WHITNEY APPROXIMATION: DOMAINS AND BOUNDS

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Abstract. We investigate properties of holomorphic extensions in the one-variable case of Whitney's Approximation Theorem on intervals. Improving a result of Gauthier-Kienzle, we construct tangentially approximating functions which extend holomorphically to domains of optimal size. For approximands on unbounded closed intervals, we also bound the growth of holomorphic extensions, in the spirit of Arakelyan, Bernstein, Keldych, and Kober.

Introduction

In this paper we let a, b, s, t range over $\mathbb{R}$, m, n over $\mathbb{N} = \{0, 1, 2, \dots\}$, r over $\mathbb{N} \cup \{\infty\}$, and z over $\mathbb{C}$. Let $I = (\alpha, \beta)$, where $-\infty \leqslant \alpha < \beta \leqslant \infty$, be an open interval. Every $\mathcal{C}^r$-function $I \to \mathbb{R}$ and its derivatives can be arbitrarily well approximated by an analytic function and its derivatives; more precisely:

Theorem (Whitney). *Let $f \in \mathcal{C}^r(I)$ and $\varepsilon, \rho \in \mathcal{C}(I)$ be such that $\varepsilon > 0$ and $r \geqslant \rho \geqslant 0$ on I. Then there is a $g \in \mathcal{C}^\omega(I)$ such that*

$$(1) \qquad |(f-g)^{(n)}(t)| < \varepsilon(t) \qquad \text{for all } t \in I \text{ and } n \leqslant \rho(t).$$

This theorem is implicit in the proof of a (more general) lemma in Whitney's seminal 1934 paper on extending multivariate differentiable functions [25]. In [3, Appendix A] we included a self-contained proof of this fact, which makes it apparent that there is an open subset $U = U_I$ of $\mathbb{C}$ which contains I, independent of f, such that the function g in the conclusion of the theorem extends to a holomorphic function $U \to \mathbb{C}$. If I is bounded, then the open set U obtained through the proof of Whitney's theorem as in loc. cit. is also bounded. (See [3, Corollary A.4].) However, if $I = \mathbb{R}$, then this procedure yields $U = \mathbb{C}$, so g can be taken as the restriction of an entire function. Independently of Whitney, this fact had also been discovered by Hoischen [14], and for $r = 0$ had already been shown by Carleman [8] in 1927 and for $r = 1$ by Kaplan [15] in 1955. (See [20] for the history of such approximation theorems, going back to Weierstrass.) By a theorem of Gauthier and Kienzle [13, Theorem 1.3], in the case $r < \infty$ one can always take $U = I \cup (\mathbb{C} \setminus \mathbb{R})$. Our first main result strengthens this fact by showing that it is possible to choose $U = \mathbb{C} \setminus \{\alpha, \beta\}$ (even when $r = \infty$):

Theorem 1. *For all f, ε, ρ as in Whitney's Theorem, there is a $g \in \mathcal{C}^\omega(I)$ satisfying (1) which extends to a holomorphic function $\mathbb{C} \setminus \{\alpha, \beta\} \to \mathbb{C}$.*

The proof of this theorem, given in Section 2 below, involves a simple reduction to Whitney's theorem in the case $I = \mathbb{R}$, by employing compositions with suitably chosen rational functions and some estimates involving Faà di Bruno's Formula. In this section we also generalize Theorem 1 to other kinds of intervals:

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Corollary 1. *Let $I \subseteq \mathbb{R}$ be connected and $f \in \mathcal{C}^r(I)$ (i.e., f extends to a function in $\mathcal{C}^r(J)$ for some open interval $J \supseteq I$), and let $\varepsilon, \rho \in \mathcal{C}(I)$ be such that $\varepsilon > 0$ and $r \geqslant \rho \geqslant 0$ on I. Then there is a $g \in \mathcal{C}^\omega(I)$ satisfying* (1) *which extends to a holomorphic function $\mathbb{C} \setminus \operatorname{fr}(I) \to \mathbb{C}$, where $\operatorname{fr}(I) = \operatorname{cl}(I) \setminus I$ is the frontier of I.*

The second topic of this note are versions of Whitney's theorem which include explicit bounds on the growth of the holomorphic extension of g to U. The following example, due to Arakelyan [2, pp. 14–15], indicates why in the case of Carleman's Theorem ($I = \mathbb{R}$ and $r = 0$) one cannot expect to bound the growth of the entire extension of g solely from information about that of f:

Example. Let $\beta \in \mathcal{C}(\mathbb{R})$ be even and strictly increasing on $\mathbb{R}^{\geqslant}$ such that $\beta(0) = 0$ and $\beta(t) \to \infty$ as $t \to \infty$, and let $\varepsilon \in \mathcal{C}(\mathbb{R})$ be bounded with $\varepsilon > 0$. Then the continuous function $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) := \varepsilon(t)\cos\beta(t)$ is also bounded. Call an entire function g *real* if $g(\mathbb{R}) \subseteq \mathbb{R}$. Suppose now that g is a real entire function such that $|f(t) - g(t)| < \varepsilon(t)$ for each t. Then for each t we have

$$-1 + \cos\beta(t) < \frac{g(t)}{\varepsilon(t)} < 1 + \cos\beta(t),$$

so between any two reals a, b with $\cos\beta(a) = 1$, $\cos\beta(b) = -1$ there is a zero of g. This implies that each interval $[-t, t]$ ($t \geqslant 0$) contains $\geqslant (2/\pi)\beta(t) - 2$ zeros of g. Using Jensen's Formula [6, 1.2.1] we obtain a $C \in \mathbb{R}$ such that

$$\log \|g\|_t \geqslant \frac{2}{\pi}\int_1^t \frac{\beta(s)}{s}\,ds - 2\log t + C \qquad \text{for each } t \geqslant 1.$$

Here and below $\|g\|_t := \sup\big\{|g(z)| : |z| \leqslant t\big\}$ for $t \geqslant 0$.

However, Bernstein [5] and Kober [17] discovered that in this situation, for constant error function ε, the growth of the entire function g can be limited, *provided* that f is not only assumed to be bounded, but also uniformly continuous. (See [11, Theorem 2.1].) Keldych [16] proved a similar result, assuming that f is $\mathcal{C}^1$ and the asymptotics of both f and f' are known (see [12, p. 163]), and this was refined and extended to more general error functions by Arakelyan [1, 2]. Our second theorem provides a partial generalization of these facts to the case $r > 0$, under some natural restrictions on ε, ρ. To formulate it, we introduce some notation. Let $f \in \mathcal{C}^m(\mathbb{R})$. For $t \geqslant 0$ and $\rho \in \mathbb{R}$ with $0 \leqslant \rho \leqslant m$ put

$$\|f\|_{t;\,\rho} := \sup\big\{|f^{(n)}(s)| : |s| \leqslant t,\ n \leqslant \rho\big\} \in \mathbb{R}^{\geqslant}, \qquad \|f\|_t := \|f\|_{t;\,0}.$$

Note that $\|f\|_{t_1;\,\rho_1} \leqslant \|f\|_{t_2;\,\rho_2}$ if $0 \leqslant t_1 \leqslant t_2$ and $0 \leqslant \rho_1 \leqslant \rho_2 \leqslant m$. By convention we set $\infty + 1 := \infty$, and for $\phi\colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$ we let $\Delta\phi\colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$ denote the difference function $t \mapsto \phi(t) - \phi(t+1)$. (If ϕ is strictly decreasing, then $\Delta\phi(t) > 0$ for $t \geqslant a$, and if ϕ is convex, then $\Delta\phi$ is decreasing: see Section 1.) As usual $\log^+ t := \max\{0, \log t\}$, where $\log t := -\infty$ for $t \leqslant 0$.

Theorem 2. *For each $f \in \mathcal{C}^{r+1}(\mathbb{R})$ and $\varepsilon, \rho \in \mathcal{C}(\mathbb{R}^{\geqslant})$ where $\varepsilon > 0$ is strictly decreasing and convex with $\varepsilon(t) \to 0$ as $t \to \infty$, and ρ with $r \geqslant \rho \geqslant 0$ is increasing, there are some $C, D \in \mathbb{R}^{>}$ and a real entire function g such that*

$$|(f-g)^{(n)}(t)| < \varepsilon(|t|) \quad \textit{for } n \leqslant \rho(|t|),$$

and

$$\|g\|_t \leqslant \exp\big(C \cdot s^2 \cdot \big(1 + \log^+\|f\|_{s+3\sqrt{2}} + \lambda(s + 3\sqrt{2})\big)\big) \quad \textit{for } t \geqslant 0,\ s = \sqrt{2}t + 1,$$

where

$$\lambda(s) := \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s} \cdot \left(\frac{\|f\|_{s;\,\rho(s)+1}}{\Delta\varepsilon(s)}\right)^3.$$

Here C can be chosen independently of ρ, and D independently of f, ε, ρ, r.

The proof of this theorem is given in Section 5, and is obtained by making the already quite constructive argument of Whitney completely explicit. This requires some bounds for derivatives of bump functions and for the approximation of functions of bounded support by Weierstrass transforms, computed in Sections 3 and 4 after some preliminaries in Section 1. In the rest of this introduction we collect some applications of the previous theorem, and also state a variant for functions $f \in \mathcal{C}^{r+1}(\mathbb{R}^{\geqslant})$.

First, for $r < \infty$, by Theorem 2 applied to the constant function $\rho(t) := r$:

Corollary 2. *Suppose $r < \infty$. Then for each $f \in \mathcal{C}^{r+1}(\mathbb{R})$ and $\varepsilon \in \mathcal{C}(\mathbb{R}^{\geqslant})$ such that $\varepsilon > 0$ is strictly decreasing and convex with $\varepsilon(t) \to 0$ as $t \to \infty$, there are $C, D \geqslant 1$ and a real entire function g such that $|(f-g)^{(n)}(t)| < \varepsilon(|t|)$ for $n \leqslant r$, and*

$$\|g\|_t \leqslant \exp\left(C \cdot s^2 \cdot \left(1 + \log^+\|f\|_s + D^s \left(\frac{\|f\|_{s;\,r+1}}{\Delta\varepsilon(s)}\right)^3\right)\right)$$

for $t \geqslant 0$ and $s = \sqrt{2}(t+4)$. Here D can be chosen to only depend on r.

For $r = 0$ and certain ε, Arakelyan [2] has a qualitatively better bound:

Remark. Let $\varepsilon \in \mathcal{C}^1(\mathbb{R}^{\geqslant})$ be decreasing with $\varepsilon > 0$, such that $\lim\limits_{t\to\infty} t\varepsilon'(t)/\varepsilon(t) \in \mathbb{R}^{\leqslant}$ exists, and let $f \in \mathcal{C}^1(\mathbb{R})$. Then by [2, Theorem 6] there are a $C \in \mathbb{R}^{>}$ and a real entire function g such that $|f(t) - g(t)| < \varepsilon(|t|)$ for each t and

$$\|g\|_t \leqslant \exp\left(C \cdot s \cdot \left(1 + \log^+\left(\frac{\|f\|_s}{\varepsilon(t)}\right) + \frac{\|f'\|_s}{\varepsilon(t)}\right)\right) \quad \text{if } t \geqslant 0,\ s = \sqrt{2}(t+4).$$

To facilitate the comparison with the bound in Corollary 2, note that if in addition ε is assumed to be $\mathcal{C}^2$, strictly decreasing, and convex, then we have $1/\Delta\varepsilon(s) = O\big((s+1)/\varepsilon(s+1)\big)$ as $s \to \infty$. (Lemma 1.2.)

