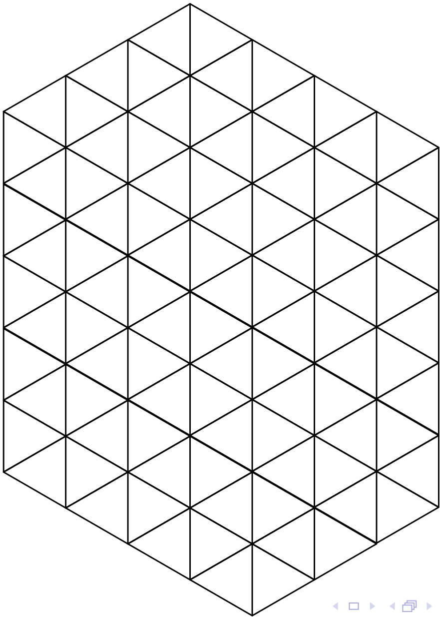
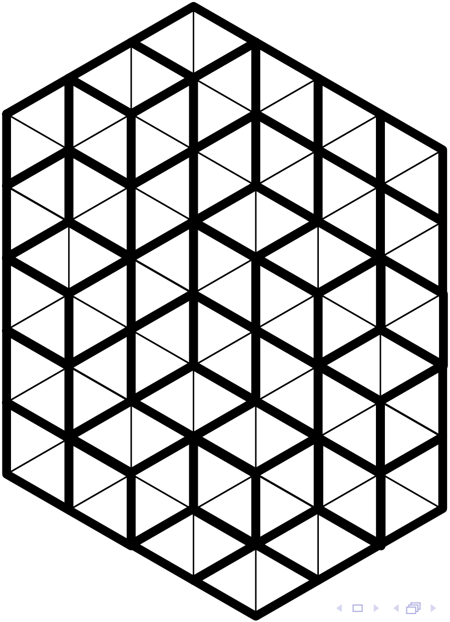
Enumeration of Tilings

Christian Krattenthaler

Universität Wien

Rhombus Tilings of a Hexagon





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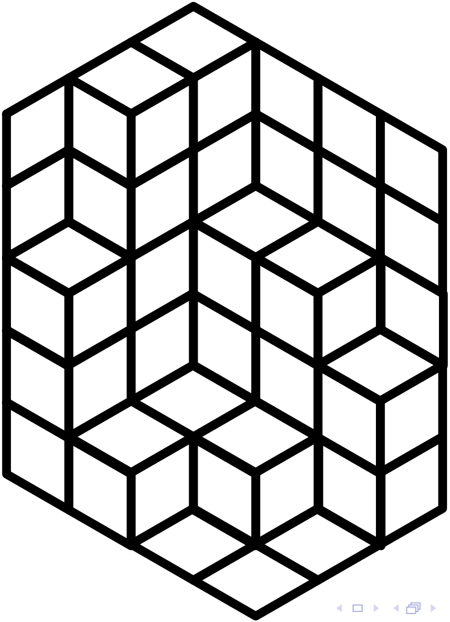

Theorem (MacMahon 1915)

Let a, b, c be positive integers. The number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c is given by

$$\prod_{i=1}^{c} \frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}.$$

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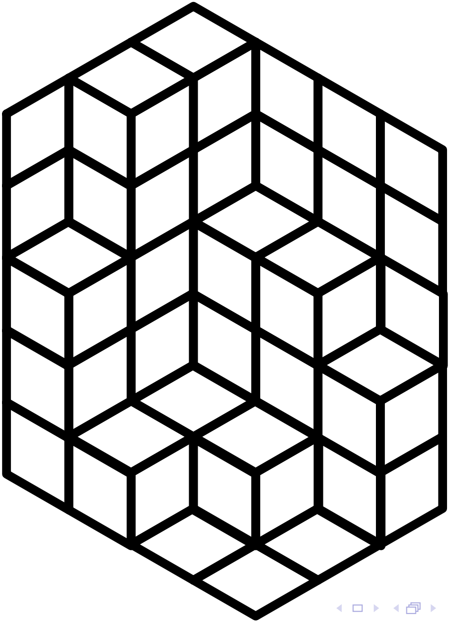

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Domino Tilings of a Rectangle

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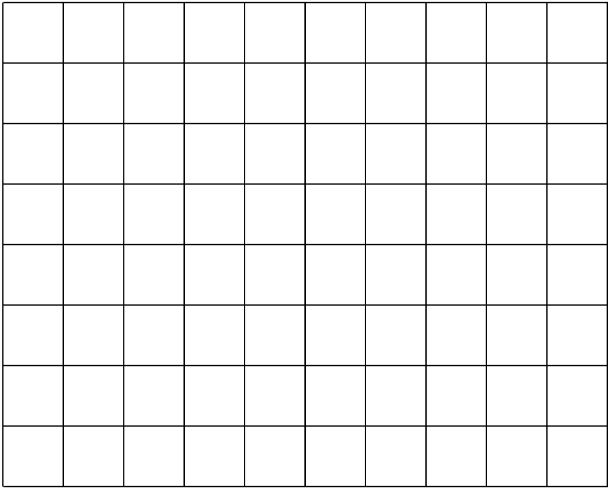

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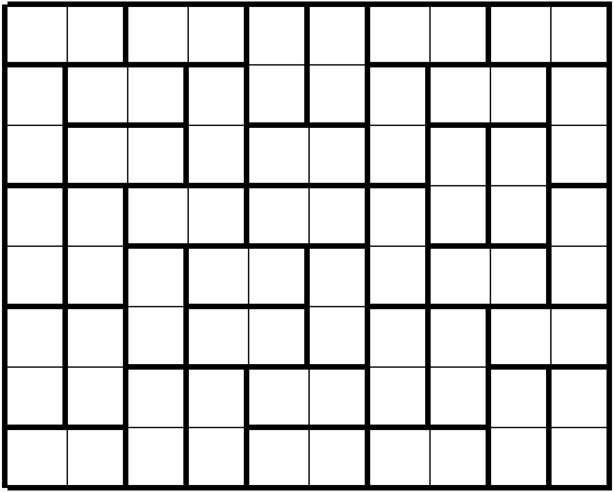

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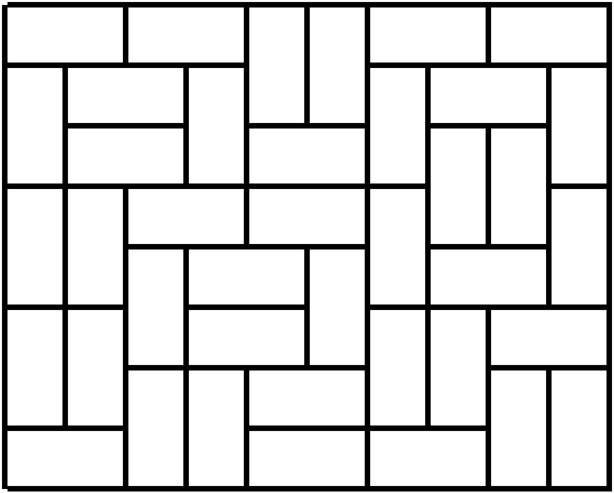

Domino Tilings of a Rectangle

Theorem (Kasteleyn 1961, Temperley and Fisher 1961)

Let m and n be positive integers, with m being even. The number of domino tilings of an $m \times n$-rectangle is given by

$$2^{mn/2} \prod_{i=1}^{m/2} \prod_{j=1}^{n} \left(\cos^2 \frac{\pi i}{m+1} + \cos^2 \frac{\pi j}{n+1} \right)^{1/2}.$$

Domino Tilings of the Aztec Diamond

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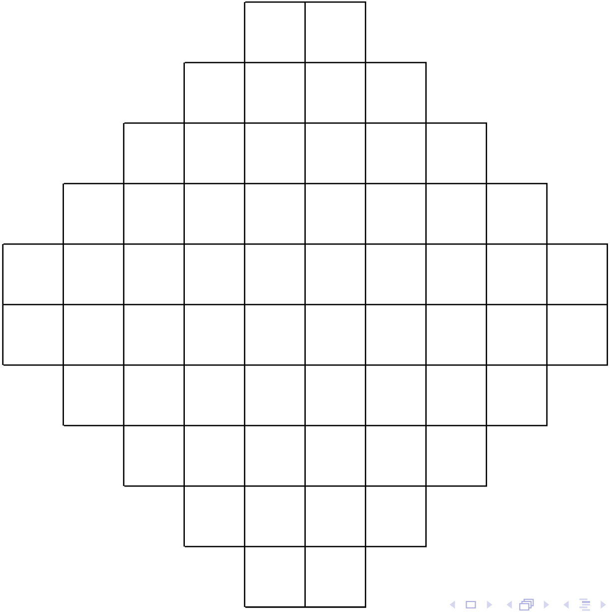
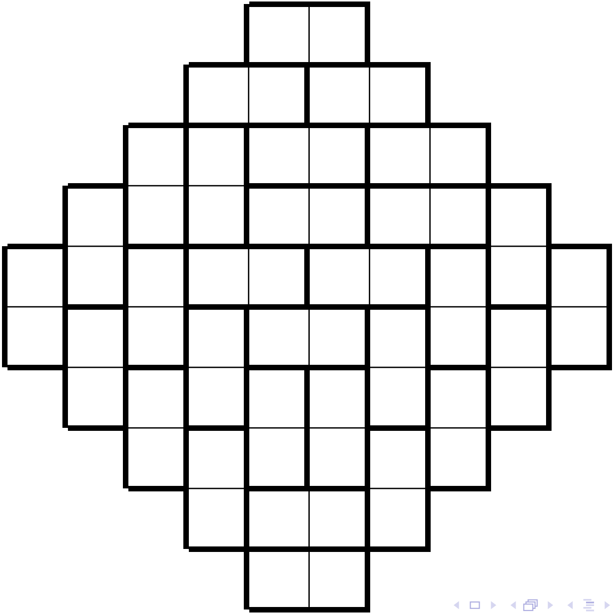


Domino Tilings of the Aztec Diamond



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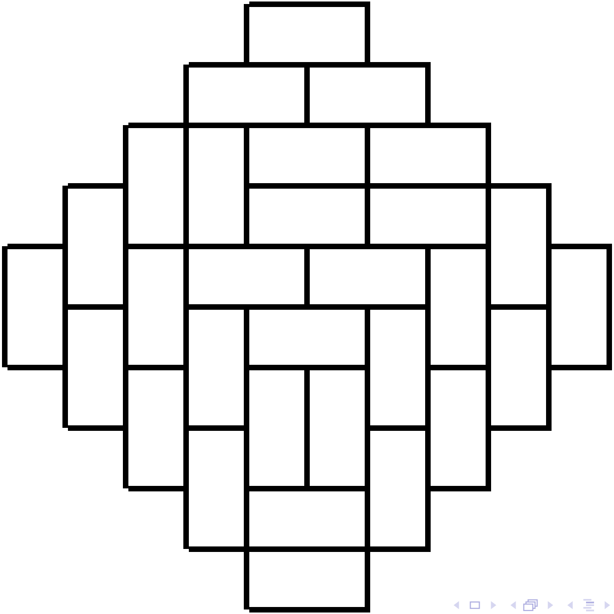

The Aztec Diamond Theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

Proof Methods I: Non-Intersecting Paths

Theorem (MacMahon 1915)

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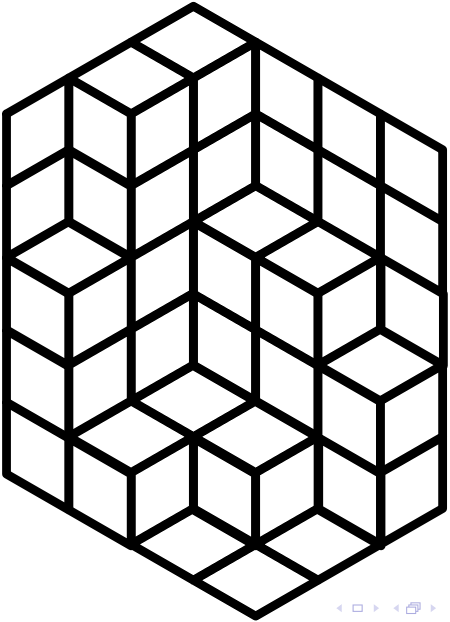

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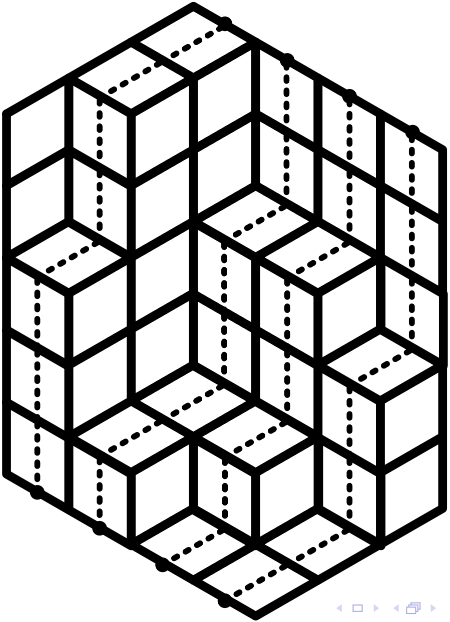

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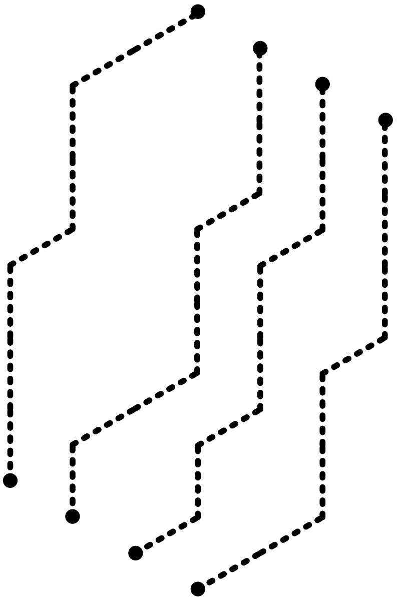

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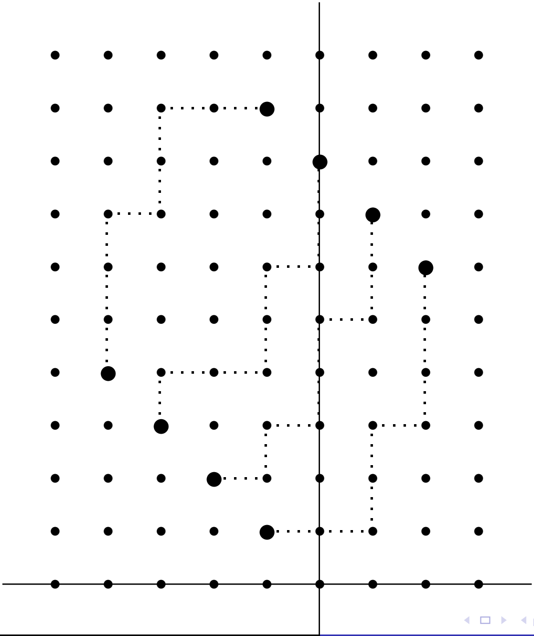

