Refined Enumeration of Two-Rowed Set-Valued Standard Tableaux via Two-Coloured Motzkin Paths

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Set-Valued Standard Tableaux

Definition

A *set-valued standard tableau* of shape λ is a filling of the cells of the Young diagram of λ with $1, 2, \ldots, n$, for some positive integer n, each being used exactly once, such that each cell is filled with at least one number, and such that a number in a cell ρ is less than *all* numbers in cells that are weakly to the right and weakly below ρ (ρ excluded).

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EXAMPLE. A set-valued tableau of shape $(4, 3, 3, 1)$ with $n = 15$

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Theorem (Lazar and Linusson)

Let n and m be positive integers. The number of set-valued standard tableaux of rectangular shape (e, e), for some non-negative integer e, with m entries in the first row and n entries in total is given by

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$$\frac{1}{n}\binom{2n-2}{n-1}.$$

Conjecture (Lazar and Linusson)

Let n be a positive integer ≥ 3. The expected value of the number of columns in set-valued standard tableaux with n entries and rectangular shape (e, e) for some e is given by

$$\frac{\binom{n}{2} + n - 3}{2n - 3}.$$

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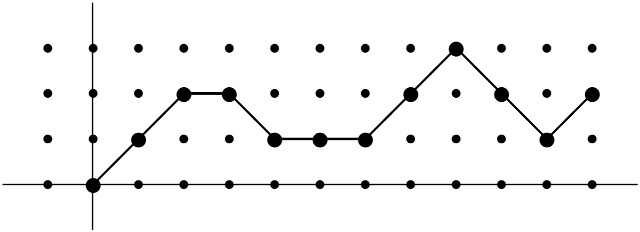
EXAMPLE. A Motzkin path of length 11.



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Given a two-rowed set-valued tableaux, construct a two-coloured Motzkin path from $(f, 0)$ to (n, t) step by step by letting $i = 1, 2, \ldots, n$, in that order, and:

- if i is an entry in the first row and the smallest number in its cell, we add an up-step;
- if i is an entry in the second row and the smallest number in its cell, we add a down-step;
- if i is an entry in the first row but not the smallest number in its cell, we add an umber horizontal step;
- if i is an entry in the second row but not the smallest number in its cell, we add a denim horizontal step.

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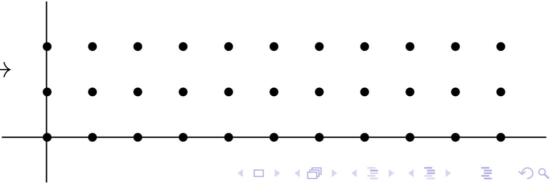

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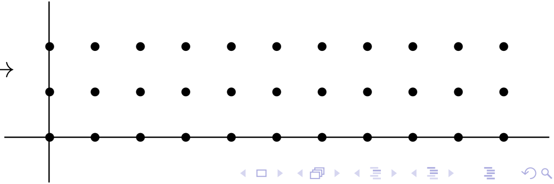

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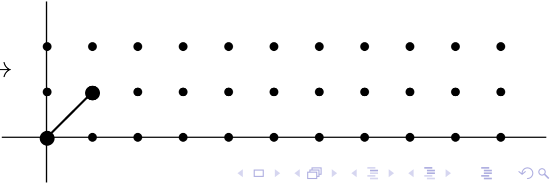

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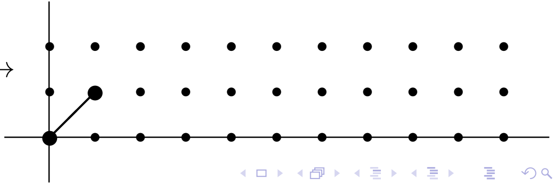

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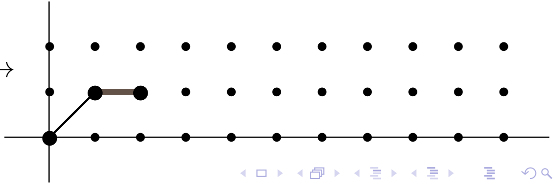

