Homework 1

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Solutions

1. (20 pts.) We have $0+1=1+0=1$ and $0\cdot 1=1\cdot 0=0$, hence addition and multiplication on $\mathbb{F}_2$ are commutative. Clearly $a+0=a\cdot 1=a$ for $a=0,1$, hence 0 and 1 are identity elements for $+$ and $\cdot$. The additive inverse of 0 is 0 and the additive inverse of 1 is 1 (since $1+1=0$); the multiplicative inverse of 1 (the only nonzero element of $\mathbb{F}_2$) is 1 (since $1\cdot 1=1$).

2. (10 pts.) We have $\mathbb{F}_2^2=\{(0,0),(0,1),(1,0),(1,1)\}$. Substituting each element of this set for (x,y) in g shows that $g(x,y)=0$ for every $(x,y)\in\mathbb{F}_2^2$. Since the field $\mathbb{F}_2$ is finite, and although the polynomial g is not the zero polynomial, this does not contradict what we proved in class, where we showed that for infinite coefficient fields k only the zero polynomial in $k[x,y]$ defines the zero function on k^2.

3. (20 pts.) Let $(a_1,\dots,a_n)\in k^n$, and consider $V=V(x_1-a_1,\dots,x_n-a_n)$. Then $(a_1,\dots,a_n)\in V$, and if $(b_1,\dots,b_n)\in V$, then $b_i=a_i$ for each i, so $(b_1,\dots,b_n)=(a_1,\dots,a_n)$, and thus $V=\{(a_1,\dots,a_n)\}$ is an affine variety. In class we proved that the union of two affine varieties in k^n is an affine variety in k^n. By induction on m, this implies that the union of m affine varieties in k^n is an affine variety. Hence in particular, every finite subset of k^n is an affine variety.

4. (20 pts.) For a contradiction, suppose that X is an affine variety, so $X=V(f_1,f_2,\dots,f_s)$ for some polynomials $f_1,f_2,\dots,f_s\in\mathbb{R}[x,y]$. Let $f\in\langle f_1,f_2,\dots,f_s\rangle$. The polynomial f vanishes on X. Let $g(t)=f(t,t)$. Then $g(t)$ is a polynomial which vanishes away from $t=1$. Using the fact that a nonzero polynomial $g(t)$ defined over an infinite field has only finitely many zeros, we conclude that $g=0$. This implies that $f(1,1)=0$ for any $f\in\langle f_1,\dots,f_s\rangle$, contradicting $(1,1)\notin X$.

5. (a) (20 pts.) Multiplying the equation $xy-1=0$ on both sides by $xy+1$, we obtain
$$x^2y^2-1=0.$$
From the equation $x^2+y^2=1$, we also have $y^2=1-x^2$, and thus substituting in for y^2 we get
$$x^2(1-x^2)-1=0.$$
   (b) (10 pts.) We have
$$x^2(1-x^2)-1=(xy-1)(xy+1)-(x^2+y^2-1)x^2\in\langle x^2+y^2-1,xy-1\rangle.$$