, also denoted by $f$, such that $|f(t)| = 1$ for all $t \geqslant a$. The proof of [21, (9.8.1)] shows that for $b \in (a, +\infty)$ and $\phi_a \in \mathbb{R}$ with $f(a) = \mathrm{e}^{\phi_a \mathrm{i}}$ there is a unique continuous function $\phi\colon [a, b] \to \mathbb{R}$ such that $\phi(a) = \phi_a$ and $f(t) = \mathrm{e}^{\phi(t)\mathrm{i}}$ for all $t \in [a, b]$, and if also $f|_{[a,b]}$ is of class $\mathcal{C}^1$, then so is this $\phi$ with $\mathrm{i}\phi'(t) = f'(t)/f(t)$ for all $t \in [a, b]$. With $b \to +\infty$ this yields the desired result. □

**Lemma 1.10.** *Suppose $H \supseteq \mathbb{R}$ is Liouville closed and $f \in \mathcal{C}^1[\mathrm{i}]^\times$. Then $f^\dagger \in K$ iff $|f| \in H^>$ and $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ for some $\phi \in H$. If in addition $f \in K^\times$, then $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ for some $\phi \preccurlyeq 1$ in $H$.*

*Proof.* Take $\phi \in \mathcal{C}$ as in Lemma 1.9. Then $\phi \in \mathcal{C}^1$ and $\operatorname{Re} f^\dagger = |f|^\dagger$, $\operatorname{Im} f^\dagger = \phi'$. If $f \in K^\times$, then the remarks preceding Lemma 1.6 give $\phi' \in \mathrm{I}(H)$, so $\phi \preccurlyeq 1$. □

**Corollary 1.11.** *Suppose $H \supseteq \mathbb{R}$ is Liouville closed with $\mathrm{I}(K) \subseteq K^\dagger$. Let $L$ be a differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ containing $K$. Then $L^\dagger \cap K = K^\dagger$.*

*Proof.* Let $f \in L^\times$ satisfy $f^\dagger \in K$. Then $f = |f|\,\mathrm{e}^{\phi \mathrm{i}}$ with $|f| \in H^>$, $\phi \in H$, by Lemma 1.10. Hence $\mathrm{e}^{\phi \mathrm{i}}, \mathrm{e}^{-\phi \mathrm{i}} \in L$, thus $\cos\phi = \frac{1}{2}(\mathrm{e}^{\phi \mathrm{i}} + \mathrm{e}^{-\phi \mathrm{i}}) \in L$. In particular, $\cos\phi$ doesn't oscillate, so $\phi \preccurlyeq 1$. Thus $f = |f|(\cos\phi + \mathrm{i}\sin\phi) \in K$ by Proposition 1.8. □

## 2. Hensel's Lemma for Analytic Functions on Hahn Fields

Let $\boldsymbol{k}$ be a field and $\mathfrak{M}$ a (multiplicatively written) ordered abelian group, with the ordering of $\mathfrak{M}$ denoted by $\preccurlyeq$. Let $K = \boldsymbol{k}[[\mathfrak{M}]]$ be the corresponding Hahn field over $\boldsymbol{k}$. Its (Hahn) valuation has valuation ring $\mathcal{O} := \boldsymbol{k}[[\mathfrak{M}^{\preccurlyeq 1}]]$, with maximal ideal $\smallo := \boldsymbol{k}[[\mathfrak{M}^{\prec 1}]]$. Let $Q(Z)$ be a power series $\sum_n a_n Z^n$ with all $a_n \in \mathcal{O}$. Then for $z \in \smallo$ the sum $\sum_n a_n z^n$ exists with value $Q(z) := \sum_n a_n z^n \in \mathcal{O}$. Let $Q'(Z) := \sum_n (n+1)a_{n+1} Z^n$ be the formal derivative of $Q(Z)$. Here is a Hensel type lemma:

**Lemma 2.1.** *If $Q(0) \prec 1$ and $Q'(0) \asymp 1$, then $Q(z) = 0$ for a unique $z \in \smallo$.*

*Proof.* Assume $Q(0) \prec 1$, $Q'(0) \asymp 1$. This means $a_0 \prec 1$ and $a_1 \asymp 1$. Multiplying with $a_1^{-1}$ we reduce to the case $a_1 = 1$, so for $z \in \smallo$ we have

$$Q(z) = 0 \iff z - Q(z) = -a_0 - a_2 z^2 - a_3 z^3 - \cdots = z.$$

Now for distinct $z_1, z_2 \in \smallo$ we have $z_1^n - z_2^n \prec z_1 - z_2$ for $n \geqslant 2$, so the map $z \mapsto z - Q(z) \colon \smallo \to \smallo$ is contracting: $\big(z_1 - Q(z_1)\big) - \big(z_2 - Q(z_2)\big) \prec z_1 - z_2$ for distinct $z_1, z_2 \in \smallo$. Thus this map has a unique fixpoint $z \in \smallo$ by [ADH, 2.2.12]. □

The valued field $\mathbb{T}$ of transseries is not a Hahn field, but it is a direct union of Hahn subfields over the coefficient field $\mathbb{R}$, and in this way the above lemma applies to $\mathbb{T}$. For $c \in \mathbb{R}$ and $f \in \mathbb{T}^>$ we set $f^c := \exp(c \log f) \in \mathbb{T}$. Then $(f^c)^\dagger = c f^\dagger$ and $z \in \mathbb{T}$ with $z \prec 1$ gives $(1+z)^c = \sum_n \binom{c}{n} z^n \in \mathbb{T}$: for this, note that for real $t \in (-1,1)$,

$$(1+t)^c \ = \ \sum_{n=0}^{\infty} \binom{c}{n} t^n \ = \ \mathrm{e}^{c\log(1+t)} \ = \ \sum_{n=0}^{\infty} \frac{c^n}{n!} \left( \sum_{m=1}^{\infty} \frac{(-1)^{m-1} t^m}{m} \right)^n,$$

so the formal power series $\sum_{n=0}^{\infty} \binom{c}{n} Z^n$ and $\sum_{n=0}^{\infty} \frac{c^n}{n!} \big( \sum_{m=1}^{\infty} \frac{(-1)^{m-1} Z^m}{m} \big)^n$ in $Z$ are equal, from which the identity claimed about $z$ follows by substitution.

**Corollary 2.2.** *Let $c \in \mathbb{R} \setminus \{-1\}$ and $\varepsilon \in \smallo_{\mathbb{T}}$. Then there is a unique $z \in \smallo_{\mathbb{T}}$ with*

$$(1+z)^c \cdot (1 + \varepsilon + z) \ = \ 1.$$

*Proof.* Note that $Q(Z) := \big(\sum_n \binom{c}{n} Z^n\big)(1 + \varepsilon + Z) - 1$ satisfies the assumption of Lemma 2.1: its constant term is $\varepsilon$ and its term of degree 1 is $(1 + c + c\varepsilon)Z$. □

## 3. Relative Differential Closure

Let $K \subseteq L$ be an extension of differential fields, and let $r$ range over $\mathbb{N}$. We say that $K$ is **$r$-differentially closed** in $L$ for every $P \in K\{Y\}^{\neq}$ of order $\leqslant r$, each zero of $P$ in $L$ lies in $K$. We also say that $K$ is **weakly $r$-differentially closed** in $L$ if every $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ with a zero in $L$ has a zero in $K$. We abbreviate "$r$-differentially closed" by "$r$-d-closed." Thus

$$K \text{ is } r\text{-d-closed in } L \quad \Longrightarrow \quad K \text{ is weakly } r\text{-d-closed in } L,$$

and

$$\text{(3.1)}\qquad K \text{ is 0-d-closed in } L \iff K \text{ is weakly 0-d-closed in } L \iff K \text{ is algebraically closed in } L.$$

Hence, with $C$ the constant field of $K$ and $C_L$ the constant field of $L$:

$$\text{(3.2)}\qquad K \text{ is weakly 0-d-closed in } L \implies C \text{ is algebraically closed in } C_L.$$

Also, if $K$ is weakly 0-d-closed in $L$ and $L$ is algebraically closed, then $K$ is algebraically closed, and similarly with "real closed" in place of "algebraically closed". In [ADH, 5.8] we defined $K$ to be *weakly $r$-d-closed* if every $P \in K\{Y\} \setminus K$ of order $\leqslant r$ has a zero in $K$. Thus if $K$ is weakly 0-d-closed, then $K$ is algebraically closed, and

$$K \text{ is weakly } r\text{-d-closed} \iff \begin{cases} K \text{ is weakly } r\text{-d-closed in every differential field} \\ \text{extension of } K. \end{cases}$$

If $K$ is weakly $r$-d-closed in $L$, then $P(K) = P(L) \cap K$ for all $P \in K\{Y\}$ of order $\leqslant r$; in particular,

$$\text{(3.3)}\qquad K \text{ is weakly 1-d-closed in } L \implies \partial K = \partial L \cap K.$$

