Abstract

Diese Arbeit hat das Ziel, eine Einleitung in das Thema „chromatische Funktionen“ auf dem Niveau eines Masterstudenten zu liefern. Bei diesen handelt es sich um Funktionen, welche die Färbungen von endlichen Graphen abzählen. Als Grundlage dafür werden die Definition und grundlegende Eigenschaften des chromatische Polynoms besprochen. Der größte Fokus der Arbeit liegt bei der von Stanley eingeführten symmetrischen chromatischen Funktion, welche eine Verallgemeinerung des chromatischen Polynoms gibt. Diese wird in einigen der bekannten Basen für symmetrische Funktionen ausgedrückt. Außerdem werden einige Verallgemeinerungen der symmetrischen chromatischen Funktion besprochen, unter anderem die von Shareshian und Wachs eingeführte quasisymmetrische chromatische Funktion.

The goal of this work is to give an introduction into the topic of “chromatic functions” on the level of a masters student. These are functions which enumerate colorings of a finite graph. As a basis the definition and basic properties of the chromatic polynomial will be discussed. The main focus of this work is the symmetric chromatic function introduced by Stanley, which serves as a generalization of the chromatic polynomial. It will be expanded into some of the well known bases of symmetric functions. Additionally, several generalizations of the chromatic function will be discussed, including the quasisymmetric chromatic function introduced by Shareshian and Wachs.

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1. The Chromatic Polynomial

In this chapter we will look at a classical object of graph theory, the chromatic polynomial. We will give a brief overview of the definitions and some basic results as well as applications in order to motivate the generalization to Stanley's chromatic function in Chapter 2.

1.1. Introduction

Before we can start to define the object of our interest we need to briefly discuss our framework. In this chapter we will work with graphs, that is, a tuple $G = (V, E)$. For reasons which will soon be explained it makes sense to consider graphs which are loopless, meaning no edge connects a vertex to itself, and that there are no multiple edges. Notice that we may identify edges as two element subsets of V in this case. Such graphs are called simple and we will refer to a simple graph with finite vertex set V by the word graph, unless stated otherwise.

Definition 1.1.1. Let $G = (V, E)$ be a graph and let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting indeterminates. A function $\kappa : V \to \mathbb{N}$ is called a *coloring* of the graph G. A coloring κ is called *proper* if no two adjacent vertices share a color, that is if $\{v_1, v_2\} \in E$ then $\kappa(v_1) \neq \kappa(v_2)$ for all $v_1, v_2 \in V$.

Considering the definition of a proper coloring it is now apparent why we disallowed loops and multiple edges. The former would render every coloring of a graph non-proper, while the latter do not matter for proper colorings at all.

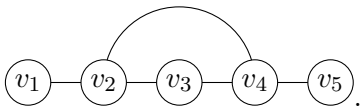
Definition 1.1.2. Let $G = (V, E)$ be a graph and $n \in \mathbb{N}$. Denote by $c(G, n)$ the number of proper colorings of G with codomain contained in $[n] = \{1, \dots, n\}$, meaning when using at most the first n colors. For any given graph G we may now define the function

$$\chi_G : \mathbb{N} \to \mathbb{N},\ n \mapsto c(G, n),$$

simply denoted by χ if no confusion arises. This function is called the *chromatic polynomial* of the graph G.

Here it is important that the graph is in fact finite, since the function χ_G would be ill defined in general.

Example 1.1.3. Consider the following graph:



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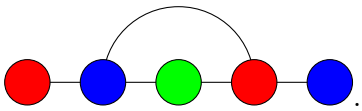
We can visualize a coloring $\kappa : V \to [n]$ by choosing for each $i \in [n]$ a unique color, hence the namesake. Playing around with the definitions in the case $n = 2$ it quickly becomes apparent that there is no proper coloring in this case, meaning $\chi_G(2) = 0$. For $n = 3$ however there are several, one of them being



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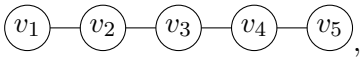
At this point it is not clear why χ would be a polynomial. To prove this, we first consider *deletion* and *contraction*. Assume that we have a graph $G = (V, E)$ and a coloring κ of G. For any given edge $e = \{v_1, v_2\} \in E$ denote by $G\backslash e$ the graph with vertex set V and edge set $E\backslash\{e\}$. Furthermore, denote by G/e the graph with vertex set $(V\backslash\{v_1, v_2\})\dot{\cup}\{u\}$, where u is a new vertex, and edge set starting with $E\backslash e$ and replacing every edge of the form $\{v_i, x\}$ by $\{u, x\}$ for $i = 1, 2$ and $x \in V$. Less rigorously, $G\backslash e$ is the deletion of the edge e from G, while G/e contracts the edge e, merging the connected vertices in the process.

Example 1.1.4. Consider the graph from example 1.1.3 and choose $e = \{v_2, v_4\}$, so that deletion of e will yield the line graph of length 5



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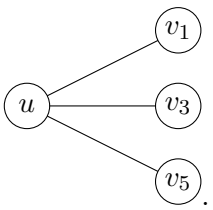
whereas the contraction along e gives us the complete bipartite graph on groups of size 1 and 3 denoted $K_{1,3}$ (also called the claw graph)



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With these operations in mind we may now begin to prove that χ is a polynomial.

Lemma 1.1.5. *Let $G = (V, E)$ be a graph. The function $\chi_G(n)$ satisfies the recursion*

$$\chi_G(n) = \chi_{G\backslash e}(n) - \chi_{G/e}(n).$$

Proof. For any given edge $e = \{v_1, v_2\} \in E$ and proper coloring κ of $G\backslash e$ we can construct a proper coloring of G in the natural way, by simply coloring the vertices the same, given it is the case that v_1 and v_2 have distinct colors. On the other hand, if v_1 and v_2 are colored the same then we may consider κ as a proper coloring of G/e instead. Since these two cases are exclusive and exhaustive each proper coloring of $G\backslash e$ with colors in $[n]$ can either be interpreted as a proper coloring of G or of G/e with colors in $[n]$, so that if we rearrange the terms slightly the claim follows. □

Corollary 1.1.6. *Let $G = (V, E)$ be a graph. Then the function $\chi_G(n)$ is a polynomial in n.*

Proof. We prove this by induction on the number of edges k.
If $k = 0$, then our graph is completely disconnected, meaning any coloring will be proper. Therefore we get $\chi_G(n) = n^{|V|}$, a polynomial in n.
Now suppose that $k > 0$. Consider the recursion in Lemma 1.1.5 and notice that both $G\backslash e$ and G/e have $k - 1$ or less edges and therefore satisfy the induction hypothesis, which concludes the proof. □

Remark 1.1.7. Since χ_G is a polynomial we may in fact uniquely extend it as a polynomial $\chi_G : \mathbb{C} \to \mathbb{C}$, which will be convenient later.

Example 1.1.8. To illustrate the usefulness of the above recursion consider the graph of example 1.1.3. It is possible to derive χ_G just by looking at the graph, but this is not feasible for more complicated cases. Hence we consider instead its respective deletion and contraction along $\{v_2, v_4\}$ found in example 1.1.4. It is almost trivial to extract $\chi_{G\backslash e}(n)$ and $\chi_{G/e}(n)$, since we can choose any color for v_1 respectively u so that all following vertices have to be chosen among $n-1$ colors, meaning $\chi_{G\backslash e}(n) = n(n-1)^4$ and $\chi_{G/e}(n) = n(n-1)^3$. This approach, in fact, works for all tree graphs by choosing any of its vertices as a root which will be colored first. We can conclude $\chi_G(n) = n(n-1)^3(n-2)$ using Lemma 1.1.5. If one is unable to infer $\chi_{G\backslash e}$ or $\chi_{G/e}$ in a more complicated graph it is always possible to apply deletion and contraction again in a recursive process. For large graphs choosing convenient edges is mandatory to reduce the computational power required to calculate χ_G, but we will not discuss this aspect.