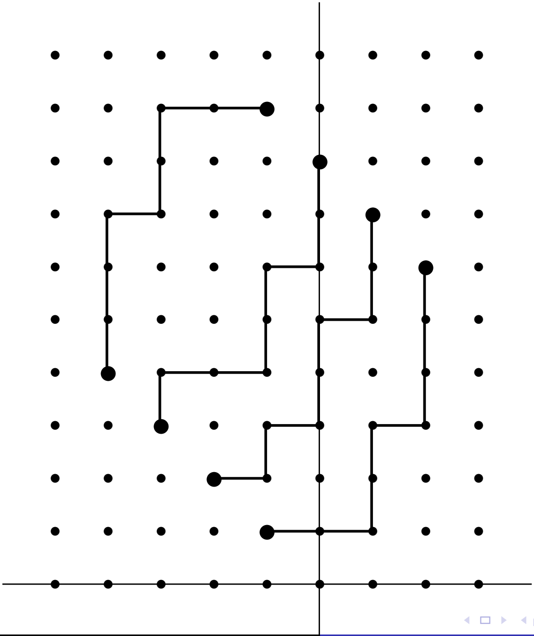

Proof Methods I: Non-Intersecting Paths

Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \dots, P_n)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \dots, n$, is given by

$$\det_{1\le i,j\le n}\left(|\mathcal{P}(A_j \to E_i)|\right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

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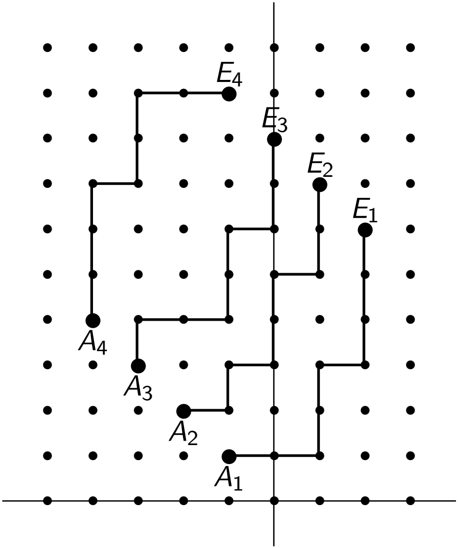

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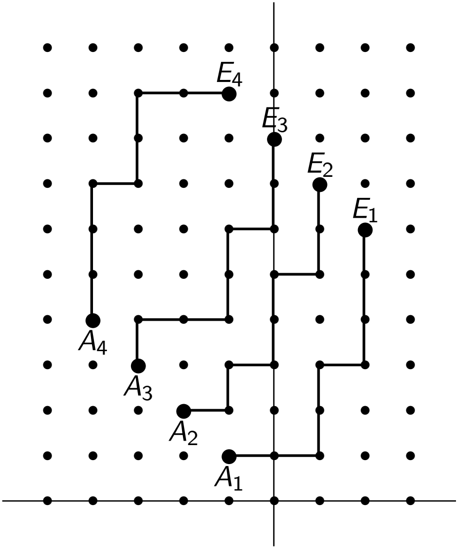

We have $A_i = (-i, i)$ and $E_i = (a - i, c + i)$, $i = 1, 2, \ldots, b$.

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The resulting determinant in our case:

$$\det_{1\le i,j\le c}\left(\binom{a+b}{a-i+j}\right).$$

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The result is:

Theorem (MacMahon 1915)

The number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c whose angles are 120° is equal to

$$\prod_{i=1}^{c}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}.$$

Proof Methods I: Non-Intersecting Paths

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

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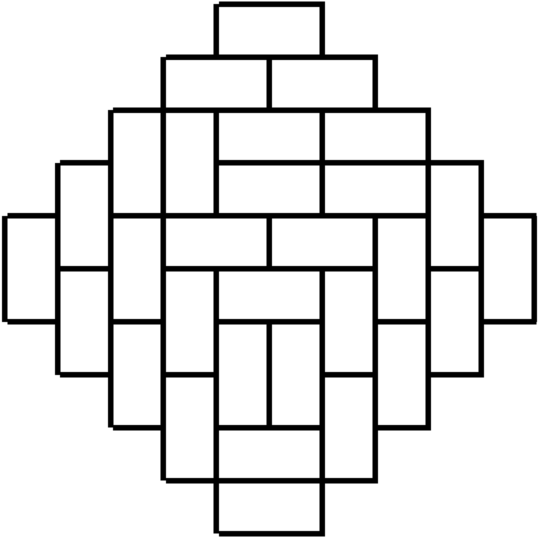

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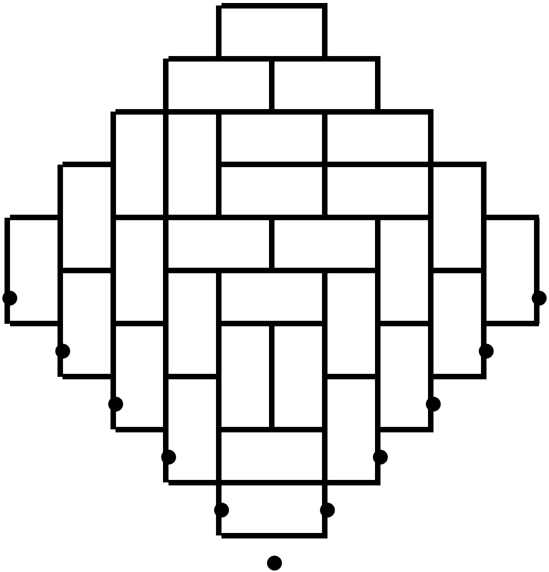

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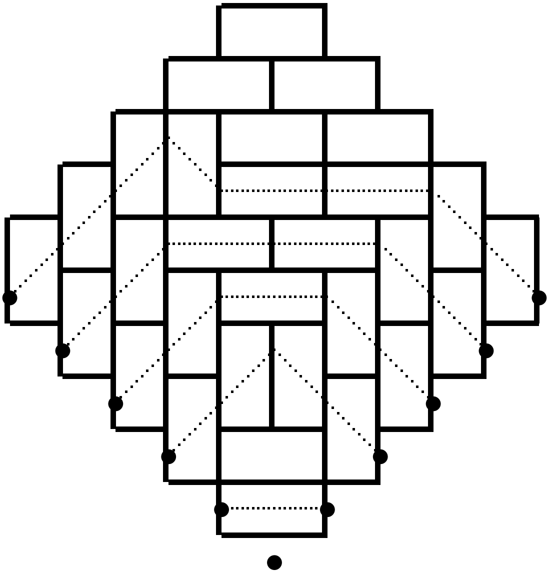

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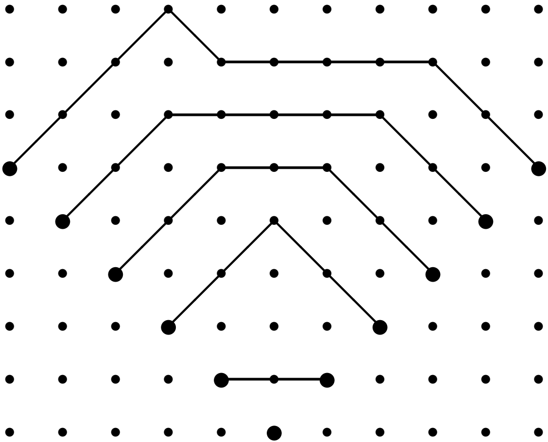

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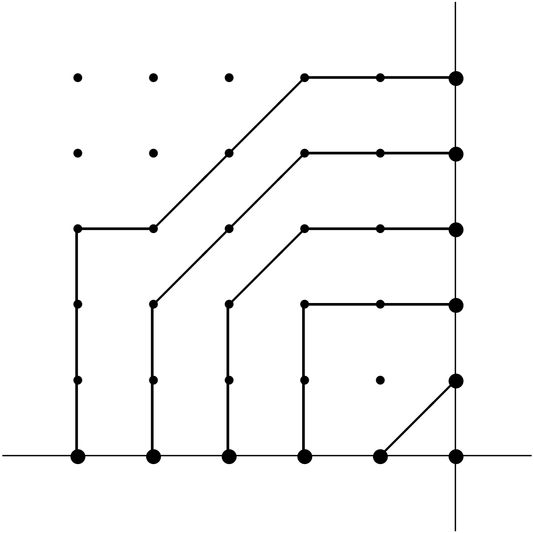

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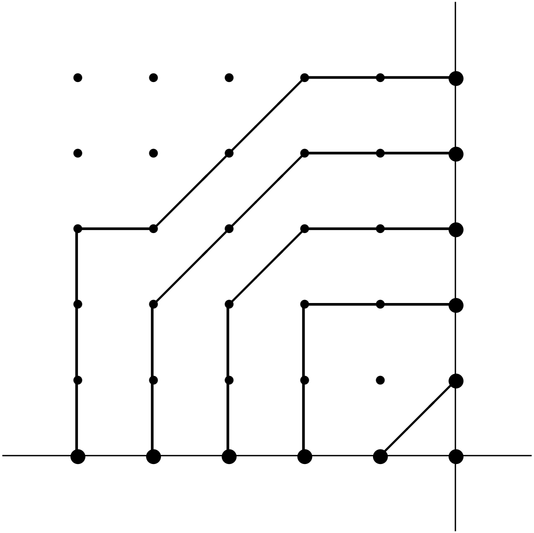
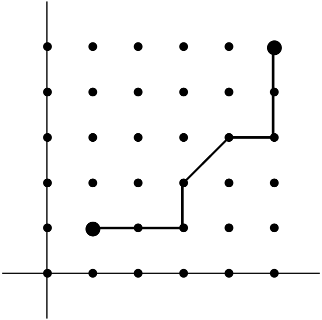


These are Delannoy paths.

Proof Methods I: Non-Intersecting Paths

Delannoy paths are paths on $\mathbb{Z}^2$ using only steps $(1,0)$, $(0,1)$ or $(1,1)$.



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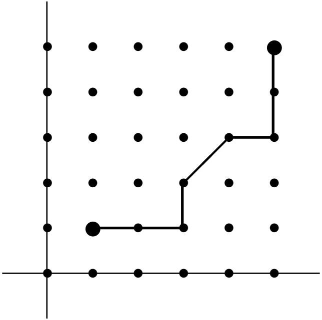

Let $D(i,j)$ denote the number of Delannoy paths from $(0,0)$ to (i,j).

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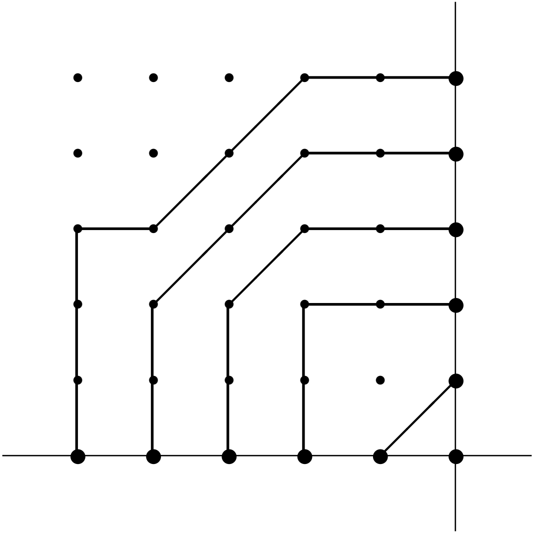

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Proof Methods I: Non-Intersecting Paths

By the Lindström–Gessel–Viennot theorem on non-intersecting lattice paths, we obtain

$$\mathrm{M}(AD_n) = \det \big(D(i,j)\big)_{0 \le i,j \le n},$$

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Claim. We have

$$\left(D(i,j)\right)_{0\le i,j\le n} = L_n \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 4 & & \\ & & & \ddots & \\ & & & & 2^n \end{pmatrix} L_n^t,$$

with $L_n = \left(\binom{i}{j}\right)_{0\le i,j\le n}$.

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The claim is equivalent to $D(i,j) = \sum_{k\ge 0}\binom{i}{k}\binom{j}{k}2^k$.

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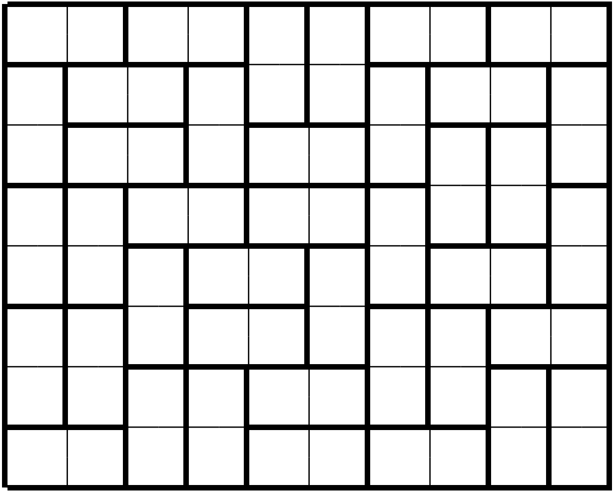
Proof Methods II: Kasteleyn Matrices

domino tilings



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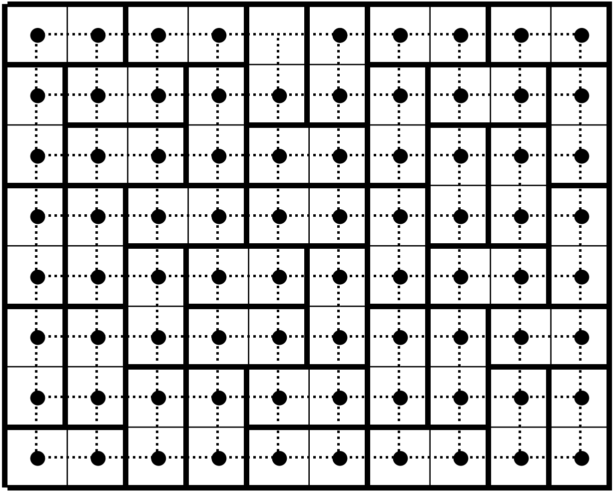
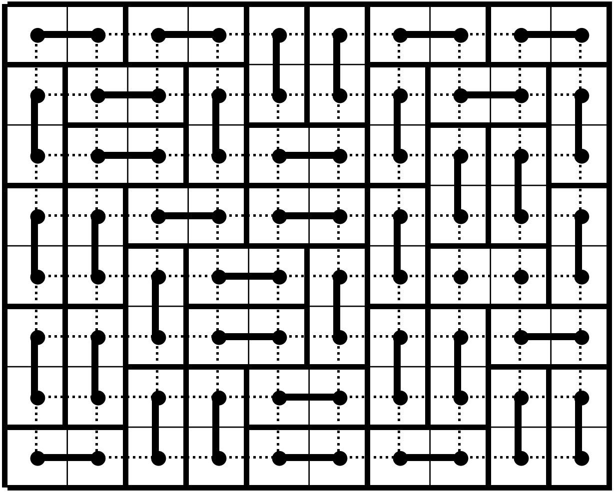