Before stating the next corollary, we define some quantities measuring the growth of an entire function g introduced in [22]. First recall that except in the case where g is a constant, the function $t \mapsto \|g\|_t \colon \mathbb{R}^{\geqslant} \to \mathbb{R}^{\geqslant}$ is strictly increasing with $\|g\|_t \to \infty$ as $t \to \infty$. If g is given by a polynomial of degree n then $\|g\|_t = O(t^n)$ as $t \to \infty$, and if $\lambda \in \mathbb{R}^{\geqslant}$ is such that $\|g\|_t = O(t^\lambda)$ as $t \to \infty$, then g is given by a polynomial of degree at most $\lfloor \lambda \rfloor$. Suppose now that g is non-constant. With $\log_m$ denoting the m-fold iterated logarithm, define

$$\lambda_m(g) := \limsup_{t\to\infty} \frac{\log_m \|g\|_t}{\log t} \in [0, \infty].$$

If $\lambda_m(g) < \infty$ for some m, then g is said to have *finite index,* and in this case the smallest such m is called the *index* of g. Note that if g has finite index m, then $m \geqslant 1$. By convention, constant functions $\mathbb{C} \to \mathbb{C}$ have index 0. The entire functions of index 1 are exactly the non-constant polynomial functions. The entire functions of index $\leqslant 2$ are also known as the entire functions of finite order, and $\lambda_2(g)$ is called the *order* of g. (See [6, Chapter 2].) By [22, Theorem 1], if g has index $m \geqslant 2$, then $\lambda = \lambda_m(g)$ may be computed from the Taylor coefficients $g_n := g^{(n)}(0)/n!$ of g at 0 as follows:

$$\lambda = \limsup_{n\to\infty} \frac{n \log_{m-1} n}{-\log|g_n|}.$$

For $f \in \mathcal{C}(\mathbb{R})$ put $\|f\| := \sup_{t\geqslant 0}\|f\|_t \in [0,\infty]$. From Corollary 2 we obtain:

Corollary 3. *Suppose $r < \infty$, and let $f \in \mathcal{C}^{r+1}(\mathbb{R})$ be such that $\|f^{(n)}\| < \infty$ for $n \leqslant r+1$, and $\varepsilon \in \mathbb{R}^{>}$. Then there is a real entire function g of index $\leqslant 3$ such that $|(f-g)^{(n)}(t)| < \varepsilon$ for $n \leqslant r$.*

Proof. Consider $\varepsilon_0 \in \mathcal{C}(\mathbb{R}^{\geqslant})$, $\varepsilon_0(t) := \varepsilon/(t+1)$; then $\Delta\varepsilon_0(t) = \frac{\varepsilon}{(t+1)(t+2)}$ for $t \geqslant 0$. Take C, D, g as in Corollary 2 applied to ε_0 in place of ε. Then with $M := \max\{1, \|f\|_0, \dots, \|f\|_{r+1}\}$, for $t \geqslant 0$ and $s = \sqrt{2}(t+4)$ we have

$$\|g\|_t \leqslant \exp\left(C \cdot s^2 \cdot \left(1 + \log M + D^s \left(\frac{M(s+1)(s+2)}{\varepsilon}\right)^3\right)\right).$$

For suitable $E \in \mathbb{R}^{\geqslant 1}$ (only depending on C, D, ε, M) we thus have $\|g\|_t \leqslant \exp(E^s)$ for all sufficiently large $t \geqslant 0$, and this yields $\lambda_3(g) \leqslant 1$. $\square$

Remarks. Let $f \in \mathcal{C}^1(\mathbb{R})$ and $M \in \mathbb{R}^{\geqslant 1}$ be such that $\|f\|, \|f'\| \leqslant M$, and $\varepsilon > 0$. Then by the remark following Corollary 2, there is a $C \in \mathbb{R}^{>}$ and a real entire function g such that $|(f-g)(t)| < \varepsilon$ for all t and

$$\|g\|_t \leqslant \exp\left(C \cdot s \cdot \left(1 + \log^+\left(\frac{M}{\varepsilon}\right) + \frac{M}{\varepsilon}\right)\right) \quad \text{if } t \geqslant 0,\ s = \sqrt{2}(t+4),$$

thus $\lambda_2(g) \leqslant 1$. Therefore in Corollary 3, in the case $r = 0$ one can replace "index $\leqslant 3$" by "order $\leqslant 1$". (This is originally due to Bernstein [5] and Kober [17].) It would be interesting to know whether in the case $r > 0$, one can improve "index $\leqslant 3$" to "finite order" in the previous corollary.

The case $r = \infty$ of Theorem 2 leads to an application to rapidly decreasing functions: Let $f \in \mathcal{C}^\infty(\mathbb{R})$, and put $\|f\|_{m,n} := \sup\{|t^m f^{(n)}(t)| : t \in \mathbb{R}\} \in [0,\infty]$. Then f is said to be *rapidly decreasing* if $\|f\|_{m,n} < \infty$ for all m, n; equivalently, if for all m, n we have $t^m f^{(n)}(t) \to 0$ as $|t| \to \infty$. Examples: if f has bounded support, then it is rapidly decreasing; and for each $\lambda \in \mathbb{R}^{>}$ and m, the function $t \mapsto t^m\, \mathrm{e}^{-\lambda t^2} : \mathbb{R} \to \mathbb{R}$ is rapidly decreasing. The rapidly decreasing functions form a subalgebra $\mathcal{S}(\mathbb{R})$ of the $\mathbb{R}$-algebra $\mathcal{C}^\infty(\mathbb{R})$. The collection of sets

$$\{g \in \mathcal{S}(\mathbb{R}) : \|f-g\|_{m,n} < \varepsilon \text{ for } m,n \leqslant N\} \quad \text{where } f \in \mathcal{S}(\mathbb{R}),\ \varepsilon \in \mathbb{R}^{>}, \text{ and } N \in \mathbb{N},$$

is the basis of a topology on $\mathcal{S}(\mathbb{R})$, which makes the $\mathbb{R}$-vector space $\mathcal{S}(\mathbb{R})$ into a topological $\mathbb{R}$-vector space (in fact, a Fréchet space), known as the Schwartz space on $\mathbb{R}$, which plays an important role in the theory of distributions (cf. [23]). It is well-known that the set of rapidly decreasing functions which extend to entire functions is dense in $\mathcal{S}(\mathbb{R})$; see [23, Theorem 15.5, p. 160]. From Theorem 2 we obtain an effective version of this fact:

Corollary 4. *Let $f \in \mathcal{S}(\mathbb{R})$, $\varepsilon \in \mathbb{R}^{>}$, and $N \in \mathbb{N}$. Then there is a real entire function g such that $g|_{\mathbb{R}} \in \mathcal{S}(\mathbb{R})$ and $\|f-g\|_{m,n} \leqslant \varepsilon$ for each $m, n \leqslant N$. Moreover, we can choose g so that for some $C, D \in \mathbb{R}^{\geqslant 1}$ we have*

$$\|\widehat{g}\|_t \leqslant \exp\left(C \cdot s^2 \cdot \left(1 + \log^+\|f\|_{s+3\sqrt{2}} + \lambda(s+3\sqrt{2})\right)\right) \quad \text{for } t \geqslant 0,\ s = \sqrt{2}t + 1,$$

where

$$\lambda(s) := \big(D\,(r+1)\big)^{D\,(r+1)\,s} \cdot \left(\frac{(N/\mathrm{e})^N \|f\|_{0,\lceil r+1\rceil}}{\varepsilon}\right)^3 \qquad \text{where } r = N+s.$$

Here C can be chosen independently of N, and D independently of f, ε, N.

Proof. Put $\delta := \varepsilon/(N/\mathrm{e})^N \in \mathbb{R}^{>}$ and consider $\varepsilon_0, \rho_0 \in \mathcal{C}(\mathbb{R}^{\geqslant})$ given by $\varepsilon_0(t) := \delta\,\mathrm{e}^{-t}$ and $\rho_0(t) := N+t$ for $t \geqslant 0$. Take g as in Theorem 2 applied to ε_0, ρ_0 in place of ε, ρ. Then for each m, n we have

$$|t^m g^{(n)}(t)| \leqslant |t^m f^{(n)}| + \delta|t|^m\,\mathrm{e}^{-|t|} \quad \text{if } |t| \geqslant n-N,$$

thus

$$\|g\|_{m,n} \leqslant \|x^m g^{(n)}\|_{\max\{0,n-N\}} + \|f\|_{m,n} + \delta(m/\mathrm{e})^m < \infty.$$

Hence $g|_{\mathbb{R}} \in \mathcal{S}(\mathbb{R})$; moreover, if $m, n \leqslant N$ then

$$|t^m (f-g)^{(n)}| \leqslant \delta|t|^m\,\mathrm{e}^{-|t|} \leqslant \delta|t|^N\,\mathrm{e}^{-|t|} \leqslant \varepsilon \quad \text{for each } t,$$

hence $\|f-g\|_{m,n} \leqslant \varepsilon$. The rest now follows from Theorem 2. Note that $\Delta\varepsilon_0(t) = \delta\,\mathrm{e}^{-t}(1-\mathrm{e}^{-1}) \geqslant \varepsilon_0(t)/2$ and so $1/\Delta\varepsilon_0(t) \leqslant (2/\varepsilon)(N/\mathrm{e})^N\,\mathrm{e}^t$, for each $t \geqslant 0$. □

Finally, in Section 5 we also establish a version of Theorem 2 for $f \in \mathcal{C}^{r+1}(\mathbb{R}^{\geqslant})$. For such f, given $\varepsilon, \rho \in \mathcal{C}(\mathbb{R}^{\geqslant})$ with $\varepsilon > 0$ and $r \geqslant \rho \geqslant 0$, Corollary 1 yields a real entire function g such that $|(f-g)^{(n)}(t)| < \varepsilon(t)$ for $n \leqslant \rho(t)$, $t \geqslant 0$. Here we will confine ourselves to constructing a holomorphic approximation of f with domain $U := \{z \in \mathbb{C} : |\mathrm{Im}\,z| < \mathrm{Re}\,z\}$ while at the same time controlling its growth, as in Theorem 2, on a proper subset of U. Notation: for $f \in \mathcal{C}^m(\mathbb{R}^{\geqslant})$, $t \geqslant 0$, and $\rho \in [0,m]$ put

$$\|f\|_{t;\,\rho} := \sup\{|f^{(n)}(s)| : 0 \leqslant s \leqslant t,\ n \leqslant \rho\} \in \mathbb{R}^{\geqslant}, \qquad \|f\|_t := \|f\|_{t;\,0}.$$

Theorem 3. *Let $\alpha \in \mathbb{R}^{>}$, $f \in \mathcal{C}^{r+1}(\mathbb{R}^{\geqslant})$, and $\varepsilon, \rho \in \mathcal{C}(\mathbb{R}^{>})$ where $\varepsilon > 0$ is strictly decreasing and convex with $\varepsilon(t) \to 0$ as $t \to \infty$, and ρ with $r \geqslant \rho \geqslant 0$ is increasing. Then there are some $C, D \in \mathbb{R}^{>}$ and a holomorphic $g\colon U \to \mathbb{C}$ such that $g(\mathbb{R}^{>}) \subseteq \mathbb{R}$,*

$$|(f-g)^{(n)}(t)| < \varepsilon(t) \quad \text{for } n \leqslant \rho(t) \text{ and all } t > 0,$$

and for all $z \in \mathbb{C}$ with $\mathrm{Re}\,z > 0$ and $(\mathrm{Im}\,z)^2 \leqslant (\mathrm{Re}\,z)^2 - \alpha$ we have

$$|g(z)| \leqslant \exp\big(C \cdot s \cdot \big(1 + \log^+\|f\|_s + \lambda(s)\big)\big) \quad \text{for } t \geqslant |z|,\ s = D(t^2+1),$$

where

$$\lambda(s) := \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s^2} \cdot \left(\frac{\|f\|_{s;\,\rho(s)+1}}{\Delta\varepsilon(s)}\right)^3.$$

Here C can be chosen independently of ρ, and D independently of f, ε, ρ, r.

Notations and conventions. Let $U \subseteq \mathbb{R}$ be open and $\emptyset \neq S \subseteq U$. For $f \in \mathcal{C}(U)$ we set

$$\|f\|_S \ :=\ \sup\{|f(s)| :\ s \in S\} \in [0,\infty],$$

so for $f, g \in \mathcal{C}(U)$ and $\lambda \in \mathbb{R}$ (and the convention $0 \cdot \infty = \infty \cdot 0 = 0$) we have

$$\|f+g\|_S \leqslant \|f\|_S + \|g\|_S, \quad \|\lambda f\|_S = |\lambda| \cdot \|f\|_S, \quad \text{and} \quad \|fg\|_S \leqslant \|f\|_S\|g\|_S.$$

If $\emptyset \neq S' \subseteq S$ then $\|f\|_{S'} \leqslant \|f\|_S$. Next, let $f \in \mathcal{C}^m(U)$. We then put

$$\|f\|_{S;\,m} \ :=\ \max\{\|f\|_S, \dots, \|f^{(m)}\|_S\} \in [0,\infty].$$

Then again for $f, g \in \mathcal{C}^m(U)$ and $\lambda \in \mathbb{R}$ we have

$$\|f+g\|_{S;m} \leqslant \|f\|_{S;m} + \|g\|_{S;m}, \quad \|\lambda f\|_{S;m} = |\lambda| \cdot \|f\|_{S;m},$$

and

$$\|fg\|_{S;\,m} \ \leqslant\ 2^m \|f\|_{S;\,m}\|g\|_{S;\,m}. \tag{0.1}$$

Also note that for $f \in \mathcal{C}^m(\mathbb{R})$, $t \geqslant 0$, and $\rho \in [0,m]$ we have $\|f\|_{t;\,\rho} = \|f\|_{[-t,t];\,n}$ where $n = \lfloor r \rfloor$.

Let $f \in \mathcal{C}^m(U)$. For $U = \mathbb{R}$ we set $\|f\|_m := \|f\|_{\mathbb{R};\,m}$ and $\|f\| := \|f\|_0$. For $k \leqslant m$ and $\emptyset \neq S' \subseteq S \subseteq U$ we have $\|f\|_{S';\,k} \leqslant \|f\|_{S;\,m}$. Moreover, $\|f\|_{S;\,m}$ does not change if S is replaced by its closure in U.