Remark 1.1.9. There are two simple properties the chromatic polynomial satisfies, which are nonetheless worth stating: If G consists only of a single vertex $G = K_1$, then $\chi_{K_1}(n) = n$. Furthermore, for two arbitrary graphs G and H, the chromatic polynomial of their disjoint union $G + H$ has the property $\chi_{G+H}(n) = \chi_G(n)\chi_H(n)$. What is interesting about these is that if we assume there is a function $f(G, n)$ having a graph and natural number as arguments and it satisfies those two properties as well as an appropriate version of Lemma 1.1.5, then it follows that $f(G, n) = \chi_G(n)$.
The reason for this is that we can do a kind of induction on all graphs, so let $n \in \mathbb{N}$ be fixed. When determining $f(G, n)$ we can separately consider the connected components of G, since $f(G_1 + G_2, n) = f(G_1, n)f(G_2, n)$, which reduces the number of vertices by at least 1. Furthermore, by $f(G, n) = f(G\backslash e, n) - f(G/e, n)$ we can decrease the number of edges by 1. Reapplying those two rules a finite amount of times will yield an expression for $f(G, n)$ strictly in terms of $f(K_1, n)$ and since we assumed $f(K_1, n) = \chi_{K_1}(n) = n$ it follows that $f(G, n) = \chi_G(n)$.

1.2. Expansions of $\chi_G(n)$

In this section we will briefly discuss three statements one can make about the chromatic polynomial in relation to combinatorial objects. Our goal here is twofold: On the one

hand we want to know what kind of statements are possible about the chromatic polynomial. On the other hand, which is arguably more relevant here, we want to establish properties that might in some way also hold for a more general object than the chromatic polynomial, which will be discussed in Chapter 2. Most of the ideas here can be found in a paper by Whitney [13], with the exception of Theorem 1.2.5 and (a more general version of) its proof which can be found in an article by Sagan [4].
We first relate the chromatic polynomial to the number of connected components of induced subgraphs.

Theorem 1.2.1. *Let $G = (V, E)$ be a graph and for $S \subseteq E$ denote by $c(S)$ the number of connected components of $G_S = (V, S)$. Then*

$$\chi_G(n) = \sum_{S \subseteq E} (-1)^{|S|} n^{c(S)}.$$

Proof. A key idea to prove this is that of inclusion exclusion. To apply it, we first need to define the base set from which we are excluding: Consider the set C of all colorings of G using the colors weakly smaller than n, including non-proper ones. Since our goal is the number $\chi(n)$, which is defined as the number of proper colorings, we start with C and count all colorings which are non-proper on a given edge $e \in E$. When we do this for each edge and add together the numbers we will have overcounted the number of non-proper colorings in this exclusion step, leading to an inclusion step where we consider pairs of edges $e_1, e_2 \in E$ and so on. After a finite amount of steps this will lead us to the desired number. To make this precise, define the sets

$$C_{\{v,w\}} = \{\kappa \in C \mid \kappa(v) = \kappa(w)\},$$

where $\{v, w\} \in E$. Now it is apparent that

$$\chi(n) = |C| - \sum_{\substack{S \subseteq E \\ S \neq \emptyset}} (-1)^{|S|} \left| \bigcap_{e \in S} C_e \right| = \sum_{S \subseteq E} (-1)^{|S|} \left| \bigcap_{e \in S} C_e \right|, \tag{1.1}$$

where by convention $\bigcap_{e \in \emptyset} C_e = C$. Now we will consider the colorings in a different way and then relate them back to equation 1.1 to find a usable expression.
For $S \subseteq E$ consider the set $C_S = \bigcap_{e \in S} C_e$: This is the set of colorings which are constant on the connected components of G_S, as the color is constant along every edge in S. The number $|C_S|$ is therefore $n^{c(S)}$, as we may choose one color for each component without restriction. Relating this back to our inclusion exclusion, the set $S \subseteq E$ will contribute exactly $(-1)^{|S|} n^{c(S)}$ to the sum. Applying this logic to all S will therefore give us the desired result

$$\chi(n) = \sum_{S \subseteq E} (-1)^{|S|} n^{c(S)}.$$

□

The next expansion we will look at has to do with broken circuits.

Definition 1.2.2. Let $G = (V, E)$ be a graph. Let $\alpha : E \to [\,|E|\,]$ be a bijective map, which we interpret as putting labels from 1 to $|E|$ on the edges of G. A *circuit* is a subset $S \subseteq E$ which is minimal with respect to inclusion and does contain a cycle. A *broken circuit* is a circuit with the edge of largest label removed. The *broken circuit complex* B_G with respect to α is the set of all subsets of edges which do not contain a broken circuit, so $B_G = \{S \subseteq E \,|\, K \subseteq S \Rightarrow K \text{ is not a broken circuit}\}$.

Theorem 1.2.3 (Broken Circuit Theorem)**.** *Let $G = (V, E)$ be a graph. Then*

$$\chi_G(n) = \sum_{S \in B_G} (-1)^{|S|} n^{|V|-|S|}.$$

Proof. We will at some points in this proof conflate between edge sets $S \subseteq E$ and their corresponding subgraphs $G_S = (V, S)$, since the two are essentially the same object.
We first want to impose an order relation on the set of broken circuits. For this purpose let $\alpha : E \to [\,|E|\,]$ be the labelling of our edges. For two broken circuits P and Q define $P < Q$ if Q contains the edge of largest label among $P \cup Q$ while P does not. Note at this point that if the edge of largest label is in both P and Q then the two sets are incomparable. Now list all broken circuits $P_1, \ldots, P_k$ in such a way that $P_i \not> P_j$ if $i < j$.
Next we will partition the set of subsets of E. For this purpose we define

$$\begin{aligned}
S_1 &= \{S \subseteq E \,|\, P_1 \subseteq S\},\\
S_i &= \{S \subseteq E \,|\, P_i \subseteq S\} \setminus \bigcup_{l=1}^{i-1} S_l \quad \text{for } i \in \{2, \ldots, k\} \text{ and}\\
S_{k+1} &= \{S \subseteq E\} \setminus \bigcup_{l=1}^{k} S_l.
\end{aligned}$$

From the construction it is apparent that these really form a partition of the set of subsets of E.
We once again consider our objects in light of the inclusion exclusion (1.1), that is for $S \subseteq E$ we will consider the impact of $\bigcap_{e \in S} C_e$ on the sum. Since we have a partition we may consider the contribution of each block to this formula independently.
Let for this purpose $S \in S_1$. First, notice that for each broken circuit there is a unique circuit which formed the broken one by removing the edge of largest label under α. Since this is the case, we may assign each broken circuit its removed edge in a well defined manner. Let e_1 be this edge in the case of P_1, such that $P_1 \cup \{e_1\}$ is a circuit. There are now two possibilities: Either $e_1 \in S$ or $e_1 \notin S$. Define a map by removing it from S in the first case and adding it in the latter. This defines an involution $S_1 \to S_1$. Now if we consider our object S to be signed (via the associated sign $(-1)^{|S|}$ of $\left|\bigcap_{e \in S} C_e\right|$ in the inclusion exclusion), since the size of the object shifts by exactly 1 under this map we end up with a sign reversing involution, which means that the object S does not contribute to the sum at all. Therefore, no element of S_1 contributes to the inclusion exclusion.