Proof Methods II: Kasteleyn Matrices

domino tilings — perfect matchings



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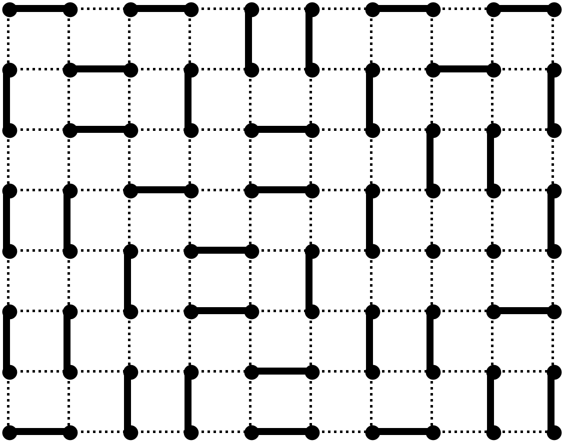
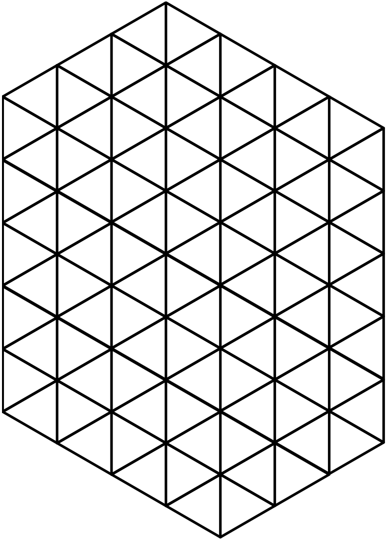


rhombus tilings



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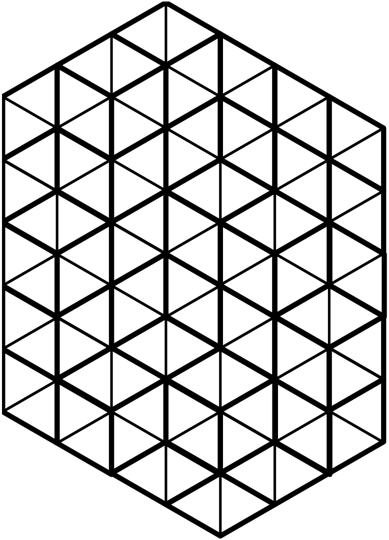

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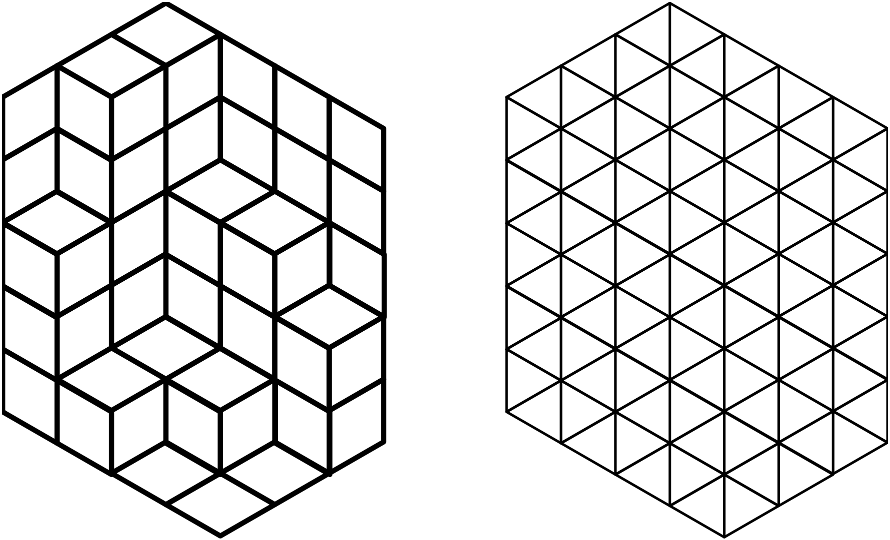

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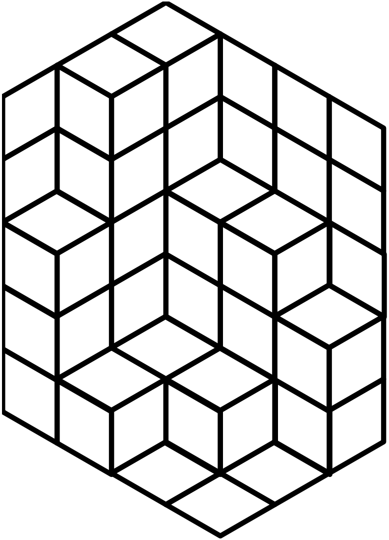

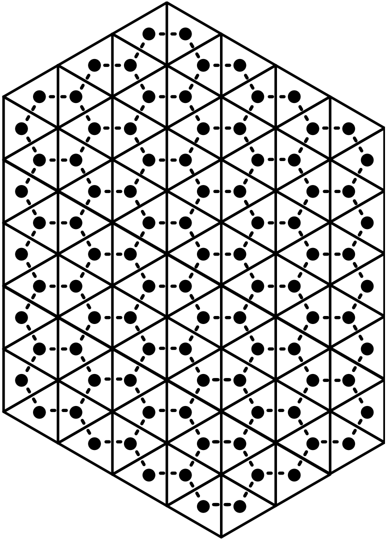

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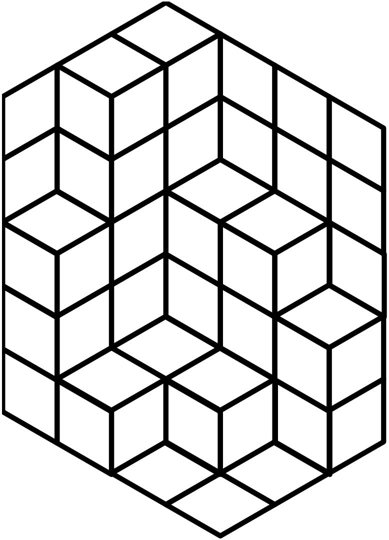

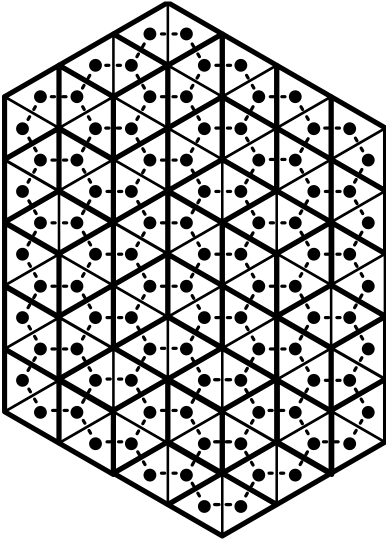

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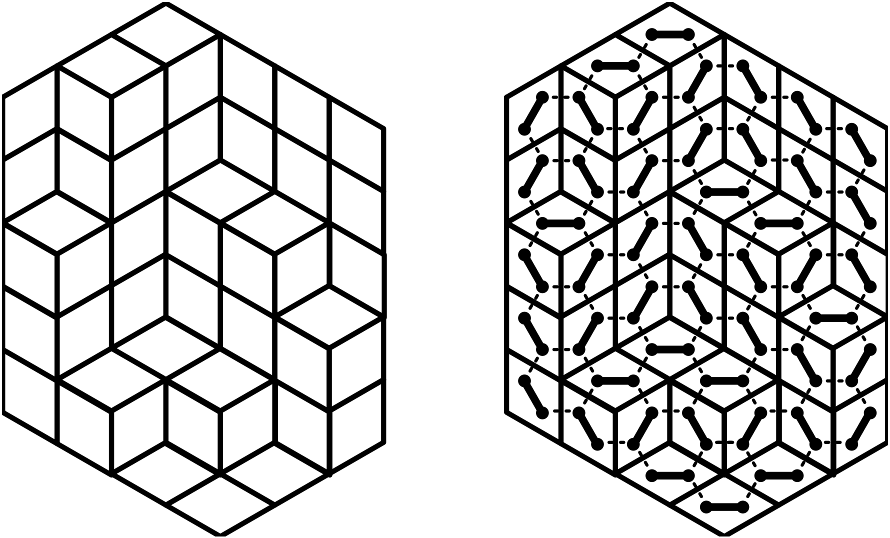

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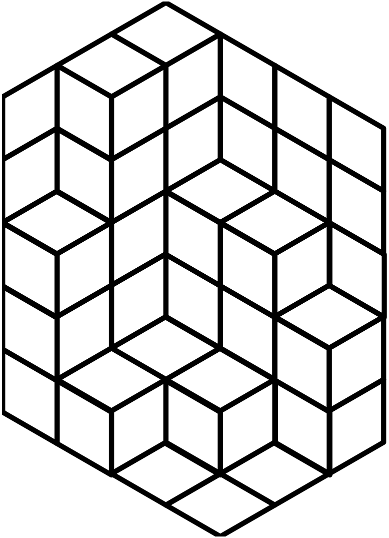
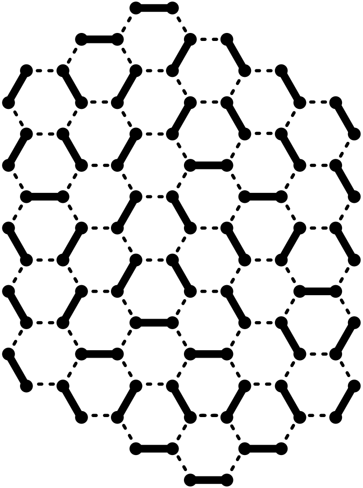


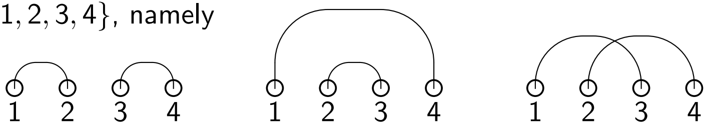


Proof Methods II: Kasteleyn Matrices

The Pfaffian of an upper triangular array $A = (A_{ij})_{1 \le i < j \le 2n}$ is defined by

$$\mathrm{Pf}(A) := \sum_{\pi \text{ a perfect matching of } \{1,2,\dots,2n\}} \operatorname{sgn} \pi \prod_{\{i,j\} \in \pi} A_{ij}.$$

For example, for $n = 2$ there are three perfect matchings of $\{1, 2, 3, 4\}$, namely



and so

$$\mathrm{Pf}\left((A_{ij})_{1 \le i < j \le 4}\right) = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}.$$

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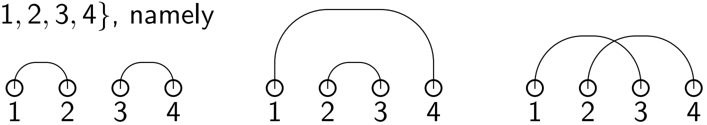

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Fact. $\operatorname{Pf}^2(A) = \det(\tilde{A})$, where $\tilde{A}$ is the completion of A to a skew-symmetric matrix.

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Let G be a graph with $2n$ vertices, with adjacency triangle $(A_{i,j})_{1\le i<j\le 2n}$. The Kasteleyn triangle arises from the adjacency triangle of G by switching some of the signs of entries in a certain systematic manner.

Let G be planar. A *Kasteleyn orientation* of the graph is one where around each face an odd number of edges is oriented clockwise. The corresponding Kasteleyn triangle $K = (K_{i,j})_{1\le i<j\le 2n}$ arises by

$$K_{i,j} := \begin{cases} A_{i,j}, & \text{if } i \to j, \\ -A_{i,j}, & \text{if } i \leftarrow j. \end{cases}$$

Theorem (Kasteleyn 1961)

The number of perfect matchings of G equals $|\operatorname{Pf} K|$.

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Lemma (Kasteleyn 1961)

A planar graph has always a Kasteleyn orientation.

Another fact.

$$\operatorname{Pf}\begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} = (-1)^{\binom{b}{2}} \det(B)$$

for any $b \times b$ matrix B.

Proof Methods II: Kasteleyn Matrices

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Given a planar bipartite graph G, let A be its bipartite adjacency matrix, that is, the rows of A are indexed by the black vertices $b_1, b_2, \dots$, and the columns are indexed by the white vertices, and

$$A_{b_i,w_j} = \begin{cases} 1 & \text{if } b_i \text{ and } w_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

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Corollary

The number of perfect matchings of G equals $|\det K|$.

Proof Methods II: Kasteleyn Matrices

In the case of the graph G being an $m \times n$-rectangle (with m even), it is possible to explicitly describe the eigenstructure (that is, eigenvalues and eigenvectors) of the corresponding Kasteleyn matrix.

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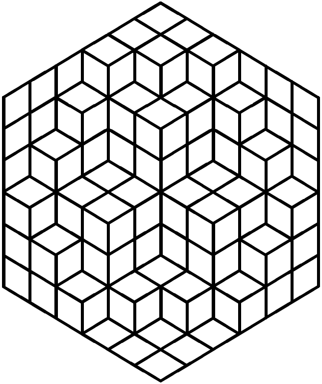
Theorem (KASTELEYN 1961, TEMPERLEY AND FISHER 1961)

Let m and n be positive integers, with m being even. The number of domino tilings of an $m \times n$-rectangle is given by

$$2^{mn/2} \prod_{i=1}^{m/2} \prod_{j=1}^{n} \left(\cos^2 \frac{\pi i}{m+1} + \cos^2 \frac{\pi j}{n+1} \right)^{1/2}.$$

Symmetry Classes of Rhombus Tilings of a Hexagon

Rhombus Tilings of a Hexagon can have several symmetries.



There are 10 possible ways to combine these symmetries and accordingly 10 symmetry classes of rhombus tilings.

Stanley proposed the programme of enumeration of the 10 symmetry classes of rhombus tilings.

Rather surprisingly, it turned out that for all the symmetry classes there are nice compact product formulae.

