Homework 4

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due Friday, November 2, 2007.

1. For $\mathbb{N}$ with its usual ordering, between any two integers there are only finitely many other integers. Is it true that for every monomial ordering of $k[x_1, \ldots, x_n]$, there are only finitely many monomials between any two given monomials? Is it true for the grlex ordering?

2. A basis $B = \{x^{\alpha(1)}, \ldots x^{\alpha(s)}\}$ for a monomial ideal I of $k[x_1, \ldots, x_n]$ is said to be **minimal** if no proper subset of B generates I. Prove that every monomial ideal has a minimal basis, and that this minimal basis is unique. What is the minimal basis of the monomial ideal

$$I = \langle x^2, x^5, xy^2, x^3y^3, y \rangle$$

of $k[x, y]$?

3. Let $u = (u_1, \ldots, u_n)$ be a vector in $\mathbb{R}^n$ such that $u_1, \ldots, u_n$ are positive and $\mathbb{Q}$-linearly independent. (We say that u is an **independent weight vector.**) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ set $u \cdot \alpha = u_1\alpha_1 + \cdots + u_n\alpha_n$. For $\alpha, \beta \in \mathbb{N}^n$ define

$$x^\alpha <_u x^\beta \qquad :\Longleftrightarrow \qquad \alpha <_u \beta.$$

   (a) Show that $<_u$ is a monomial ordering. (Hint: Where does your argument use the linear independence of $u_1, \ldots, u_n$?)

   (b) Show that $u = (1, \sqrt{2})$ is an independent weight vector. (So $<_u$ is a monomial ordering of $k[x, y]$.)

4. Let $u = (u_1, \ldots, u_n)$ be a vector in $\mathbb{R}^n$ such that $u_1, \ldots, u_n \geq 0$. Suppose $<_\sigma$ is a monomial ordering for $k[x_1, \ldots, x_n]$. Then for $\alpha, \beta \in \mathbb{N}^n$ define

$$x^\alpha <_{\sigma,u} x^\beta \qquad :\Longleftrightarrow \qquad u \cdot \alpha < u \cdot \beta, \text{ or } u \cdot \alpha = u \cdot \beta \text{ and } x^\alpha <_\sigma x^\beta.$$

   (a) Show that $<_{\sigma,u}$ is a monomial ordering.

   (b) Find u such that $<_{\text{lex},u}$ is the grlex ordering.