We may employ similar logic for $S \in S_i$, $i \in \{2, \dots, k\}$ but need to be more careful and use a convenient property. If e_i is the edge such that $P_i \cup \{e_i\}$ is the original circuit, then $e_i \notin P_j$ for $j < i$. To show this assume on the contrary that $e_i \in P_j$. Since we are dealing with a broken circuit, e_i has a label larger than any edge in P_i, so it follows that $P_j > P_i$, which contradicts the ordering of the broken circuits $P_j \not\geq P_i$ and proves the claim. Now we would like to use a similar involution as before to show that no $S \in S_i$ contributes to our sum. However, we need to take care that we do not shift between different S_i when applying the map. Removing e_i from S will clearly still land us in S_i, so the only thing which we need to consider is that a set in S_i gets mapped to a set in S_j for $j < i$ if we add e_i. For this to happen, all edges of P_j need to already be in S, with the exception of the edge we add e_i, such that the broken circuit P_j suddenly emerges. But as previously established $e_i \notin P_j$, meaning that this case in fact never occurs and our involution works as planned.
Due to be above we arrive at the conclusion that at most elements in S_{k+1} contribute to the inclusion exclusion. At this point we want to make explicit what the set S_{k+1} contains: It is the set of all subsets of E which do not contain any of the broken circuits of G. Therefore this set is by definition the broken circuit complex, $S_{k+1} = B_G$. Note that for $S \in B_G$ the induced subgraph G_S is a forest, because if it contained a circuit it would also contain a broken circuit. We can think of G_S as adding edges one step at a time to the completely disconnected graph on $|V|$ vertices (and therefore $|V|$ connected components), so we end up with $|V| - |S|$ connected components. Since we may color each connected component freely this gives us $n^{|V|-|S|}$ colorings. Now also taking into consideration the sign associated to $|\bigcap_{e \in S} C_e|$ and writing everything as a sum gives us the desired result of

$$\chi_G(n) = \sum_{S \in B_G} (-1)^{|S|} n^{|V|-|S|}.$$

□

Lastly, we will look at an expansion which relates to the Möbius function of a certain lattice.

Definition 1.2.4. Let $G = (V, E)$ be a graph. A *connected partition* π of G is a partition of V such that each block of π is connected as an induced subgraph of G. We define an order relation on the set of connected partitions via refinement, that is for partitions $\pi = \{\pi_1, \dots, \pi_n\}$ and $\rho = \{\rho_1, \dots, \rho_m\}$ we say that $\pi \leq \rho$ if for all $i \in \{1, \dots, n\}$ there exists a $j \in \{1, \dots, m\}$ such that $\pi_i \subseteq \rho_j$. If $\pi \leq \rho$ we say that π is finer than ρ and that ρ is coarser than π. In this way we define a partially ordered set which turns out to be a geometric lattice, called the *lattice of contractions* L_G. As a finite lattice L_G has a unique smallest and largest element, denoted by $\hat{0} = \{\{v\} \text{ for } v \in V\}$ and $\hat{1} = \{V\}$ respectively. Since it is geometric it by definition admits a rank function and it is easy to see that the rank of any connected partition in this lattice is inverse proportional to the amount of blocks it has, that is $\text{rank}(\pi) = |V| - |\pi|$. We will denote the Möbius function associated to L_G by μ.

Theorem 1.2.5. *Let $G = (V, E)$ be a graph. Then*

$$\chi_G(n) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) n^{|\pi|}.$$

Proof. Our goal will be to find appropriate functions to apply Möbius inversion to and then specialize the resulting formula.
Fix a number $n \in \mathbb{N}$. We use the word coloring to refer to a coloring using at most the first n colors. For a connected partition π denote by G_π the induced subgraph of G whose connected components are the blocks of π.
For $\sigma \in L_G$ let $g(\sigma)$ be the number of colorings which are constant on blocks of σ. Furthermore let $f(\sigma)$ be the number of colorings which stay constant on blocks of σ and additionally satisfy $\kappa(v) \neq \kappa(w)$ if $\{v, w\} \in E \backslash E(G_\sigma)$, that is, two blocks of σ connected by an edge may not share a color. Then we would like to show that

$$g(\sigma) = \sum_{\sigma \leq \pi} f(\pi) \quad \text{for all } \sigma \in L_G \tag{1.2}$$

so that Möbius inversion yields

$$f(\sigma) = \sum_{\sigma \leq \pi} g(\pi)\mu(\sigma, \pi) \quad \text{for all } \sigma \in L_G.$$

Notice that for any given coloring κ there is exactly one $\pi \in L_G$ such that κ gets counted in $f(\pi)$. This ensures that $\sum_{\sigma \leq \pi} f(\pi)$ counts any coloring at most once. Since π is coarser than σ a coloring associated to π in the above sense stays constant on the blocks of σ. Due to this we conclude that $\sum_{\sigma \leq \pi} f(\pi)$ is exactly the number of all colorings which stay constant on blocks of σ, which is by definition the number $g(\sigma)$. We have therefore proven equation 1.2 and applying Möbius inversion readily proves the second formula.
To prove the Theorem itself we specialize $\sigma = \hat{0}$ and notice the following things. By treating each connected component as a vertex we see that the number $g(\pi)$ is the same as the number of colorings of the completely disconnected graph with $|\pi|$ vertices, meaning $g(\pi) = n^{|\pi|}$. Additionally, the sum now ranges over all partitions in L_G. Finally, $f(\hat{0})$ is easily seen to be the number of proper colorings of G so that all in all

$$\chi_G(n) = f(\hat{0}) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) n^{|\pi|}.$$

□

Remark 1.2.6. One can draw a connection between Theorem 1.2.3 and Theorem 1.2.5. Fixing some $\pi \in B_G$ we may consider $G_\pi = (V, E_\pi)$ and its related lattice of contractions L_{G_π}. Doing so leads to a natural order isomorphism $L_{G_\pi} \to [\hat{0}, \pi] \subseteq L_G$. Rephrasing both results to the graph G_π we get $\sum_{\sigma \in L_{G_\pi}} \mu(\hat{0}, \sigma) n^{|\sigma|} = \chi_{G_\pi}(n) = \sum_{S \in B_{G_\pi}} (-1)^{|S|} n^{|V|-|S|}$, meaning that we can compare coefficients of n^k. Specifically let $k = |\pi|$ so that only $\sigma = \hat{1}$ appears in the first sum. Since we now require $|\pi| = |V| - |S|$ on the latter

sum, we are left with the coefficient $\sum_{\substack{S\in B_{G_\pi}\\ |S|=|V|-|\pi|}} (-1)^{|V|-|\pi|}$. As previously established $\operatorname{rank}(\pi) = |V| - |\pi|$ in L_G (and L_{G_π}). Using this we conclude

$$\mu(\hat{0}, \pi)(-1)^{\operatorname{rank}(\pi)} = \#\,\operatorname{rank}(\pi)\text{-element non broken circuits in } B_{G_\pi}.$$

1.3. Applications

To conclude this chapter we want to discuss applications of the chromatic polynomial, specifically in the enumeration of acyclic orientations and hyperplane arrangements. A broader overview and additional results can be found in [4].

Definition 1.3.1. Let $G = (V, E)$ be a graph. An *orientation* of G is a directed graph $\mathcal{O}$ with vertex set V and such that for each edge $\{v, w\} \in E$ either (v, w) or (w, v) is an edge of $\mathcal{O}$. An orientation $\mathcal{O}$ is called *acyclic* if it has no (directed) cycles. Note that we use the notation (v, w) to mean an edge going from v to w, as directed edges can easily be identified with ordered tuples in our framework.

Theorem 1.3.2. *Let $G = (V, E)$ be a graph and denote by $O(G)$ the number of acyclic orientations of G. Then*

$$\chi(-1) = (-1)^{|V|} O(G).$$

To prove this we first need to consider an equivalent definition of the chromatic polynomial.