Class 1: *Unrestricted Rhombus Tilings*

Theorem (MacMahon 1915)

The number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c *is given by*

$$\prod_{i=1}^{c} \frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}.$$

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Class 2: *Vertically Symmetric Rhombus Tilings*

Theorem (Andrews 1978)

The number of all vertically symmetric rhombus tilings of a hexagon with side lengths a, a, c, a, a, c is given by

$$\prod_{i=1}^{a} \frac{c+2i-1}{2i-1} \prod_{1\le i<j\le a} \frac{c+i+j-1}{i+j-1}.$$

Class 3: 120°-*Rotation-Invariant Rhombus Tilings*

Theorem (ANDREWS ∼ 1979)

The number of all rhombus tilings of a hexagon with all side lengths equal to a that are invariant under 120°*-rotation is given by*

$$\prod_{i=1}^{a}\frac{3i-1}{3i-2}\prod_{1\le i<j\le a}\frac{2i+j-1}{2i+j-2}\prod_{1\le i<j,k\le a}\frac{i+j+k-1}{i+j+k-2}.$$

Class 4: *Vertically Symmetric and* 120°*-Rotation-Invariant Rhombus Tilings*

Theorem (Stembridge 1995)

The number of all rhombus tilings of a hexagon with all side lengths equal to a that are verrtically symmetric and invariant under 120°*-rotation is given by*

$$\prod_{1 \le i \le j \le k \le a} \frac{i+j+k-1}{i+j+k-2}.$$

Class 5: 180°-*Rotation-Invariant Rhombus Tilings*

Theorem (Stanley 1986)

The number $R_5(a, b, c)$ *of all rhombus tilings of a hexagon with side lengths* a, b, c, a, b, c *that are invariant under* 180°*-rotation is given by*

$$\begin{aligned} R_5(2a, 2b, 2c) &= R_1(a, b, c)^2, \\ R_5(2a+1, 2b, 2c) &= R_1(a, b, c) R_1(a+1, b, c), \\ R_5(2a+1, 2b+1, 2c) &= R_1(a+1, b, c) R_1(a, b+1, c), \end{aligned}$$

where $R_1(a, b, c)$ *is the number of unrestricted rhombus tilings in a hexagon with side lengths* a, b, c, a, b, c.

Class 6: *Horizontally Symmetric Rhombus Tilings*

Theorem (Proctor 1984)

The number of all horizontally symmetric rhombus tilings of a hexagon with side lengths a, a, c, a, a, c *is given by*

$$\binom{c+a-1}{a-1} \prod_{1 \le i \le j \le a-2} \frac{2c+i+j+1}{i+j+1}.$$

Class 6: *Horizontally Symmetric Rhombus Tilings*

Theorem (PROCTOR 1984)

The number of all horizontally symmetric rhombus tilings of a hexagon with side lengths a, a, c, a, a, c is given by

$$\binom{c+a-1}{a-1}\prod_{1\le i\le j\le a-2}\frac{2c+i+j+1}{i+j+1}.$$

Class 7: *Vertically and Horizontally Symmetric Rhombus Tilings*

Theorem (PROCTOR 1983)

The number $R_7(a,b,c)$ of all vertically and horizontally symmetric rhombus tilings of a hexagon with side lengths a, b, c, a, b, c is given by

$$R_7(2a,2a,2c)=R_1(a,a,c),$$
$$R_7(2a+1,2a+1,2c)=R_1(a+1,a,c).$$

Class 8: *Horizontally Symmetric and* 120°*-Rotation-Invariant Rhombus Tilings*

Theorem (MILLS, ROBBINS, RUMSEY 1987)

The number of all rhombus tilings of a hexagon with all side lengths equal to $2a$ *that are horizontally symmetric and invariant under* 120°*-rotation is given by*

$$\prod_{i=0}^{a-1} \frac{(3i+1)!\,(6i)!\,(2i)!}{(4i+1)!\,(4i)!}.$$

Class 8: *Horizontally Symmetric and* 120°*-Rotation-Invariant Rhombus Tilings*

Theorem (Mills, Robbins, Rumsey 1987)

The number of all rhombus tilings of a hexagon with all side lengths equal to $2a$ *that are horizontally symmetric and invariant under* 120°*-rotation is given by*

$$\prod_{i=0}^{a-1} \frac{(3i+1)!\,(6i)!\,(2i)!}{(4i+1)!\,(4i)!}.$$

Class 9: 60°*-Rotation-Invariant Rhombus Tilings*

Theorem (Kuperberg 1994)

The number of all rhombus tilings of a hexagon with all side lengths equal to $2a$ *that are invariant under* 60°*-rotation is given by*

$$\prod_{i=0}^{a-1} \frac{(3i+1)!^2}{(a+i)!^2}.$$

Class 10: *Totally Symmetric Rhombus Tilings*

Theorem (Andrews 1994)

The number of all rhombus tilings of a hexagon with all side lengths equal to 2a that have all possible symmetries is given by

$$\prod_{i=0}^{a-1} \frac{(3i+1)!}{(a+i)!}.$$

How have these results been obtained?

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$\longrightarrow$ Non-intersecting paths and determinant evaluations.

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Class 1:

$$\det_{1\le i,j\le c}\left(\binom{a+b}{a-i+j}\right)=\prod_{i=1}^{c}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}.$$

Class 8:

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right)$$

$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n-1}{2}}\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor (i+1)/2\rfloor}\,\left(-\mu-3n+i+\frac{3}{2}\right)_{\lfloor i/2}}{(i)_i}$$

where $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m>0$ and $(\alpha)_0 := 1$, and where $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise.

Class 3:

$$\det_{0\le i,j\le n-1}\left(\delta_{ij}+\binom{2\mu+i+j}{j}\right)$$

$$=\begin{cases} 2^{\lceil n/2\rceil}\displaystyle\prod_{i=1}^{n-2}(\mu+\lceil i/2\rceil+1)_{\lfloor (i+3)/4\rfloor} \\ \quad\times\dfrac{\prod_{i=1}^{n/2}\left(\mu+\frac{3n}{2}-\lceil\frac{3i}{2}\rceil+\frac{3}{2}\right)_{\lceil i/2\rceil-1}\left(\mu+\frac{3n}{2}-\lceil\frac{3i}{2}\rceil+\frac{3}{2}\right)_{\lceil i/2\rceil}}{\prod_{i=1}^{n/2-1}(2i-1)!!\,(2i+1)!!} & \text{if } n \text{ is even,}\\ 2^{\lceil n/2\rceil}\displaystyle\prod_{i=1}^{n-2}(\mu+\lceil i/2\rceil+1)_{\lfloor (i+3)/4\rfloor} \\ \quad\times\dfrac{\prod_{i=1}^{(n-1)/2}\left(\mu+\frac{3n}{2}-\lceil\frac{3i-1}{2}\rceil+1\right)_{\lceil (i-1)/2\rceil}\left(\mu+\frac{3n}{2}-\lceil\frac{3i}{2}\rceil\right)_{\lceil i/2\rceil}}{\prod_{i=1}^{(n-1)/2}(2i-1)!!^2} & \text{if } n \text{ is odd.}\end{cases}$$

Symmetry Classes of Domino Tilings of an Aztec Diamond?

There are 5 symmetry classes, but only 3 of them have nice product formulae. These were established by Ciucu and Yang.

In his proofs, Ciucu uses his matchings factorisation theorem.

Proof Methods III: Ciucu's Matchings Factorisation Theorem

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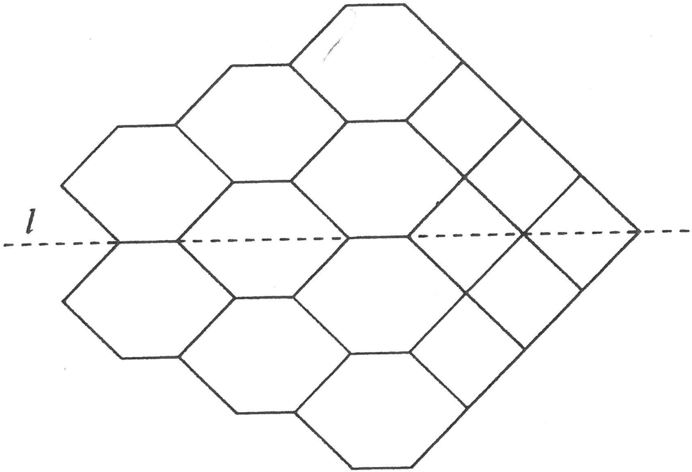
Let G be a bipartite weighted symmetric graph separated by its symmetry axis.

Proof Methods III: Ciucu's Matchings Factorisation Theorem



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Label the vertices along the symmetry axis $a_1, b_1, a_2, b_2, \dots$.
Fix the bipartition so that the left-most vertex on the symmetry axis is white.
Now delete the edges above all white a_i's and black b_i's and below all black a_i's and white b_i's.

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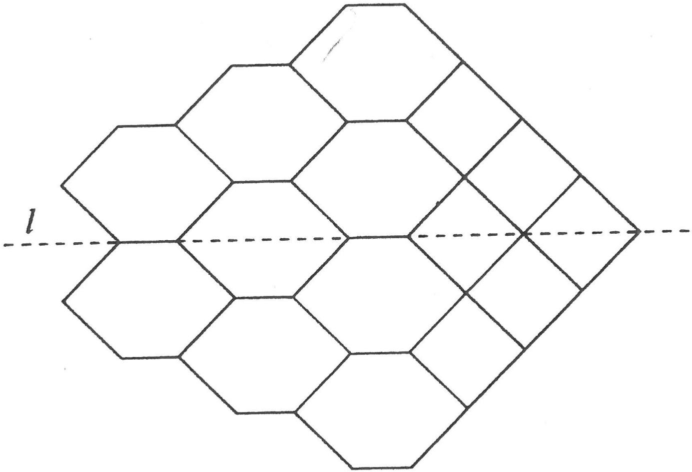

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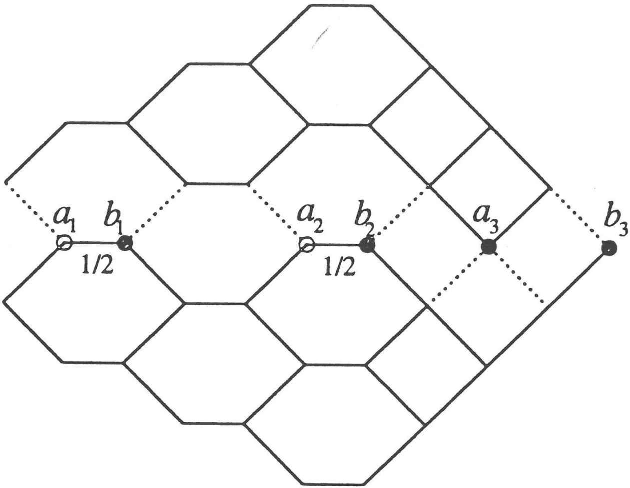

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Now delete the edges above all white a_i's and black b_i's and below all black a_i's and white b_i's.
Let G^+ be the upper part and G^- the lower part, in both cases with weight $1/2$ on the edges that lie on the symmetry axis.

Theorem (CIUCU 1997)

With the above assumptions, we have

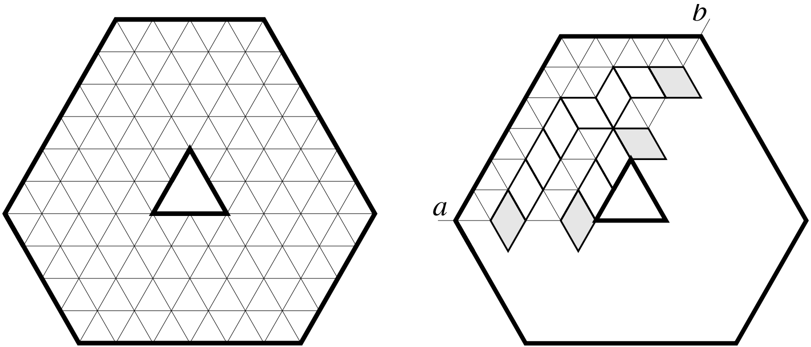
$$M(G) = 2^{w(G)} M(G^+) M(G^-),$$

where $M(H)$ denotes the (weighted) number of perfect matchings of H, and $w(G)$ is half the number of vertices of G along the symmetry axis.

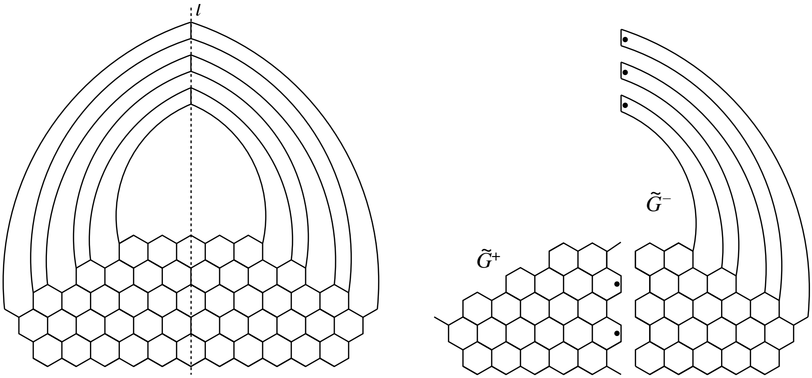
Proof Methods III: Ciucu's Matchings Factorisation Theorem

Application: A proof of the product formula for Class 3 (rhombus tilings that are invariant under 120°-rotation).

