Homework 2

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due Wednesday, October 17, 2007.

1. Let k be a field, let $I = k[x_1, \ldots, x_n]$ be an ideal, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. Prove that the following statements are equivalent:
   (a) $f_1, \ldots, f_s \in I$.
   (b) $\langle f_1, \ldots, f_s \rangle \subseteq I$.

2. Use the previous problem to prove the following equalities of ideals in $\mathbb{Q}[x, y]$.
   (a) $\langle x + xy, y + xy, x^2, y^2 \rangle = \langle x, y \rangle$;
   (b) $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle = \langle x^2 - 4, y^2 - 1 \rangle$.

3. Let k be a field. An ideal I of $k[x_1, \ldots, x_n]$ is said to be **radical** if whenever a power f^m of a polynomial f lies in I, for some positive integer m, then f itself is in I.
   (a) Prove that if V is an affine variety in k^n, then $I(V)$ is always a radical ideal of $k[x_1, \ldots, x_n]$.
   (b) Prove that $\langle x^2, y^2 \rangle \neq I(V)$ for every affine variety V in k^2.

4. The *consistency problem* asks, given $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, whether $V(f_1, \ldots, f_s) = \emptyset$. In this exercise we consider this problem for the case $k = \mathbb{C}$ and $n = 1$:
   (a) Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Show that $V(f) = \emptyset$ if and only if f is a nonzero constant.
   (b) Let $f_1, \ldots, f_s \in \mathbb{C}[x]$. Prove that $V(f_1, \ldots, f_s) = \emptyset$ if and only if $\mathrm{GCD}(f_1, \ldots, f_s) = 1$.
   (c) Describe (in words) an algorithm to determine, given $f_1, \ldots, f_s \in \mathbb{C}[x]$, whether $V(f_1, \ldots, f_s) = \emptyset$.

5. MathSciNet (`www.ams.org/mathscinet`) indexes the vast majority of mathematics papers and books published each year. Use it to find out:
   (a) How many other books have the authors of our text-book written? Does the UCLA library have any of them?
   (b) For how many papers published between 1990 and 1995 does the word "Gröbner" appear in the title? You will need to be logged on to a computer on the UCLA network to use MathSciNet.