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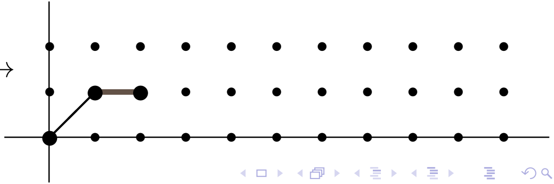

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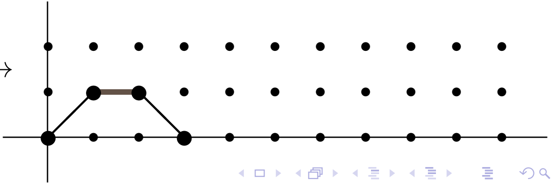

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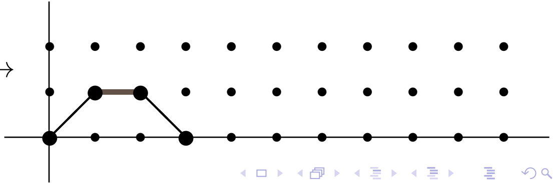

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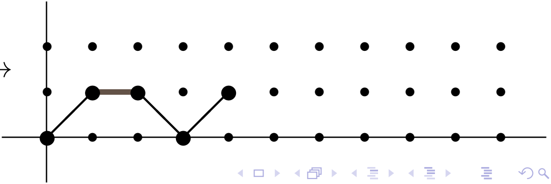

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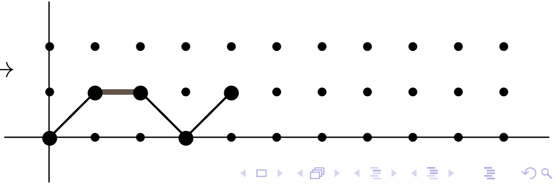

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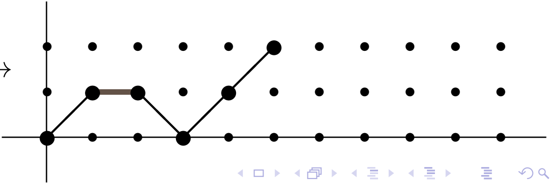

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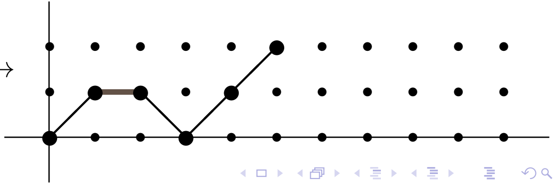

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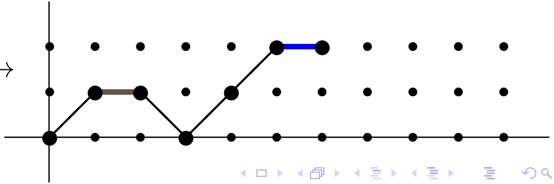

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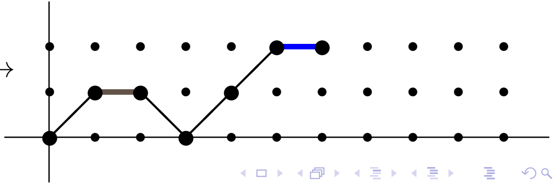

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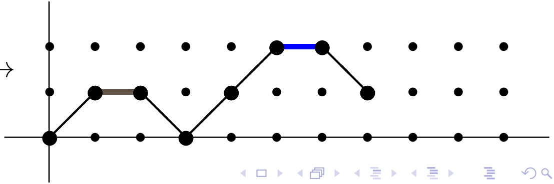

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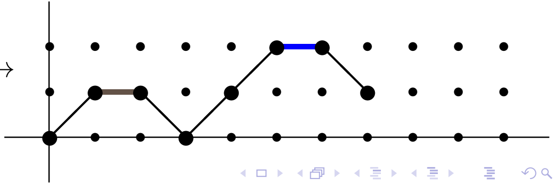

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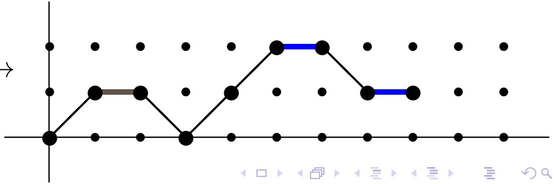