Lemma 1.3.3. *Let $G = (V, E)$ be a graph and $n \in \mathbb{N}$. Then $\chi_G(n)$ is equal to the number of pairs $(\kappa, \mathcal{O})$, where κ is any coloring with codomain contained in $[n]$ and $\mathcal{O}$ is an acyclic orientation of G which agrees with κ, in the sense that if $(v, w) \in E(\mathcal{O})$, then $\kappa(v) > \kappa(w)$.*

Proof. First recall that $\chi(n)$ simply counts the number of proper colorings with codomain contained in $[n]$. Now consider any pair $(\kappa, \mathcal{O})$ as above. Since $\mathcal{O}$ agrees with κ we know that $\kappa(v) \neq \kappa(w)$ for any edge $\{v, w\} \in E$, meaning κ is a proper coloring. Additionally, any proper coloring has a unique orientation that agrees with it (by choosing the orientation of each edge individually). Such an orientation is necessarily acyclic, meaning the number of pairs $(\kappa, \mathcal{O})$ is the same as the number of proper colorings, that is $\chi(n)$. □

What we will use in the proof of our Theorem is a function defined similarly to $\chi(n)$. For a graph G, let $\overline{\chi}_G(n) = \overline{\chi}(n)$ be the number of pairs $(\kappa, \mathcal{O})$ such that κ is a coloring of G with codomain contained in $[n]$ and $\mathcal{O}$ is an acyclic orientation of G which almost agrees with κ, in the sense that if $(v, w) \in E(\mathcal{O})$ then $\kappa(v) \geq \kappa(w)$. For ease of use we will say that $\mathcal{O}$ is compatible with κ in such a case and say that $(\kappa, \mathcal{O})$ is a compatible pair.

Proof of Theorem 1.3.2. As always we will simply use the word coloring to imply a coloring with codomain contained in $[n]$. We will first prove a relationship between χ and $\overline{\chi}$, that being

$$\bar{\chi}(n) = (-1)^{|V|}\chi(-n).$$

Recall that χ can also be defined by the following three properties:

- $\chi_{K_1}(n) = n$ for the one vertex graph K_1,
- $\chi_{G+H}(n) = \chi_G(n)\chi_H(n)$ and
- $\chi_G(n) = \chi_{G/e}(n) - \chi_{G\setminus e}(n)$ for all graphs G, H and edges $e \in E(G)$.

Proving the above relationship is therefore equivalent to proving that $\overline{\chi}$ satisfies the following three properties (which uniquely determine $\overline{\chi}$ due to an inductive argument):

- $\overline{\chi}_{K_1}(n) = n$ for the one vertex graph K_1,
- $\overline{\chi}_{G+H}(n) = \overline{\chi}_G(n)\overline{\chi}_H(n)$ and
- $\overline{\chi}_G(n) = \overline{\chi}_{G/e}(n) + \overline{\chi}_{G\setminus e}(n)$ for all graphs G, H and edges $e \in E(G)$.

On a technical note, since $\overline{\chi}$ is only defined on the set of nonnegative integers we have to consider the three properties of χ on nonpositive integers to successfully reformulate what we need to prove. That being said, we already saw that expanding the domain of χ to $\mathbb{C}$ poses no problem and furthermore these three properties translate without issues when switching the domain, which serves as a justification for the step we just took.
To check whether or not these three properties hold we have to count compatible pairs $(\kappa, \mathcal{O})$. The first equation immediately follows from the fact that there is only one acyclic orientation of K_1, that being K_1 itself considered as a directed graph. The second fact holds as we can consider colorings and acyclic orientations separately on connected components, so we take pairs $(\kappa_1, \mathcal{O}_1)$ and $(\kappa_2, \mathcal{O}_2)$ on G and H respectively and then combine them to get a pair for $G + H$ in the natural way. The third property is a little more involved.
Let $e = \{v, w\} \in E(G)$, κ be a coloring of $G\setminus e$ and $\mathcal{O}$ be an acyclic orientation of $G\setminus e$ compatible with κ. Denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the orientations we get by adding the edges (v, w) and (w, v) respectively to $\mathcal{O}$. Note that κ can be interpreted as a coloring of G itself and we will do so. The goal is now to show that for most such pairs $(\kappa, \mathcal{O})$ of $G\setminus e$ either $(\kappa, \mathcal{O}_1)$ or $(\kappa, \mathcal{O}_2)$ is a compatible pair for G and that the number of exceptions is $\overline{\chi}_{G/e}(n)$, in which case both are compatible pairs. Showing these two facts will prove that

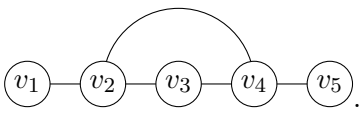
$$\overline{\chi}_G(n) = \overline{\chi}_{G/e}(n) + \overline{\chi}_{G\setminus e}(n)$$

as claimed.
Clearly there are three possibilities on how the colors of v and w could behave. Either the first is larger or the second is larger or they are equal. Therefore we distinguish the following cases.
If $\kappa(v) > \kappa(w)$, then $\mathcal{O}_2$ is not compatible with κ but $\mathcal{O}_1$ is. However $\mathcal{O}_1$ could contain a directed cycle. If that were the case, it would have to take the form $v \to w \to \cdots \to v$,

which would then imply that $\kappa(v) > \kappa(w) \geq \cdots \geq \kappa(v)$ for the colors of these vertices, a contradiction. So $\mathcal{O}_1$ is acyclic and in this case there is exactly one pair, that being $(\kappa, \mathcal{O}_1)$. The case where $\kappa(w) > \kappa(v)$ can be proven in a similar fashion.
Lastly, should $\kappa(v) = \kappa(w)$ then both $\mathcal{O}_1$ and $\mathcal{O}_2$ are compatible, but they may not be acyclic. We now show at least one of them is. Assume that both of them have a directed cycle, so that there exist paths $v \to w \to \cdots \to v$ and $w \to v \to \cdots \to w$ in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Implicitly this means that there is a directed path p_1 from v to w in $G\backslash e$, as well as a path p_2 from w to v. That being the case, $\mathcal{O}$ would then have to contain the directed cycle $v \xrightarrow{p_1} w \xrightarrow{p_2} v$, a contradiction. So at least one of $\mathcal{O}_1$ or $\mathcal{O}_2$ has to be acyclic, if not both.
Regarding our stated goal from earlier, this already shows that in most cases exactly one pair will be counted. It still remains to show that for exactly $\overline{\chi}_{G/e}(n)$ pairs the third case occurs in such a way that both $\mathcal{O}_1$ and $\mathcal{O}_2$ are acyclic.
To do this we denote a compatible pair for G/e by $(\kappa, \mathcal{O})_{G/e}$ and a compatible pair for $G\backslash e$ such that both $\mathcal{O}_1$ and $\mathcal{O}_2$ are acyclic by $(\kappa, \mathcal{O})_{G\backslash e}$. Additionally, denote the new vertex in G/e by $k_{v,w}$, where $e = \{v, w\}$. We will now construct a bijection $(\kappa, \mathcal{O})_{G\backslash e} \mapsto (\kappa', \mathcal{O}')_{G/e}$ which will show what we wanted. Define $\kappa \mapsto \kappa'$ by coloring everything the same. More precisely, this means $\kappa'(a) = \kappa(a)$ if $a \in V(G/e)\backslash\{k_{v,w}\}$ and $\kappa'(k_{v,w}) = \kappa(v) = \kappa(w)$. To get $\mathcal{O} \mapsto \mathcal{O}'$ we take the exact same orientation. This means that if $(a, b) \in E(\mathcal{O})$ then $(a, b) \in E(\mathcal{O}')$, where we use $k_{v,w}$ to substitute for v and w in $\mathcal{O}'$. The resulting orientation $\mathcal{O}'$ cannot have a two vertex cycle, since if $k_{v,w}$ was incident to one it would imply that there was an ingoing edge (a, v) and an outgoing one (w, a), or vice versa, in $\mathcal{O}$. This would further imply a cycle in either $\mathcal{O}_1$ or $\mathcal{O}_2$, contradicting our assumption. Therefore the orientation $\mathcal{O}'$ is acyclic. It is clear that both of these maps combine in such a way that $(\kappa', \mathcal{O}')$ is as described above and that they have an inverse which similarly preserves the desired properties.
With this we have proven the third property of $\overline{\chi}$, meaning that $\overline{\chi}(n) = (-1)^{|V|}\chi(-n)$ as claimed at the start. To conclude, we specialize $n = 1$: Since there is only one coloring of G with 1 color and every acyclic orientation is compatible with it, we see that $\overline{\chi}(1)$ is simply the number of acyclic orientations $O(G)$. □

Remark 1.3.4. The formula $\bar{\chi}(n) = (-1)^{|V|}\chi(-n)$ is interesting in itself, even though we only used it as a stepping stone for the enumeration of acyclic orientations. For example, one can use it to interpret the chromatic polynomial for negative integer arguments as a concrete combinatorial object. For further reading on this and the original proof see [7].