Also,

$$\text{(3.4)}\qquad K \text{ is 1-d-closed in } L \implies C = C_L \text{ and } K^\dagger = L^\dagger \cap K.$$

Moreover:

**Lemma 3.1.** *Suppose $K$ is weakly $r$-d-closed in $L$. If $L$ is $r$-linearly surjective, then so is $K$, and if $L$ is $(r+1)$-linearly closed, then so is $K$.*

*Proof.* The first claim from the remarks preceding the lemma, and the proof of the second statement is like that of [ADH, 5.8.9]. □

Sometimes we get more than we bargained for:

**Lemma 3.2.** *Suppose $K$ is not algebraically closed, $C \neq K$, and $K$ is weakly $r$-d-closed in $L$. Let $Q_1, \dots, Q_m \in K\{Y\}^{\neq}$ of order $\leqslant r$ have a common zero in $L$, $m \geqslant 1$. Then they have a common zero in $K$.*

*Proof.* We claim that some polynomial $\Phi \in K[X_1, \dots, X_m]$ has $(0, \dots, 0) \in K^m$ as its only zero in $K^m$. (This is folklore, but we didn't find a proof in the literature.) The claim is clear for $m = 1$. Since $K$ is not algebraically closed, we have a univariate polynomial $f \in K[X]$ of degree $> 1$ without a zero in $K$. Then the homogenization $g(X, Y) \in K[X, Y]$ of $f(X)$ has $(0, 0)$ as its only zero in $K^2$. This proves the claim for $m = 2$. Now use induction on $m$: if $\Phi$ has the above property, then $g\big(\Phi(X_1, \dots, X_m), X_{m+1}\big)$ has $(0, \dots, 0, 0) \in K^{m+1}$ as its only zero in $K^{m+1}$.

By the claim, the differential polynomial $P := \Phi(Q_1, \dots, Q_m) \in K\{Y\}$ is nonzero (use [ADH, 4.2.1]) and has order $\leqslant r$. For $y \in L$ we have

$$Q_1(y) = \cdots = Q_m(y) = 0 \implies P(y) = 0,$$

and for $y \in K$ the converse of this implication also holds. □

Here is a characterization of $r$-d-closedness:

**Lemma 3.3.** *The following are equivalent:*

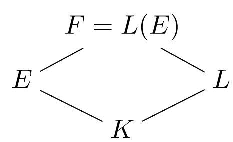
(i) *$K$ is $r$-d-closed in $L$;*
(ii) *there is no differential subfield $E$ of $L$ with $K \subset E$ and* $\operatorname{trdeg}(E|K) \leqslant r$.

*Proof.* Let $E$ be a differential subfield of $L$ with $E \supseteq K$ and $\operatorname{trdeg}(E|K) \leqslant r$. Then for $y \in E$ we have $\operatorname{trdeg}(K\langle y\rangle|K) \leqslant \operatorname{trdeg}(E|K) \leqslant r$, so [ADH, 4.1.11] gives a $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ with $P(y) = 0$, hence $y \in K$ if $K$ is $r$-d-closed in $L$. This shows (i) $\Rightarrow$ (ii). For the converse, note that for $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ and $y \in L$ with $P(y) = 0$ we have $K\langle y\rangle = K(y, y', \dots, y^{(r)})$, so $\operatorname{trdeg}(K\langle y\rangle|K) \leqslant r$. □

**Corollary 3.4.** *If $K$ is $r$-d-closed in $L$, then $K$ is $r$-d-closed in each differential subfield of $L$ containing $K$. If $M$ is a differential field extension of $L$ such that $L$ is $r$-d-closed in $M$ and $K$ is $r$-d-closed in $L$, then $K$ is $r$-d-closed in $M$.*

We say that $K$ is **differentially closed** in $L$ if $K$ is $r$-d-closed in $L$ for each $r$, and similarly we define when $K$ is **weakly differentially closed** in $L$. We also use “d-closed” to abbreviate “differentially closed”. If $K$, as a differential ring, is existentially closed in $L$, then $K$ is weakly d-closed in $L$. The elements of $L$ that are d-algebraic over $K$ form the smallest differential subfield of $L$ containing $K$ which is d-closed in $L$; we call it the **differential closure** (“d-closure” for short) of $K$ in $L$. Thus $K$ is d-closed in $L$ iff no d-subfield of $L$ properly containing $K$ is d-algebraic over $K$. For example, if $L = K\langle B\rangle$ where $B \subseteq K$ is d-algebraically independent over $K$ [ADH, p. 205], then $K$ is d-closed in $L$ by Corollary 3.4 and [ADH, 4.1.5]. This notion of being differentially closed does not seem prominent in the differential algebra literature, though the definition occurs (as “differentially algebraic closure”) in [36, p. 102]. Here is a useful fact about it:

**Lemma 3.5.** *Let $F$ be a differential field extension of $L$ and let $E$ be a subfield of $F$ containing $K$ such that $E$ is algebraic over $K$ and $F = L(E)$.*




*Then $K$ is d-closed in $L$ iff $E \cap L = K$ and $E$ is d-closed in $F$.*

*Proof.* Suppose $K$ is d-closed in $L$. Then $K$ is algebraically closed in $L$, so $L$ is linearly disjoint from $E$ over $K$. (See [37, Chapter VIII, §4].) In particular $E \cap L = K$. Now let $y \in F$ be d-algebraic over $E$; we claim that $y \in E$. Note that $y$ is d-algebraic over $K$. Take a field extension $E_0 \subseteq E$ of $K$ with $[E_0 : K] < \infty$ (so $E_0$ is a d-subfield of $E$) such that $y \in L(E_0)$; replacing $E$, $F$ by $E_0$, $L(E_0)$, respectively, we arrange that $n := [E : K] < \infty$. Let $b_1, \dots, b_n$ be a basis of the $K$-linear space $E$; then $b_1, \dots, b_n$ is also a basis of the $L$-linear space $F$. Let $\sigma_1, \dots, \sigma_n$ be the distinct field embeddings $F \to L^{\mathrm{a}}$ over $L$. Then the vectors

$$\big(\sigma_1(b_1), \dots, \sigma_1(b_n)\big), \dots, \big(\sigma_n(b_1), \dots, \sigma_n(b_n)\big) \in (L^{\mathrm{a}})^n$$

are $L^{\mathrm{a}}$-linearly independent [37, Chapter VI, Theorem 4.1]. Let $a_1, \dots, a_n \in L$ be such that $y = a_1 b_1 + \dots + a_n b_n$. Then

$$\sigma_j(y) = a_1\sigma_j(b_1) + \dots + a_n\sigma_j(b_n) \quad \text{for } j = 1, \dots, n,$$

hence by Cramer's Rule,

$$a_1, \dots, a_n \in K\big(\sigma_j(y), \sigma_j(b_i) : i, j = 1, \dots, n\big).$$

Therefore $a_1, \dots, a_n$ are d-algebraic over $K$, since $\sigma_j(y)$ and $\sigma_j(b_i)$ for $i, j = 1, \dots, n$ are. Hence $a_1, \dots, a_n \in K$ since $K$ is d-closed in $L$, so $y \in E$ as claimed. This shows the forward implication. The backward direction is clear. □

**Corollary 3.6.** *If $-1$ is not a square in $L$ and $\mathrm{i}$ in a differential field extension of $L$ satisfies $\mathrm{i}^2=-1$, then: $K$ is* d*-closed in $L$ $\Leftrightarrow$ $K[\mathrm{i}]$ is* d*-closed in $L[\mathrm{i}]$.*

The notion of d-closedness concerns one-variable differential polynomials, but under extra assumptions on the differential field extension $K \subseteq L$, it has consequences for systems of differential polynomial (in-)equalities in several indeterminates, by the next lemma. First some notation: for $P \in K\{Y\}$, $Y$ a single indeterminate, we let $\mathrm{Z}(P)$ denote the set of zeros of $P$ in $K$:

$$\mathrm{Z}(P) \;=\; \mathrm{Z}_K(P) \;:=\; \big\{y \in K : P(y)=0\big\}.$$

Let $S \subseteq K^n$ be defined in $K$ by a quantifier-free formula $\phi(y_1,\dots,y_n)$ in the language of differential fields with names for the elements of $K$, and let $S_L \subseteq L^n$ be defined in $L$ by the same formula. Then $S = S_L \cap K^n$, and if $K$ is existentially closed in $L$, then $S_L$ does not depend on the choice of $\phi$.

Suppose $C \neq K$. Then we defined in [3, Section 1] for any set $S \subseteq K^n$ its (differential) dimension $\dim S \in \{-\infty, 0, 1, 2, \dots, n\}$, with $\dim S = -\infty$ iff $S = \emptyset$. If $S \neq \emptyset$ and $S$ is finite, then $\dim S = 0$, and if $S \subseteq K$, then $\dim S = 0$ iff $S \neq \emptyset$ and $S \subseteq \mathrm{Z}(P)$ for some $P \in K\{Y\}^{\neq}$.