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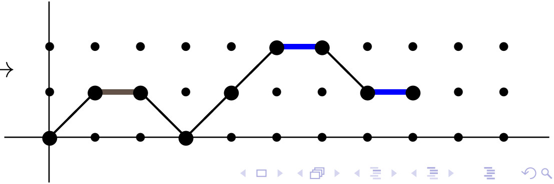

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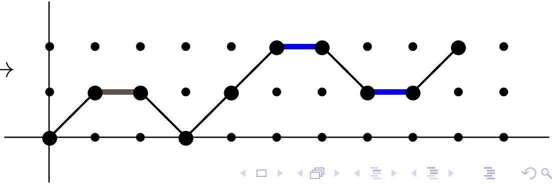

Two-Rowed Set-Valued Standard Tableaux and Motzkin Paths

Lemma

Set-valued standard tableaux of shape $(e+t, e)$ with $c+e+t$ entries in the first row and with $d+e$ entries in the second row are in bijection with two-coloured Motzkin paths from $(0,0)$ to (n,t) with c umber horizontal steps, with d denim horizontal steps, with $e+t$ up-steps, and with e down-steps, where an umber horizontal step does not occur before the first up-step and not at height 0, and a denim horizontal step does not occur before the first down-step.

Example. A set-valued tableau of shape $(4,3,3,1)$ with $n = 10$

1, 2	4	5	9
3, 6	7, 8	10	

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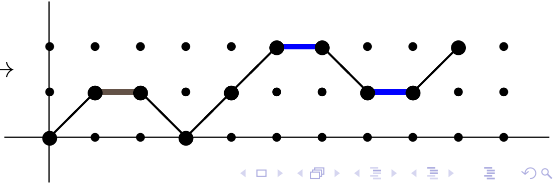

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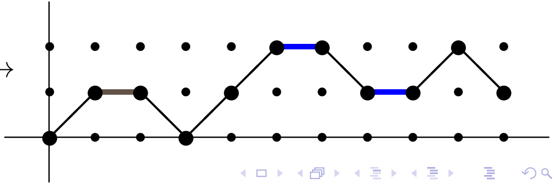

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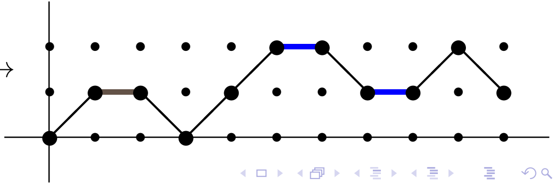

Two-Rowed Set-Valued Standard Tableaux and Motzkin Paths — How to Enumerate Them

- Enumeration problems for Motzkin paths are solved by the use of generating functions.

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- Extraction of coefficients from power series involving implicitly defined functions is solved by the — appropriate! — application of Lagrange inversion.

Refined Enumeration of Two-Rowed Set-Valued Standard Tableaux

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Theorem

Let n be a positive integer and t, c, d, e non-negative integers with $c + d + 2e + t = n$. The number of set-valued standard tableaux of shape $(e + t, e)$ with $c + e + t$ entries in the first row and with $d + e$ entries in the second row is given by

$$\chi(e = 0)\binom{n-1}{t-1} + \frac{(n-1)!}{(d+e)\cdot c!\, d!\, (e-1)!\, (e+t-1)!}$$
$$- \sum_{b=n-c-e+1}^{n-e+1} (-1)^{n-b-c-e-1} \frac{(n-b)\cdot(n-1)!}{b \cdot d!\, (b-1-d)!\, (e-1)!\, (n-b-e+1)!},$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. Here, and later, an expression containing a term $m!$ with $m < 0$ in the denominator has to be interpreted as 0.