Example 1.3.5. Consider the graph seen in example 1.1.8:



We concluded that $\chi(n) = n(n-1)^3(n-2)$. Therefore of the $2^5 = 32$ orientations exactly $\chi(-1) \cdot (-1)^5 = (-1)(-2)^3(-3) \cdot (-1) = 24$ are acyclic.

We now turn our attention towards the enumeration of hyperplane arrangements. The following Theorem can be found in a paper by Stanley [8], specificially Theorem 2.5, but we will present an alternative proof here.

Definition 1.3.6. Consider the n-dimensional vector space $\mathbb{R}^n$ with Euclidean topology. A *hyperplane* of $\mathbb{R}^n$ is an $(n-1)$-dimensional subspace. A *hyperplane arrangement* is a finite set of hyperplanes. A *region* of a hyperplane arrangement $A = \{H_1, \dots, H_k\}$ is a connected component of $\mathbb{R}^n \setminus \left(\bigcup_{H \in A} H\right)$, that is, we consider the hyperplanes to be boundaries of our regions.

Since the concepts of graphs and hyperplane arrangements have no apparent connection, we must link the two somehow. To do this we first take our graph $G = (V, E)$ and give each vertex a unique label from 1 to $n = |V|$. It is convenient to identify the vertices with their labels, which is the same as taking $V = [n]$. Now we consider for the edge $e = \{v, w\} \in E$ the equality $x_v = x_w$, imposing that the v'th and w'th coordinate of a given vector $x \in \mathbb{R}^n$ are the same, which in turn defines the hyperplane $H_e = \{x \in \mathbb{R}^n \mid x_v = x_w\}$. Doing this for every edge gives the hyperplane arangement $A(G) = \{H_e \mid e \in E\}$.
We want to give a way to think about $A(G)$ and its regions. Although the following is transferable to dimensions higher than $n = 3$, we will focus on this case as it is easy to visualize. We start with the complete graph K_3 on the vertices [3]. To consider all areas of $A(K_3)$ it suffices to do so in a suitable neighborhood around the origin. Furthermore, since the vector $v = (1, \dots, 1)$ lies within each of the hyperplanes we may instead consider a suitably oriented plane with v as its normal vector. Putting these two ideas together, we take a circle of radius 1 centered in the origin with the normal vector v. The 3 hyperplanes cut it up in such a way that 6 sectors emerge. These of course correspond to the regions of the hyperplane arrangement, so when thinking about an arrangement we may as well only consider this circle. To visualize $A(G)$ for any 3 vertex graph we simply remove edges from K_3 and their corresponding cuts from the circle.
Doing a similar construction for K_n leads to an $(n-1)$-dimensional sphere with $\binom{n}{2}$ cuts in the form of $(n-2)$-dimensional planes. So to visualize $A(G)$ for any n-vertex graph G we start with this sphere and remove cuts according to the edges we need to remove from K_n to arrive at G.

Theorem 1.3.7. *Let $G = (V, E)$ be a graph with $V = [n]$ and $R(G)$ be the number of regions in the hyperplane arrangement $A(G)$. Then*

$$\chi_G(-1) = (-1)^n R(G).$$

Proof. Comparing our claim with Theorem 1.3.2 one recognizes that if we can show that there are as many regions of $A(G)$ as there are acyclic orientations of G we are done. We will use induction based on the amount of edges one removes from K_n to arrive at G.
In the case $G = K_n$ each region of $A(K_n)$ is uniquely defined by a linear ordering $x_{\sigma(1)} > \cdots > x_{\sigma(n)}$, where $\sigma \in S_n$ is a permutation. Similarly, each acyclic orientation

viewed as a partially ordered set is a linear ordering $\sigma(1) > \cdots > \sigma(n)$. Mapping $i \mapsto x_i$ in these two posets establishes a bijection between acyclic orientations and regions.
Now assume that $G = H \backslash e$ and that there is a bijection for H linking acyclic orientations to regions via their associated posets as in the base step. For each region R in $A(H)$ there is a unique minimal set of inequalities $I_R = \{\text{“}x_i > x_j\text{”} \mid (i,j) \in B_R\}$ which defines it, where $B_R \subseteq [n]^2$. Minimal here also implies that if “$x_i > x_j$”$\in I_R$ and “$x_j > x_k$”$\in I_R$ then “$x_i > x_k$”$\notin I_R$, since that relation already follows by transitivity. We consider I_R to be the cover relations of a partially ordered set P_R on $\{x_1, \ldots, x_n\}$, meaning P_R is the transitive closure of the relations found in I_R. If we instead start with an acyclic orientation $\mathcal{O}$ we can similarly interpret it as a partially ordered set $P_\mathcal{O}$ on $[n]$ with cover relations $I_\mathcal{O}$. It should be clear from construction that these two processes have a well defined inverse.
By assumption there is a bijection linking each region R to an acyclic orientation $\mathcal{O}$ such that $P_\mathcal{O} = P_R$ and consequently $I_R = I_\mathcal{O}$. Going from H to G means removing the edge $e = \{i, j\}$ from H and the corresponding hyperplane from $A(H)$. This extends to removing the inequalities $i > j$, $j > i$ respectively $x_i > x_j$, $x_j > x_i$ from I_R and $I_\mathcal{O}$. Let us denote the sets derived from I_R and $I_\mathcal{O}$ in this way by I'_R and $I'_\mathcal{O}$. There will always be regions R_1, R_2 such that $I_{R_1} \neq I_{R_2}$ but $I'_{R_1} = I'_{R_2}$, meaning the number of regions decreases. But since $I_{R_1} = I_{\mathcal{O}_1}$ and $I_{R_2} = I_{\mathcal{O}_2}$ via our pairing this happens exactly when $I'_{\mathcal{O}_1} = I'_{\mathcal{O}_2}$, meaning the number of acyclic orientations decreases by the same value and the number of regions and acyclic orientations is once again the same. □

Remark 1.3.8. Although recursively constructed bijections are usually quite opaque due to the number of steps involved, the one used in this proof is actually very direct. The reason we used induction was not to construct the bijection itself, but rather to prove that the map we used is well defined and injective. We explicitly write down the bijection here for readability.
Let $\mathcal{O}$ be an orientation of $G = ([n], E)$ and $P_\mathcal{O}$ its transitive closure considered as a poset. Swap each i for x_i in $P_\mathcal{O}$ to get P_R for some uniquely defined region R of $A(G)$, which can be determined by noting the cover relations in P_R. The inverse should be clear up until after $P_\mathcal{O}$ is constructed. After that orient each edge of G according to $P_\mathcal{O}$ to recover $\mathcal{O}$.

