Homework 4

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due Friday, November 30, 2007.

1. Compute $S(f,g)$ using the lex ordering:
   (a) $f = 4x^2z - 7y^2$, $g = xyz^2 + 3xz^4$;
   (b) $f = xy + z^3$, $g = z^2 - 3z$.
2. Find a Gröbner basis, w.r.t. the lex ordering, of $I = \langle x - z^4, y - z^5 \rangle$.
3. Determine whether $f = xy^3 - z^2 + y^5 - z^3$ is in $I = \langle -x^3 + y, x^2y - z \rangle$.
4. Let I be a **principal ideal** of $S = k[x_1, \ldots, x_n]$, that is, I is generated by a single polynomial $f \in S$. Show that any finite subset of I containing f is a Gröbner basis for I.
5. Let $S = k[a, b, c, d]$. Let $I = \langle c^5 - b^3d^2, a^2d - bc^2, a^2c^3 - b^4d, a^4c - b^5 \rangle$.
   (a) Show that the given generating set is a Gröbner basis for I with respect to the reverse lexicographic ordering with $a > b > c > d$.
   (b) Compute the Gröbner basis for I with respect to the lexicographic ordering of the variables with $a > b > c > d$.
6. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ equipped with the lexicographic monomial ordering. Let I be an ideal of S.
   (a) Show that if I contains a polynomial which only involves powers of x_n, then there must be such a polynomial in the reduced Gröbner basis for I.
   (b) Suppose that $V(I)$ is a finite set. First show that $I(V(I))$ must contain a polynomial only involving powers of x_n. We will prove later in the course that (since we work over the complex numbers) the ideal $I(V(I))$ is the radical of I. (This is the Hilbert Nullstellensatz.) Assuming this fact, also show that I contains a polynomial containing only powers of x_n.
   (c) If we know a polynomial in I containing only powers of one variable, we can solve for the roots of this polynomial (symbolically or numerically), and use this to reduce to a simpler problem. Use this idea to solve the system of equations:

$$\begin{aligned} x^2 - 3xy + y^2 &= 0 \\ x^3 - 8x + 3y &= 0 \\ x^2y - 3x + y &= 0 \end{aligned}$$

   Give your answer symbolically (that is, in terms of radicals).