**Lemma 3.7.** *Suppose that $C \neq K$, that $K$ is both existentially closed in $L$ and* d*-closed in $L$, and that $\dim S = 0$. Then $S = S_L$.*

*Proof.* Since $K$ is existentially closed in $L$, [ADH, B.8.5] yields a differential field extension $K^*$ of $L$ such that $K \preccurlyeq K^*$. For $i = 1,\dots,n$ and the $i$th coordinate projection $\pi_i \colon K^n \to K$ we have $\dim \pi_i(S) = 0$ and therefore $\pi_i(S) \subseteq \mathrm{Z}(P_i)$ with $P_i \in K\{Y\}^{\neq}$, by [3, Lemma 1.2], so $S \subseteq \mathrm{Z}(P_1) \times \cdots \times \mathrm{Z}(P_n)$, hence

$$S_{K^*} \subseteq \mathrm{Z}_{K^*}(P_1) \times \cdots \times \mathrm{Z}_{K^*}(P_n) \qquad \text{since } K \preccurlyeq K^*\text{, and thus}$$
$$S_L = S_{K^*} \cap L^n \subseteq \mathrm{Z}_L(P_1) \times \cdots \times \mathrm{Z}_L(P_n) \subseteq K^n \qquad \text{since } K \text{ is d-closed in } L.$$

Thus $S = S_L$. □

Recall from [ADH, 4.7] that $K$ is said to be d-*closed* if for all $P \in K[Y,\dots,Y^{(r)}]^{\neq}$ and $Q \in K[Y,\dots,Y^{(r-1)}]^{\neq}$ such that $Y^{(r)}$ occurs in $P$ there is a $y \in K$ such that $P(y)=0$ and $Q(y) \neq 0$. (For $r=0$ this says that $K$ is algebraically closed.) If $K$ is d-closed in $L$ and $L$ is d-closed, then $K$ is d-closed. Therefore, since $K$ has a d-closed differential field extension (cf. [ADH, remark after 4.7.1]): if $K$ is d-closed in each differential field extension, then $K$ is d-closed. The reverse implication, however, is false: if $C$ is algebraically closed, then $K$ has a proper d-algebraic differential field extension $L$, even with $C_L = C$. To see this we use the theorem of Rosenlicht [44, Proposition 2] below. We equip the field $C(Y)$ of rational functions over the constant field $C$ of $K$ with the derivation $\partial = \partial/\partial Y$ [ADH, p. 200].

**Theorem 3.8** (Rosenlicht)**.** *Let $R \in C(Y)^\times$ be such that $1/R$ is neither of the form $a\frac{\partial U}{U}$ ($a \in C, U \in C(Y)^\times$), nor of the form $\partial V$ ($V \in C(Y)^\times$). Let $y$ be an element of a differential field extension of $K$ such that $y$ is transcendental over $K$ and $y' = R(y)$. Then $C_{K\langle y\rangle} = C$.*

Let $c \in C$ and consider the differential polynomial $P = P_c \in K\{Y\}$ and the rational function $R = R_c \in C(Y)$ given by

$$P(Y) := Y'\big(c(Y+1)+Y\big) - Y(Y+1) \in K\{Y\}, \qquad R(Y) := \frac{Y(Y+1)}{c(Y+1)+Y} \in C(Y).$$

For $y \notin K$ in a differential field extension of $K$ we have $P(y) = 0$ iff $y' = R(y)$. We identify $\mathbb{Q}$ with the prime field of $C$, as usual. *Suppose $c \notin \mathbb{Q}$.* Then $R$ satisfies the hypotheses of Theorem 3.8: First,

$$\frac{1}{R} \;=\; \frac{1}{Y+1} + \frac{c}{Y}.$$

Next, if $\frac{1}{R} = a\,\frac{\partial U}{U}$, $a \in C^\times$, $U \in C(Y)^\times$, then by [ADH, 4.1.3] and [37, Chapter V, Theorem 5.2] both $1/a$ and $c/a$ are integers, hence $c \in \mathbb{Q}$, a contradiction. Equip $E := C(Y)$ with the valuation $v\colon E^\times \to \mathbb{Z}$ with $v(C^\times) = \{0\}$ and $v(Y) = 1$. Then $E$ is d-valued with $\Psi_E = \{-1\}$. Since $v(1/R) = -1$, there is also no $V \in E^\times$ with $\partial V = 1/R$. For the constant field $C$ of $K$ this yields by Theorem 3.8:

**Corollary 3.9.** *Let $c \in C \setminus \mathbb{Q}$. Then $P(y) = 0$ for some $y$ in a differential field extension of $K$ and transcendental over $K$; for any such $y$ we have $C_{K\langle y\rangle} = C$.*

## 4. Revisiting [ADH, 16.0.3]

The following important fact about closed $H$-fields stands in marked contrast to Corollary 3.9. It is [ADH, 16.0.3], and was applied in [3] to gain information about the zero sets of differential polynomials in closed $H$-fields:

**Theorem 4.1.** *Every closed $H$-field is* d-*closed in every $H$-field extension with the same constant field.*

Any pre-$H$-field extension of an $H$-field with the same residue field is automatically an $H$-field with the same constant field, so in Theorem 4.1 we can replace "$H$-field extension with the same constant field" by "pre-$H$-field extension with the same residue field". But we cannot further weaken this to "pre-$H$-field extension with the same constant field" as we show in the next subsection. (This justifies a remark in [ADH, "Notes and comments" on p. 684].)

**How not to use Theorem 4.1.** We work in the differential field $\mathbb{T}$ of transseries. Let $c \in \mathbb{R}^>$, and let $P = P_c \in \mathbb{R}\{Y\}$ be as introduced at the end of the previous section[11]. Note that $P(0) = P(-1) = 0$. Let $y \in \mathbb{T} \setminus \{0, -1\}$ and put

$$U(y) \;:=\; |y|^c(y+1) \in \mathbb{T}^\times,$$

so

$$U(y)^\dagger \;=\; c\frac{y'}{y} + \frac{y'}{y+1} \;=\; y' \cdot \frac{c(y+1)+y}{y(y+1)}$$

and therefore

$$\begin{aligned} P(y) = 0 \quad &\iff \quad U(y)^\dagger = 1 \\ &\iff \quad U(y) \in \mathbb{R}^\times\, \mathrm{e}^x \\ &\iff \quad |y|^c(y+1) = a\,\mathrm{e}^x \text{ for some } a \in \mathbb{R}^\times. \end{aligned}$$

Hence $y \succ 1$ whenever $P(y) = 0$. More precisely:

**Lemma 4.2.** *Let $a \in \mathbb{R}^\times$, $y \in \mathbb{T}^\times$ with $|y|^c(y+1) = a\,\mathrm{e}^x$. Then $y \sim b\,\mathrm{e}^{x/(c+1)}$ where $b = a^{1/(c+1)}$ if $a > 0$ and $b = -(-a)^{1/(c+1)}$ if $a < 0$.*

*Proof.* Note that $|y|^c y \sim |y|^c(1+y) = a\,\mathrm{e}^x$. Hence if $a > 0$, then $y > 0$ and $y \sim b\,\mathrm{e}^{x/(c+1)}$ with $b := a^{1/(c+1)}$, and if $a < 0$, then $y < 0$ and $(-y)^{c+1} \sim -a\,\mathrm{e}^x$, thus $y \sim b\,\mathrm{e}^{x/(c+1)}$ for $b := -(-a)^{1/(c+1)}$. ∎

In the next lemma we let $b \in \mathbb{R}^\times$, and for $z \in \mathbb{T}$, $z \prec 1$ we set

$$Q(z) \ := \ (1+z)^c(1+\varepsilon+z)-1 \qquad \text{where } \varepsilon := b^{-1}\,\mathrm{e}^{-x/(c+1)} \prec 1.$$

**Lemma 4.3.** *Let $y = b\,\mathrm{e}^{x/(c+1)}(1+z)$ where $z \prec 1$. Then $P(y) = 0 \Longleftrightarrow Q(z) = 0$.*

*Proof.* We have $|y|^c(y+1) = |b|^c b\,\mathrm{e}^x\big(Q(z)+1\big)$ and $Q(z)+1 \sim 1$. Assume $P(y) = 0$, and take $a \in \mathbb{R}^\times$ such that $|y|^c(y+1) = a\,\mathrm{e}^x$. Then $a = |b|^c b$ and $Q(z) = 0$. Conversely, if $Q(z) = 0$, then $|y|^c(y+1) = a\,\mathrm{e}^x$ for $a := |b|^c b$ and so $P(y) = 0$. □

Combining Corollary 2.2 with the previous lemma yields:

**Corollary 4.4.** *For each $b \in \mathbb{R}^\times$ there is a unique $y \in \mathbb{T}^\times$ such that $P(y) = 0$ and $y \sim b\,\mathrm{e}^{x/(c+1)}$.*

Let $H$ be a prime model of the theory of closed $H$-fields with small derivation, and identify $H$ with an ordered valued differential subfield of $\mathbb{T}$; see [ADH, p. 705]. The constant field $C$ of $H$ is the real closure of $\mathbb{Q}$ in $\mathbb{R}$. Take $c \in C^>$. Since $H$ is Liouville closed, we have $d \in \mathbb{R}^>$ with $f := d\,\mathrm{e}^{x/(c+1)} \in H^>$. Suppose also $c \notin \mathbb{Q}$ and let $b \in \mathbb{R}\setminus C$. Then Corollary 4.4 gives $y \in \mathbb{T}^\times$ such that $P(y) = 0$ and $y \sim bf$. So $H\langle y\rangle$ is a pre-$H$-subfield of $\mathbb{T}$ with $y \notin H$, and $C_{H\langle y\rangle} = C$ by Corollary 3.9. Hence $H\langle y\rangle \supseteq H$ is an example of a proper d-algebraic pre-$H$-field extension of a closed $H$-field with the same constant field, as promised after Theorem 4.1. A similar argument gives an analogue of Corollary 3.9 in the category of $H$-fields:

**Corollary 4.5.** *Let $H \subseteq E$ be an extension of closed $H$-fields with small derivation such that $C_E \neq C$, and let $c \in C^>$. Then $P_c(y) = 0$ for some $y \in E \setminus H$, and if $c \notin \mathbb{Q}$, then for any such $y$ we have $C_{H\langle y\rangle} = C$.*

*Proof.* Take $f \in H^\times$ with $f^\dagger = 1/(c+1)$. Since $E \equiv \mathbb{T}$, there is for each $b \in C_E^\times$ a unique $y \in E^\times$ with $P_c(y) = 0$ and $y \sim bf$. Taking $b \in C_E \setminus C$, this $y$ satisfies $y \notin H$, with $C_{H\langle y\rangle} = C$ if also $c \notin \mathbb{Q}$, by Corollary 3.9. □

*Example.* Let $H := \mathbb{T}_x := \mathbb{R}[[[x]]]$ be the field of transseries in $x$ over $\mathbb{R}$ and let $E := \mathbb{T}_u[[[x]]]$ be the field of transseries in $x$ over the field of transseries $\mathbb{T}_u := \mathbb{R}[[[u]]]$ in a second variable $u$ (elements in $\mathbb{T}_u$ being constants for the derivation $\frac{d}{dx}$). We refer to [33] for the construction of transseries[12] over an arbitrary exponential ordered constant field like $\mathbb{T}_u$[13]. It is also shown there that both $H$ and $E$ are Liouville closed and satisfy DIVP and so are $H$-closed by [5, Theorem 2.7]. Now by what precedes, there is a unique $y \in E\setminus H$ with $P_c(y) = 0$, $y \sim u\,\mathrm{e}^{x/(c+1)}$, and $C_{H\langle y\rangle} = \mathbb{R}$.

In the rest of this subsection $H$ is a closed $H$-field with small derivation and constant field $C$. In [3, Section 5] we called a definable set $S \subseteq H^n$ *parametrizable by constants* if there are a semialgebraic set $X \subseteq C^m$, for some $m$, and a definable bijection $X \to S$. Lemma 4.2, Corollary 4.4, and $H \equiv \mathbb{T}$ yield:

**Corollary 4.6.** *For every $c \in C^>$ the definable set $\mathrm{Z}(P_c) \subseteq H$ is parametrizable by constants.*

In [3, Section 5] an irreducible differential polynomial $Q \in H\{Y\}^{\neq}$ was said to *create a constant* if for some element $y$ in a differential field extension of $H$ with minimal annihilator $Q$ over $H$ we have $C_{H\langle y\rangle} \neq C$. We showed that for such $Q$, if $\operatorname{order}(Q) = 1$, then $\mathrm{Z}(Q)$ is parametrizable by constants [3, Proposition 5.4]. If $c \notin \mathbb{Q}$ in Corollary 4.6, then the irreducible differential polynomial $P_c \in H\{Y\}$ of order 1 is parametrizable by constants, yet does not create a constant. Nonetheless, it does create a constant in the following more liberal sense.

**Proposition 4.7.** *Let $E$ be a closed $H$-field extension of $H$ and let $y \in E \setminus H$ be d-algebraic over $H$. Then some $c \in C_E \setminus C$ is definable in $E$ over $H \cup \{y\}$.*

*Proof.* The definable closure $K$ of $H \cup \{y\}$ in $E$ is a differential subfield, even an $H$-subfield of $E$: if $u \in K$ and $u \asymp 1$, then $u \sim c$ for a unique $c \in C_E$, so $c \in K$ for this $c$. It remains to note that the constant field of $K$ is strictly larger than that of $H$, by Theorem 4.1 and our assumption that $y \in K \setminus H$ is d-algebraic over $H$. □

**Generalizing Theorem 4.1.** In this subsection $K$ is a d-valued field of $H$-type with algebraically closed constant field $C$ and divisible group $K^\dagger$ of logarithmic derivatives. We use spectral extensions to prove an analogue of Theorem 4.1:

**Theorem 4.8.** *Suppose $K$ is $\omega$-free and newtonian. Then $K$ is* d*-closed in each* d*-valued field extension $L$ of $H$-type with $C_L = C$ and $L^\dagger \cap K = K^\dagger$.*

This yields a generalization of Theorem 4.1:

**Corollary 4.9.** *Let $H$ be an $\omega$-free newtonian real closed $H$-field and $E$ be an $H$-field extension of $H$. Then $H$ is* d*-closed in $E$ iff $C_E = C_H$ and $E^\dagger \cap H = H^\dagger$.*

*Proof.* The "only if" direction holds by (3.4). Suppose $C_E = C_H$ and $E^\dagger \cap H = H^\dagger$; we need to show that $H$ is d-closed in $E$. Replacing $E$ by its real closure and using Lemma 1.5 and Corollary 3.6, it suffices to show that the d-valued field $K := H[\mathrm{i}]$ is d-closed in its d-valued field extension $L := E[\mathrm{i}]$. By [ADH, 11.7.23, 14.5.7], $K$ is $\omega$-free and newtonian. Also $C_L = C_E[\mathrm{i}] = C_H[\mathrm{i}] = C$. Moreover, by Lemma 1.6,

$$L^\dagger \cap K \;=\; (E^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i} \;=\; H^\dagger \oplus \mathrm{I}(H)\mathrm{i} \;=\; K^\dagger.$$

Now use Theorem 4.8. □

In the same way that [ADH, 16.0.3] follows from [ADH, 16.1.1], Theorem 4.8 follows from an analogue of [ADH, 16.1.1]:

**Lemma 4.10.** *Let $K$ be an $\omega$-free newtonian* d*-valued field, $L$ a* d*-valued field extension of $K$ of $H$-type with $C_L = C$ and $L^\dagger \cap K = K^\dagger$, and let $f \in L \setminus K$. Suppose there is no $y \in K\langle f\rangle \setminus K$ such that $K\langle y\rangle$ is an immediate extension of $K$. Then the $\mathbb{Q}$-linear space $\mathbb{Q}\Gamma_{K\langle f\rangle}/\Gamma$ is infinite-dimensional.*

The proof of Lemma 4.10 is much like that of [ADH, 16.1.1], except where the latter uses that any $b$ in a Liouville closed $H$-field equals $a^\dagger$ for some nonzero $a$ in that field. This might not work with elements of $K$, and the remedy is to take instead for every $b \in K$ an element $a$ in $\mathrm{U}^\times$ with $b = a^\dagger$. The relevant computation should then take place in the differential fraction field $\Omega_L$ of $\mathrm{U}_L$ instead of in $L$ where $\Omega_L$ is equipped with a spectral extension of the valuation of $L$. For all this to make sense, we first take an active $\phi$ in $K$ and replace $K$ and $L$ by $K^\phi$ and $L^\phi$, arranging in this way that the derivation of $L$ (and of $K$) is small. Next we replace $L$ by its algebraic closure, so that $L^\dagger$ is divisible, while preserving $L^\dagger \cap K = K^\dagger$ by [8, Lemma 1.2.1], and also preserving the other conditions on $L$ in Lemma 4.10, as well as the derivation of $L$ being small. This allows us to identify $\mathrm{U}$ with a differential subring of $\mathrm{U}_L$ and accordingly $\Omega$ with a differential subfield of $\Omega_L$. We equip $\Omega_L$ with a spectral extension of the valuation of $L$ and make $\Omega$ a valued subfield of $\Omega_L$. Then the valuation of $\Omega$ is a spectral extension of the valuation of $K$ to $\Omega$, so we have the following inclusions of d-valued fields:

$$\begin{array}{ccc} L & \longrightarrow & \Omega_L \\ \uparrow & & \uparrow \\ K & \longrightarrow & \Omega \end{array}$$

With these preparations we can now give the proof of Lemma 4.10:

*Proof.* As we just indicated we arrange that $L$ is algebraically closed with small derivation, and with an inclusion diagram of d-valued fields involving $\Omega$ and $\Omega_L$, as above. (This will not be used until we arrive at the Claim below.)