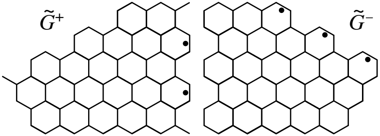
Cyclically symmetric tilings

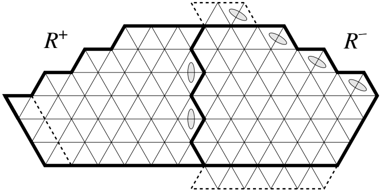


Application of the factorisation theorem

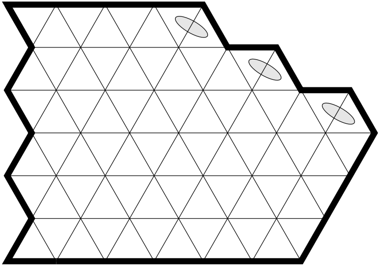


The resulting regions

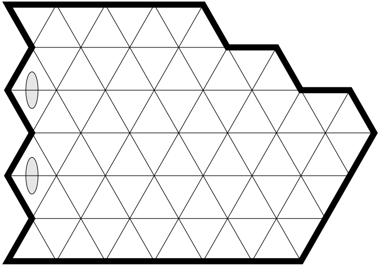




Two special regions on the triangular grid



$A_{3,4}$



$B_{3,4}$

Proof Methods III: Ciucu's Matchings Factorisation Theorem

Using non-intersecting lattice paths, determinantal formulae can be obtained for both (weighted) numbers of matchings:

$$M(A_{n,x}) = \frac{1}{2^n} \det_{0\le i,j\le n-1} \left(\frac{(x+i+j-1)!}{(x+2i-j)!\,(2j-i)!} \right) \prod_{i=0}^{n-1} (x+3i),$$

$$M(B_{n,x}) = \frac{1}{2^n} \det_{0\le i,j\le n-1} \left(\frac{(x+i+j-1)!}{(x+2i-j)!\,(2j-i)!} \right) \prod_{i=0}^{n-1} (2x+3i).$$

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A more general determinant had been earlier evaluated (C.K. 1997)

$$\det_{0\le i,j\le n-1} \left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!} \right) = \prod_{i=0}^{n-1} \frac{i!\,(x+y+i-1)!\,(2x+y+2i)_i\,(x+2y+2i)_i}{(x+2i)!\,(y+2i)!}.$$

Theorem (ANDREWS ∼ 1979)

The number of all rhombus tilings of a hexagon with all side lengths equal to a that are invariant under 120°*-rotation is given by*

$$\prod_{i=1}^{a} \frac{3i-1}{3i-2} \prod_{1\le i<j\le a} \frac{2i+j-1}{2i+j-2} \prod_{1\le i<j,k\le a} \frac{i+j+k-1}{i+j+k-2}.$$

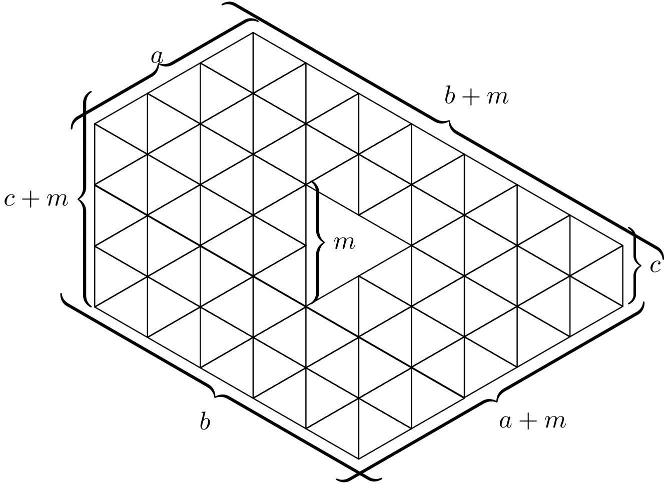
Propp's Open Problems

In 1996, Jim Propp circulated a manuscript of the title

> *"Twenty Open Problems on Enumeration of Matchings"*.

This inspired a lot of work on rhombus and domino tilings of regions "with defects."

Rhombus Tilings with Defects



Theorem (Ciucu, Eisenkölbl, C.K., Zare 2001)

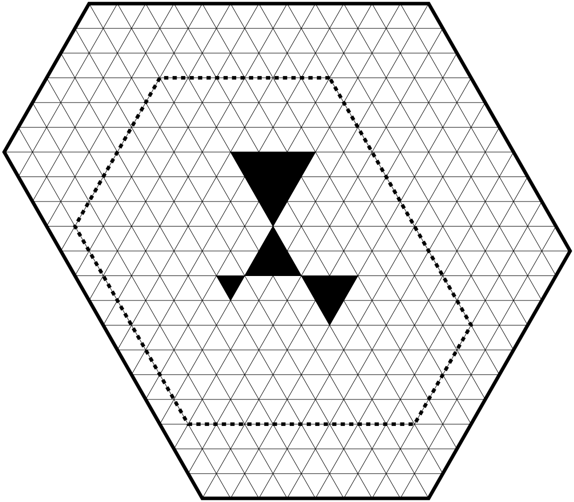
] *Let a, b, c, m be non-negative integers, a, b, c of the same parity. The number of rhombus tilings of a hexagon with side lengths $a, b+m, c, a+m, b, c+m$, where a triangular hole of size m is removed from the centre is equal to*

$$\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)}$$
$$\times\frac{\mathrm{H}(m+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(m+\lfloor\frac{a+b+c}{2}\rfloor)}{\mathrm{H}(\frac{a+b}{2}+m)\,\mathrm{H}(\frac{a+c}{2}+m)\,\mathrm{H}(\frac{b+c}{2}+m)}$$
$$\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{a+c+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2})\,\mathrm{H}(\frac{a+c}{2})\,\mathrm{H}(\frac{b+c}{2})},$$

où

$$\mathrm{H}(n) := \begin{cases} \prod_{k=0}^{n-1}\Gamma(k+1) & \text{for } n \text{ integral,}\\ \prod_{k=0}^{n-\frac{1}{2}}\Gamma(k+\frac{1}{2}) & \text{for } n \text{ half-integral.}\end{cases}$$

The region $SC_{x,y,z}(a,b,c,m)$:



Proof Methods IV: Kuo Condensation

Theorem (Kuo 2006)

Let $G = (V_1, V_2, E)$ be a planar bipartite graph in which $|V_1| = |V_2|$. Let vertices α, β, γ and δ appear cyclically on a face of G. If $\alpha, \gamma \in V_1$ and $\beta, \delta \in V_2$, then

$$\begin{aligned}&\mathrm{M}(G)\,\mathrm{M}(G - \{\alpha, \beta, \gamma, \delta\})\\ &= \mathrm{M}(G - \{\alpha, \beta\})\,\mathrm{M}(G - \{\gamma, \delta\}) + \mathrm{M}(G - \{\alpha, \delta\})\,\mathrm{M}(G - \{\beta, \gamma\}),\end{aligned}$$

where $\mathrm{M}(H)$ denotes the number of perfect matchings of the bipartite graph H.

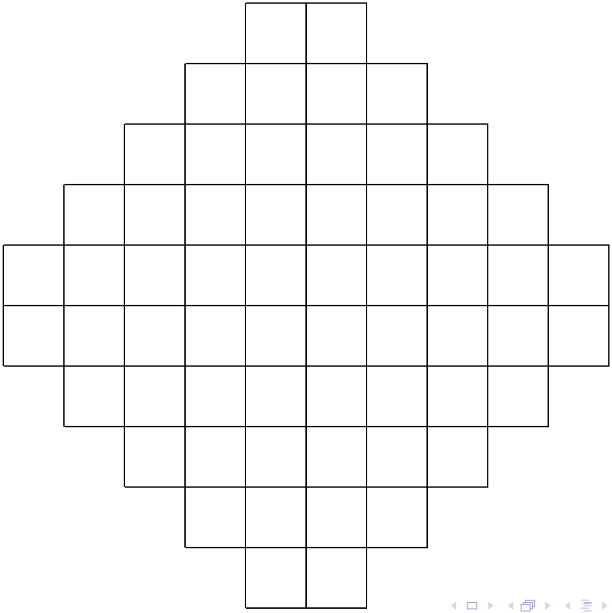
Application: Another proof of the Aztec diamond theorem.

Theorem (ELKIES, KUPERBERG, LARSEN, PROPP 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

Proof Methods IV: Kuo Condensation



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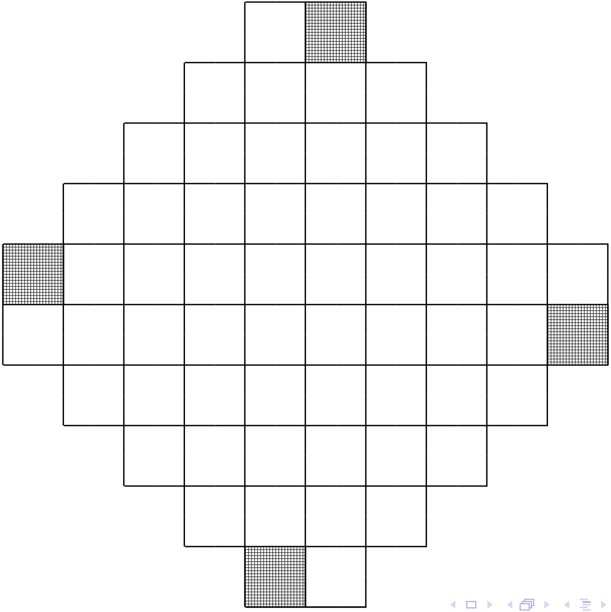

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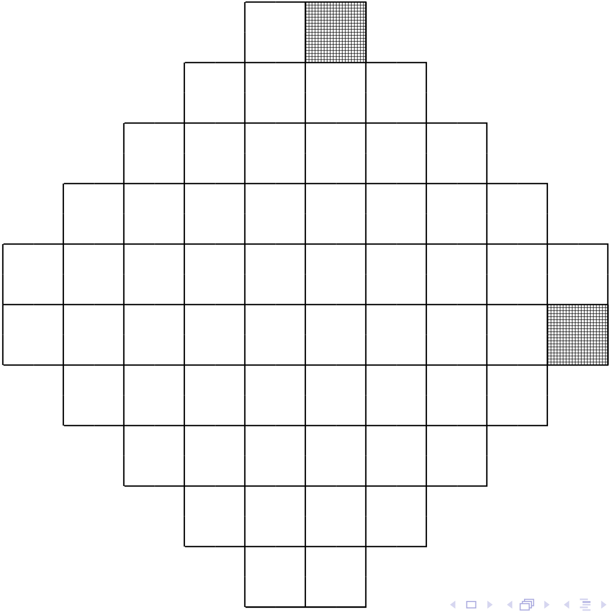

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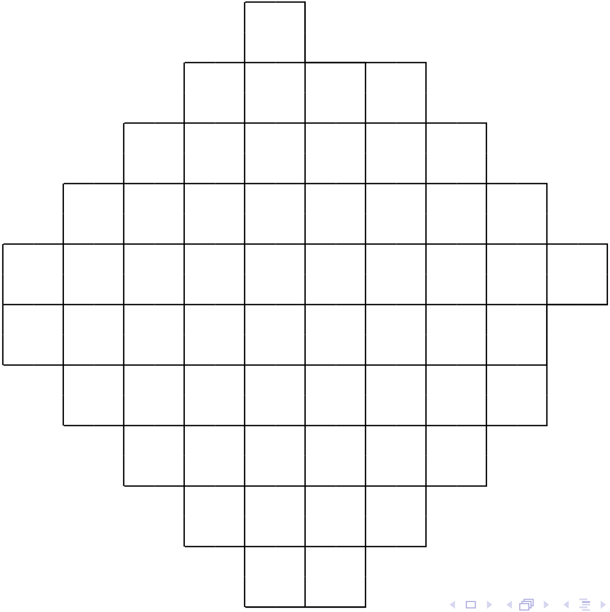

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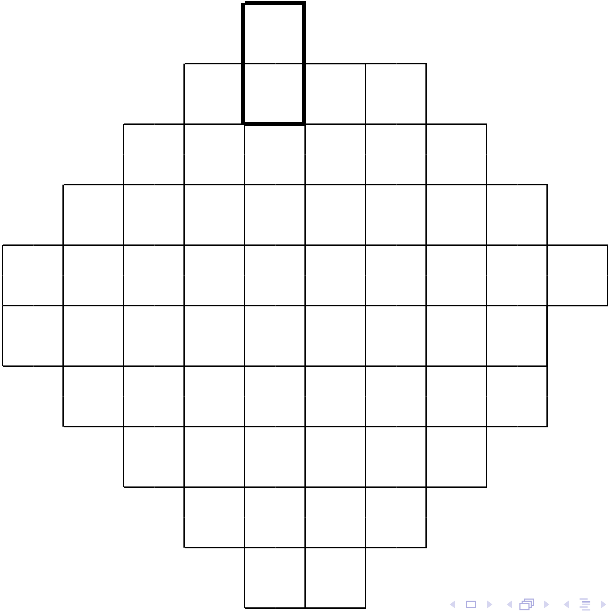
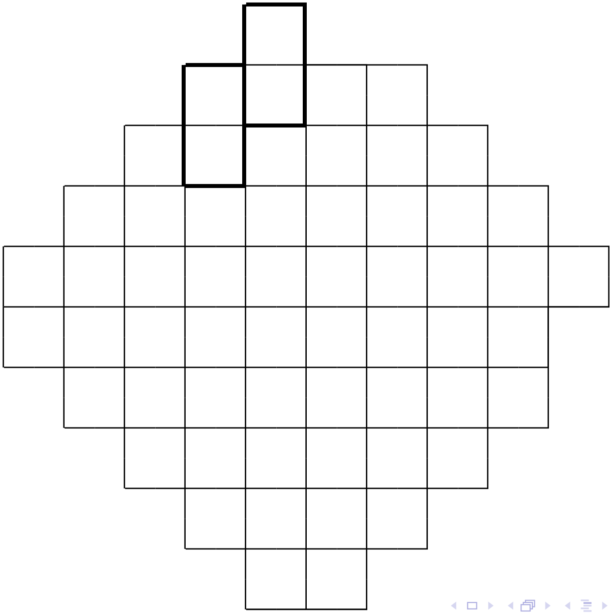
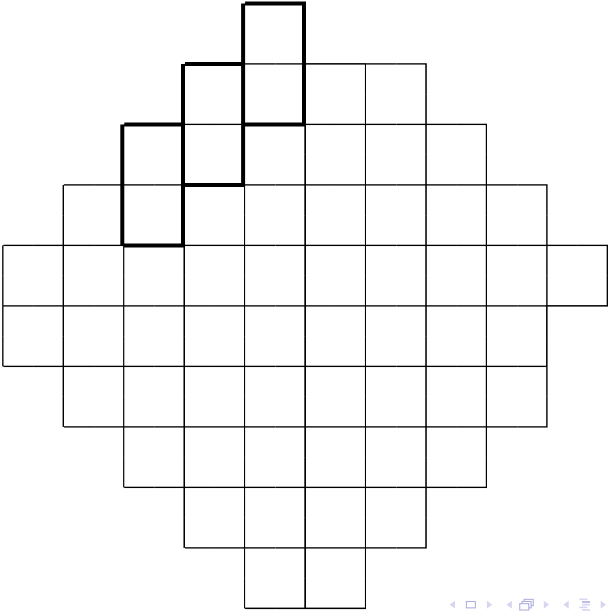