Corollary

Let n be a positive integer and t, e non-negative integers. The number of set-valued standard tableaux of shape $(e+t, e)$ with n entries is given by

$$\chi(e=0)\binom{n-1}{t-1}+\sum_{d=0}^{n-2e-t}\frac{(n-1)!}{(n-d-2e-t)!\,d!\,(e-1)!\,(e+t-1)!}\cdot\left(\frac{1}{d+e}-\frac{n-d-e-t-1}{(d+e+t+1)(d+e+t)}\right).$$

Refined Enumeration of Two-Rowed Set-Valued Standard Tableaux

Corollary

Let n and m be positive integers and t a non-negative integer. The number of set-valued standard tableaux of shape $(e, e+t)$, for some non-negative integer e, with m entries in the first row and n entries in total is given by

$$\chi(m=n)\binom{n-1}{t-1} + \chi(m<n)\frac{t}{n-1}\binom{n}{m}\binom{n-1}{m-t-1} \\ + \frac{1}{n-1}\binom{n-1}{m}\binom{n-1}{m-t-1}.$$

Refined Enumeration of Two-Rowed Set-Valued Standard Tableaux

Corollary

Let n be a positive integer and t a non-negative integer. The number of set-valued standard tableaux of shape $(e, e+t)$, for some non-negative integer e, and n entries in total is given by

$$\binom{2n-2}{n-t-1} - \binom{2n-2}{n-t-2} + \binom{n-2}{t-2}.$$

Corollary

Let n be a positive integer and t a non-negative integer. The expected value of the length of the second row in set-valued standard tableaux with n entries with a (straight) shape where the first row is by t longer than the second row is given by

$$\frac{\binom{2n-4}{n-t-1} + (n-2)\binom{2n-4}{n-t-3} - (n+1)\binom{2n-4}{n-t-4} - \binom{n-3}{t-2}}{\binom{2n-2}{n-t-1} - \binom{2n-2}{n-t-2} + \binom{n-2}{t-2}}.$$

Motzkin Path Generating Functions

We consider generating functions of the form

$$\sum_{P\in\mathcal{P}} z^{\#(\text{steps of } P)} x^{\#(\text{umber horizontal steps of } P)} y^{\#(\text{denim horizontal steps of } P)} \cdot \alpha^{\#(\text{down-steps of } P)}$$

for the sets $\mathcal{P}$ of two-coloured Motzkin paths that are relevant here.

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Let $M(z) = M(x, y, \alpha; z)$ be the above generating function where $\mathcal{P}$ is the set of two-coloured Motzkin paths starting at $(0, 0)$ and ending on the x-axis.

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We have the following combinatorial decomposition:

$$\mathcal{M} = \{\emptyset\} \sqcup h_u\mathcal{M} \sqcup h_d\mathcal{M} \sqcup u\mathcal{M}d\mathcal{M},$$

where $\mathcal{M}$ is the set of these two-coloured Motzkin paths, u denotes an up-step, d denotes a down-step, h_u denotes an umber horizontal step, h_d denotes a denim horizontal step, and $\emptyset$ stands for the zero-length path from $(0, 0)$ to $(0, 0)$.

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This translates into the functional equation

$$M(z) = 1 + (x + y)zM(z) + \alpha z^2 M^2(z).$$

Let $\mathcal{M}_0$ denote the set of all two-coloured Motzkin paths in $\mathcal{M}$ with no umber horizontal step on the x-axis.

Motzkin Path Generating Functions

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The corresponding generating function is denoted by $M_0(z)$. We have

$$\mathcal{M}_0 = \{\emptyset\} \sqcup h_d \mathcal{M}_0 \sqcup u \mathcal{M} d \mathcal{M}_0.$$

This decomposition translates into the equation

$$M_0(z) = 1 + yzM_0(z) + \alpha z^2 M(z) M_0(z),$$

or, equivalently,

$$M_0(z) = \frac{1}{1 - yz - \alpha z^2 M(z)}.$$

Motzkin Path Generating Functions

The generating function for the two-coloured Motzkin paths starting at $(0,0)$ and ending at height t, where an umber horizontal step does not occur at height 0 and a denim horizontal step does not occur before the first down-step is given by

$$\left(\frac{z}{1-xz}\right)^t + \sum_{s\ge0}\left(\frac{z}{1-xz}\right)^{s+1} \alpha z\,(M(z)\alpha z)^s\,M_0(z)\,(M(z)z)^t$$
$$+\sum_{s=0}^{t-1}\left(\frac{z}{1-xz}\right)^{s+1} \alpha z\sum_{h=1}^{s}(M(z)\alpha z)^{s-h}\,M(z)\,(M(z)z)^{t-h}$$
$$+\sum_{s\ge t}\left(\frac{z}{1-xz}\right)^{s+1} \alpha z\sum_{h=1}^{t}(M(z)\alpha z)^{s-h}\,M(z)\,(M(z)z)^{t-h}.$$