By [ADH, 14.0.2], $K$ is asymptotically d-algebraically maximal. Using this and the assumption about $K\langle f\rangle$ it follows as in the proof of [ADH, 16.1.1] that there is no divergent pc-sequence in $K$ with a pseudolimit in $K\langle f\rangle$. Thus every $y$ in $K\langle f\rangle\setminus K$ has a *a best approximation in* $K$, that is, an element $b \in K$ such that $v(y-b) = \max v(y-K)$. For such $b$ we have $v(y-b) \notin \Gamma$, since $C_L = C$.

Now pick a best approximation $b_0$ in $K$ to $f_0 := f$, and set $f_1 := (f_0 - b_0)^\dagger$. Then $f_1 \in K\langle f\rangle \setminus K$, since $L^\dagger \cap K = K^\dagger$ and $C = C_L$. Thus $f_1$ has a best approximation $b_1$ in $K$, and continuing this way, we obtain a sequence $(f_n)$ in $K\langle f\rangle \setminus K$ and a sequence $(b_n)$ in $K$, such that $b_n$ is a best approximation in $K$ to $f_n$ and $f_{n+1} = (f_n - b_n)^\dagger$ for all $n$. Thus $v(f_n - b_n) \in \Gamma_{K\langle f\rangle} \setminus \Gamma$ for all $n$.

**Claim:** $v(f_0 - b_0), v(f_1 - b_1), v(f_2 - b_2), \ldots$ *are* $\mathbb{Q}$*-linearly independent over* $\Gamma$.

To prove this claim, take $a_n \in \mathrm{U}^\times$ with $a_n^\dagger = b_n$ for $n \geqslant 1$. Then in $\Omega_L$,

$$f_n - b_n \;=\; (f_{n-1} - b_{n-1})^\dagger - a_n^\dagger \;=\; \left(\frac{f_{n-1} - b_{n-1}}{a_n}\right)^\dagger \qquad (n \geqslant 1).$$

With $\psi := \psi_{\Omega_L}$ and $\alpha_n = v(a_n) \in \Gamma_\Omega \subseteq \Gamma_{\Omega_L}$ for $n \geqslant 1$, we get

$$v(f_n - b_n) \;=\; \psi\big(v(f_{n-1} - b_{n-1}) - \alpha_n\big), \quad \text{so by an easy induction on } n,$$

$$v(f_n - b_n) \;=\; \psi_{\alpha_1,\ldots,\alpha_n}\big(v(f_0 - b_0)\big) \qquad (n \geqslant 1).$$

(The definition of the functions $\psi_{\alpha_1,\ldots,\alpha_n}$ is given just before [ADH, 9.9.2].) Suppose towards a contradiction that $v(f_0 - b_0), \ldots, v(f_n - b_n)$ are $\mathbb{Q}$-linearly dependent over $\Gamma$. Then we have $m < n$ and $q_1, \ldots, q_{n-m} \in \mathbb{Q}$ such that

$$v(f_m - b_m) + q_1 v(f_{m+1} - b_{m+1}) + \cdots + q_{n-m} v(f_n - b_n) \in \Gamma.$$

For $\gamma := v(f_m - b_m) \in \Gamma_L \setminus \Gamma$ this gives

$$\gamma + q_1 \psi_{\alpha_{m+1}}(\gamma) + \cdots + q_{n-m} \psi_{\alpha_{m+1},\ldots,\alpha_n}(\gamma) \in \Gamma.$$

By [ADH, 14.2.5] we have $\mathrm{I}(K) \subseteq K^\dagger$, so the $H$-asymptotic couple of $\Omega$ is closed with $\Psi_\Omega \subseteq \Gamma$, by Lemma 1.7. Hence $\gamma \in \Gamma_\Omega$ by [ADH, 9.9.2]. Together with $\Psi_\Omega \subseteq \Gamma$ and $\alpha_{m+1}, \ldots, \alpha_n \in \Gamma_\Omega$ this gives $\psi_{\alpha_{m+1}}(\gamma), \ldots, \psi_{\alpha_{m+1},\ldots,\alpha_n}(\gamma) \in \Gamma$ and thus $\gamma \in \Gamma$, a contradiction. □

We augment the language $\mathcal{L}_\preccurlyeq = \{0, 1, -, +, \cdot, \partial, \preccurlyeq\}$ of valued differential rings by a unary relation symbol to obtain the language $\mathcal{L}_c$. We construe each valued differential field as an $\mathcal{L}_c$-structure by interpreting the new relation symbol as its constant field. In [3, Proposition 6.2] we showed that if $H$ is a closed $H$-field, then a nonempty definable set $S \subseteq H^n$ is co-analyzable (relative to $C_H$) iff $\dim S = 0$. (See Section 6 of [3] for the definition of "co-analyzable"[14].) This used Theorem 4.1

and a model-theoretic test for co-analyzability from [31] (cf. [3, Proposition 6.1]). From Corollary 4.9 we obtain a partial generalization of this fact:

**Corollary 4.11.** *Let $H$ be an $\omega$-free newtonian real closed $H$-field. If $S \subseteq H^n$ and* $\dim S = 0$*, then $S$ is co-analyzable.*

*Proof.* If $E$ is an elementary extension of $H$, then $E^\dagger \cap H = H^\dagger$. Hence for each $P \in H\{Y\}^{\neq}$, the zero set $\mathrm{Z}(P) \subseteq H$ is co-analyzable by Corollary 4.9 and by [3, Proposition 6.1] applied to the $\mathcal{L}_{c,A}$-theory $\mathrm{Th}(H_A)$ where $A$ is the finite set of nonzero coefficients of $P$. This special case implies the general case, since for each $S \subseteq H^n$ with $\dim S = 0$ there are $P_1, \dots, P_n \in H\{Y\}^{\neq}$ such that $S$ is contained in $\mathrm{Z}(P_1) \times \cdots \times \mathrm{Z}(P_n)$. □

Likewise, using Theorem 4.8 instead of Corollary 4.9 yields:

**Corollary 4.12.** *If $K$ is $\omega$-free and newtonian, then each set $S \subseteq K^n$ such that* $\dim S = 0$ *is co-analyzable.*

Thus for $K$, $S$ as in Corollary 4.12 and countable $C$, all $S \subseteq K^n$ of dimension 0 are countable, by [3, Proposition 6.1]. Note that Corollary 4.11 applies to the valued differential field $\mathbb{T}_{\log}$ of logarithmic transseries from [ADH, Appendix A][15].

## 5. Relative Differential Closure in $H$-fields

In this section we turn to relative differential closure in the $H$-field setting, and we use Theorem 4.1 to relate d-closedness to the elementary substructure property and d-closure to Newton-Liouville closure.

Let $\mathcal{L}_\partial = \{0, 1, -, +, \cdot, \partial\}$ be the language of differential rings, a sublanguage of the language $\mathcal{L} := \mathcal{L}_\partial \cup \{\leqslant, \preccurlyeq\}$ of ordered valued differential rings (see [ADH, p. 678]). In this section we also let $M$ be an $H$-closed field and $H$ a pre-$H$-subfield of $M$ whose valuation ring and constant field we denote by $\mathcal{O}$ and $C$. Construing $H$ and $M$ as $\mathcal{L}$-structures in the usual way, $H$ is an $\mathcal{L}$-substructure of $M$. We also use the sublanguage $\mathcal{L}_\preccurlyeq := \mathcal{L}_\partial \cup \{\preccurlyeq\}$ of $\mathcal{L}$, so $\mathcal{L}_\preccurlyeq$ is the language of valued differential rings. We expand the $\mathcal{L}_\partial$-structure $H[\mathrm{i}]$ to an $\mathcal{L}_\preccurlyeq$-structure by interpreting $\preccurlyeq$ as the dominance relation associated to the valuation ring $\mathcal{O} + \mathcal{O}\mathrm{i}$ of $H[\mathrm{i}]$; we expand likewise $M[\mathrm{i}]$ to an $\mathcal{L}_\preccurlyeq$-structure by interpreting $\preccurlyeq$ as the dominance relation associated to the valuation ring $\mathcal{O}_{M[\mathrm{i}]} = \mathcal{O}_M + \mathcal{O}_M\mathrm{i}$ of $M[\mathrm{i}]$. Then $H[\mathrm{i}]$ is an $\mathcal{L}_\preccurlyeq$-substructure of $M[\mathrm{i}]$. By $H \preccurlyeq_\mathcal{L} M$ we mean that $H$ is an elementary $\mathcal{L}$-substructure of $M$, and we use expressions like "$H[\mathrm{i}] \preccurlyeq_{\mathcal{L}_\preccurlyeq} M[\mathrm{i}]$" in the same way; of course, the two uses of the symbol $\preccurlyeq$ in the latter are unrelated.

