Midterm Examination

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due at the beginning of class on Monday, November 5, 2007.

You may consult the text-book, your notes, and your old homework while attempting the exam. You may also use any computer system, as long as the use is acknowledged and relevant printouts are attached. You may not discuss the exam with anyone else during the exam period. Turning in the exam will be taken as agreement with these terms. Show all work on all questions. Answers without full justification will receive little or no credit.

We let $S = k[x_1, \dots, x_n]$, where k is a field.

1. Let $I = \langle x^3 - 3x + 2, x^4 + 2x^3 - 4x^2 - 2x + 3\rangle \subseteq \mathbb{Q}[x]$.
   (a) Find a single polynomial $f \in \mathbb{Q}[x]$ for which $I = \langle f\rangle$.
   (b) Does $x^2 + 4x - 5$ lie in I?
   (c) What is $V(I)$? $I(V(I))$?
2. Fix a monomial ordering $<$. Below, I, J are ideals of S.
   (a) Show that a monomial ideal of S is radical if and only if it has a squarefree generating set (that is, one where each generator is not divisible by the square of any variable).
   (b) Show that $\sqrt{I} := \{f \in S : f^m \in I \text{ for some } m > 0\}$ is an ideal of S containing I (called the **radical** of I).
   (c) Suppose $I \subseteq J$ and $\langle \mathrm{lm}(I)\rangle = \langle \mathrm{lm}(J)\rangle$. Show that $I = J$. (Hint: assume otherwise, and choose $f \in J \setminus I$ such that $\mathrm{lm}(f)$ is minimal.)
   (d) Show: if $\langle \mathrm{lm}(J)\rangle$ is radical, then J is radical. (Hint: use (b) and (c).)
   (e) Is the converse true? Prove or give a counterexample. You may (or may not) want to experiment with a computer for this.
3. In class we defined a monomial ordering of the monomials in S to be any total ordering $<$ of the set M of those monomials which satisfies
   (a) $x^\alpha < x^\beta \Rightarrow x^\alpha x^\gamma < x^\beta x^\gamma$ for all multi-indices α, β, γ;
   (b) $1 < x_i$ for $i = 1, \dots, n$.

   We then proved that any such monomial ordering is a well-ordering. Show conversely: if $<$ is a well-ordering of M which satisfies (a), then (b) holds.
4. Let $f \in \mathbb{C}[x]$, $f \neq 0$, all of whose coefficients are rational. Let f_{red} be the generator of $I(V(f))$ (with $\mathrm{lc}(f_{\mathrm{red}}) = 1$). Show that all coefficients of f_{red} are rational.