Proof Methods IV: Kuo Condensation





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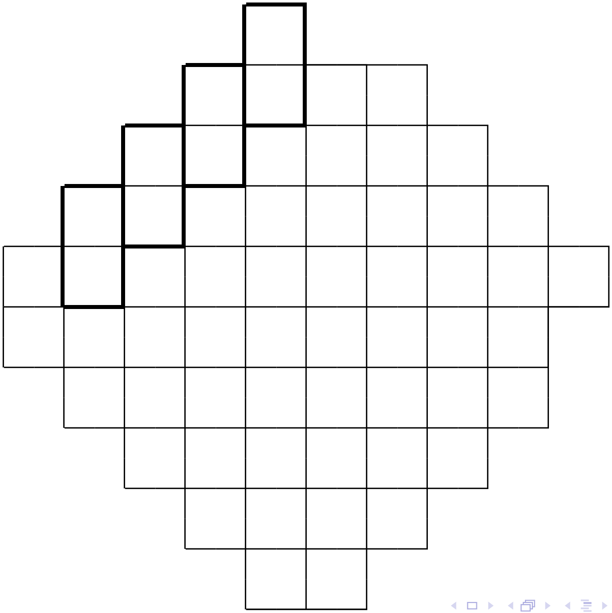

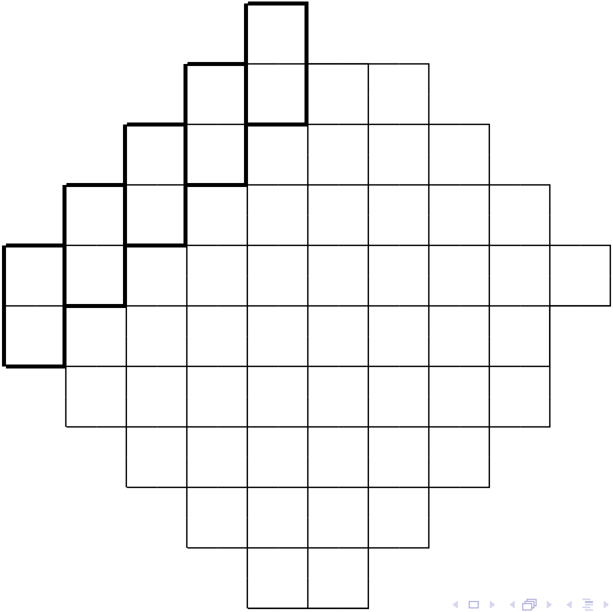

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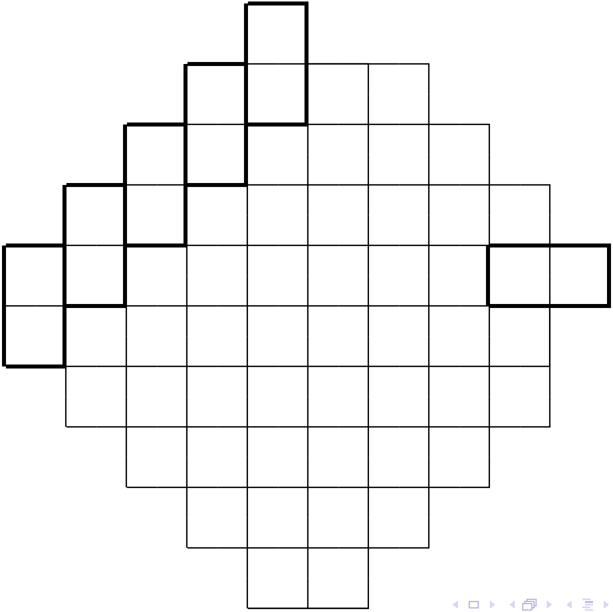

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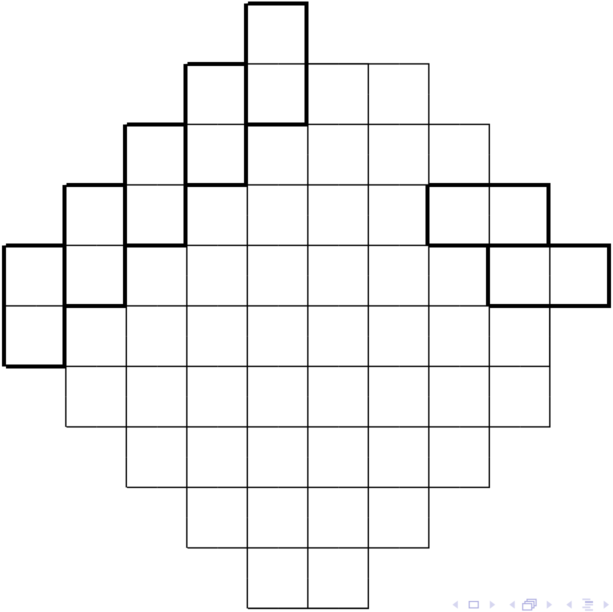

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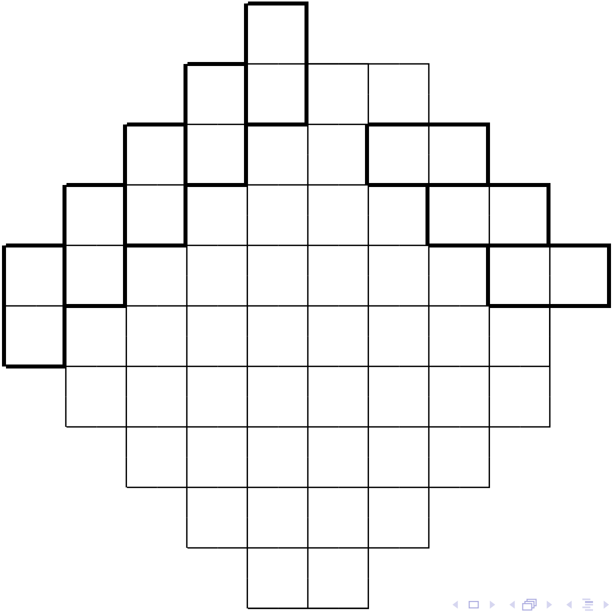

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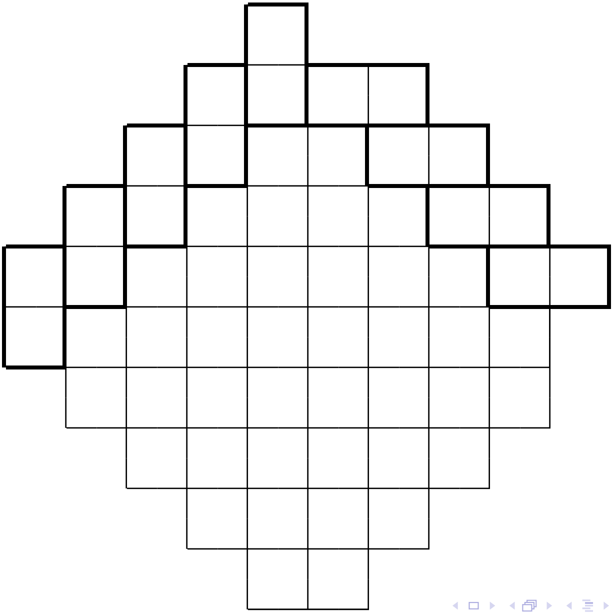
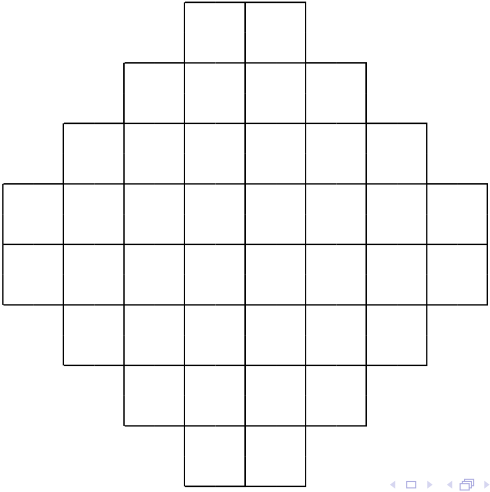


Proof Methods IV: Kuo Condensation



Proof Methods IV: Kuo Condensation

$$\begin{aligned}&\mathsf{M}(G)\,\mathsf{M}(G-\{\alpha,\beta,\gamma,\delta\})\\ &=\mathsf{M}(G-\{\alpha,\beta\})\,\mathsf{M}(G-\{\gamma,\delta\})+\mathsf{M}(G-\{\alpha,\delta\})\,\mathsf{M}(G-\{\beta,\gamma\}).\end{aligned}$$

$$
\begin{aligned}
&\mathrm{M}(G)\,\mathrm{M}(G-\{\alpha,\beta,\gamma,\delta\})\\
&=\mathrm{M}(G-\{\alpha,\beta\})\,\mathrm{M}(G-\{\gamma,\delta\})+\mathrm{M}(G-\{\alpha,\delta\})\,\mathrm{M}(G-\{\beta,\gamma\}).
\end{aligned}
$$

Let $G = A_n$, with A_n denoting the Aztec diamond of size n. Then we get

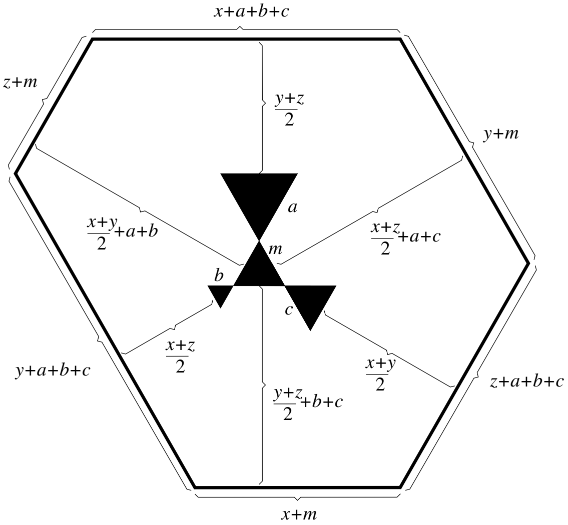
$$\mathrm{M}(A_n)\,\mathrm{M}(A_{n-2}) = \mathrm{M}(A_{n-1})\,\mathrm{M}(A_{n-1}) + = 2\,\mathrm{M}^2(A_{n-1}).$$

Hence, by induction,

$$M(A_n) = \frac{1}{2^{\binom{n-1}{2}}} 2 \cdot \left(2^{\binom{n}{2}}\right) = 2^{-\binom{n-1}{2}+1+n(n-1)} = 2^{\binom{n+1}{2}}.$$

Proof Methods IV: Kuo Condensation

The region $SC_{x,y,z}(a,b,c,m)$:



Proof Methods IV: Kuo Condensation

Theorem (Ciucu, C.K. 2013)

Let x, y, z, a, b, c and m be nonnegative integers. If x, y and z have the same parity, we have

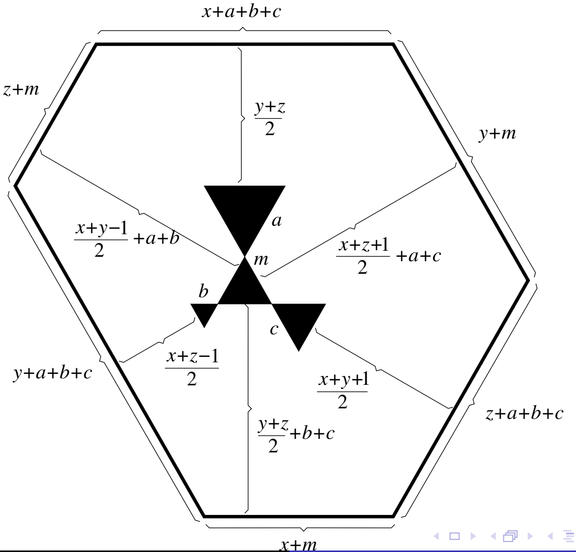
$$\mathrm{M}(SC_{x,y,z}(a,b,c,m)) = \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(\frac{x+y}{2}+m+a+b)\,\mathrm{H}(\frac{x+z}{2}+m+a+c)\,\mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\frac{x+y}{2}+m+c)\,\mathrm{H}(\frac{x+z}{2}+m+b)\,\mathrm{H}(\frac{y+z}{2}+m+a)}$$
$$\times \frac{\mathrm{H}(\frac{x+y}{2}+c)\,\mathrm{H}(\frac{x+z}{2}+b)\,\mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\frac{x+y}{2}+a+b)\,\mathrm{H}(\frac{x+z}{2}+a+c)\,\mathrm{H}(\frac{y+z}{2}+b+c)}$$
$$\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}$$
$$\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)}$$

$$\times \frac{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + m+a+b+c)}{\mathrm{H}(\frac{x+y}{2} + m+a+b+c)\,\mathrm{H}(\frac{x+z}{2} + m+a+b+c)}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + m+a+b+c)\,\mathrm{H}(\frac{y+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\frac{y+z}{2} + m+a+b+c)\,\mathrm{H}(\frac{x+y}{2})\,\mathrm{H}(\frac{x+z}{2})\,\mathrm{H}(\frac{y+z}{2})}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x}{2} \rceil\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor)\,\mathrm{H}(\lceil \frac{y}{2} \rceil)}{\mathrm{H}(\lceil \frac{x}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{y}{2} \rceil + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{y}{2} \rfloor)\,\mathrm{H}(\lceil \frac{z}{2} \rceil)\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor)}{\mathrm{H}(\lfloor \frac{y}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\;\mathrm{H}(\frac{x+y}{2} + \frac{m+a+b+c}{2})^2\;\mathrm{H}(\frac{x+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + \frac{m+a+b+c}{2})},$$

where

$$\mathrm{H}(n) := \begin{cases} \prod_{k=0}^{n-1} \Gamma(k+1), & \textit{for } n \textit{ a positive integer,} \\ \prod_{k=0}^{n-\frac{1}{2}} \Gamma(k+\frac{1}{2}), & \textit{for } n \textit{ a positive half-integer,} \end{cases}$$

The region $SC_{x,y,z}(a,b,c,m)$:



Theorem (Ciucu, C.K. 2013)

Let x, y, z, a, b, c and m be nonnegative integers. If x has parity different from the parity of y and z, we have

$$\begin{aligned}
\mathrm{M}(SC_{x,y,z}(a,b,c,m)) &= \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}\\
&\times \frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+c)\,\mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+m+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+m+b)\,\mathrm{H}(\frac{y+z}{2}+m+a)}\\
&\times \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+b)\,\mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+a+b)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+a+c)\,\mathrm{H}(\frac{y+z}{2}+b+c)}\\
&\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}\\
&\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)}
\end{aligned}$$

$$\times \frac{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + m + a + b + c)}{\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + m + a + b + c)\,\mathrm{H}(\lceil \frac{x+z}{2} \rceil + m + a + b + c)}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + m + a + b + c)}{\mathrm{H}(\frac{y+z}{2} + m + a + b + c)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x}{2} \rceil)\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor)\,\mathrm{H}(\lceil \frac{y}{2} \rceil)}{\mathrm{H}(\lceil \frac{x}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{y}{2} + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{y}{2} \rfloor)\,\mathrm{H}(\lceil \frac{z}{2} \rceil)\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor)}{\mathrm{H}(\lfloor \frac{y}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\;\mathrm{H}(\lceil \frac{x+y}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + \frac{m+a+b+c}{2})}{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{x+y}{2} \rceil)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+z}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\frac{y+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\lfloor \frac{x+z}{2} \rfloor)\,\mathrm{H}(\frac{y+z}{2})}.$$

Proof Methods IV: Kuo Condensation

Since then, a whole industry of discovery and proof of exact enumeration results for rhombus tilings with defects developed, the proofs being based on Kuo condensation.