By Corollary 3.6, $H$ is d-closed in $M$ iff $H[\mathrm{i}]$ is d-closed in $M[\mathrm{i}]$.

**Lemma 5.1.** *Suppose $M$ has small derivation. Then*

$$H \preccurlyeq_{\mathcal{L}_\partial} M \iff H[\mathrm{i}] \preccurlyeq_{\mathcal{L}_\partial} M[\mathrm{i}].$$

*Also, if $H \preccurlyeq_{\mathcal{L}_\partial} M$, then $H \preccurlyeq_\mathcal{L} M$ and $H[\mathrm{i}] \preccurlyeq_{\mathcal{L}_\preccurlyeq} M[\mathrm{i}]$.*

*Proof.* The forward direction in the equivalence is obvious. For the converse, let $H[\mathrm{i}] \preccurlyeq_{\mathcal{L}_\partial} M[\mathrm{i}]$. We have $M \equiv_{\mathcal{L}_\partial} \mathbb{T}$ by [ADH, 16.6.3]. Then [ADH, 10.7.10] yields an $\mathcal{L}_\partial$-formula defining $M$ in $M[\mathrm{i}]$, so the same formula defines $M \cap H[\mathrm{i}] = H$ in $H[\mathrm{i}]$, and thus $H \preccurlyeq_{\mathcal{L}_\partial} M$. For the "also" part, use that the squares of $M$ are the nonnegative elements in its ordering, that $\mathcal{O}_M$ is then definable as the convex hull of $C_M$ in $M$ with respect to this ordering, and if $H \preccurlyeq_{\mathcal{L}_\partial} M$, then each $\mathcal{L}_\partial$-formula defining $\mathcal{O}_M$ in $M$ also defines $\mathcal{O} = \mathcal{O}_M \cap H$ in $H$. □

By [ADH, 14.5.7, 14.5.3], if $H$ is $H$-closed, then $H[i]$ is weakly d-closed. The next proposition complements Theorem 4.1 and [ADH, 16.2.5]:

**Proposition 5.2.** *The following are equivalent:*

(i) $H$ *is* d*-closed in* $M$*;*
(ii) $C = C_M$ *and* $H \preccurlyeq_{\mathcal{L}} M$*;*
(iii) $C = C_M$ *and* $H$ *is* $H$*-closed.*

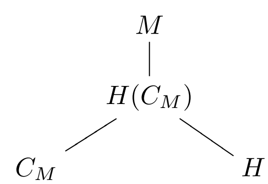
First a lemma, where *extension* refers to an extension of valued differential fields, and where $r \in \mathbb{N}$.

**Lemma 5.3.** *Let* $K$ *be a* $\lambda$*-free* $H$*-asymptotic field which is* $r$-d*-closed in an* $r$*-newtonian ungrounded* $H$*-asymptotic extension* $L$*. Then* $K$ *is also* $r$*-newtonian.*

*Proof.* Let $P \in K\{Y\}^{\neq}$ be quasilinear of order $\leqslant r$. Then $P$ remains quasilinear when viewed as differential polynomial over $L$, by [8, Lemma 1.7.9]. Hence $P$ has a zero $y \preccurlyeq 1$ in $L$, which lies in $K$ since $K$ is $r$-d-closed in $L$. □

*Proof of Proposition 5.2.* Assume (i). Then $C = C_M$ and $H$ is a Liouville closed $H$-field, by (3.1), (3.3), and (3.4). We have $\omega(M) \cap H = \omega(H)$ since $H$ is weakly 1-d-closed in $M$, and $\sigma\big(\Gamma(M)\big) \cap H = \sigma\big(\Gamma(M) \cap H\big) = \sigma\big(\Gamma(H)\big)$ since $H$ is 2-d-closed in $M$ and $\Gamma(M) \cap H = \Gamma(H)$ by [ADH, p. 520]. Now $M$ is Schwarz closed [ADH, 14.2.20], so $M = \omega(M) \cup \sigma\big(\Gamma(M)\big)$, hence also $H = \omega(H) \cup \sigma\big(\Gamma(H)\big)$, thus $H$ is Schwarz closed [ADH, 11.8.33]; in particular, $H$ is $\omega$-free. By Lemma 5.3, $H$ is newtonian. This shows (i) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is Theorem 4.1, and (iii) $\Leftrightarrow$ (ii) follows from [ADH, 16.2.5]. □

Next a consequence of [ADH, 16.2.1], but note first that $H(C_M)$ is an $H$-subfield of $M$ and d-algebraic over $H$, and recall that each $\omega$-free $H$-field has a Newton-Liouville closure, as defined in [ADH, p. 669].




**Corollary 5.4.** *If* $H$ *is* $\omega$*-free, then the differential closure of* $H$ *in* $M$ *is a Newton-Liouville closure of the* $\omega$*-free* $H$*-subfield* $H(C_M)$ *of* $M$*.*

For (pre-)$\Lambda\Omega$-fields, see [ADH, 16.3], and for the Newton-Liouville closure of a pre-$\Lambda\Omega$-field see [ADH, 16.4.8]. Let $\boldsymbol{M}$ be the expansion of $M$ to a $\Lambda\Omega$-field, and let $\boldsymbol{H}$, $\boldsymbol{H}(C_M)$ be the expansions of $H$, $H(C_M)$, respectively, to pre-$\Lambda\Omega$-subfields of $\boldsymbol{M}$; then $\boldsymbol{H}(C_M)$ is a $\Lambda\Omega$-field. By Proposition 5.2, the d-closure $H^{\text{da}}$ of $H$ in $M$ is $H$-closed and hence has a unique expansion $\boldsymbol{H}^{\text{da}}$ to a $\Lambda\Omega$-field. Then $\boldsymbol{H} \subseteq \boldsymbol{H}(C_M) \subseteq \boldsymbol{H}^{\text{da}} \subseteq \boldsymbol{M}$.

**Corollary 5.5.** *The* $\Lambda\Omega$*-field* $\boldsymbol{H}^{\text{da}}$ *is a Newton-Liouville closure of* $\boldsymbol{H}(C_M)$*.*

*Proof.* Let $\boldsymbol{H}(C_M)^{\text{nl}}$ be a Newton-Liouville closure of $\boldsymbol{H}(C_M)$. Since $\boldsymbol{H}^{\text{da}}$ is $H$-closed and extends $\boldsymbol{H}(C_M)$, there is an embedding $\boldsymbol{H}(C_M)^{\text{nl}} \to \boldsymbol{H}^{\text{da}}$ over $\boldsymbol{H}(C_M)$, and any such embedding is an isomorphism, thanks to Theorem 4.1. □

## 6. Relative Differential Closure in Hardy Fields

Specializing now to Hardy fields, assume in this section that $H$ is a Hardy field and set $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$, an $H$-asymptotic extension of $H$. By definition, $H$ is d-maximal iff $H$ is d-closed in every Hardy field extension of $H$. Moreover:

**Corollary 6.1.** *The following are equivalent:*

(i) $H \supseteq \mathbb{R}$ *and* $H$ *is* $H$*-closed;*
(ii) $H$ *is* d*-maximal;*
(iii) $H$ *is* d*-closed in some* d*-maximal Hardy field extension of* $H$*.*

*If one of these conditions holds, then* $K$ *is weakly* d*-closed.*

*Proof.* The equivalence (i) ⇔ (ii) is part of Theorem 1.3, (ii) ⇒ (iii) is trivial, and (iii) ⇒ (i) follows from the implication (i) ⇒ (iii) of Proposition 5.2. For the rest, use the remark before Proposition 5.2. □

Thus by Corollary 3.6:

**Corollary 6.2.** *If* $H$ *is* d*-maximal and* $E$ *is a Hardy field extension of* $H$*, then* $K$ *is* d*-closed in* $E[\mathrm{i}]$*.*

Using Theorem 4.8 we can strengthen Corollary 6.2:

**Corollary 6.3.** *Suppose* $H$ *is* d*-maximal and* $L \supseteq K$ *is a differential subfield of* $\mathcal{C}^{<\infty}[\mathrm{i}]$ *such that* $L$ *is a* d*-valued* $H$*-asymptotic extension of* $K$ *with respect to some dominance relation on* $L$*. Then* $K$ *is* d*-closed in* $L$*.*

*Proof.* The d-valued field $K$ is $\boldsymbol{\omega}$-free and newtonian by [ADH, 11.7.23, 14.5.7]. Also $L^\dagger \cap K = K^\dagger$ by Corollary 1.11. Now apply Theorem 4.8. □

We do not require the dominance relation on $L$ in Corollary 6.3 to be the restriction to $L$ of the relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$.

