Homework 2

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Solutions.

1. Since $f_i = 1 \cdot f_i + \sum_{j \neq i} 0 \cdot f_j \in \langle f_1, \ldots, f_s \rangle$, we have that (b) implies (a). Now we show that (a) implies (b). Every element $g \in \langle f_1, \ldots, f_s \rangle$ can be written as $g = \sum_{i=1}^{s} h_i f_i$ for some $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$; since $f_1, \ldots, f_s \in I$, and ideals are closed under addition and polynomial multiplication, $g = \sum_{i=1}^{s} h_i f_i \in I$. Thus $\langle f_1, \ldots, f_s \rangle \subseteq I$.

2. By the previous question we just need to show that each set of generators is in the ideal generated by the other set of generators.

   (a) $\langle x+xy, y+xy, x^2, y^2 \rangle \subseteq \langle x, y \rangle$ since $x+xy, y+xy, x^2$, and y^2 are all visibly in $\langle x, y \rangle$. For the other direction, note that

   $$x = (x + xy) - (y + xy)x + x^2 y,$$
   $$y = (y + xy) - (x + xy)y + y^2 x.$$

   (b) The fact that

   $$2x^2 + 3y^2 - 11 = 2(x^2 - 4) + 3(y^2 - 1)$$

   and

   $$x^2 - y^2 - 3 = (x^2 - 4) - (y^2 - 1)$$

   implies that

   $$\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle \subseteq \langle x^2 - 4, y^2 - 1 \rangle.$$

   In the same way,

   $$x^2 - 4 = \frac{1}{5}(2x^2 + 3y^2 - 11) + \frac{3}{5}(x^2 - y^2 - 3)$$

   and

   $$y^2 - 1 = \frac{1}{5}(2x^2 + 3y^2 - 11) - \frac{2}{5}(x^2 - y^2 - 3)$$

   imply that

   $$\langle x^2 - 4, y^2 - 1 \rangle \subseteq \langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle.$$

3. (a) Let V be an affine variety in k^n, let f be a polynomial and $m > 0$ an integer such that $f^m \in I(V)$. Then $(f(x))^m = f^m(x) = 0$ for every $x \in V$. The only element a of k with the property that $a^m = 0$ is $a = 0$ (why?), so we conclude that $f(x) = 0$ for all $x \in V$. Thus $f \in I(V)$. This means that $I(V)$ is radical.

(b) If $\langle x^2, y^2 \rangle = I(V)$ for some affine variety $V \subseteq k^2$, then $\langle x^2, y^2 \rangle$ would be a radical ideal by (a). However $x^2 \in \langle x^2, y^2 \rangle$, but any polynomial of the form $f(x,y)x^2 + g(x,y)y^2$ where $f, g \in k[x,y]$ has total degree at least two, so $x \notin \langle x^2, y^2 \rangle$. Thus $\langle x^2, y^2 \rangle$ is not radical, and so not the ideal of any variety.

4. (a) By the Fundamental Theorem of Algebra, if $f \in \mathbb{C}[x]$ is nonconstant, then f has a zero, so $V(f) \neq \emptyset$. Conversely, if f is a constant, then $V(f) = \emptyset$.

   (b) Let $h = \mathrm{GCD}(f_1, f_2, \ldots, f_s)$. Then $\langle h \rangle = \langle f_1, f_2 \cdots, f_s \rangle$. The variety of an ideal does not depend on the choice of generators, so $V(f_1, \ldots, f_s) = V(h)$, and thus by part (a) we have $V(f_1, \ldots, f_s) = \emptyset$ if and only if $\mathrm{GCD}(f_1, \ldots, f_s) = 1$ (up to multiplication by a nonzero constant from $\mathbb{C}$).

   (c) The following algorithm determines, given $f_1, \ldots, f_s \in \mathbb{C}[x]$, whether $V(f_1, \ldots, f_s) = \emptyset$: Given $f_1, \ldots, f_s$, compute $\mathrm{GCD}(f_1, f_2, \ldots, f_s)$ using the Extended Euclidean Algorithm (as explained in class). If the answer is a nonzero constant, output "yes", otherwise output "no."

5. (a) The authors have written one other joint book, entitled "Using Algebraic Geometry," which appeared in two editions. Our library owns both; I placed the second edition on reserve for this class.

   (b) There were 99 papers published between 1990 and 1995 with the word "Gröbner" in the title.