2. The Chromatic Function

Our goal in this chapter will be to generalize the chromatic polynomial in some sense and to study the properties and applications of that object. As was the case in Chapter 1, we will deal with finite, simple graphs. Additionally we will make heavy use of symmetric functions. Readers who are unfamiliar with them or simply wish to refresh their memory can read about the necessary concepts in Appendix B. Here we need to specify in which set the coefficients of our functions lie. Since we are dealing with enumeration it seems reasonable to take a commutative ring which contains $\mathbb{N}$, most commonly $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We will therefore work with the algebra $\Lambda_{\mathbb{Q}}$ and not mention the base ring any further unless necessary.

2.1. Definition and Basic Properties

First we need to define which object we are working with.

Definition 2.1.1. Let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting variables and $G = (V, E)$ be a graph. We define the formal power series

$$X_G(\mathbf{x}) = \sum_{\kappa \in K} \mathbf{x}^{\kappa(V)},$$

where $\mathbf{x}^{\kappa(V)} = \prod_{v \in V} x_{\kappa(v)}$, similar to $\mathbf{x}^{\pi}$ for an integer partition π, and K is the set of proper colorings of G. As with the chromatic polynomial we will simply denote this function by $X(\mathbf{x})$ if no confusion arises. Because of properties which we will discuss shortly we call this the *chromatic function* of G. For convenience we may sometimes omit the dependence on $\mathbf{x}$ for any formal power series, with the implicit understanding that it still exists.

The chromatic function was first introduced and studied by Richard P. Stanley in [6] and his paper serves as a foundation for this work. As such the bulk of the concepts discussed in this chapter and further reading can be found there.
Given the structure of this text it will be no surprise that the chromatic function of a graph is strongly linked to the chromatic polynomial of the same. As we will now see, the chromatic function already contains all the information the chromatic polynomial does.

Proposition 2.1.2. *Let $G = (V, E)$ be a graph. Then*

$$X_G(1^m) = \chi_G(m),$$

where $X_G(1^m) = X_G(\mathbf{x})\Big|_{\substack{x_i=1,\ i\leq m \\ x_i=0,\ i>m}}$.

Proof. This follows immediately from the definitions. Denote K_m the set of proper colorings with codomain in $[m]$ so that

$$X(1^m) = \sum_{\kappa \in K} \mathbf{x}^{\kappa(V)}\Big|_{\mathbf{x}=(1^m)} = \sum_{\kappa \in K_m} \mathbf{x}^{\kappa(V)}\Big|_{\mathbf{x}=(1^m)} = \sum_{\kappa \in K_m} 1 = \chi(m).$$

□

Although immediately apparent, we will also state the following fact because of its significance.

Proposition 2.1.3. *The power series $X_G(\mathbf{x})$ is a symmetric function.*

Proof. We will show that $X(x_1, x_2, x_3, \dots) = X(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, \dots)$ for all transpositions $\tau \in S_{\mathbb{N}}$. Since the transpositions generate all permutations on $\mathbb{N}$ the claim then follows. Let $\tau = (i\, j)$, so that the above equality is equivalent to the statement, that if in all proper colorings of G we swap the colors i and j, each monomial $\mathbf{x}^\alpha$ (where α is some weak composition) will appear in both sums in equal number. Since swapping i and j is an involution on the set of proper colorings we are done. □

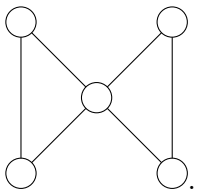
Another property which follows immediately from the definitions is the following.

Proposition 2.1.4. *Let G and H be graphs and denote their disjoint union by $G + H$. Then we have*

$$X_{G+H} = X_G X_H.$$

Proof. Any proper coloring κ of $G + H$ consists of two independent proper colorings κ_1 on G and κ_2 on H. Additionally, each combination of some κ_1 and κ_2 will result in a unique κ. Algebraically this translates to $\mathbf{x}^\kappa = \mathbf{x}^{\kappa_1}\mathbf{x}^{\kappa_2}$ from which the claim follows. □

Example 2.1.5. Consider the graph



We need at least 3 and can use at most 5 colors for a proper coloring. Consider the case of 3 colors: We will look at essentially different colorings, meaning that switching two colors which appear in equal amount will be considered the same coloring. Due to us disregarding this symmetry, we need to adjust the true number of colorings at the end of our reasoning.
The middle vertex will have a unique color. If we then choose a second color for the upper left vertex, the lower left one will be of the third color. Those same two colors then need to be used for the right vertices, we do however have to decide which of them will be on top, giving us a choice of 2.

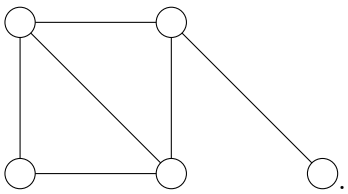
Thinking about X_G, we quickly realize that all monomials corresponding to such colorings must be of the form $x_i^2 x_j^2 x_k$ with pairwise distinct $i, j, k \in \mathbb{N}$. Considering this monomial as the partition $(2, 2, 1)$, there are 3 possible rearrangements of the same as compositions, those being $(2, 2, 1)$, $(2, 1, 2)$ and $(1, 2, 2)$. The reason to consider these compositions is that we may see them as all possible powers of x_i, x_j, x_k under the assumption $i < j < k$.
Since we only worked with essentially different colorings until now, the two 2's do not need to be permuted, as it would result in the same coloring. That being said, to get from essentially different colorings to proper ones we do need to consider this symmetry. Instead of doing this by counting all new colorings, we instead realize that we only need to multiply what we already have by 2, as the newly added colorings would all have monomials we already considered. Putting all of this information together and relabeling the indices we get a contribution to X_G of

$$4 \sum_{i<j<k \in \mathbb{N}} (x_i^2 x_j^2 x_k + x_i^2 x_j x_k^2 + x_i x_j^2 x_k^2) = 4m_{221},$$

a (multiple of a) monomial symmetric function. We can apply similar reasoning to colorings with 4 and 5 colors to arrive at

$$X_G = 4m_{221} + 24m_{2111} + 120m_{11111}.$$

As it turns out, there is another graph with this chromatic function, meaning that X_G does not fully characterize G. This other graph is given by



In this example it was very natural to expand X_G in terms of the basis of monomial symmetric functions. This leads one to the question if the same can be said for arbitrary graphs and whether something similar can be done in other popular bases for symmetric functions. Furthermore, one might wonder if the coefficients appearing in those expansions can be given some kind of meaning.

Remark 2.1.6. Depending on one's viewpoint the following sections could be viewed as backwards. Usually one would be faced with a given problem, say, the enumeration of acyclic orientations, and then use the chromatic polynomial or function to solve the problem, as we did in Section 1.3. Although the following can (and should) be similarly understood as applications of the chromatic function, since the focus of this work is to study X_G in its own right we will instead frame the results in terms of an expansion of X_G in bases of $\Lambda_{\mathbb{Q}}$.

2.2. Expansion in m_λ

One of the most common bases used for symmetric functions is the one of monomial symmetric functions, which we will look at first. One known way to express the chromatic function of a graph in this basis is by considering stable partitions.

Definition 2.2.1. Let $G = (V, E)$ be a graph. A *stable partition* π of G is a partition of V in which each block is totally disconnected as an induced subgraph of G. We can assign to any set partition $\pi = \{\pi_1, \dots, \pi_l\}$ of a finite set (where without loss of generality $|\pi_i| \geq |\pi_j|$ if $i < j$) an integer partition λ, called the *type* of π and denoted $\lambda = \text{type}(\pi)$, by recording for each block π_i its size $\lambda_i = |\pi_i|$.