This simplifies to:

Proposition

The generating function for the two-coloured Motzkin paths starting at $(0,0)$ *and ending at height* t*, where an umber horizontal step does not occur at height 0 and a denim horizontal step does not occur before the first down-step is given by*

$$\left(\frac{z}{1-xz}\right)^t+\frac{\alpha\,(zM(z))^{t+2}}{(1+yzM(z))(1+xzM(z))}\\+\frac{\alpha zM(z)\left((zM(z))^t-\left(\frac{z}{1-xz}\right)^t\right)}{(y+\alpha zM(z))\,(1+yzM(z))}.$$

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Equivalently, this is

$$\frac{zM(z)}{1 + (x+y)zM(z) + \alpha z^2 M^2(z)} = z.$$

In other words, the series $zM(z)$ is the compositional inverse of

$$\frac{z}{1 + (x+y)z + \alpha z^2}.$$

Lagrange Inversion

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Theorem (LAGRANGE INVERSION)

Let $f(z)$ be a formal power series with $f(0)=0$ and $f'(0)\neq 0$, and let $F(z)$ be its compositional inverse. If $g(z)$ is a Laurent series with only finitely many terms containing negative powers of z, then

$$\langle z^n\rangle\, g(F(z)) = \frac{1}{n}\left\langle z^{-1}\right\rangle g'(z) f^{-n}(z).$$

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Lagrange inversion — Application

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Lagrange inversion says: if $f(z)$ and $F(z)$ are compositional inverses of each other, then

$$\langle z^n \rangle\, g(F(z)) = \left\langle z^{-1} \right\rangle g(z) f^{-n-1}(z) f'(z).$$

We apply this with $F(z) = zM(z)$, $f(z) = \frac{z}{1+(x+y)z+\alpha z^2}$, and $g(z) = \frac{\alpha z^{t+2}}{(1+yz)(1+xz)}$.

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Let $f(z)$ be a formal power series with $f(0)=0$ and $f'(0)\neq 0$, and let $F(z)$ be its compositional inverse. If $g(z)$ is a Laurent series with only finitely many terms containing negative powers of z, then

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We have

$$f'(z) = \frac{1-\alpha z^2}{\left(1+(x+y)z+\alpha z^2\right)^2}.$$

Using the second version of the Lagrange inversion formula, we obtain

$$\begin{aligned}\langle z^n\rangle &\frac{\alpha\left(zM(z)\right)^{t+2}}{(1+xzM(z))(1+yzM(z))}\\ &=\left\langle z^{-1}\right\rangle \frac{\alpha z^{t+2}}{(1+xz)(1+yz)}\cdot\frac{\left(1+(x+y)z+\alpha z^2\right)^{n+1}}{z^{n+1}}\\ &\qquad\cdot\frac{1-\alpha z^2}{(1+(x+y)z+\alpha z^2)^2}\\ &=\left\langle z^{n-t-2}\right\rangle \frac{\alpha(1-\alpha z^2)\left(1+(x+y)z+\alpha z^2\right)^{n-1}}{(1+xz)(1+yz)}.\end{aligned}$$

Etc.

In the end this leads to:

Theorem

Let n be a positive integer and t, c, d, e non-negative integers with $c + d + 2e + t = n$. The number of set-valued standard tableaux of shape $(e + t, e)$ with $c + e + t$ entries in the first row and with $d + e$ entries in the second row is given by

$$\chi(e=0)\binom{n-1}{t-1} + \frac{(n-1)!}{(d+e)\cdot c!\, d!\, (e-1)!\, (e+t-1)!}$$
$$-\sum_{b=n-c-e+1}^{n-e+1} (-1)^{n-b-c-e-1} \frac{(n-b)\cdot(n-1)!}{b\cdot d!\,(b-1-d)!\,(e-1)!\,(n-b-e+1)!},$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. Here, and later, an expression containing a term $m!$ with $m < 0$ in the denominator has to be interpreted as 0.