A differential subfield $L$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ is said to *come from a Hardy field* if $L = E[\mathrm{i}]$ for some Hardy field $E$. By [7, Section 2] this is equivalent to: $i \in L$ and $\operatorname{Re} f, \operatorname{Im} f \in L$ for all $f \in L$. Corollary 6.3 for $L$ coming from a Hardy field falls under Corollary 6.2. However, not every differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ containing $\mathbb{C}$ comes from a Hardy field: [7, Section 5] has an example of a differential subfield $L \supseteq \mathbb{C}$ of $\mathcal{C}^{\omega}[\mathrm{i}]$ not coming from a Hardy field, yet the relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on $L$ making $L$ a d-valued field of $H$-type. In the next example (not used later) we employ a variant of this construction to obtain $H$, $K$, $L$ as in Corollary 6.3 where $L$ equipped with the restriction of $\preccurlyeq$ is a d-valued field of $H$-type, but $L$ doesn't come from a Hardy field:

*Example.* Let $M$ be a maximal analytic Hardy field. Then $M$ is $H$-closed by Corollary 1.4, the d-closure $H$ of $\mathbb{R}$ in $M$ is a d-maximal Hardy field by Corollary 6.1, and no $h \in H$ is transexponential by [7, Lemma 5.1]. Theorem 1.3 in [16] and [7, Corollary 5.24]) yield a transexponential $z \in M$. Take any $h \in \mathbb{R}(x)$ with $0 \neq h \prec 1$, and put $y := z(1 + h\,\mathrm{e}^{x\mathrm{i}}) \in \mathcal{C}^{\omega}[\mathrm{i}]$. Now $z$ is $H$-hardian, so by [7, Lemma 5.16], $y$ generates a differential subfield $L_0 := H\langle y\rangle$ of $\mathcal{C}^{\omega}[\mathrm{i}]$, and $\preccurlyeq$ restricts to a dominance relation on $L_0$ which makes $L_0$ a d-valued field of $H$-type with constant field $\mathbb{R}$. Then $L := L_0[\mathrm{i}]$ is a differential subfield of $\mathcal{C}^{\omega}[\mathrm{i}]$ with constant field $\mathbb{C}$, and $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on $L$ that makes $L$ a d-valued field of $H$-type extending $K$ as a d-valued field and which does not come from a Hardy field, since $\operatorname{Im} y = zh \sin x$ oscillates. By Corollary 6.3, $K$ is d-closed in $L = K\langle y\rangle$.

The following is Theorem B from the introduction:

**Corollary 6.4.** *Suppose $H \supset \mathbb{R}$ has* DIVP*, and $E \supseteq H$ is a Hardy field. Then*

$$H \text{ is d-closed in } E \iff E \cap \exp(H) \subseteq H.$$

*Proof.* By [5, Corollary 2.6], $H$ is $\boldsymbol{\omega}$-free and newtonian. In particular, $H$ is closed under integration, by [ADH, 14.2.2]. Applying DIVP to ordinary polynomials, we see that $H$ is also real closed. It is clear that if $H$ is d-closed in $E$, then $E \cap \exp(H) \subseteq H$. Conversely, suppose that $E \cap \exp(H) \subseteq H$. If $y \in E^{>}$ with $y^\dagger \in H$, then $\log y \in H$ and so $y \in E \cap \exp(H) \subseteq H$. This yields $E^\dagger \cap H = H^\dagger$. Hence $H$ is d-closed in $E$ by Corollary 4.9. □

*Remarks.* Suppose $H \supset \mathbb{R}$ has DIVP. Then $H$ is closed under integration and real closed by the proof of Corollary 6.4, so $H$ is Liouville closed iff $\exp(H) \subseteq H$. Thus by [5, Corollary 2.6], if $\exp(H) \subseteq H$, then $H$ is $H$-closed. For an example of an analytic $H \supset \mathbb{R}$ with DIVP that is not Liouville closed, consider the specialization $F$ of $\mathbb{T}$ with respect to the convex subgroup

$$\Delta := \{\gamma \in \Gamma_{\mathbb{T}} : |\gamma| \leqslant n\, v(x^{-1}) \text{ for some } n\}$$

of its value group $\Gamma_{\mathbb{T}} = v(\mathbb{T}^\times)$. Then $F$ is an $H$-field with constant field $\mathbb{R}$ and value group $\Delta$, and $F$ has DIVP but is not Liouville closed: see [1, Lemma 14.3 and subsequent remark]. By [ADH, 14.1.2, 15.0.2], the $\Delta$-coarsening of the valued differential field $\mathbb{T}$ is d-henselian, hence [ADH, 7.1.3] yields an embedding $F \to \mathbb{T}$ of differential fields that is the identity on $\mathbb{R}$, and $F$ and $\mathbb{T}$ being real closed, this is even an embedding of ordered valued differential fields. Now $\mathbb{T}$ is isomorphic over $\mathbb{R}$ to an analytic Hardy field containing $\mathbb{R}$ by [2, Corollary 7.9]. Hence $F$ is isomorphic to an analytic Hardy field $H \supset \mathbb{R}$ with DIVP that is not Liouville closed.

Recall from Section 1 that the d-perfect hull $\mathrm{D}(H)$ of $H$ is defined as the intersection of all d-maximal Hardy field extensions of $H$. By the next result we only need to consider here d-algebraic Hardy field extensions of $H$:

**Corollary 6.5.**

$$\mathrm{D}(H) = \bigcap \{M : M \text{ is a d-maximal d-algebraic Hardy field extension of } H\}.$$

*Proof.* We only need to show the inclusion "$\supseteq$". So let $f$ be an element of every d-maximal d-algebraic Hardy field extension of $H$, and let $M$ be any d-maximal Hardy field extension of $H$; we need to show $f \in M$. Let $E$ be the d-closure of $H$ in $M$. Then $E$ is d-algebraic over $H$, and by Corollary 6.1, $E$ is d-maximal. Hence $f \in E$, and thus $f \in M$ as required. □

We can now also prove a variant of Lemma 1.1 for $\mathcal{C}^\infty$- and $\mathcal{C}^\omega$-Hardy fields:

**Corollary 6.6.** *Suppose $H$ is a $\mathcal{C}^\infty$-Hardy field. Then*

$$\begin{aligned} \mathrm{D}(H) &= \bigcap \{M : M \supseteq H \text{ is a d-maximal } \mathcal{C}^\infty\text{-Hardy field}\} \\ &= \{f \in \mathrm{E}^\infty(H) : f \text{ is d-algebraic over } H\}. \end{aligned}$$

*Likewise with $\omega$ in place of $\infty$.*

*Proof.* With both equalities replaced by "$\subseteq$", this follows from the definitions and the remarks following Proposition 1.2. Let $f \in \mathrm{E}^\infty(H)$ be d-algebraic over $H$; we claim that $f \in \mathrm{D}(H)$. To prove this claim, let $E$ be a d-maximal Hardy field extension of $H$; it is enough to show that then $f \in E$. Now $F := E \cap \mathcal{C}^\infty$ is a $\mathcal{C}^\infty$-Hardy field extension of $H$ which is d-closed in $E$, by Proposition 1.2, and hence d-maximal by Corollary 6.1. Thus we may replace $E$ by $F$ to arrange that $E \subseteq \mathcal{C}^\infty$, and then take a $\mathcal{C}^\infty$-maximal Hardy field extension $M$ of $E$. Now $f \in \mathrm{E}^\infty(H)$ gives $f \in M$, and $E$ being d-maximal and $f$ being d-algebraic over $E$ yields $f \in E$. The proof for $\omega$ in place of $\infty$ is similar. □

We say that $H$ is *bounded* if there is a germ $\phi \in \mathcal{C}$ such that $h \leqslant \phi$ for all $h \in H$. No maximal Hardy field, no maximal smooth Hardy field, and no maximal analytic Hardy field is bounded. However, if $H$ has countable cofinality (as ordered set), then $H$ is bounded. (See [7, Section 5].) If $H$ is bounded, then $\mathrm{E}(H)$ is d-algebraic over $H$ (see [7, Theorem 5.20]) and hence $\mathrm{D}(H) = \mathrm{E}(H)$ by Lemma 1.1. If in addition $H \subseteq \mathcal{C}^\infty$, then $\mathrm{E}^\infty(H)$ is also d-algebraic over $H$, and likewise with $\omega$ in place of $\infty$ [7, Theorem 5.20]. Combined with Corollary 6.6, this yields:

**Corollary 6.7.** *If $H \subseteq \mathcal{C}^\infty$ is bounded, then $\mathrm{D}(H) = \mathrm{E}(H) = \mathrm{E}^\infty(H)$. Likewise with $\omega$ in place of $\infty$.*

Let $\mathrm{E} := \mathrm{E}(\mathbb{Q})$ be the perfect hull of the Hardy field $\mathbb{Q}$. From Corollary 6.7 we obtain the next result, which establishes Theorem A from the introduction:

**Corollary 6.8.** $\mathrm{E} = \mathrm{E}^\infty(\mathbb{Q}) = \mathrm{E}^\omega(\mathbb{Q}) = \mathrm{D}(\mathbb{Q})$.

