Homework 1

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due Monday, October 8, 2007

1. Let $\mathbb{F}_2 = \{0, 1\}$, and define two operations $+$ and $\cdot$ on $\mathbb{F}_2$ by

$$0 + 0 = 1 + 1 = 0, \qquad 0 + 1 = 1 + 0 = 1$$

and

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \qquad 1 \cdot 1 = 1.$$

Verify that $\mathbb{F}_2 = \{0, 1\}$, equipped with these operations, forms a field. (You may skip the verification of the associative and distributive properties.)

2. Let $\mathbb{F}_2$ be the field from Problem 1. Consider the polynomial

$$g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y].$$

Show that $g(x, y) = 0$ for every $(x, y) \in \mathbb{F}_2^2$. Why does this not contradict what we proved in class?

3. Let k be a field. Prove that every single point $(a_1, \dots, a_n) \in k^n$ is an affine variety. Use this to prove that every finite subset of k^n is an affine variety.

4. Show that the set

$$X = \big\{(x, x) : x \in \mathbb{R}, x \neq 1\big\} \subseteq \mathbb{R}^2$$

is not an affine variety. (This is Exercise 1.2.8 in the textbook, where you can find some hints.)

5. Consider the equations:

$$\begin{aligned} x^2 + y^2 &= 1 \\ xy &= 1 \end{aligned}$$

which describe the intersection of a circle and a hyperbola.

   (a) Use algebra to eliminate y from the above equation.

   (b) Show that the polynomial you found in part (a) lies in the ideal $(x^2 + y^2 - 1, xy - 1)$.