Theorem 2.2.2. *Let $G = (V, E)$ be a graph with $|V| = d$. For a partition $\lambda \vdash d$ denote by S_λ the number of stable partitions of G of type λ. Then*

$$X_G = \sum_{\lambda \vdash d} S_\lambda \tilde{m}_\lambda.$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_l) = \langle 1^{r_1} \dots w^{r_w} \rangle$ be a partition of d. The coefficient of any monomial of the form $x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}$ in X_G, where $i_1, \dots, i_l$ are distinct, is by definition the number of ways to properly color G by using the color i_k exactly λ_k times.
We give a perspective on how this relates to stable partitions: Fix $i_1, \dots, i_l$ distinct and let $\pi = \{\pi_1, \dots, \pi_l\}$ be a stable partition of type λ. Color every vertex in π_j with the color i_j, such that the corresponding coloring is by definition proper and produces the monomial $x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}$. If $\lambda_j = \lambda_{j+1}$ for some j then we could alternatively swap the colors used for the blocks π_j and π_{j+1}. If there are instead k blocks of equal size, then clearly we have $k!$ choices on how to attribute their colors and still get the same monomial. Generalizing this further, we get $r_1! \cdots r_w!$ possible ways to assign colors and still end up with the desired monomial. Varying this construction over all distinct $i_1, \dots, i_l$ shows that each stable partition of type λ contributes $r_1! \cdots r_w! m_\lambda = \tilde{m}_\lambda$ to X_G. Since the number of such partitions is S_λ and each coloring appearing in X_G can be associated to a unique stable partition we may rearrange the sum as

$$X_G = \sum_{\pi \text{ stable}} \tilde{m}_{\text{type}(\pi)} = \sum_{\lambda \vdash d} \sum_{\substack{\pi \text{ stable} \\ \text{type}(\pi) = \lambda}} \tilde{m}_\lambda = \sum_{\lambda \vdash d} S_\lambda \tilde{m}_\lambda.$$

□

Remark 2.2.3. Thinking back on our reasoning in example 2.1.5, what we then called "essentially different colorings" is accounted for by a combination of the amount of stable partitions and the prefactor found in the augmented monomial function. Given this and the previous Theorem, the way in which we calculated X_G becomes significantly more methodical.

2.3. Expansion in p_λ

The next basis we will consider is that of the power sum symmetric functions. There are a few objects which are relevant for expanding X_G in this case. The first one we will look at is edge subsets.

Theorem 2.3.1. *Let $G = (V, E)$ be a graph and $S \subseteq E$. Consider the induced subgraph $G_S = (V, S)$ and interpret it as a connected partition of G. Denote by $\lambda(S) = \text{type}(G_S)$ the integer partition recording the size of its components. Then*

$$X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}.$$

Proof. Let $S \subseteq E$ and $G_S = (V, S)$. Denote by K the set of all colorings of G and by K_S the subset of all colorings which stay constant on connected components of G_S. By definition we can express

$$p_{\lambda(S)}(\mathbf{x}) = \sum_{\kappa \in K_S} \mathbf{x}^\kappa.$$

As a result

$$\begin{aligned} \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}(\mathbf{x}) &= \sum_{S \subseteq E} (-1)^{|S|} \sum_{\kappa \in K_S} \mathbf{x}^\kappa \\ &= \sum_{\kappa \in K} \mathbf{x}^\kappa \sum_{S \subseteq E_\kappa} (-1)^{|S|}, \end{aligned}$$

where E_κ is the set of edges for which the incident vertices have the same color with respect to κ. In this way the last step is simply a double counting argument. If for some coloring κ the set E_κ is not empty then $\sum_{S \subseteq E_\kappa} (-1)^{|S|} = 0$, meaning those expressions do not contribute overall. In the other case E_κ is empty, meaning $\sum_{S \subseteq E_\kappa} (-1)^{|S|} = (-1)^{|\emptyset|} = 1$. Note that E_κ is empty if and only if κ is a proper coloring, meaning we are summing the terms $\mathbf{x}^\kappa$ over all proper colorings κ of G, which is the definition of X_G. □

Remark 2.3.2. In Theorem 1.2.1 we showed that $\chi_G(n) = \sum_{S \subseteq E} (-1)^{|S|} n^{c(S)}$, where $c(S)$ was the number of connected components of $G_S = (V, S)$, which looks vaguely similar to what we just proved. In fact, one can see that $p_{\lambda(S)}(1^n) = n^{c(S)}$, since $p_{\lambda(S)}(1^n)$ can be interpreted to count the number of all colorings $\kappa : V \mapsto [n]$ which stay constant on connected components of G_S. Applying this to our new result yields

$$\chi(n) = X(1^n) = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}(1^n) = \sum_{S \subseteq E} (-1)^{|S|} n^{c(S)}.$$

We will see that X_G also generalizes other properties of χ_G in a similar fashion.

Another way to express X_G in our current basis requires the Möbius function of the lattice of contractions L_G, see Definition 1.2.4.

Theorem 2.3.3. *Let G be a graph. Then*

$$X_G = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}.$$

Proof. Let $\sigma \in L_G$ and define

$$X_\sigma(\mathbf{x}) = \sum_{\kappa \in K_\sigma} \mathbf{x}^\kappa,$$

where K_σ is the set of all colorings such that κ is constant on the blocks of σ and for two connected vertices v, w in different blocks we have $\kappa(v) \neq \kappa(w)$. For any fixed coloring κ there is a unique partition $\pi \in L_G$ such that $\mathbf{x}^\kappa$ appears in X_π: By the two rules we imposed any pair of connected vertices must belong to the same block if they share a color and must belong to different blocks if they do not. Since this fixes the connected components of π it is already uniquely determined as a connected partition. It follows for any $\pi \in L_G$ that $p_{\mathrm{type}(\pi)}$ consists of monomials which are consistent with the partition π, in the sense that if $\sigma \geq \pi$ then all monomials of X_σ appear in $p_{\mathrm{type}(\pi)}$. It is easy to see that this characterization in fact covers all monomials in $p_{\mathrm{type}(\pi)}$.
Since each $\mathbf{x}^\kappa$ belongs to a unique X_σ we may simply write

$$p_{\mathrm{type}(\pi)} = \sum_{\sigma \geq \pi} X_\sigma.$$

Now applying Möbius inversion on X_σ and $p_{\mathrm{type}(\sigma)}$ viewed as functions $L_G \to \Lambda_{\mathbb{Q}}$ results in

$$X_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_{\mathrm{type}(\sigma)}.$$

Finally set $\pi = \hat{0}$ and notice $X_{\hat{0}} = X_G$, finishing the proof. □

Remark 2.3.4. Once again we see a striking resemblance to a result about the chromatic polynomial, this time Theorem 1.2.5. In a similar fashion to before one may specialize $X_G(1^n)$ and interpret $p_{\mathrm{type}(\pi)}$ to link the two statements.

The last way to expand X_G in the basis of power sum symmetric functions we will discuss here comes in the form of broken circuits, see Definition 1.2.2.

Theorem 2.3.5. *Let $G = (V, E)$ be a graph and for $S \subseteq E$ define $\lambda(S)$ as in Theorem 2.3.1. Then*

$$X_G = \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)}.$$

Proof. Let $\pi \in L_G$ with $\mathrm{rank}(\pi) = k$, that is $|\pi| = |V| - k$. Denote by G_π the spanning subgraph of G which encompasses all edges $\{v, w\} \in E$ for which v and w are in the same block of π. By Remark 1.2.6 we have that $(-1)^k \mu(\hat{0}, \pi)$ is equal to the number of k-element subsets $S \subseteq E(G_\pi)$ which contain no broken circuit of G_π (with edge labels inherited from G). Those S then also contain no broken circuits of G by extension. Denote the set of those k-element subsets by $B(\pi)$ and let $S \in B(\pi)$. Since $|S| = k$ and S

contains no broken circuit (and therefore no circuit) of G we conclude that G_S has $|V|-k$ connected components. The only way this can be the case is if the connected components of G_S are the blocks of π, so $\lambda(S) = \text{type}(\pi)$ and by assumption $|S| = \text{rank}(\pi)$. Due to this reasoning we conclude that each $S \in B_G$ is contained in a unique $B(\pi)$. Using all of these facts and applying Theorem 2.3.3 gives us

$$\begin{aligned} X_G = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\text{type}(\pi)} &= \sum_{\pi \in L_G} (-1)^{\text{rank}(\pi)} p_{\text{type}(\pi)} \sum_{S \in B(\pi)} 1 \\ &= \sum_{\pi \in L_G} \sum_{S \in B(\pi)} (-1)^{|S|} p_{\lambda(S)} \\ &= \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)}. \end{aligned}$$

□

Remark 2.3.6. This result is a generalization of the Broken Circuit Theorem 1.2.3. As usual specialize $X_G(1^n)$ and interpret $p_{\lambda(S)}$ to see this.