Tri Lai alone wrote over 10 such papers; other authors include Byun, Ciucu, Rohatgi.

Ciucu's Electrostatic Conjecture

It predicts that, in a scaling limit, the correlation of (microscopic) holes is given by the 2D Coulomb energy of the system of charges obtained by regarding each hole as a point charge of magnitude equal to the number of right-pointing triangles minus the number of left-pointing triangle in the hole.

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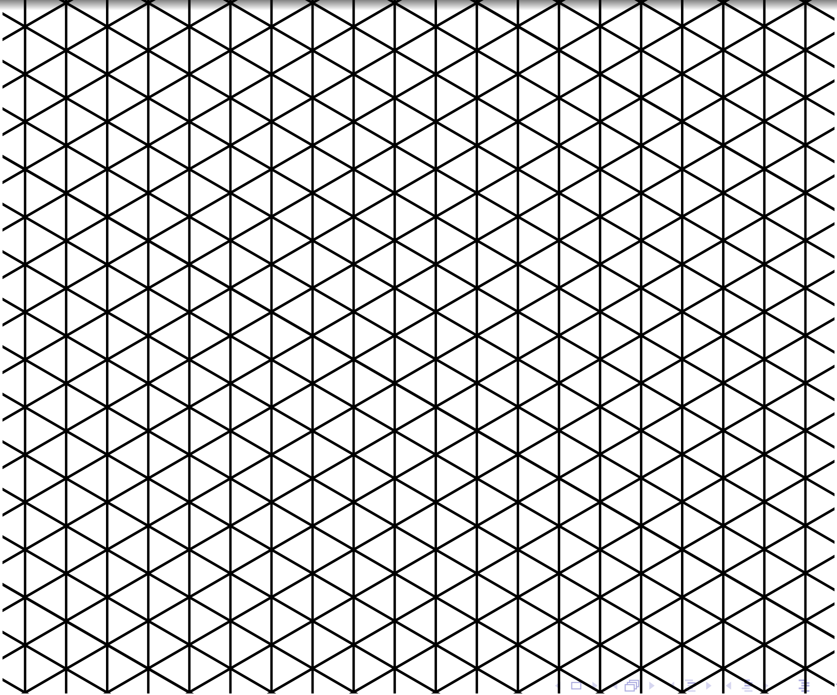
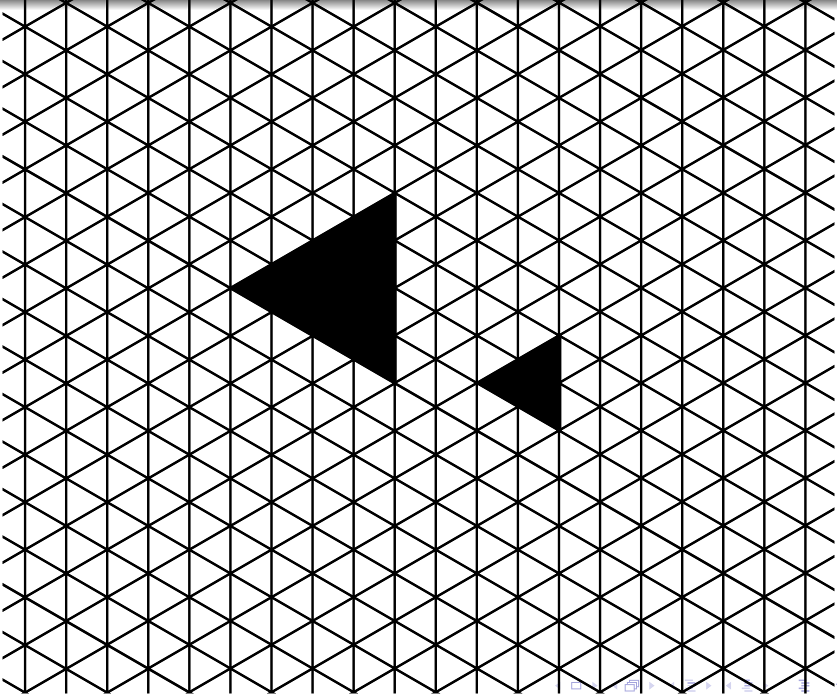
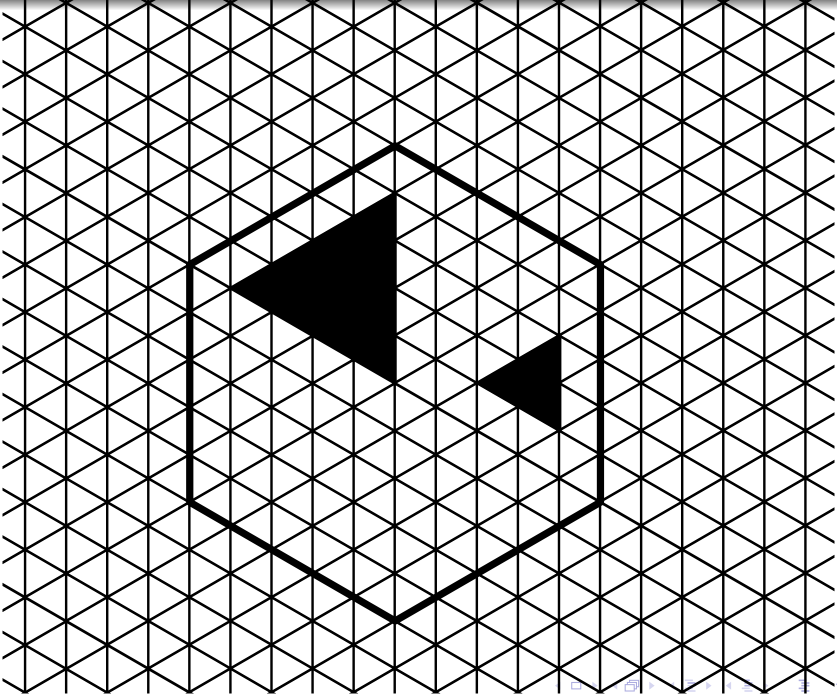
Consider holes $O_1, O_2, \dots, O_m$ in a triangular grid, and put them into a finite subregion (for example, a hexagon).







Ciucu's Electrostatic Conjecture

$$\frac{\text{number of rhombus tilings}\atop\text{of hexagon with holes}}{\text{number of rhombus tilings}\atop\text{of hexagon without holes}}$$

Ciucu's Electrostatic Conjecture

The correlation of the holes O_1, O_2, ..., O_m is defined as

$$\omega(O_1, \dots, O_m) = \text{limit } \frac{\substack{\text{number of rhombus tilings} \\ \text{of hexagon with holes}}}{\substack{\text{number of rhombus tilings} \\ \text{of hexagon without holes}}}$$

when the size of the hexagon tends to infinity, leaving the position of the holes fixed.

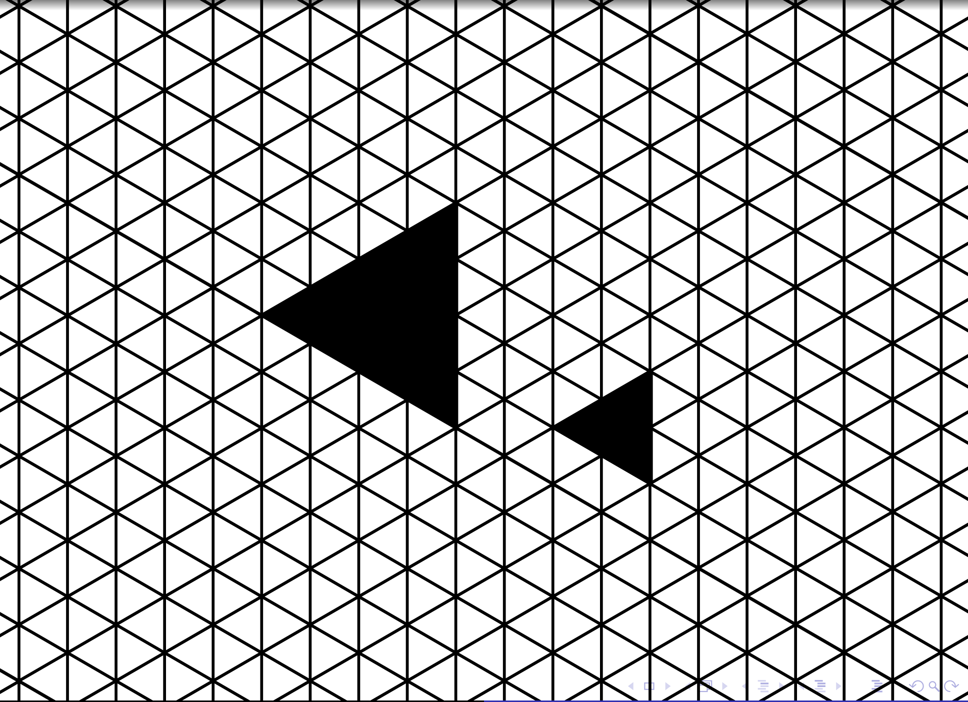
Ciucu's Electrostatic Conjecture

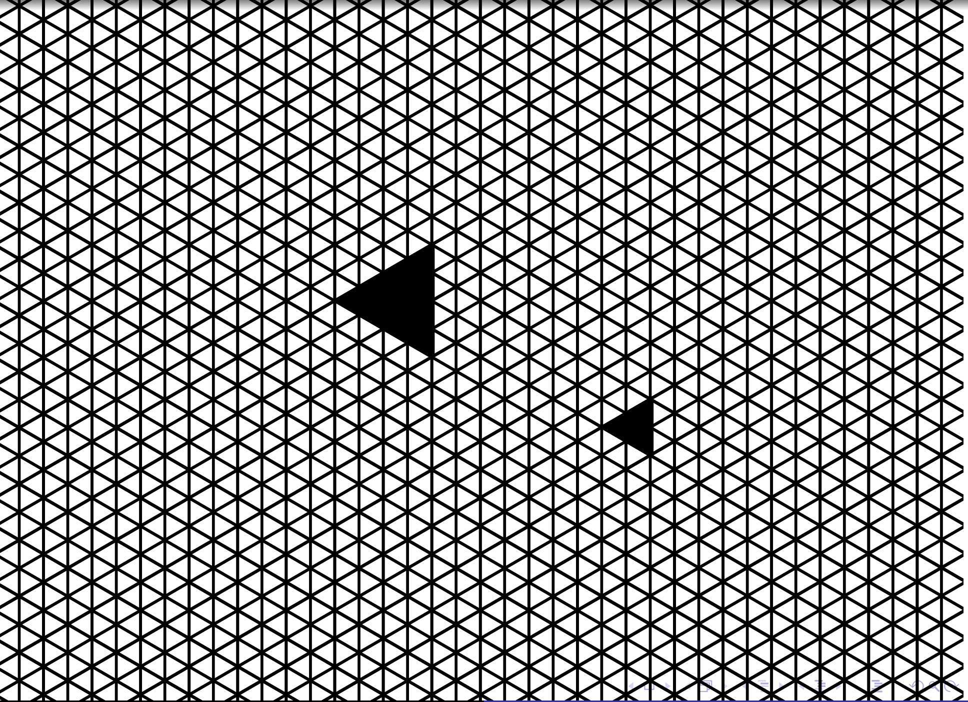
The correlation of the holes O_1, O_2, ..., O_m is defined as

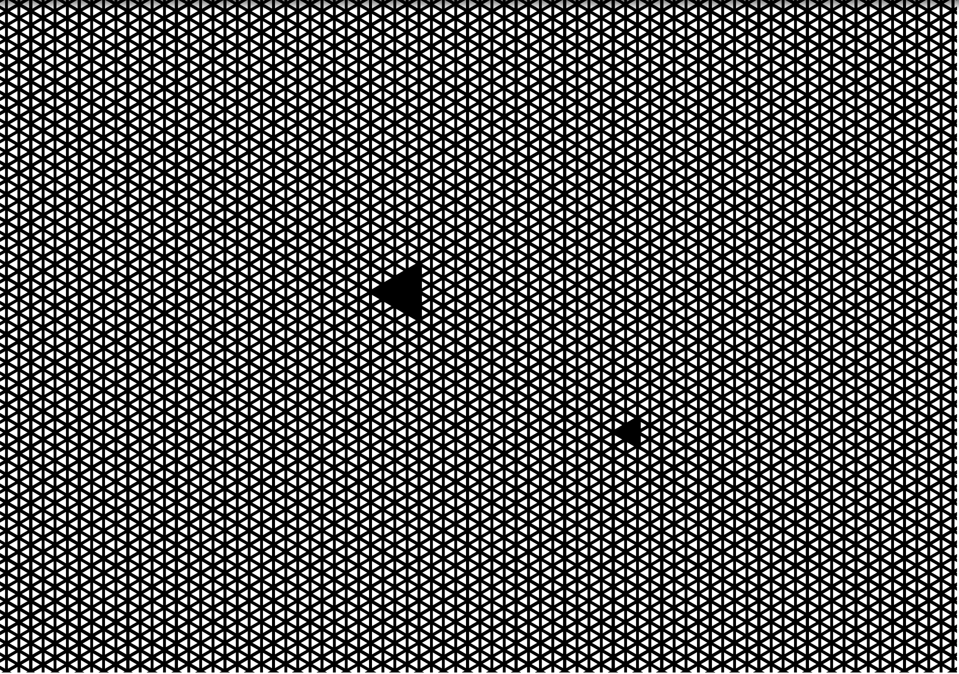
$$\omega(O_1,\dots,O_m) = \text{limit}\ \frac{\begin{array}{c}\text{number of rhombus tilings}\\ \text{of hexagon with holes}\end{array}}{\begin{array}{c}\text{number of rhombus tilings}\\ \text{of hexagon without holes}\end{array}}$$

when the size of the hexagon tends to infinity, leaving the position of the holes fixed.

Subsequently, one performs a "scaling limit," which one may think of making the grid finer and finer, and at the same time shrink the holes to points.