1. Preliminaries

In this section we collect various auxiliary results used later in the paper: some remarks on difference functions of convex functions, estimates concerning factorials, and an easy bound for a sum of square roots.

The difference function of a convex function. In the next lemma we let I be an interval (of any kind) in $\mathbb{R}$, and we let $\phi\colon I \to \mathbb{R}$ be a convex function, that is,

$$\phi\big((1-\lambda)s + \lambda t\big) \leqslant (1-\lambda)\phi(s) + \lambda\phi(t) \quad \text{for each } \lambda \in [0,1] \text{ and } s,t \in I.$$

We consider the function

$$\Delta\colon D := \big\{(s,t) \in I \times I : s \neq t\big\} \to \mathbb{R}, \qquad \Delta(s,t) := \frac{\phi(s)-\phi(t)}{s-t}.$$

Note that D and Δ are symmetric. The following fact is well-known:

Lemma 1.1. *Let $s < t < u$ be in I; then $\Delta(s,t) \leqslant \Delta(s,u) \leqslant \Delta(t,u)$.*

Proof. We have $t = \lambda s + \mu u$ where $\lambda := (u-t)/(u-s)$ and $\mu := 1-\lambda = (t-s)/(u-s)$. Then $\phi(t) \leqslant \lambda\phi(s) + \mu\phi(u)$ by convexity of ϕ. Subtracting $\phi(s)$ from both sides of this inequality yields

$$\phi(t) - \phi(s) \leqslant \mu\big(\phi(s) - \phi(u)\big) = (t-s)\Delta(s,u),$$

and so we get $\Delta(s,t) \leqslant \Delta(s,u)$, whereas subtracting $\phi(t) + \lambda\big(\phi(s) - \phi(u)\big)$ from both sides of this inequality yields

$$(u-t)\Delta(s,u) = \lambda\big(\phi(u) - \phi(s)\big) \leqslant \phi(u) - \phi(t)$$

and so $\Delta(s,u) \leqslant \Delta(t,u)$. □

Suppose now that I is not bounded from above. Then by Lemma 1.1, the function

$$t \mapsto \Delta\phi(t) := -\Delta(t,t+1) = \phi(t) - \phi(t+1)\colon I \to \mathbb{R}$$

is decreasing. Also note that if $\phi \in \mathcal{C}^2(I)$, then $\phi'' \geqslant 0$ by convexity of ϕ, so $\Delta\phi(t) \geqslant -\phi'(t+1)$ for $t \in I$, by the Mean Value Theorem. This yields:

Lemma 1.2. *Suppose $\phi \in \mathcal{C}^2(I)$, $\phi > 0$, and $C \in \mathbb{R}^>$ are such that $\phi'(t)/\phi(t) \leqslant -C/t$ for each $t \in I$ with $t > 0$. Then $\Delta\phi(t) \geqslant C\phi(t+1)/(t+1)$ for $t \in I$, $t > 0$.*

Cheap approximations to the factorial. The following estimates for $n!$ are not as tight as Stirling's approximation $n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$ (as $n \to \infty$), but good enough for our purposes:

Lemma 1.3. *Suppose $n \geqslant 1$. Then*

$$\mathrm{e}\left(\frac{n}{\mathrm{e}}\right)^n \leqslant n! \leqslant \frac{\mathrm{e}^2}{4}\left(\frac{n+1}{\mathrm{e}}\right)^{n+1}.$$

Proof. For $t > 1$, from the inequalities $t-1 \leqslant \lfloor t \rfloor \leqslant t$ we obtain $\log(t-1) \leqslant \log\lfloor t \rfloor \leqslant \log t$. Hence by integration

$$\int_2^{n+1} \log(t-1)\,dt \leqslant \int_2^{n+1} \log\lfloor t \rfloor\,dt = \sum_{k=2}^{n} \log k \leqslant \int_2^{n+1} \log t\,dt$$

and thus

$$n\big(\log(n)-1\big)+1 \leqslant \log(n!) \leqslant (n+1)\big(\log(n+1)-1\big) - \big(2\log(2)-2\big).$$

and this yields the claimed inequality. □

Corollary 1.4. *Let $\rho \in \mathbb{R}$ with $\rho \geqslant \mathrm{e}\,t > 0$. Then*

$$t^\rho \frac{4}{\mathrm{e}^2}\left(\frac{\mathrm{e}}{\rho+2}\right)^{\rho+2} \leqslant \frac{t^n}{n!} \qquad \textit{for } n \leqslant \rho.$$

Proof. Put $m := \lceil \rho \rceil$. Then $0 \leqslant m-1 < \rho \leqslant m$ and thus

$$\frac{4}{\mathrm{e}^2}\left(\frac{\mathrm{e}}{\rho+2}\right)^{\rho+2} \leqslant \frac{4}{\mathrm{e}^2}\left(\frac{\mathrm{e}}{m+1}\right)^{m+1} \leqslant \frac{1}{m!}, \tag{1.1}$$

where the second inequality holds by Lemma 1.3. Since $m \geqslant \mathrm{e}\,t$, by Lemma 1.3 again:

$$\frac{t^m}{m!} \leqslant t^m \frac{1}{\mathrm{e}}\left(\frac{\mathrm{e}}{m}\right)^m \leqslant \left(\frac{\mathrm{e}\,t}{m}\right)^m \leqslant 1.$$

Moreover, $t^n/n! \leqslant t^{n+1}/(n+1)!$ for $n \leqslant t-1$, and $t^n/n! \geqslant t^{n+1}/(n+1)!$ for $n \geqslant t-1$. Hence $\min\{t^n/n! : n \leqslant m\} = t^m/m!$, so in particular,

$$t^m/m! \leqslant t^n/n! \quad \text{ for } n \leqslant \rho. \tag{1.2}$$

Now if $t \leqslant 1$ then for $n \leqslant \rho$ we have $t^\rho \leqslant t^n$ and thus by (1.1):

$$t^\rho \frac{4}{\mathrm{e}^2}\left(\frac{\mathrm{e}}{\rho+2}\right)^{\rho+2} \leqslant \frac{t^n}{m!} \leqslant \frac{t^n}{n!}.$$

If $t > 1$, then $t^\rho \leqslant t^m$ and so

$$t^\rho \frac{4}{\mathrm{e}^2}\left(\frac{\mathrm{e}}{\rho+2}\right)^{\rho+2} \leqslant \frac{t^m}{m!} \leqslant \frac{t^n}{n!},$$

using both (1.1) and (1.2). □

Bounding a sum of square roots. For all $\alpha, \beta \in \mathbb{R}^{\geqslant}$ we have

$$\sqrt{\alpha+\beta} \leqslant \sqrt{\alpha} + \sqrt{\beta} \leqslant \sqrt{2(\alpha+\beta)}.$$

This observation easily yields the following fact, recorded here for later use:

Lemma 1.5. *If $a, b, c \in \mathbb{R}^{\geqslant}$, then $\sqrt{a+b} + \sqrt{c} \leqslant \sqrt{a} + \sqrt{2(b+c)}$.*

2. Proof of Theorem 1

We begin with a general estimate coming out of a case of Faà di Bruno's Formula [9, §3.4, Theorem B]. For this, let $f \in \mathcal{C}^\infty(I)$ where I is an open interval in $\mathbb{R}$ and $g \in \mathcal{C}^r(J)$ where J is an open interval with $f(I) \subseteq J$, and set $h := g \circ f \in \mathcal{C}^r(I)$. Let $B_{mn}(y_1,\dots,y_{n-m+1}) \in \mathbb{Q}[y_1,\dots,y_{n-m+1}]$ $(m \leqslant n)$ be the Bell polynomials as defined in [4, 12.5]. Then for $n \leqslant r$ we have

$$h^{(n)} = \sum_{m=0}^{n} (g^{(m)} \circ f) \cdot B_{mn}(f', f'', \dots, f^{(n-m+1)}). \tag{2.1}$$

See, e.g., [9, §3.3] for basic facts about the B_{mn}. For example, they have coefficients in $\mathbb{N}$, with $B_{00} = 1$, $B_{0n} = 0$ for $n \geqslant 1$, and for $1 \leqslant m \leqslant n$, B_{mn} is homogeneous of degree m. Let $\left\{ {n \atop m} \right\}$ denote the Stirling numbers of the second kind, that is, the number of equivalence relations on an n-element set with exactly m equivalence classes; cf. [4, p. 576]. They obey the recurrence relations

$$\left\{ {n+1 \atop m} \right\} = m \left\{ {n \atop m} \right\} + \left\{ {n \atop m-1} \right\}$$

with side conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop m} \right\} = 0$ for $m, n \geqslant 1$. The numbers

$$B_n := \sum_{m=0}^{n} \left\{ {n \atop m} \right\}$$

are known as the Bell numbers. We have $B_n \leqslant n^n$ for $n \geqslant 1$. To see this note that B_n is the number of equivalence relations on $[n] := \{1, \dots, n\}$, and there is a surjection from the set $[n]^{[n]}$ of all maps $\lambda \colon [n] \to [n]$ to the collection of equivalence relations on $[n]$, which maps λ to the equivalence relation $\sim_\lambda$ on $[n]$ given by $i \sim_\lambda j :\Leftrightarrow \lambda(i) = \lambda(j)$, for $i, j \in [n]$. (We won't need this here, but much more is known about the asymptotics of B_n, see, e.g., [7, p. 108].) We have $B_{mn}(1, 1, \dots, 1) = \left\{ {n \atop m} \right\}$; cf. [9, §3.3, Theorem B]. These observations yield:

Lemma 2.1. *Suppose $1 \leqslant n \leqslant r$ and $t \in I$, and let $F, G \in \mathbb{R}$, $F \geqslant 1$ be such that*

$$|f^{(k)}(t)| \leqslant F \quad \text{for } k = 1, \dots, n, \qquad |g^{(m)}(f(t))| \leqslant G \quad \text{for } m = 1, \dots, n.$$

Then $|h^{(n)}(t)| \leqslant G \cdot (nF)^n$.

Proof. By (2.1) we have

$$\begin{aligned} |h^{(n)}(t)| &\leqslant \sum_{m=1}^{n} G \cdot |B_{mn}(f'(t), f''(t), \dots, f^{(n-m+1)}(t))| \\ &\leqslant \sum_{m=1}^{n} G \cdot B_{mn}(1, 1, \dots, 1) \cdot F^m \leqslant G \cdot B_n \cdot F^n \leqslant G \cdot (nF)^n. \end{aligned}$$ □

We now prove Theorem 1. Thus let I be an open interval, $f \in \mathcal{C}^r(I)$, and $\varepsilon, \rho \in \mathcal{C}(I)$ where $\varepsilon > 0$, $r \geqslant \rho \geqslant 0$. We first assume that I is bounded, say $I = (a, b)$ where $a < b$, and we need to show the existence of a holomorphic function $g \colon \mathbb{C} \setminus \{a, b\} \to \mathbb{C}$ such that $g(I) \subseteq \mathbb{R}$ and $|(f-g)^{(n)}(t)| < \varepsilon(t)$ for all $t \in I$ and $n \leqslant \rho(t)$. It is straightforward to arrange that $I = (-1, 1)$: Set $\alpha := \frac{b-a}{2}$, $\beta := \min\{\alpha, 1\}$, and $J := (-1, 1)$, and consider the holomorphic bijection $\phi \colon \mathbb{C} \to \mathbb{C}$, $\phi(z) := \alpha \cdot z + \frac{1}{2}(a+b)$, with compositional inverse ϕ^{inv}. Put $f_0 := (f \circ \phi)|_J \colon J \to \mathbb{R}$ and let $\varepsilon_0, \rho_0 \in \mathcal{C}(J)$ be given by $\varepsilon_0(s) := \beta^{\rho(\phi(s))} \varepsilon(\phi(s))$ and $\rho_0(s) := \rho(\phi(s))$, respectively. Suppose we have a holomorphic function $g_0 \colon \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ such that $g_0(s) \in \mathbb{R}$ and

$$|(f_0 - g_0)^{(n)}(s)| < \varepsilon_0(s) \quad \text{for } s \in J,\ n \leqslant \rho_0(s).$$

Consider the holomorphic function $g := g_0 \circ \phi^{\text{inv}} \colon \mathbb{C} \setminus \{a, b\} \to \mathbb{C}$. Then for $t \in I$ and $n \leqslant \rho(t)$, setting $s := \phi^{\text{inv}}(t) \in J$ we have $g(t) = g_0(s) \in \mathbb{R}$ and

$$(f_0 - g_0)^{(n)}(s) = \big((f-g) \circ \phi\big)^{(n)}(s) = (f-g)^{(n)}(t) \cdot \alpha^n$$

and so

$$|(f-g)^{(n)}(t)| < \beta^{\rho(t)} \alpha^{-n} \varepsilon(t) \leqslant \varepsilon(t).$$