*Question.* Do the following implications hold for all $H$?

$$H \subseteq \mathcal{C}^\infty \implies \mathrm{E}(H) \subseteq \mathrm{E}^\infty(H), \qquad H \subseteq \mathcal{C}^\omega \implies \mathrm{E}(H) \subseteq \mathrm{E}^\infty(H) \subseteq \mathrm{E}^\omega(H).$$

These implications hold if $\mathrm{D}(H) = \mathrm{E}(H)$, but we don't know whether $\mathrm{D}(H) = \mathrm{E}(H)$ for all $H$; see also [15, p. 144].

By Theorem 6.9 below, each d-perfect Hardy field is 1-d-closed in all its Hardy field extensions. Here, $Y$ and $Z$ are distinct indeterminates.

**Theorem 6.9.** *Let $P \in H[Y,Z]^{\neq}$, and suppose $y \in \mathcal{C}^1$ lies in a Hausdorff field extension of $H$ and $P(y,y') = 0$. Then $y \in \mathrm{D}(H)$.*

This is stated in [15, Theorem 11.8], where the proof is only indicated; for a detailed proof, see [6, Theorem 6.3.14]. Every d-maximal Hardy field is 1-newtonian (Theorem 1.3 or [9, Lemma 11.12]). Together with Lemma 5.3, this yields:

**Corollary 6.10.** *Every* d*-perfect Hardy field is* 1*-newtonian.*

By Theorem 6.9 and Corollary 6.10, E is 1-d-closed in all its Hardy field extensions and 1-newtonian. However, E is not 2-linearly surjective by [17, Proposition 3.7], so E is not weakly 2-d-closed in any d-maximal Hardy field extension of E (see Lemma 3.1) and E is not 2-linearly newtonian (see [ADH, 14.2.2]).

The material at the end of Section 5 has consequences for the relationship between Newton-Liouville closure and d-closure in the context of Hardy fields. For this, we recall from [9, Section 12] that every d-maximal Hardy field $M$ has a unique expansion to a $\Lambda\Omega$-field $\boldsymbol{M}$, and that every $H$ has a unique expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{H}$ such that $\boldsymbol{H} \subseteq \boldsymbol{M}$ for all d-maximal Hardy fields $M \supseteq H$; we call $\boldsymbol{H}$ the *canonical* $\Lambda\Omega$*-expansion* of $H$.

Let now $M$ be a d-maximal Hardy field extension of $H$ and $H^{\mathrm{da}}$ the d-closure of $H$ in $M$, so $H(\mathbb{R}) \subseteq H^{\mathrm{da}} \subseteq M$. From Corollary 5.4 we obtain:

**Corollary 6.11.** *If $H$ is $\omega$-free, then $H^{\mathrm{da}}$ is a Newton-Liouville closure of $H(\mathbb{R})$.*

Next, let $\boldsymbol{H}(\mathbb{R})$, $\boldsymbol{H}^{\mathrm{da}}$, $\boldsymbol{M}$ be the canonical $\Lambda\Omega$-expansions of the Hardy fields $H(\mathbb{R})$, $H^{\mathrm{da}}$, $M$, respectively, so $\boldsymbol{H}(\mathbb{R}) \subseteq \boldsymbol{H}^{\mathrm{da}} \subseteq \boldsymbol{M}$. Corollary 5.5 then yields:

**Corollary 6.12.** *$\boldsymbol{H}^{\mathrm{da}}$ is a Newton-Liouville closure of $\boldsymbol{H}(\mathbb{R})$.*

We finish with an example justifying the remark after Theorem B:

*Example.* Let $M$ be a maximal Hardy field, and let $\boldsymbol{H}$ and $\boldsymbol{F}$ be Newton-Liouville closures of the canonical $\Lambda\Omega$-expansions of the Hardy fields $\mathbb{Q}$ and $\mathbb{Q}(\mathrm{e})$, respectively. We embed $\boldsymbol{H}$ and $\boldsymbol{F}$ in $\boldsymbol{M}$ in such a way that upon identifying $\boldsymbol{H}$ and $\boldsymbol{F}$ with their images in $\boldsymbol{M}$ we have $H \subseteq F \subseteq M$ for the corresponding underlying differential fields. Now $H \cap F^{\dagger} = H \cap F = H = H^{\dagger}$, since $H$, $F$ are Liouville closed. The constant fields of $H$, $F$ are the real closures in $\mathbb{R}$ of $\mathbb{Q}$, $\mathbb{Q}(\mathrm{e})$, respectively (see [ADH, proof of 16.4.9]), so $C_H \neq C_F$. Corollary 4.5 yields a $y \in F \setminus H$ that is d-algebraic over $H$ with $C_E = C_H$ for $E := H\langle y \rangle$. If $g \in E \cap \exp(H)$, then $g^{\dagger} \in H = H^{\dagger}$, so $g = ch$ where $c \in C_E^{\times}$ and $h \in H^{\times}$, and hence $g \in H$ since $C_E = C_H$. Thus $E \cap \exp(H) \subseteq H$.

## Notes

1. The first edition (1910) of [30] was reviewed for this journal by Gödel's PhD advisor Hahn [28].

2. Boshernitzan [14] (with details in [26]) first suggested an example of a non-smooth Hardy field. Rolin, Speissegger, Wilkie [42] construct o-minimal expansions $\widetilde{\mathbb{R}}$ of the ordered field of real numbers such that the Hardy field $H$ consisting of the germs of functions $\mathbb{R} \to \mathbb{R}$ that are definable in $\widetilde{\mathbb{R}}$ is smooth but not analytic. Le Gal and Rolin [38] construct such expansions such that the corresponding Hardy field $H$ is not smooth.

3. Gödel's Completeness Theorem from his PhD thesis [25] is in the background of the model theoretic tools used in proving the completeness of the theory of maximal Hardy fields, which in particular entails the existence of a decision procedure for this theory.

4. The existence of a solution to this equation in each maximal Hardy field $H$ follows from the above theorem with $P(Y) = Y'' + Y - \mathrm{e}^{x^2}$, $f = 0$, and $g = \mathrm{e}^{x^2}$.

5. The difference of two distinct solutions to the equation $y'' + y = \mathrm{e}^{x^2}$ is a nonzero solution to the homogeneous equation $y + y'' = 0$ and thus oscillating.

6. This argument also suggests that there exist many maximal Hardy fields: in [10], we show that there are actually $2^{2^{\aleph_0}}$ many.

7. For a construction and basic properties of the field of transseries, see [ADH, Appendix A]. A more leisurely exposition is in [24].

8. A brief summary of work on **No** past Conway is in [50]. We also refer to [11] for an interpretation of surreal numbers in terms of growth rates at infinity.

9. Let us mention here Hamkin's historical thought experiment [29]: an early acceptance of CH as a standard set-theoretic axiom alongside the usual ZFC axioms would presumably have promoted a more wide-spread use of infinitesimals in the style of non-standard analysis, because CH guarantees the existence of up-to-isomorphism unique saturated elementary extensions of the real field and its expansions of size $2^{\aleph_0}$. (Without CH there is no such uniqueness.) We also note that Cantor [19, VI] was hostile to Paul du Bois-Reymond's infinitesimals, which are at the root of Hardy fields and which put the "actual infinitesimal" on as solid a footing as Cantor's "actual infinite". Maybe Cantor saw this as unwelcome competition.

10. See also [8] or the regularly updated page `https://www.mat.univie.ac.at/~maschenbrenner/pdf/mt-errata.pdf` for a list of errata to [ADH].

11. The differential polynomial $P_c$ studied here was inspired by an example attributed to McGrail and Marker in [34, Example 2.20], and suggested to us by James Freitag.

12. The transseries from [33] are *grid-based*, subject to a stronger restriction than the *well-based* transseries from [ADH, Appendix A]. In both cases these transseries form $H$-closed fields.

13. The requirement that the constant field comes with an exponential function is natural from the transseries perspective in [33], but not required from an $H$-field perspective: *mutatis mutandis*, the construction goes through for general ordered constant fields $C$, but the resulting $H$-field of transseries $\mathbb{T}$ is just no longer closed under exponentiation. Nevertheless, it remains closed under exponential integration and exponentiation for transseries without constant terms.

14. This terminology, coming from [31], is slightly unfortunate in our context in light of the important role played by Écalle's analyzable functions [23] in the theory of transseries.

15. Allen Gehret has a different proof that the zero set of each nonzero univariate differential polynomial over $\mathbb{T}_{\log}$ is co-analyzable.

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