Ciucu's Electrostatic Conjecture

Conjecture (Ciucu 2008)

For holes O_1, O_2, ..., O_m, in this scaling limit we have

$$\omega\big(R\cdot O_1,\dots,R\cdot O_n\big) \sim \text{const.}\times\prod_{1\le i<j\le n} d\big(R\cdot O_i,R\cdot O_j\big)^{\chi(O_i)\chi(O_j)/2}$$

as $R\to\infty$, where $\chi(O)$ is the charge of the hole O and $d(R\cdot O_i,R\cdot O_j)$ is the distance between the (shifted) holes $R\cdot O_i$ and $R\cdot O_j$.

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Why "Electrostatic Conjecture"?

Ciucu's Electrostatic Conjecture

Conjecture (Ciucu 2008)

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Why "Electrostatic Conjecture"?

Coulomb's Law of 2-Dimensional Electrostatics. *Given point charges $C_1,C_2,\dots,C_n$ in the plane, the electrostatic potential energy of the point charges is proportional to*

$$\sum_{1\le i<j\le n} \frac{\chi(C_i)\chi(C_j)}{2}\log d(C_i,C_j),$$

The conjecture is expected to hold as well for domino tilings of square grid graphs. Moreover, it is expected to hold more generally for dimer configurations on (infinite) periodic graphs.

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In general, the conjecture is wide open.
However, it has been proven in several special instances.

Ciucu's Electrostatic Conjecture: proved special cases

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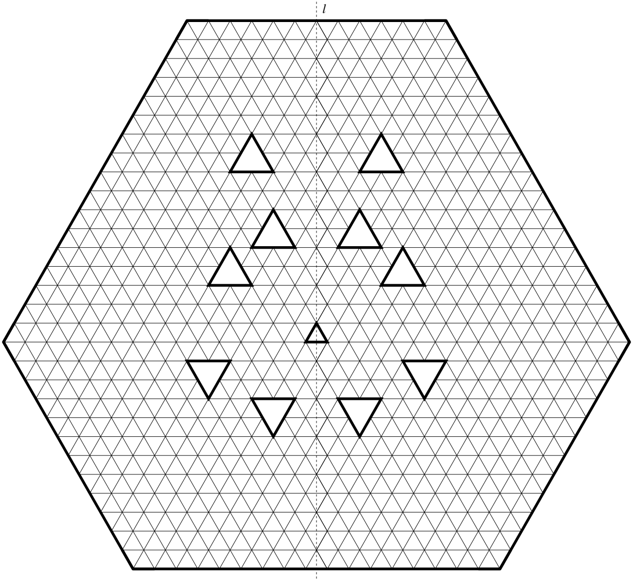
A symmetric case (Ciucu 2005):



Ciucu's Electrostatic Conjecture: proved special cases

A symmetric case (CIUCU 2005):

Ciucu's Electrostatic Conjecture: proved special cases

A symmetric case (Ciucu 2005):

Method of proof:

- Use Ciucu's Matchings Factorisation Theorem
- Apply non-intersecting lattice paths to the factor regions
- Do an asymptotic analysis of the resulting closed-form product formula

Ciucu's Electrostatic Conjecture: proved special cases

Rhombus tilings on the infinite torus with even-size holes (Ciucu 2009):

Ciucu's Electrostatic Conjecture: proved special cases

Rhombus tilings on the infinite torus with even-size holes (CIUCU 2009):

Method of proof:

- Use an asymptotic formula of KENYON for the entries of the inverse Kasteleyn matrix
- Do heavy determinant evaluations

Ciucu's Electrostatic Conjecture: proved special cases

Rhombus tilings on the infinite torus (Ciucu 2009):

Theorem

Let $s_1, s_2, \dots, s_m \ge 1$ *and* $t_1, t_2, \dots, t_n \ge 1$ *be integers. Write* $S = \sum_{i=1}^{m} s_i$, $T = \sum_{j=1}^{n} t_j$, *and assume* $S \ge T$. *Let* $x_1, x_2, \dots, x_m$ *and* $y_1, y_2, \dots, y_n$ *be indeterminates. Define* N *to be the* $S \times S$ *matrix*

$$N = \begin{bmatrix} A & B \end{bmatrix}$$

whose blocks are given by

$$A=$$

$$\begin{pmatrix}
\frac{\binom{0}{0}}{y_1-x_1} & \frac{\binom{1}{0}}{(y_1-x_1)^2} & \cdots & \frac{\binom{t_1-1}{0}}{(y_1-x_1)^{t_1}} & & \frac{\binom{0}{0}}{y_n-x_1} & \frac{\binom{1}{0}}{(y_n-x_1)^2} & \cdots & \frac{\binom{t_n-1}{0}}{(y_n-x_1)^{t_n}} \\
\frac{\binom{1}{1}}{(y_1-x_1)^2} & \frac{\binom{2}{1}}{(y_1-x_1)^3} & \cdots & \frac{\binom{t_1}{1}}{(y_1-x_1)^{t_1+1}} & & \frac{\binom{1}{1}}{(y_n-x_1)^2} & \frac{\binom{2}{1}}{(y_n-x_1)^3} & \cdots & \frac{\binom{t_n}{1}}{(y_n-x_1)^{t_n+1}} \\
\vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
\frac{\binom{s_1-1}{s_1-1}}{(y_1-x_1)^{s_1}} & \frac{\binom{s_1}{s_1-1}}{(y_1-x_1)^{s_1+1}} & \cdots & \frac{\binom{s_1+t_1-2}{s_1-1}}{(y_1-x_1)^{s_1+t_1-1}} & & \frac{\binom{s_1-1}{s_1-1}}{(y_n-x_1)^{s_1}} & \frac{\binom{s_1}{s_1-1}}{(y_n-x_1)^{s_1+1}} & \cdots & \frac{\binom{s_1+t_n-2}{s_1-1}}{(y_n-x_1)^{s_1+t_n-1}} \\
 & & \vdots & & & & & \vdots & \\
\frac{\binom{0}{0}}{y_1-x_m} & \frac{\binom{1}{0}}{(y_1-x_m)^2} & \cdots & \frac{\binom{t_1-1}{0}}{(y_1-x_m)^{t_1}} & & \frac{\binom{0}{0}}{y_n-x_m} & \frac{\binom{1}{0}}{(y_n-x_m)^2} & \cdots & \frac{\binom{t_n-1}{0}}{(y_n-x_m)^{t_n}} \\
\frac{\binom{1}{1}}{(y_1-x_m)^2} & \frac{\binom{2}{1}}{(y_1-x_m)^3} & \cdots & \frac{\binom{t_1}{1}}{(y_1-x_m)^{t_1+1}} & & \frac{\binom{1}{1}}{(y_n-x_m)^2} & \frac{\binom{2}{1}}{(y_n-x_m)^3} & \cdots & \frac{\binom{t_n}{1}}{(y_n-x_m)^{t_n+1}} \\
\vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots
\end{pmatrix}$$

and

$$B=\begin{pmatrix}
\binom{0}{0}x_1^0 & \binom{1}{0}x_1 & \cdots & \binom{S-T-1}{0}x_1^{S-T-1}\\
\binom{0}{1}x_1^{-1} & \binom{1}{1}x_1^0 & \cdots & \binom{S-T-1}{1}x_1^{S-T-2}\\
. & . & & .\\
. & . & & .\\
. & . & & .\\
\binom{0}{s_1-1}x_1^{1-s_1} & \binom{1}{s_1-1}x_1^{2-s_1} & \cdots & \binom{S-T-1}{s_1-1}x_1^{S-T-s_1}\\
 & & . & \\
 & & . & \\
 & & . & \\
\binom{0}{0}x_m^0 & \binom{1}{0}x_m & \cdots & \binom{S-T-1}{0}x_m^{S-T-1}\\
\binom{0}{1}x_m^{-1} & \binom{1}{1}x_m^0 & \cdots & \binom{S-T-1}{1}x_m^{S-T-2}\\
. & . & & .\\
. & . & & .\\
. & . & & .\\
\binom{0}{s_m-1}x_m^{1-s_m} & \binom{1}{s_m-1}x_m^{2-s_m} & \cdots & \binom{S-T-1}{s_m-1}x_m^{S-T-s_m}
\end{pmatrix}.$$

Then we have

$$\det N = \frac{\prod_{1\le i<j\le m}(x_j - x_i)^{s_i s_j} \prod_{1\le i<j\le n}(y_i - y_j)^{t_i t_j}}{\prod_{i=1}^{m}\prod_{j=1}^{n}(y_j - x_i)^{s_i t_j}}.$$

Ciucu's Electrostatic Conjecture: proved special cases

Dimer model on isoradial graphs with singleton gaps (Dubédat 2014):

Ciucu's Electrostatic Conjecture: proved special cases

Dimer model on isoradial graphs with singleton gaps (Dubédat 2014):

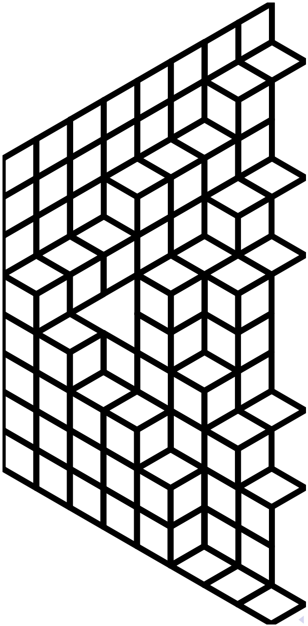
Method of proof:

- Kasteleyn method
- ???

Ciucu's Electrostatic Conjecture: proved special cases

Rhombus tilings with free boundary (Ciucu, C.K. 2011): the result is analogous to the situation of a charge near a conductor, which in electrostatics is solved by the method of images.

Ciucu's Electrostatic Conjecture: proved special cases



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Method of proof:

- Non-intersecting lattice paths
- Heavy determinant evaluation
- Asymptotic analysis of the result

The key result:

Theorem (Ciucu, C.K. 2011)

For all positive integers n, x and nonnegative integers $k \le n-1$, we have

$$M(F_{n,x} \backslash \triangleleft_2(k)) = \binom{4k+1}{2k} \frac{(n+k)!}{(x+n-k)_{2k+1}} \prod_{s=1}^{n} \frac{(2x+2s)_{4n-4s+1}}{(2s)_{4n-4s+1}}$$

$$\times \sum_{i=0}^{n-k-1} \frac{(\frac{1}{2})_i}{i!\,(n-k-i-1)!^2\,(n+k-i+1)_{n-k}\,(n+k-i+1)_i\,(2n-i+\frac{1}{2})_i}$$

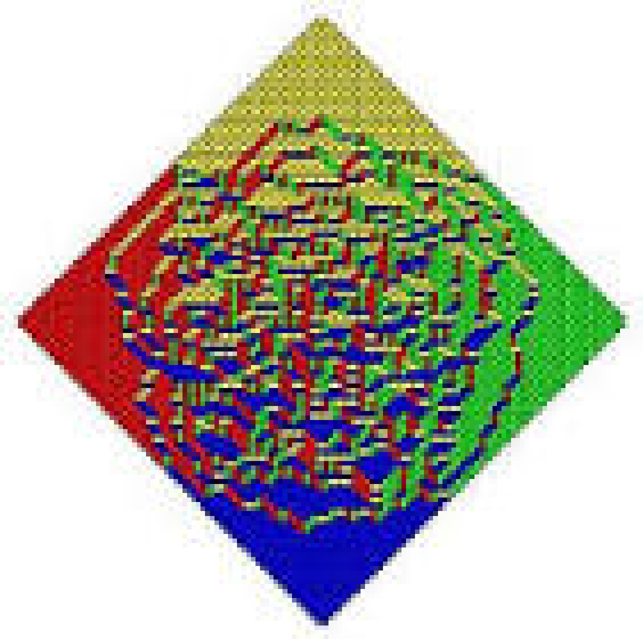
$$\cdot \Big((x)_i\,(x+i+1)_{n-k-i-1}\,(x+n+k+1)_{n-k}$$

$$- (x)_{n-k}\,(x+n+k+1)_{n-k-i-1}\,(x+2n-i+1)_i \Big).$$

Ciucu's Electrostatic Conjecture: proved special cases

There are a few more special cases and variations that have been proved (Ciucu, Fischer, Gilmore, C.K.), only very few concern domino tilings.

Random Tilings: Arctic Curve Theorems



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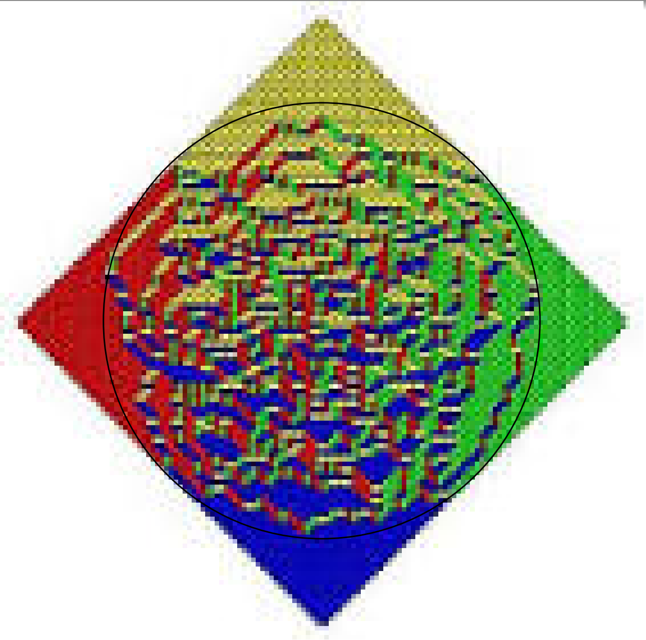
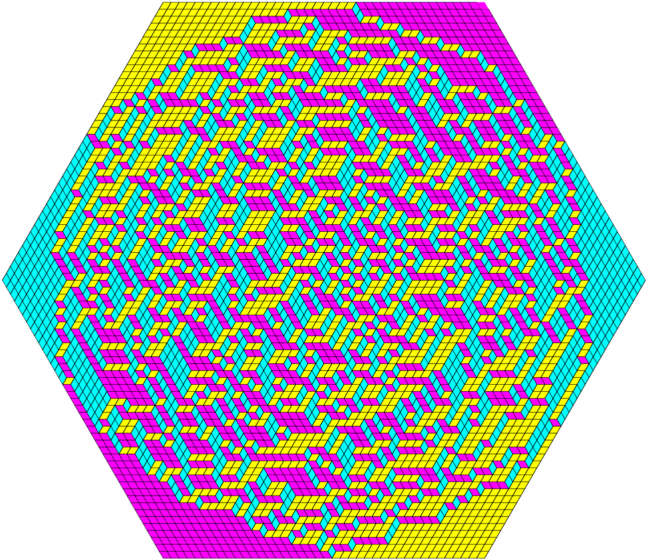


Random Tilings: Arctic Curve Theorems



Random Tilings: Arctic Curve Theorems