Hence replacing f, ε, ρ by f_0, ε_0, ρ_0, respectively, we may arrange $I = (-1, 1)$, which we assume from now on. The rational function

$$\Phi \colon \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}, \qquad \Phi(z) := \frac{2z}{1-z^2} = \frac{1}{1-z} - \frac{1}{z+1}$$

restricts to an analytic bijection $\phi\colon I \to \mathbb{R}$. Let $\phi^{\text{inv}}\colon \mathbb{R} \to I$ be its (analytic) compositional inverse. An easy induction on n shows that for each n and $t \in I$,

$$\phi^{(n)}(t) = n!\left(\frac{1}{(1-t)^{n+1}} + (-1)^{n+1}\frac{1}{(t+1)^{n+1}}\right)$$

and thus

$$|\phi^{(n)}(t)| \leqslant n!\,\frac{(t+1)^{n+1} + (1-t)^{n+1}}{(1-t^2)^{n+1}} \leqslant \frac{n!\cdot 2^{n+1}}{(1-t^2)^{n+1}}. \tag{2.2}$$

Consider the function $\varepsilon_* \in \mathcal{C}(I)$ given by

$$\varepsilon_*(t) := \varepsilon(t)\cdot\left(\frac{4}{(\rho(t)+1)\,\mathrm{e}^2}\cdot\left(\frac{\mathrm{e}(1-t^2)}{2(\rho(t)+1)}\right)^{\rho(t)+1}\right)^{\rho(t)} \qquad \text{for } t \in I.$$

Whitney's Theorem (in the case $I = \mathbb{R}$) applied to $f_* := f \circ \phi^{\text{inv}} \in \mathcal{C}^r(\mathbb{R})$ as well as $\varepsilon_* \circ \phi^{\text{inv}}, \rho \circ \phi^{\text{inv}} \in \mathcal{C}(\mathbb{R})$ in place of f, ε, ρ, respectively, yields an entire function $g_*\colon \mathbb{C} \to \mathbb{C}$ such that $g_*(\mathbb{R}) \subseteq \mathbb{R}$ and, with $h_* := f_* - g_*|_{\mathbb{R}}$:

$$|h_*^{(n)}(s)| < \varepsilon_*(\phi^{\text{inv}}(s)) \qquad \text{whenever } n \leqslant \rho(\phi^{\text{inv}}(s)). \tag{2.3}$$

Consider the holomorphic function $g := g_* \circ \Phi\colon \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ and put $h := f - g|_I$, so $h_* = h \circ \phi$. Then for $t \in I$ we have $|h(t)| < \varepsilon(t)$, and if $1 \leqslant n \leqslant \rho(t)$, then

$$\begin{aligned}
|h^{(n)}(t)| &< \varepsilon_*(t)\cdot\left(\frac{n\cdot n!\cdot 2^{n+1}}{(1-t^2)^{n+1}}\right)^n \\
&\leqslant \varepsilon_*(t)\cdot\left(\frac{n\,\mathrm{e}^2}{4}\cdot\left(\frac{2(n+1)}{\mathrm{e}(1-t^2)}\right)^{n+1}\right)^n \\
&\leqslant \varepsilon_*(t)\cdot\left(\frac{\big(\rho(t)+1\big)\,\mathrm{e}^2}{4}\cdot\left(\frac{2(\rho(t)+1)}{\mathrm{e}(1-t^2)}\right)^{\rho(t)+1}\right)^{\rho(t)} = \varepsilon(t),
\end{aligned}$$

using (2.2) and (2.3) in combination with Lemma 2.1 for the first inequality and Lemma 1.3 for the second inequality.

This shows Theorem 1 in the case of a bounded interval. Now suppose $I \subseteq \mathbb{R}$ is unbounded, say $I = (a, \infty)$. Replacing a, f, ε, ρ by 0, $t \mapsto f(t+a)$, $t \mapsto \varepsilon(t+a)$, and $t \mapsto \rho(t+a)$, respectively, we first arrange $a = 0$, so $I = \mathbb{R}^>$. Now consider the rational function

$$\Psi\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}, \qquad \Psi(z) := z - \frac{1}{z},$$

which restricts to an analytic bijection $\psi\colon \mathbb{R}^> \to \mathbb{R}$ with (analytic) compositional inverse $\psi^{\text{inv}}\colon \mathbb{R} \to \mathbb{R}^>$. For $t > 0$ we have

$$\psi'(t) = 1 + \frac{1}{t^2}, \qquad \psi^{(n)}(t) = \frac{n!}{(-t)^{n+1}} \quad \text{if } n > 1.$$

For $t > 0$, putting

$$\rho_+(t) := \rho(t) + \mathrm{e}\,t, \qquad \beta(t) := 1 + t^{-(\rho_+(t)+1)}\frac{\mathrm{e}^2}{4}\left(\frac{\rho_+(t)+2}{\mathrm{e}}\right)^{\rho_+(t)+2}$$

we obtain $|\psi^{(n)}(t)| \leqslant \beta(t)$ for $1 \leqslant n \leqslant \rho_+(t)$, by Corollary 1.4. Consider

$$\varepsilon_* \in \mathcal{C}(\mathbb{R}^>), \qquad \varepsilon_*(t) := \varepsilon(t)\cdot\big(\rho(t)\beta(t)\big)^{-\rho(t)} \quad \text{for } t > 0.$$

As in the previous case, Whitney's Theorem applied to $f_* := f \circ \psi^{\text{inv}} \in \mathcal{C}^r(\mathbb{R})$ as well as $\varepsilon_* \circ \psi^{\text{inv}}, \rho \circ \psi^{\text{inv}} \in \mathcal{C}(\mathbb{R})$ in place of f, ε, ρ, respectively, yields an entire function $g_* \colon \mathbb{C} \to \mathbb{C}$ such that $g_*(\mathbb{R}) \subseteq \mathbb{R}$ and, with $h_* := f_* - g_*|_{\mathbb{R}}$:

$$|h_*^{(n)}(s)| < \varepsilon_*(\psi^{\text{inv}}(s)) \qquad \text{whenever } n \leqslant \rho(\psi^{\text{inv}}(s)).$$

Let $g := g_* \circ \Psi$, a holomorphic function $\mathbb{C} \setminus \{0\} \to \mathbb{C}$, and $h := f - g|_{\mathbb{R}^>} \in \mathcal{C}^r(\mathbb{R}^>)$, so $h_* = h \circ \psi$. Then for $t > 0$ we have $|h(t)| < \varepsilon(t)$, and if $1 \leqslant n \leqslant \rho(t)$, then

$$|h^{(n)}(t)| < \varepsilon_*(t) \cdot \big(n\beta(t)\big)^n \leqslant \varepsilon_*(t) \cdot \big(\rho(t)\beta(t)\big)^{\rho(t)} = \varepsilon(t)$$

as required. This finishes the proof of Theorem 1. □

Next we prove Corollary 1. Let I, f, ε, ρ be as in the corollary. The case where I is open is covered by Theorem 1, and the case where I is closed and bounded is a consequence of the Weierstrass Approximation Theorem [18, p. 33]. Suppose $I = [\alpha, \beta)$ where $-\infty < \alpha < \beta \leqslant \infty$. (The remaining case where $I = (\alpha, \beta]$ with $-\infty \leqslant \alpha < \beta < \infty$ is treated in a similar way.) Then $\operatorname{fr}(I) = \{\beta\}$. We set $I_1 := (-\infty, \beta)$ and now use a well-known fact:

Claim: *There is a $\mathcal{C}^r$-function $f_1 \colon I_1 \to \mathbb{R}$ which extends f.*

To see this take $\delta \in \mathbb{R}^>$ and an extension of f to a $\mathcal{C}^r$-function $(\alpha - \delta, \beta) \to \mathbb{R}$, also denoted by f. Let $\theta \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\theta(t) = 0$ for $t \leqslant \alpha - (\delta/2)$ and $\theta(t) = 1$ for $t \geqslant \alpha$. (See, e.g., Section 3 below.) Then $f_1 \colon I_1 \to \mathbb{R}$ given by $f_1(t) := f(t)\theta(t)$ if $t \in (\alpha - \delta, \beta)$ and $f_1(t) := 0$ if $t \leqslant \alpha - \delta$ is $\mathcal{C}^r$ and extends $f|_I$.

To finish the proof, take f_1 as in the claim and extend ε, ρ to $\varepsilon_1, \rho_1 \in \mathcal{C}(I_1)$ by $\varepsilon_1(t) := \varepsilon(\alpha)$, $\rho_1(t) := \rho(\alpha)$ for $t \leqslant \alpha$; then Theorem 1 applied to f_1, ε_1, ρ_1 in place of f, ε, ρ yields $g_1 \in \mathcal{C}^\omega(I_1)$ which extends to a holomorphic function $\mathbb{C} \setminus \{\beta\} \to \mathbb{C}$ such that $|(f_1 - g_1)^{(n)}(t)| < \varepsilon_1(t)$ for each $t \in I_1$ and $n \leqslant \rho_1(t)$. Then $g := g_1|_I \in \mathcal{C}^\omega(I)$ has an extension to a holomorphic function $\mathbb{C} \setminus \{\beta\} \to \mathbb{C}$ and satisfies $|(f - g)^{(n)}(t)| < \varepsilon(t)$ for $t \in I$ and $n \leqslant \rho(t)$ as required. □

3. Bounds on the Derivatives of Bump Functions

By a *bump function* we mean here a $\mathcal{C}^\infty$-function $\alpha \colon \mathbb{R} \to \mathbb{R}$ which is 0 on $(-\infty, 0]$, strictly increasing on $[0, 1]$, and 1 on $[1, +\infty)$. Such bump functions play an important role in many constructions in analysis, e.g., partitions of unity. In this section we give a construction of a bump function α with controlled growth of its derivatives $\alpha^{(n)}$. As an auxiliary result we first establish the following formula for the derivatives of reciprocals of nonzero elements of a differential field K:

Proposition 3.1. *For $\phi \in K^\times$ and $n \geqslant 1$ we have*

$$(\phi^{-1})^{(n)} = \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} (\phi^{-1})^{k+1} (\phi^k)^{(n)}. \tag{3.1}$$

Proof. We work in the setting of [4, Chapter 12] and assume familiarity with the concepts and the notations introduced there. Let ϕ range over $K^\times$ and i, j, k, l over $\mathbb{N}$. Put

$$h_\phi := \sum_n \left(\frac{\phi^{(n)}}{\phi}\right) \frac{z^n}{n!} \in 1 + zK[[z]] \subseteq K[[z]]^\times.$$

By the generalized Leibniz Rule, $\phi \mapsto h_\phi \colon K^\times \to K[[z]]^\times$ is a group morphism, so we obtain a group morphism $\phi \mapsto [h_\phi] \colon K^\times \to \mathfrak{tr}_K$ whose image is contained in the

Appell group $\mathcal{A}$ over K (cf. [4, p. 565]). For $i \leqslant j$ we have $[h_\phi]_{ij} = \binom{j}{i}\phi^{(j-i)}/\phi$. Since $\mathcal{A} \subseteq 1 + \mathfrak{tr}_K^1$, we have $\big([h_\phi]-1\big)^k \in \mathfrak{tr}_K^k$ for $k \geqslant 1$, and

$$[h_{\phi^{-1}}] = [h_\phi]^{-1} = \sum_k (-1)^k \big([h_\phi]-1\big)^k = \sum_k (-1)^k \left(\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} [h_{\phi^l}] \right).$$

For $n \geqslant 1$ this yields

$$\begin{aligned}
(\phi^{-1})^{(n)}\phi = [h_{\phi^{-1}}]_{0n} &= \sum_{k=1}^{n} (-1)^k \left(\big([h_\phi]-1\big)^k \right)_{0n} \\
&= \sum_{k=1}^{n} (-1)^k \left(\sum_{l=1}^{k} \binom{k}{l} (-1)^{k-l} [h_{\phi^l}]_{0n} \right) \\
&= \sum_{l=1}^{n} (-1)^l \left(\sum_{k=l}^{n} \binom{k}{l} \right) (\phi^{-1})^l (\phi^l)^{(n)} \\
&= \sum_{l=1}^{n} (-1)^l \binom{n+1}{l+1} (\phi^{-1})^l (\phi^l)^{(n)}
\end{aligned}$$

where for the last equality we used the well-known identity

$$\sum_{k=l}^{n} \binom{k}{l} = \binom{n+1}{l+1},$$

which has an easy proof by induction on n. □

Remark. The previous proposition also holds if K is any differential *ring* (in the sense of [4, 4.1]). To see this let Y be a differential indeterminate over $\mathbb{Q}$. Then the identity (3.1) holds for $\mathbb{Q}\langle Y\rangle$, Y in place of K, ϕ, respectively. It now suffices to note that given any differential ring K and a unit $\phi \in K^\times$ we have the morphism $S^{-1}\mathbb{Q}\{Y\} \to K$ of differential rings with $Y \mapsto \phi$, where

$$S^{-1}\mathbb{Q}\{Y\} = \big\{ Y^{-n}P : P \in \mathbb{Q}\{Y\}, n \geqslant 0 \big\} \subseteq \mathbb{Q}\langle Y\rangle$$

is the localization of $\mathbb{Q}\{Y\}$ at its multiplicative subset $S := \{1, Y, Y^2, \dots\}$.