Set-Valued Standard Tableaux of Skew Shape

Set-valued standard tableaux can be also defined for skew shapes. For example, here is a set-valued standard tableau of shape $(4,3,3,1)/(2,1,0,0)$.

		$3,8$	12
	$2,9,10,11$	14	
$1,4,5$	13	15	
$6,7$			

Theorem

Let n and f be positive integers and t, c, d, e non-negative integers with $c + d + 2e - f + t = n$.
If $t < f$, the number of set-valued standard tableaux of shape $(e + t, e)/(f, 0)$ with $c + e + t - f$ entries in the first row and with $d + e$ entries in the second row is given by

$$\chi(t = 0 \text{ and } e = f)\binom{n-1}{f-1}$$
$$-\frac{(c+e-f+t)\cdot(n-1)!}{(c+e-f)(c+e+t)\cdot c!\,d!\,(e-f-1)!\,(e+t-1)!}$$
$$+\frac{n!}{(c+e-f+t)(d+e)\cdot c!\,d!\,(e-1)!\,(e-f+t-1)!}$$
$$+\frac{t\cdot(n-1)!\,(n-d-f-1)!}{(n-d-e)(c+e-f)\cdot c!\,d!\,(n-d-1)!\,(e-f-1)!\,(e-f+t-1)!}$$
$$+\sum_{b=n-c-e+f+1}^{n-e+f+1}(-1)^{n-b-c-e+f}\frac{(n-b)\cdot(n-1)!}{b\cdot d!\,(b-1-d)!\,(e-f-1)!\,(n-b-e+f+1)!}.$$

If $t \geq f$, then this number is given by

$$\chi(d=e=0)\binom{n-1}{t-f-1}$$
$$+\frac{n!}{(c+e-f+t)(d+e)\cdot c!\,d!\,(e-1)!\,(e-f+t-1)!}$$
$$-\frac{(n-1)!}{(c+e+t)\cdot c!\,d!\,(e-f-1)!\,(e+t-1)!}$$
$$+\sum_{b=n-c-e+f+1}^{n-e+f+1}(-1)^{n-b-c-e+f}$$
$$\cdot\frac{(n-b)\cdot(n-1)!}{b\cdot d!\,(b-1-d)!\,(e-f-1)!\,(n-b-e+f+1)!}.$$

Theorem

Let n and f be positive integers and t a non-negative integer. If $t < f$, the number of set-valued standard tableaux of shape $(e+t,e)/(f,0)$, for some non-negative integer e, with a total number of n entries is given by

$$\chi(t=0)\binom{n-1}{f-1}+\binom{2n-2}{n-f+t-1}-\binom{2n-3}{n-f-1}$$
$$-\binom{2n-2}{n-f-t-2}+\binom{2n-3}{n-f-t-1}$$
$$+\sum_{k\ge f-t}(-1)^{k-f+t}\binom{k-1}{f-t-1}\left(-\binom{2n+k-f+t-3}{n-k-1}\right.$$
$$\left.+\binom{2n+k-f+t-3}{n-k-2}+\binom{2n+k-f+t-3}{n-k-t-1}-\binom{2n+k-f+t-3}{n-k-2t-1}\right).$$

If $t \ge f$, then this number is given by

$$\binom{n-1}{t-f-1} + 2\binom{2n-3}{n+f-t-2} - \binom{2n-2}{n-f-t-2}$$
$$+ \sum_{k \ge t-f+1} (-1)^{k-t-f-1} \binom{k-1}{t-f-1}$$
$$\cdot \left(\binom{2n+k+f-t-3}{n-k-1} - \binom{2n+k+f-t-3}{n-k-2} \right).$$

Corollary

The number of set-valued standard tableaux of shape $(e+t, e)/(t, 0)$, *for some non-negative integer* e, *with a total number of* n *entries is given by*

$$2\binom{2n-3}{n-2} - \binom{2n-2}{n-2t-2}.$$