In the rest of this section we prove:

Proposition 3.2. *There are a bump function α and constants $c, d \in \mathbb{R}^>$ such that $\|\alpha\|_n \leqslant c\, n^{dn}$ for each $n \geqslant 1$.*

We begin our construction by studying the function $\theta\colon \mathbb{R} \to \mathbb{R}^{\geqslant}$ given by $\theta(t) := \mathrm{e}^{-1/t}$ if $t > 0$ and $\theta(t) := 0$ otherwise. For each n and $t > 0$ we have

$$\theta^{(n)} = \frac{p_n(t)}{t^{2n}}\theta(t),$$

where $p_n \in \mathbb{R}[T]$ is the polynomial given recursively by $p_0 = 1$ and

$$p_{n+1} = T^2 p_n' - (2nT-1)p_n. \tag{3.2}$$

Thus

$$p_1 = 1, \quad p_2 = -2T+1, \quad p_3 = 6T^2 - 6T + 1, \quad p_4 = -24T^3 + 36T^2 - 12T + 1, \quad \dots.$$

In general, $\deg p_n = n-1$ for $n \geqslant 1$. Since for each m we have $t^{-m}\,\mathrm{e}^{-1/t} \to 0$ as $t \to 0^+$, this yields that $\theta^{(n)}(t) \to 0$ as $t \to 0^+$, so θ is $\mathcal{C}^\infty$. We now want to bound the quantities $\|\theta^{(n)}\|_{[0,1]}$. First, some notation and a lemma.

Notation. For a polynomial $p = a_0 + a_1T + \cdots + a_nT^n \in \mathbb{R}[T]$ ($a_0, \dots, a_n \in \mathbb{R}$) we let $|p| := \max\big\{|a_0|, \dots, |a_n|\big\}$.

Lemma 3.3. *$|p_n| \leqslant 2^{n-1}n!$ for $n \geqslant 1$.*

Proof. This is clear for $n = 1$. Suppose we have shown the inequality for some $n \geqslant 1$. Let $a_0, \dots, a_{n-1}, b_0, \dots, b_n \in \mathbb{R}$ with $p_n = a_0 + a_1T + \cdots + a_{n-1}T^{n-1}$ and $p_{n+1} = b_0 + b_1T + \cdots + b_nT^n$. Then

$$\begin{aligned} T^2p_n' &= \sum_{m=1}^{n}(m-1)a_{m-1}T^m, \\ (2nT-1)p_n &= -a_0 + \sum_{m=1}^{n-1}(2na_{m-1} - a_m)T^m + 2na_nT^n. \end{aligned}$$

Hence by the recursion relation (3.2) we have

$$b_0 = -a_0, \quad b_m = a_m + (-2n+m-1)a_{m-1} \quad \text{for } m = 1, \dots, n-1, \quad b_n = -2na_{n-1}$$

and this yields $|p_{n+1}| \leqslant 2^n(n+1)!$ as required. □

Next, given $\lambda \in \mathbb{R}^>$, consider the function $t \mapsto \zeta(t) := t^{-\lambda}\,\mathrm{e}^{-1/t} \colon \mathbb{R}^> \to \mathbb{R}^>$. Then

$$\zeta'(t)/\zeta(t) = -\lambda/t + 1/t^2 = (-\lambda + 1/t)/t,$$

so $\max\zeta(\mathbb{R}^>) = \zeta(1/\lambda) = \lambda^\lambda\,\mathrm{e}^{-\lambda}$. Since the function $\lambda \mapsto \lambda^\lambda\,\mathrm{e}^{-\lambda} \colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}^>$ is strictly increasing, this yields, for $t \in [0,1]$ and $n \geqslant 1$:

$$|\theta^{(n)}(t)| = \left|\frac{p_n(t)}{t^{2n}}\,\mathrm{e}^{-1/t}\right| \leqslant n \cdot |p_n| \cdot (n+1)^{n+1}\,\mathrm{e}^{-(n+1)}$$

and thus

$$\log|\theta^{(n)}(t)| \leqslant \log n + (n-1)\log 2 + n\log n + (n+1)\big(\log(n+1) - 1\big) \leqslant 3n\log n.$$

Therefore

$$\|\theta^{(n)}\|_{[0,1]} \leqslant n^{3n} \qquad \text{for } n \geqslant 1. \tag{3.3}$$

Let $\mu \in \mathbb{R}^>$. Then $\theta(t)^\mu = \theta(t/\mu)$ and thus $(\theta^\mu)^{(n)}(t) = \theta^{(n)}(t/\mu)/\mu^n$, hence

$$\|(\theta^\mu)^{(n)}\|_{[0,1]} = \mu^{-n}\|\theta^{(n)}\|_{[0,1/\mu]} \leqslant \|\theta^{(n)}\|_{[0,1]} \quad \text{if } \mu \geqslant 1.$$

Also using (3.3), this yields:

$$\|\theta^\mu\|_{[0,1];\,n} \leqslant n^{3n} \quad \text{if } \mu, n \geqslant 1. \tag{3.4}$$

We note that (3.4) also holds with θ replaced by the $\mathcal{C}^\infty$-function $\theta_* \colon \mathbb{R} \to \mathbb{R}^{\geqslant}$ given by $\theta_*(t) := \theta(1-t)$. Since $\theta + \theta_* > 0$, we obtain the $\mathcal{C}^\infty$-function

$$\alpha := \theta/(\theta + \theta_*) \colon \mathbb{R} \to \mathbb{R},$$

and this is a bump function: we have $\alpha(t) = 0$ for $t \leqslant 0$ and $\alpha(t) = 1$ for $t \geqslant 1$, and since for $t \in (0,1)$ we also have

$$\alpha'(t) = \frac{\theta'(t)\theta(1-t) + \theta(t)\theta'(1-t)}{\big(\theta(t) + \theta(1-t)\big)^2} = \theta(t)\theta(1-t)\frac{t^{-2} + (1-t)^{-2}}{\big(\theta(t) + \theta(1-t)\big)^2} > 0,$$

the restriction of α to $[0,1]$ is strictly increasing. Our goal is now to compute bounds on $\|\alpha\|_n$, using (3.4). For this we apply the remark after Proposition 3.1 to the differential ring $K = \mathcal{C}^\infty(\mathbb{R})$ and the unit $\phi := \theta + \theta_*$ of K. Suppose $n \geqslant 1$. We first note that

$$\phi^k = \sum_{l=0}^{k} \binom{k}{l} \theta^{k-l}\theta_*^l$$

where by (0.1) and (3.4) we have

$$\|\theta^{k-l}\theta_*^l\|_{[0,1];\,n} \leqslant 2^n n^{6n} \quad \text{for } l = 0,\dots,k \text{ and } m = 0,\dots,n,$$

hence $\|\phi^k\|_{[0,1];\,n} \leqslant 2^{k+n} n^{6n}$. Next we note that for $t \in (0,1)$ we have

$$\phi'(t) = \frac{\theta(t)}{t^2} - \frac{\theta_*(t)}{(1-t)^2},$$

hence $\phi'(t) < 0$ if $t < 1/2$ and $\phi'(t) > 0$ if $t > 1/2$. This yields $\phi \geqslant \phi(1/2) = \mathrm{e}^{-2}$ and so $\|(\phi^{-1})^{k+1}\|_{[0,1]} \leqslant \mathrm{e}^{2(k+1)} \leqslant 2^{3(k+1)}$. Using (3.1) we thus conclude:

$$\begin{aligned} \|(\phi^{-1})^{(n)}\|_{[0,1]} &\leqslant \sum_{k=1}^{n} \binom{n+1}{k+1} \cdot \|(\phi^{-1})^{k+1}\|_{[0,1]} \cdot \|(\phi^k)^{(n)}\|_{[0,1]} \\ &\leqslant \sum_{k=1}^{n} \binom{n+1}{k+1} \cdot 2^{3(k+1)} \cdot 2^{k+n} n^{6n} \leqslant 2^{6n+4} n^{6n}. \end{aligned}$$

Now $\alpha = \theta \cdot \phi^{-1}$ and thus

$$\begin{aligned} \|\alpha\|_n = \|\alpha\|_{[0,1];\,n} &\leqslant 2^n \cdot \|\theta\|_{[0,1];\,n} \cdot \|\phi^{-1}\|_{[0,1];\,n} \\ &\leqslant 2^n \cdot n^{3n} \cdot 2^{6n+4} n^{6n} = 2^{7n+4} n^{9n} \leqslant c n^{dn} \end{aligned}$$

where $c := 2^{11}$, $d := 16$. This concludes the proof of Proposition 3.2. □

Remark. Note that the bound in this proposition is qualitatively optimal in the following sense: there is no function $\gamma\colon \mathbb{N} \to \mathbb{R}^>$ with $\gamma(n) = o(n)$ such that $\|\alpha^{(n)}\| = O(n^{\gamma(n)})$. To see this suppose β is any bump function. Let n be given and put $M := \|\beta^{(n)}\|$. If $n \geqslant 1$ then $|\beta^{(n-1)}(t)| \leqslant Mt$ for $t \in (0,1)$, hence if $n \geqslant 2$ then $|\beta^{(n-2)}(t)| \leqslant Mt^2/2$ for $t \in (0,1)$, etc., thus $|\beta(t)| \leqslant Mt^n/n!$ for $t \in (0,1)$. Hence there is no $c \in \mathbb{R}^>$ such that $\|\beta^{(n)}\| \leqslant c^n n!$ for all n, since for such c we would have $\beta(t) = 0$ for each $t \in [0, 1/c]$, contradicting that $\beta|_{[0,1]}$ is strictly increasing. Now the claim follows from this observation and Lemma 1.3.

Let α, c, d be as in Proposition 3.2, where we may assume $c \geqslant 1$. With $0^0 := 1$, put $C_n := cn^{dn}$, so $1 = C_0 \leqslant C_1 \leqslant \cdots$. For $a < b$ in $\mathbb{R}$, we define the increasing $\mathcal{C}^\infty$-function $\alpha_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by

$$\alpha_{a,b}(t) \;:=\; \alpha\!\left(\frac{t-a}{b-a}\right), \tag{3.5}$$

so $\alpha_{a,b}(t) = 0$ for $t \leqslant a$ and $\alpha_{a,b}(t) = 1$ for $t \geqslant b$. Also,

$$\left|\alpha_{a,b}^{(m)}(t)\right| \;\leqslant\; \frac{C_m}{(b-a)^m} \quad \text{for all } m \text{ and } t.$$

The case most relevant for us later is when $b - a \leqslant 1$; then we have

$$\|\alpha_{a,b}\|_n \leqslant \frac{C_n}{(b-a)^n}. \tag{3.6}$$

Next let $a < b < a_* < b_*$ be in $\mathbb{R}$, and with $\varepsilon := \frac{1}{3}(b-a), \varepsilon_* := \frac{1}{3}(b_* - a_*) \in \mathbb{R}^>$ define the (hump) function $\alpha_{a,b,a_*,b_*} \colon \mathbb{R} \to \mathbb{R}$ by

$$\alpha_{a,b,a_*,b_*}(t) \ := \ \begin{cases} \alpha_{a+\varepsilon,b-\varepsilon}(t) & \text{if } t \leqslant b, \\ 1 - \alpha_{a_*+\varepsilon_*,b_*-\varepsilon_*}(t) & \text{otherwise.} \end{cases} \tag{3.7}$$

Then $\alpha_{a,b,a_*,b_*}(t) = 0$ if $t \notin [a, b_*]$, $\alpha_{a,b,a_*,b_*}(t) = 1$ if $t \in [b, a_*]$, and α_{a,b,a_*,b_*} is increasing on $[a, b]$ and decreasing on $[a_*, b_*]$. Suppose $\varepsilon, \varepsilon_* \leqslant 1$. Then by (3.6),

$$\|\alpha_{a,b,a_*,b_*}\|_n \ \leqslant \ 3^n C_n \max\left\{ \frac{1}{(b-a)^n}, \frac{1}{(b_*-a_*)^n} \right\}. \tag{3.8}$$

In particular, for suitable reals $c_1, d_1 \geqslant 1$ (depending only on $b-a$, $b_* - a_*$, c, d) we have $\|\alpha_{a,b,a_*,b_*}\|_n \leqslant c_1 n^{d_1 n}$ for each n.

4. Weierstrass Transforms

Recall that the support $\operatorname{supp} f$ of a function $f \colon \mathbb{R} \to \mathbb{R}$ is the closure in $\mathbb{R}$ of the set $\big\{t \in U : f(t) \neq 0\big\}$. Let $f \in \mathcal{C}^m(\mathbb{R})$ be such that $\operatorname{supp} f$ is bounded; let also λ range over $\mathbb{R}^>$. The function $f_\lambda \colon \mathbb{R} \to \mathbb{R}$ given by

$$f_\lambda(t) := (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f(s)\, \mathrm{e}^{-\lambda(s-t)^2}\, ds \tag{4.1}$$

is known as the (generalized) *Weierstrass transform* with parameter λ; below we sometimes denote f_λ also by $W_\lambda(f)$. We could have replaced the bounds $-\infty$, ∞ in this integral by any a, b such that $\operatorname{supp}(f) \subseteq [a, b]$. A change of variables gives

$$f_\lambda(t) \ = \ (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f(t-s)\, \mathrm{e}^{-\lambda s^2}\, ds.$$

From the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\, ds = \pi^{1/2}$ we get $(\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda s^2}\, ds = 1$, hence $\|f_\lambda\| \leqslant \|f\|$. Also, f_λ extends to an entire function; in particular, $f_\lambda \in \mathcal{C}^\omega(\mathbb{R})$. For $k \leqslant m$ we have $(f_\lambda)^{(k)} = (f^{(k)})_\lambda$, so $\|f_\lambda\|_m \leqslant \|f\|_m$. See [3, Appendix A] for proofs of these facts. (Entire functions which arise as Weierstrass transforms have been studied extensively [10, 19, 21, 24].) Next we record an explicit version of [3, Lemma A.2], for which we let $f \in \mathcal{C}^{m+1}(\mathbb{R})$ have bounded support.

Lemma 4.1. *Let $\varepsilon \in \mathbb{R}^>$, and set*

$$\lambda(\varepsilon) := 8(\|f\|_{m+1}\varepsilon^{-1})^2 \log^+\big(2\sqrt{2}\, \|f\|_m\, \varepsilon^{-1}\big) \in \mathbb{R}^{\geqslant}.$$

Then for each $\lambda > \lambda(\varepsilon)$ we have $\|f_\lambda - f\|_m \leqslant \varepsilon$. (In particular, $\|f_\lambda - f\|_m \leqslant \varepsilon$ provided $\lambda \geqslant 16\sqrt{2}(\|f\|_{m+1}\varepsilon^{-1})^3 + 1$.)

Proof. Put $\delta := (\varepsilon/2)/\|f\|_{m+1}$. By the Mean Value Theorem we have

$$|f^{(k)}(s) - f^{(k)}(t)| \leqslant \varepsilon/2 \quad \text{whenever } |s-t| \leqslant \delta \text{ and } k \leqslant m.$$

An easy computation shows that $\lambda > \lambda(\varepsilon)$ implies $\sqrt{2}\|f\|_m\, \mathrm{e}^{-(\lambda/2)\delta^2} \leqslant \varepsilon/2$, and by the argument in the proof of Lemma A.2, this guarantees $\|f_\lambda - f\| \leqslant \varepsilon$. □

5. Whitney Approximation with Bounds

Let (a_n), (b_n), (ε_n) be sequences in $\mathbb{R}$ and (r_n) in $\mathbb{N}$ such that

(i) $a_0 = b_0$, (a_n) is strictly decreasing, (b_n) is strictly increasing,
(ii) $\varepsilon_n > \varepsilon_{n+1}$ with $\varepsilon_n \to 0$ as $n \to \infty$, and
(iii) $r_n \leqslant r_{n+1} \leqslant r$ for each n.

Set

$$K_n \ := \ [a_n, b_n], \qquad L_n \ := \ K_{n+1} \setminus K_n \ = \ [a_{n+1}, a_n) \cup (b_n, b_{n+1}].$$

Then $I := \bigcup_n K_n$ is an open interval in $\mathbb{R}$. Let $f \in \mathcal{C}^r(I)$. The proof of Whitney's Approximation Theorem given in [3, Appendix A] then produces a $g \in \mathcal{C}^\omega(I)$ such that $\|f-g\|_{L_n;\,r_n} < \varepsilon_n$ for each n. We recall the main lines of this argument. First, we introduce some hump functions φ_n as follows: for $n \geqslant 1$, employing the $\mathcal{C}^\infty$-functions introduced in (3.7), we set

$$\alpha_n := \alpha_{a_{n+2},a_{n+1},a_n,a_{n-1}}, \qquad \beta_n := \alpha_{b_{n-1},b_n,b_{n+1},b_{n+2}}, \qquad \varphi_n := \alpha_n + \beta_n,$$

and also set $\varphi_0 := \alpha_{a_2,a_1,b_1,b_2} \in \mathcal{C}^\infty(\mathbb{R})$. Then for each n we have $\varphi_n = 0$ on a neighborhood of K_{n-1} (satisfied automatically for $n=0$, by convention), $\varphi_n = 1$ on a neighborhood of $\operatorname{cl}(L_n) = [a_{n+1}, a_n] \cup [b_n, b_{n+1}]$, and $\operatorname{supp}\varphi_n \subseteq K_{n+2}$. With $M_n := 1 + 2^{r_n}\|\varphi_n\|_{r_n}$, choose $\delta_n \in \mathbb{R}^>$ so that for all n,

$$2\delta_{n+1} \leqslant \delta_n, \qquad \sum_{m=n}^{\infty} \delta_m M_{m+1} \leqslant \varepsilon_n/4. \tag{5.1}$$

We inductively define sequences (λ_n) in $\mathbb{R}^>$ and (g_n) in $\mathcal{C}^\omega(\mathbb{R})$ as follows: Let $\lambda_m \in \mathbb{R}^>$ and $g_m \in \mathcal{C}^\omega(\mathbb{R})$ for $m < n$; then consider $h_n \in \mathcal{C}^r(\mathbb{R})$ given by

$$h_n(t) \ := \ \begin{cases} \varphi_n(t)\cdot\big(f(t) - \big(g_0(t) + \cdots + g_{n-1}(t)\big)\big) & \text{if } t \in I, \\ 0 & \text{otherwise.}\end{cases}$$

Thus $\operatorname{supp} h_n \subseteq \operatorname{supp}\varphi_n \subseteq K_{n+2}$ is bounded. Put

$$g_n := W_{\lambda_n}(h_n) \in \mathcal{C}^\omega(\mathbb{R})$$

where we take $\lambda_n \in \mathbb{R}^>$ such that $\|g_n - h_n\|_{r_n} < \delta_n$: any sufficiently large λ_n will do, by Lemma 4.1. The argument in [3, Appendix A] then yields a function

$$g\colon I \to \mathbb{R}, \qquad g(t) = \sum_{i=0}^{\infty} g_i(t) \text{ for each } t \in I$$

with $g \in \mathcal{C}^{r_n}(I)$ and $\|f-g\|_{L_n;\,r_n} < \varepsilon_n$ for each n. Next set

$$H_m := 2(\lambda_m/\pi)^{1/2}\|h_m\|(b_{m+2} - a_{m+2}) \in \mathbb{R}^{\geqslant},$$

and fix a sequence (c_m) of positive reals such that $M := \sum_{m\geqslant 1} c_m < \infty$. Then we can and do choose the sequences (g_m), (λ_m) so that in addition

$$H_m \exp(-\lambda_m/m) \ \leqslant \ c_m \quad \text{for all } m \geqslant 1.$$

Therefore

$$\sum_n H_n \exp(-\lambda_n\rho) < \infty \qquad \text{for each } \rho \in \mathbb{R}^>.$$

Now g_n is the restriction to $\mathbb{R}$ of the entire function $\widehat{g}_n$ given by

$$\widehat{g}_n(z) \ = \ (\lambda_n/\pi)^{1/2}\int_{a_{n+2}}^{b_{n+2}} h_n(s)\,\mathrm{e}^{-\lambda_n(s-z)^2}\,ds.$$

In the following we let $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Then $\operatorname{Re}(s-z)^2 = (s-x)^2 - y^2$ and so

$$|\widehat{g}_n(z)| \leqslant (\lambda_n/\pi)^{1/2} \int_{-\infty}^{\infty} |h_n(s)|\, \mathrm{e}^{-\lambda_n \operatorname{Re}(s-z)^2}\, ds \leqslant \|h_n\|\, \mathrm{e}^{\lambda_n y^2}. \tag{5.2}$$

Put

$$\rho_n := \tfrac{1}{2} \min\bigl\{(a_n - a_{n+1})^2, (b_{n+1} - b_n)^2\bigr\} \in \mathbb{R}^{>}$$

and

$$U_n := \bigl\{z : a_{n+1} < \operatorname{Re} z < b_{n+1},\ \operatorname{Re}\bigl((z-a_{n+1})^2\bigr),\ \operatorname{Re}\bigl((z-b_{n+1})^2\bigr) > \rho_n\bigr\},$$

an open subset of $\mathbb{C}$ containing K_n such that $\operatorname{Re}\bigl((s-z)^2\bigr) > \rho_n$ for all $z \in U_n$ and $s \in \mathbb{R} \setminus K_{n+1}$. If $\rho_n \geqslant \rho_{n+1}$, then $U_n \subseteq U_{n+1}$. The argument in [3, Appendix A] shows that for each n the series $\sum_m \widehat{g}_m$ converges uniformly on compact subsets of U_n, and so we obtain a holomorphic function

$$z \mapsto \widehat{g}(z) := \sum_m \widehat{g}_m(z) \ :\ U = \bigcup_n U_n \to \mathbb{C}$$

which extends g, as claimed. Indeed, if $z \in U_n$ and $m \geqslant n+2$, then $\operatorname{supp} h_m \subseteq K_{m+2} \setminus K_{m-1} \subseteq \mathbb{R} \setminus K_{n+1}$ and so $|\widehat{g}_m(z)| \leqslant H_m\, \mathrm{e}^{-\lambda_m \rho_n}$, hence

$$|\widehat{g}_m(z)| \leqslant H_m\, \mathrm{e}^{-\lambda_m/m} \leqslant c_m \qquad \text{if } m \geqslant k_n := \max\bigl\{\lceil 1/\rho_n \rceil, n+2\bigr\} \tag{5.3}$$

and so $\sum_{m \geqslant k_n} |\widehat{g}_m(z)| \leqslant M$.

In the proofs of Theorems 2 and 3 (given at the end of this section) we need to control the growth of $|\widehat{g}(z)|$ for $z \in U$, and this requires us to choose the quantities δ_n, λ_n, c_n in a more explicit way. With this in mind we focus now in particular on the two cases $I = \mathbb{R}$ and $I = \mathbb{R}^{>}$, where we assume that a_n, b_n are chosen as follows, for some $\delta \in \mathbb{R}^{>}$:

($\mathbb{R}$) $a_n = -b_n = -\delta n$ for each n;
($\mathbb{R}^{>}$) $a_n = \delta/(n+1)$, $b_n = \delta(n+1)$ for each n.

Below we will label various displayed statements pertaining to these two cases accordingly. For example, note that

$$\rho_n = \delta^2/2, \qquad U = \mathbb{C}. \tag{5.4 $\mathbb{R}$}$$

This follows by observing that given z, for sufficiently large n we have $a_{n+1} = -\delta(n+1) < x < \delta(n+1) = b_{n+1}$ as well as

$$(x - a_{n+1})^2 - y^2 - \rho_n = \bigl(x + \delta(n+1)\bigr)^2 - y^2 - (\delta^2/2) > 0$$

and

$$(x - b_{n+1})^2 - y^2 - \rho_n = \bigl(x - \delta(n+1)\bigr)^2 - y^2 - (\delta^2/2) > 0,$$

so $z \in U_n$. We also have

$$\rho_n = \frac{\delta^2}{2} \left(\frac{1}{(n+1)(n+2)}\right)^2, \qquad U = \bigl\{z : |\operatorname{Im} z| < \operatorname{Re} z\bigr\}. \tag{5.4 $\mathbb{R}^{>}$}$$

To see the latter note that if $x > 0$ and $y^2 < x^2$, then

$$(x - a_{n+1})^2 - y^2 - \rho_n \to x^2 - y^2 > 0 \text{ as } n \to \infty,$$

and thus $\operatorname{Re}\bigl((z - a_{n+1})^2\bigr) = (x - a_{n+1})^2 - y^2 > \rho_n$ for sufficiently large n, and likewise, $\operatorname{Re}\bigl((z - b_{n+1})^2\bigr) > \rho_n$ for all sufficiently large n. Conversely, if $z \in U_n$, then $x > a_{n+1} > 0$ and $x^2 - y^2 \geqslant (x - a_{n+1})^2 - y^2 > \rho_n > 0$.

It will be convenient to assume, in addition to (i)–(iii) above:

(iv) $\varepsilon_n + \varepsilon_{n+2} \geqslant 2\varepsilon_{n+1}$ for each n, and

(v) $b_n - a_n = O(n)$.

(Here (iv) will hold in the settings of the proofs of Theorems 2 and 3, and (v) clearly holds both in the cases ($\mathbb{R}$) and ($\mathbb{R}^>$).) We shall also assume $f \neq 0$ below, and let $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

We now estimate the growth of the derivatives of the φ_n. Let $C_n \in \mathbb{R}$ be as defined in Section 3. Suppose first we are in the case ($\mathbb{R}$). By the remark after (3.8) we can take $c, d \in \mathbb{R}^{\geqslant 1}$ such that

$$\text{(5.5 }\mathbb{R}) \qquad \|\varphi_n\|_m \leqslant c\, m^{dm} \quad \text{for all } m,\, n.$$

In the case ($\mathbb{R}^>$) we note $a_n - a_{n+1} = \frac{\delta}{(n+1)(n+2)}$ for each n, hence for $n \geqslant 1$ we have $\|\alpha_n\|_m \leqslant (3/\delta)^m C_m\big((n+2)(n+3)\big)^m$ and $\|\beta_n\|_m \leqslant (3/\delta)^m C_m$, so $\|\varphi_n\|_m \leqslant (3/\delta)^m C_m\big((n+2)(n+3)\big)^m$; we also have $\|\varphi_0\|_m \leqslant (18/\delta)^m C_m$. Thus in this case we obtain $c, d \in \mathbb{R}^>$ such that

$$\text{(5.5 }\mathbb{R}^>) \qquad \|\varphi_0\|_m \leqslant c\, m^{dm}, \qquad \|\varphi_n\|_m \leqslant c\,(mn)^{dm} \quad \text{if } n \geqslant 1.$$

Going forward we assume that we have real numbers D_{mn} such that

$$D_{0n} = 1, \quad 2^m\|\varphi_n\|_m \leqslant D_{mn} \leqslant D_{m+1,n}, D_{m,n+1} \qquad \text{for all } m,\, n.$$

(In the case ($\mathbb{R}$), by (5.5 $\mathbb{R}$), for suitable $c, d \in \mathbb{R}^{\geqslant 1}$ we can take $D_{mn} := c\, m^{dm}$. In the case ($\mathbb{R}^>$) we may similarly take $D_{m0} := c\, m^{dm}$ and $D_{mn} := c\,(mn)^{dm}$ if $n \geqslant 1$, for suitable $c, d \in \mathbb{R}^{\geqslant 1}$.) Now put

$$D_n := D_{r_n n}, \qquad N_n := 2^{n+1} D_n.$$

Then $N_n \geqslant 2D_n$, thus $M_n = 1 + 2^{r_n}\|\varphi_n\|_{r_n} \leqslant 1 + D_n \leqslant N_n$. Also note: $2N_n \leqslant N_{n+1}$. *We now assume that in the construction of $\widehat{g}$ above we chose*

$$\delta_n := \frac{\varepsilon_n - \varepsilon_{n+1}}{4N_{n+1}} > 0.$$

Note that these δ_n indeed satisfy the requirements (5.1): we have

$$\sum_{m=n}^{\infty} \delta_m M_{m+1} \leqslant \sum_{m=n}^{\infty} \delta_m N_{m+1} = \frac{1}{4}\sum_{m=n}^{\infty} (\varepsilon_m - \varepsilon_{m+1}) = \frac{1}{4}\lim_{k\to\infty} (\varepsilon_n - \varepsilon_{k+1}) \leqslant \frac{1}{4}\varepsilon_n,$$

and we also have $2\delta_{n+1} \leqslant \delta_n$ since by (iv):

$$\frac{N_{n+2}}{2N_{n+1}} \geqslant 1 \geqslant \frac{\varepsilon_{n+1} - \varepsilon_{n+2}}{\varepsilon_n - \varepsilon_{n+1}}.$$

In order to deduce from Lemma 4.1 a lower bound on just how large λ_n needs to be taken, we need an upper bound on $\|h_n\|_m$ for $m \leqslant r$, which we establish next. For this we note that $\operatorname{supp}\varphi_n \subseteq K_{n+2}$ and $\|g_n\|_m \leqslant \|h_n\|_m$ for each n yields

$$\|h_n\|_m \leqslant 2^m\|\varphi_n\|_m\big(\|f\|_{K_{n+2};\,m} + \|h_0\|_m + \cdots + \|h_{n-1}\|_m\big).$$

An easy induction on n now shows that

$$\|h_n\|_m \leqslant G_{mn} := 2^n (D_{mn})^{n+1} \cdot \|f\|_{K_{n+2};\,m}.$$

In particular $\|h_n\| \leqslant G_{0n} = 2^n\|f\|_{K_{n+2}}$. Also note that

$$\|f\|_{K_{n+2}} \leqslant G_{mn} \leqslant G_{m+1,n}, G_{m,n+1}.$$

Next put

$$\mu_n := 128\sqrt{2}(\delta_n^{-1} G_{r_n+1,n})^3 + 1.$$

Then $\mu_{n+1} \geqslant \mu_n \geqslant 1$, and for each $\lambda \geqslant \mu_n$ we have $\|W_\lambda(h_n) - h_n\|_{r_n} \leqslant \delta_n/2 < \delta_n$ by Lemma 4.1. *Below we assume that in our construction of the sequences* (λ_n), (g_n) *we always chose* $\lambda_n = \mu_n$. Note that $\delta_n \leqslant \delta_0 \leqslant \varepsilon_0/2^{n+3}$ and so

$$\lambda_n \geqslant (\delta_0^{-1} G_{0n})^3 \geqslant 8^{n+3} \varepsilon_0^{-3} G_{0n}^3.$$

Since $f \neq 0$ by assumption, we can take n_0 with $G_{0n_0} > 0$. Then with $c := \varepsilon_0^3/G_{0n_0}^2$ we have $c\lambda_n \geqslant n^5 G_{0n}$ for each n. The function $t \mapsto t^2\,\mathrm{e}^{-t} \colon \mathbb{R}^{>} \to \mathbb{R}$ takes on its maximum value $4\,\mathrm{e}^{-2} < 1$ at $t = 2$, so $t\,\mathrm{e}^{-t} < t^{-1}$ for each $t > 0$. Hence for $n \geqslant 1$:

$$\lambda_n^{1/2}\,\mathrm{e}^{-\lambda_n/n} \leqslant n \cdot (\lambda_n/n)\,\mathrm{e}^{-\lambda_n/n} \leqslant n^2/\lambda_n$$

and so $\lambda_n^{1/2}\,\mathrm{e}^{-\lambda_n/n}\, G_{0n} \leqslant c/n^3$. We have $H_n = O\big(\lambda_n^{1/2} G_{0n}\,(n+2)\big)$ by (v), so we obtain a constant $c_* > 0$ (not depending on the sequence r_n) such that

$$H_n \exp(-\lambda_n/n) \leqslant c_*/n^2 \quad \text{for } n \geqslant 1.$$

Hence in the construction of $\widehat{g}$ *above we may choose* $c_n := c_*/n^2$ *for each* $n \geqslant 1$.

Proof of Theorem 2. Let f, ε, ρ be as in the statement of Theorem 2. We choose a_n, b_n as in ($\mathbb{R}$) above, with $\delta := \sqrt{2}$, and set $\varepsilon_n := \varepsilon(b_{n+1}) = \varepsilon\big(\sqrt{2}(n+1)\big)$ and $r_n := \lfloor \rho(b_{n+1}) \rfloor$. Then conditions (i)–(v) hold, with (iv) a consequence of the convexity of ε. We also have $\rho_n = 1$ and $U = \mathbb{C}$ by (5.4 $\mathbb{R}$). We now construct the real entire function $\widehat{g}$ and its restriction $g \in \mathcal{C}^\omega(\mathbb{R})$ as described above, in the process choosing the quantities D_n, δ_n, λ_n, c_n as indicated, in the case ($\mathbb{R}$). (In particular, $c_n = c_*/n^2$ for $n \geqslant 1$, where the constant $c_* > 0$ doesn't depend on ρ.) Then for $t \in L_n$ and $k \leqslant \rho(|t|)$ we have $|t| \leqslant b_{n+1}$, thus $k \leqslant r_n$ and so

$$|(f-g)^{(k)}(t)| \leqslant \|f - g\|_{L_n;\, r_n} < \varepsilon_n = \varepsilon(b_{n+1}) \leqslant \varepsilon(|t|).$$

Thus $|(f-g)^{(k)}(t)| < \varepsilon(|t|)$ for all t and $k \leqslant \rho(|t|)$. We now aim to estimate $|\widehat{g}(z)|$ when $|z| \leqslant t$. We need two lemmas. In the first one we bound

$$\lambda_n = 128\sqrt{2}\big(\delta_n^{-1} G_{r_n+1,n}\big)^3 + 1$$

from above:

Lemma 5.1. *There is a* $D \in \mathbb{R}^{\geqslant 1}$, *independent of* f, ε, ρ, *such that for each* n,

$$\lambda_n \leqslant \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s} \cdot \left(\frac{\|f\|_{s;\,\rho(s)+1}}{\Delta\varepsilon(s)}\right)^3 + 1 \qquad \text{where } s := \sqrt{2}(n+2).$$

Proof. As we are in case ($\mathbb{R}$), we can assume that we have $c, d \in \mathbb{R}^{\geqslant 1}$ (independent of f, ε, ρ) such that $D_{mn} = cm^{dm}$ for all m, n. We have

$$\delta_n^{-1} = \frac{2^{n+3} D_{n+1}}{\varepsilon_n - \varepsilon_{n+1}} \qquad \text{where } D_{n+1} = D_{r_{n+1},n+1}.$$

Since ε is convex and decreasing,

$$\varepsilon_n - \varepsilon_{n+1} = \varepsilon(s - \sqrt{2}) - \varepsilon(s) \geqslant \varepsilon(s) - \varepsilon(s+1) = \Delta\varepsilon(s).$$

Now $r_{n+1} \leqslant \rho(s)$, thus

$$\delta_n^{-1} = \frac{c 2^{n+3} (r_{n+1})^{dr_{n+1}}}{\varepsilon_n - \varepsilon_{n+1}} \leqslant \frac{2c\, 2^{s/\sqrt{2}} \rho(s)^{d\rho(s)}}{\Delta\varepsilon(s)}$$

and so

$$\delta_n^{-3} \leqslant \frac{c_0\, 2^{(3/\sqrt{2})s} \rho(s)^{d_0\rho(s)}}{\Delta\varepsilon(s)^3} \qquad \text{with } c_0 := (2c)^3,\ d_0 := 3d.$$

Moreover, using that ρ is increasing:

$$\begin{aligned} G_{r_n+1,n} &= 2^n (D_{r_n+1,n})^{n+1} \cdot \|f\|_{K_{n+2;\, r_n+1}} \\ &\leqslant 2^n \big(c(r_n+1)\big)^{d(r_n+1)(n+1)} \cdot \|f\|_{K_{n+2;\, r_n+1}} \\ &\leqslant 2^{s/\sqrt{2}-2} \big(c(\rho(s)+1)\big)^{d(\rho(s)+1)(s/\sqrt{2}-1)} \cdot \|f\|_{s;\, \rho(s)+1} \\ &\leqslant \big(2c(\rho(s)+1)\big)^{d(\rho(s)+1)(s/\sqrt{2}-1)} \cdot \|f\|_{s;\, \rho(s)+1} \end{aligned}$$

So with $c_1 := 2c$, $d_1 := 3d/\sqrt{2}$ we have

$$(G_{r_n+1,n})^3 \leqslant \big(c_1\big(\rho(s)+1\big)\big)^{d_1(\rho(s)+1)s} \cdot \|f\|^3_{s;\, \rho(s)+1}.$$

Combining these estimates for δ_n^{-3} and $(G_{r_n+1,n})^3$ yields D with the required properties. $\square$

In the following we set $U_{-1} := \emptyset$.

Lemma 5.2. *Suppose $z \notin U_{n-1}$. Then $\sqrt{2}n \leqslant |x| + \sqrt{1+y^2} \leqslant 1 + \sqrt{2}|z|$.*

Proof. The first inequality is clear if $|x| \geqslant \sqrt{2}n$, so suppose $|x| < \sqrt{2}n$. Then $n \geqslant 1$, and thus $z \notin U_{n-1}$ yields $\operatorname{Re}(z-\sqrt{2}n)^2 \leqslant 1$ or $\operatorname{Re}(z+\sqrt{2}n)^2 \leqslant 1$. In the first case $(\sqrt{2}n - x)^2 \leqslant 1 + y^2$ where $\sqrt{2}n - x > 0$ and thus

$$\sqrt{2}n \leqslant x + \sqrt{1+y^2} \leqslant |x| + \sqrt{1+y^2}.$$

In the second case, similarly $(\sqrt{2}n + x)^2 \leqslant 1 + y^2$ where $\sqrt{2}n + x > 0$ and thus $\sqrt{2}n \leqslant -x + \sqrt{1+y^2} \leqslant |x| + \sqrt{1+y^2}$. The second inequality is a consequence of Lemma 1.5. $\square$

Now take n such that $z \in U_n \setminus U_{n-1}$. From (5.3) and the remark after it recall that $\sum_{m \geqslant k_n} |\widehat{g}_m(z)| \leqslant M$ where $k_n = n+2$, thus

$$|\widehat{g}(z)| \leqslant \sum_{m \leqslant n+1} |\widehat{g}_m(z)| + \sum_{m \geqslant n+2} |\widehat{g}_m(z)|$$

where the second (infinite) sum is $\leqslant M$, so we focus on the first sum. With $s := \sqrt{2}t+1$ we have $\sqrt{2}n \leqslant s$ by Lemma 5.2, so $n \leqslant s/\sqrt{2} \leqslant t+1 \leqslant s$. Recalling that $K_{n+3} = \big[-\sqrt{2}(n+3), \sqrt{2}(n+3)\big]$, we get $G_{0,n+1} = 2^{n+1}\|f\|_{K_{n+3}} \leqslant 2^{s+1}\|f\|_{s+3\sqrt{2}}$, hence by (5.2), for $m \leqslant n+1$:

$$|\widehat{g}_m(z)| \leqslant G_{0m} \cdot \exp(\lambda_m t^2) \leqslant G_{0,n+1} \cdot \exp(\lambda_{n+1} t^2) \leqslant 2^{s+1}\|f\|_{s+3\sqrt{2}} \cdot \exp(\lambda_{n+1} t^2).$$

This yields

$$\sum_{m \leqslant n+1} |\widehat{g}_m(z)| \leqslant (s+2) \cdot 2^{s+1}\|f\|_{s+3\sqrt{2}} \cdot \exp(\lambda_{n+1} t^2),$$

hence

$$\begin{aligned} \|\widehat{g}\|_t &\leqslant (s+2) \cdot 2^{s+1}\|f\|_{s+3\sqrt{2}} \cdot \exp(\lambda_{n+1} t^2) + M \\ &\leqslant \exp\big((s+2) + \log^+\|f\|_{s+3\sqrt{2}} + \lambda_{n+1} t^2\big) + M \\ &\leqslant \exp\big(N + \log^+\|f\|_{s+3\sqrt{2}} + \lambda_{n+1} s^2\big) \quad \text{where } N := \log(1+M) + 3, \\ &\leqslant \exp\big(N \cdot s^2 \cdot (1 + \log^+\|f\|_{s+3\sqrt{2}} + \lambda_{n+1})\big). \end{aligned}$$

Next take D as in Lemma 5.1. Since ρ and $1/\Delta\varepsilon$ are increasing and $s + 3\sqrt{2} \geqslant \sqrt{2}(n+3)$, we obtain $\lambda_{n+1} \leqslant \lambda(s+3\sqrt{2}) + 1$ where

$$\lambda(s) := \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s} \cdot \left(\frac{\|f\|_{s;\,\rho(s)+1}}{\Delta\varepsilon(s)}\right)^3.$$

Hence with $C := 2N$ we have

$$\|\widehat{g}\|_t \leqslant \exp\big(C \cdot s^2 \cdot \big(1 + \log^+ \|f\|_{s+3\sqrt{2}} + \lambda(s+3\sqrt{2})\big)\big) \quad \text{for } t \geqslant 0,\ s = \sqrt{2}t + 1.$$

Here C does not depend on ρ, and D does not depend on f, ε, ρ. This concludes the proof of the theorem. $\square$

Proof of Theorem 3. Let α, f, ε, ρ, V be as in Theorem 3, so

$$V = \big\{z : \operatorname{Re} z > 0,\ (\operatorname{Im} z)^2 \leqslant (\operatorname{Re} z)^2 - \alpha\big\}.$$

We choose (a_n), (b_n) as in $(\mathbb{R}^>)$ with $\delta := \sqrt{\alpha/2}$, and we set $\varepsilon_n := \varepsilon(b_{n+1}) = \varepsilon\big(\delta(n+2)\big)$ and $r_n := \lfloor \rho(b_{n+1}) \rfloor$ for each n. Then (i)–(v) hold, and by (5.4 $\mathbb{R}^>$) we have $V \subseteq U = \big\{z : |\operatorname{Im} z| < \operatorname{Re} z\big\}$ and

$$\rho_n = \frac{\alpha}{4}\left(\frac{1}{(n+1)(n+2)}\right)^2.$$

We now construct the holomorphic function $\widehat{g}\colon U \to \mathbb{C}$ with $\widehat{g}(\mathbb{R}^>) \subseteq \mathbb{R}$ as described before, and set $g := \widehat{g}|_{\mathbb{R}^>} \in \mathcal{C}^\omega(\mathbb{R}^>)$. As in the proof of Theorem 2, we see that for $t \in L_n$ and $k \leqslant \rho(t)$ we have $|(f-g)^{(k)}| < \varepsilon(t)$. Since $\bigcup_n L_n = \mathbb{R}^>$, this yields $|(f-g)^{(k)}(t)| < \varepsilon(t)$ for all $t > 0$ and $k \leqslant \rho(t)$. Now put $V_n := U_n \cap V$, so $V = \bigcup_n V_n$, and set $V_{-1} := \emptyset$.

Lemma 5.3. *Let $z \in V \setminus V_{n-1}$. Then $n \leqslant (2/\sqrt{\alpha})|z|$.*

Proof. We may assume $n \geqslant 1$. Note that $z \in V$ gives $x^2 \geqslant y^2 + \alpha \geqslant \alpha$ and $x > 0$, so $x \geqslant \sqrt{\alpha} > \delta > a_n$. If $x \geqslant b_n = \delta(n+1)$, then $n < n+1 \leqslant (1/\delta)x$, hence $n \leqslant (2/\sqrt{\alpha})|z|$; thus suppose $x < b_n$. We first show that $\operatorname{Re}\big((z-a_n)^2\big) > \rho_{n-1}$. To see this note $z \in V$ also yields $y^2 \leqslant x^2 - \alpha$, and together with $a_n x < a_n b_n = \delta^2 = \alpha/2$,

$$\begin{aligned}
\operatorname{Re}\big((z-a_n)^2\big) = (x-a_n)^2 - y^2 &\geqslant a_n^2 - 2a_n x + \alpha \\
&> a_n^2 = \frac{\delta^2}{(n+1)^2} \\
&\geqslant \frac{\delta^2}{2}\left(\frac{1}{n(n+1)}\right)^2 = \rho_{n-1}
\end{aligned}$$

as claimed. Now $z \notin V_{n-1}$ yields $\operatorname{Re}\big((z-b_n)^2\big) \leqslant \rho_{n-1}$, and so

$$\big(x - \delta(n+1)\big)^2 - y^2 = \operatorname{Re}\big((z-b_n)^2\big) \leqslant \rho_{n-1} \leqslant \alpha/16,$$

hence $\delta(n+1) - x \leqslant \sqrt{(\alpha/16) + y^2}$. Using Lemma 1.5 we get $\delta(n+1) \leqslant (\sqrt{\alpha}/4) + \sqrt{2}|z|$ and thus $\delta n < \sqrt{2}|z|$, hence $n \leqslant (2/\sqrt{\alpha})|z|$ follows. $\square$

Lemma 5.4. *There is a $D \in \mathbb{R}^{\geqslant 1}$, independent of f, ε, ρ, such that for each $n \geqslant 1$,*

$$\lambda_n \leqslant \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s^2} \cdot \left(\frac{\|f\|_{s;\,\rho(s)+1}}{\Delta\varepsilon(s)}\right)^3 + 1 \qquad \textit{where } s := \delta(n+3).$$

Proof. Since we are in the case ($\mathbb{R}^{>}$), we can assume to have $c, d \in \mathbb{R}^{\geqslant 1}$ (independent of f, ε, ρ) such that $D_{mn} = c(mn)^{dm}$ for all m and $n \geqslant 1$. Suppose $n \geqslant 1$, and recall again that

$$\delta_n^{-1} = \frac{2^{n+3} D_{n+1}}{\varepsilon_n - \varepsilon_{n+1}} \qquad \text{where } D_{n+1} = D_{r_{n+1},n+1}.$$

Using Lemma 1.1, from the convexity of ε we obtain

$$\varepsilon_n - \varepsilon_{n+1} = \varepsilon(s-2\delta) - \varepsilon(s-\delta) \geqslant \delta\big(\varepsilon(s) - \varepsilon(s+1)\big) = \delta \cdot \Delta\varepsilon(s).$$

We also have $r_{n+1} \leqslant \rho(b_{n+2}) = \rho(s)$ and thus

$$\delta_n^{-1} \leqslant \frac{c 2^{s/\delta}\big(\rho(s)(s/\delta - 2)\big)^{d\rho(s)}}{\varepsilon_n - \varepsilon_{n+1}} \leqslant \frac{(c/\delta)\, 2^{s/\delta}\, \big(\rho(s)(s/\delta)\big)^{d\rho(s)}}{\Delta\varepsilon(s)},$$

so

$$\delta_n^{-3} \leqslant \frac{c_0 2^{(3/\delta)s}\big(\rho(s)(s/\delta)\big)^{d_0\rho(s)}}{\Delta\varepsilon(s)^3} \qquad \text{with } c_0 := (c/\delta)^3,\ d_0 := 3d.$$

Moreover,

$$\begin{aligned} G_{r_n+1,n} &= 2^n (D_{r_n+1,n})^{n+1} \cdot \|f\|_{K_{n+2};\, r_n+1} \\ &\leqslant 2^n \big(c(r_n+1)n\big)^{d(r_n+1)(n+1)} \cdot \|f\|_{K_{n+2};\, r_n+1} \\ &\leqslant \big(2c(\rho(s)+1)\big)^{d(\rho(s)+1)(s/\delta-3)(s/\delta-2)} \cdot \|f\|_{s;\, \rho(s)+1} \end{aligned}$$

and thus

$$(G_{r_n+1,n})^3 \leqslant \big(c_1\big(\rho(s)+1\big)\big)^{d_1(\rho(s)+1)s^2} \cdot \|f\|^3_{s;\, \rho(s)+1}$$

where $c_1 := 2c$, $d_1 := 3d/\delta^2$, and this yields the lemma. $\square$

Recall that

$$k_n = \max\big\{\lceil 1/\rho_n \rceil, n+2\big\} \leqslant \max\Big\{(4/\alpha)\big((n+1)(n+2)\big)^2, n+1\Big\} + 1,$$

so we obtain a constant $c_0 \in \mathbb{R}^{>}$, only depending on α, such that

$$k_n + 2 \leqslant c_0(t^2+1) \qquad \text{for all } n \text{ and } t \geqslant (\delta/\sqrt{2})n.$$

Let now $z \in V$ with $|z| \leqslant t$; we aim to estimate $|\widehat{g}(z)|$ in terms of t. Take n with $z \in V_n \setminus V_{n-1}$, so $t \geqslant (\delta/\sqrt{2})n$ by Lemma 5.3. We have $\sum_{m \geqslant k_n} |\widehat{g}(z)| \leqslant M$ by (5.3). Also $K_{m+2} = \big[\delta/(m+3), \delta(m+3)\big]$ and so $G_{0m} = 2^m \|f\|_{K_{m+2}} \leqslant 2^m \|f\|_{\delta(m+3)}$, hence if $m < k_n$ then $G_{0m} \leqslant 2^{c_0(t^2+1)} \|f\|_{c_1(t^2+1)}$ where $c_1 := \delta c_0$. This yields

$$\begin{aligned} \sum_{m<k_n} |\widehat{g}_m(z)| &\leqslant k_n \cdot 2^{c_0(t^2+1)} \|f\|_{\delta c_0(t^2+1)} \cdot \exp(\lambda_{k_n-1} t^2) \\ &\leqslant c_0(t^2+1) \cdot 2^{c_0(t^2+1)} \|f\|_{c_1(t^2+1)} \cdot \exp(\lambda_{k_n-1} t^2), \end{aligned}$$

and hence for suitable $N \in \mathbb{R}^{>}$, not depending on ρ:

$$\begin{aligned} |\widehat{g}(z)| &\leqslant \sum_{m<k_n} \|\widehat{g}_m(z)\| + \sum_{m \geqslant k_n} \|\widehat{g}_m(z)\| \\ &\leqslant c_0(t^2+1) \cdot 2^{c_0(t^2+1)} \|f\|_{\delta c_0(t^2+1)} \cdot \exp(\lambda_{k_n-1} t^2) + M \\ &\leqslant \exp\big(N \cdot (t^2+1) \cdot (1 + \log^+ \|f\|_{c_1(t^2+1)} + \lambda_{k_n-1})\big). \end{aligned}$$

Now put $C := 2N/c_1$, take D as in Lemma 5.4, and let

$$s := c_1(t^2+1), \qquad \lambda(s) := \big(D(\rho(s)+1)\big)^{D(\rho(s)+1)s^2} \cdot \left(\frac{\|f\|_{s;\, \rho(s)+1}}{\Delta\varepsilon(s)}\right)^3.$$

Then $\lambda_{k_n-1} \leqslant \lambda(s)+1$, and so $|\widehat{g}(z)| \leqslant \exp\big(C\cdot s\cdot\big(1+\log^+\|f\|_s+\lambda(s)\big)\big)$. Here C does not depend on ρ, and c_1, D only on α. This concludes the proof of Theorem 3. $\square$

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