2.4. Expansion in e_λ

Lastly, we will expand X_G in the basis of elementary symmetric functions. Until now there were ways to expand X_G in a basis and give the coefficients meaning. However, for e_λ it is not quite so simple. We will go into further detail and consider open problems regarding this topic in Chapter 3, but here we want to focus on the results found in the original paper by Stanley [6]. We assume from this point forward that the reader is familiar with the concept of and the notation surrounding quasisymmetric functions. A basic introduction of what we need here can be found in Appendix B.2.
We now spend some time defining a quasisymmetric function which will be helpful in a proof later.

Definition 2.4.1. Let P be any partially ordered set of size d. Then define

$$X_P(\mathbf{x}) = \sum_{\mu} x_{\mu(p_1)} \cdots x_{\mu(p_d)},$$

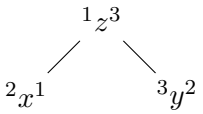
where the sum ranges over all order preserving maps $\mu : P \to \mathbb{N}$, meaning each μ can be considered as a linear extension of P. One can see that X_P is quasisymmetric and we wish to expand it in the basis of fundamental quasisymmetric functions. Based on on the theory of P-partitions (see [9] Section 4.5 and [10] Section 7.19) it is possible to expand X_P in $\{Q_S \,|\, S \subseteq [d-1]\}$. For this fix any order reversing map $\omega : P \to [d]$. Then when considering a linear extension $\alpha : P \to [d]$ we can interpret α as a permutation of $[d]$ by considering $a = \big(\omega(\alpha^{-1}(1)), \ldots, \omega(\alpha^{-1}(d))\big)$ (in one line notation).
For any permutation $\sigma \in S_d$ define its *descent set* as $D(\sigma) = \{j \in [d-1] \,|\, \sigma(j) > \sigma(j+1)\}$, which naturally extends to α by interpreting it as a permutation. Denote by $\mathcal{L}(P, \omega)$ the

set of linear extensions α of P when considered as permutations a through ω. In the case that P is a poset on $[d]$ we will call a map $\mathbb{N} \supseteq [d] \to [d] = P$ a *sequencing*, while a map $P = [d] \to [d] \subseteq \mathbb{N}$ will be called a *labeling* to distinguish the two concepts. In this sense we use the labeling ω and sequencing α^{-1} to construct our permutation. The following can be shown by using the theory of P-partitions, see [10] Corollary 7.19.5.

Theorem 2.4.2. *Let P be a partially ordered set of size d and ω be as above. Then*

$$X_P = \sum_{\alpha \in \mathcal{L}(P,\omega)} Q_{D(\alpha),d}.$$

Example 2.4.3. Consider the partially ordered set on $\{x, y, z\}$ given by the relations $z > x$ and $z > y$. There are exactly two order reversing maps, since z needs to be mapped to 1. We pick the one for which $\omega(x) = 2$ and $\omega(y) = 3$. In a similar fashion there are exactly two linear orderings, since z will necessarily be mapped to 3. They are given by $(\alpha_1(x), \alpha_1(y), \alpha_1(z)) = (1, 2, 3)$ and $(\alpha_2(x), \alpha_2(y), \alpha_2(z)) = (2, 1, 3)$. Now by definition $\mathcal{L}(P, \omega)$ consists of the permutations $a_1 = (\omega(\alpha_1^{-1}(1)), \omega(\alpha_1^{-1}(2)), \omega(\alpha_1^{-1}(3))) = (\omega(x), \omega(y), \omega(z)) = (2, 3, 1)$ and $a_2 = (3, 2, 1)$. One way to easily visualize the construction of each permutation is by considering P and labeling it according to ω and α in the format ${}^{\omega(p)}p^{\alpha(p)}$ for each $p \in P$, which gives



for a_1 in our example. The way to read of the permutation from this is that $a_1(1) = 2$ since we have the node ${}^2x^1$.
Clearly we have the descent sets $D_1 = D(a_1) = \{2\}$ and $D_2 = D(a_2) = \{1, 2\}$, so by Theorem 2.4.2 we get

$$X_P = Q_{\{2\},3} + Q_{\{1,2\},3} = Q_{\{2\},3} + e_3.$$

With this construction we are finally ready to prove our first result.

Theorem 2.4.4. *Let $G = (V, E)$ be a graph and denote $d = |V|$. Let furthermore*

$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$$

be the expansion of X_G in the basis of elementary symmetric functions. Denote by $\mathrm{sink}(G, j)$ *the number of acyclic orientations of G with exactly j sinks. Then*

$$\mathrm{sink}(G, j) = \sum_{\substack{\lambda \vdash d \\ l(\lambda) = j}} c_\lambda.$$

Proof. Remember that for an acyclic orientation $\mathcal{O}$ we say that a proper coloring κ agrees with $\mathcal{O}$, if for all edges $(v,w) \in E(\mathcal{O})$ we have $\kappa(v) > \kappa(w)$. It is easy to see that for each proper coloring κ there is a unique acyclic orientation $\mathcal{O}$ such that κ agrees with $\mathcal{O}$. This means that we have a partition of the set of all proper colorings K of G into sets of $\mathcal{O}$-agreeable proper colorings $K_\mathcal{O}$ by $K = \bigcup K_\mathcal{O}$. Due to this, by defining $X_\mathcal{O} = \sum_{\kappa \in K_\mathcal{O}} x_\kappa$, we also get that $X_G = \sum_\mathcal{O} X_\mathcal{O}$ when we sum over all acyclic orientations. We would like to interpret $\mathcal{O}$ as a partially ordered set by $(v,w) \in E(\mathcal{O})$ if and only if $v > w$. This will fail since the relations one gets are generally not transitive. So we instead consider the transitive closure $\bar{\mathcal{O}}$ of $\mathcal{O}$, that is $V(\mathcal{O}) = V(\bar{\mathcal{O}})$, $E(\mathcal{O}) \subseteq E(\bar{\mathcal{O}})$ and if $(v_1,v_2),(v_2,v_3) \in E(\bar{\mathcal{O}})$ then also $(v_1,v_3) \in E(\bar{\mathcal{O}})$. Since $\mathcal{O}$ is acyclic we may interpret $\bar{\mathcal{O}}$ as a poset in the above mentioned way. Notice that all proper colorings agreeing with $\mathcal{O}$ can be considered as order preserving maps $\bar{\mathcal{O}} \to \mathbb{N}$ and vice versa, so that by the definition of $X_{\bar{\mathcal{O}}}$ for a poset $X_\mathcal{O} = X_{\bar{\mathcal{O}}}$, meaning ultimately $X_G = \sum_\mathcal{O} X_{\bar{\mathcal{O}}}$.
We now define a function which will extract precisely the information we wish to prove. Let t be an indeterminate and define on the set of quasisymmetric functions of degree d the map $\varphi : \mathcal{Q}_d \to \mathbb{Q}[t]$ as

$$\varphi(Q_D) = \begin{cases} t(1-t)^i & \text{if } D = \{i+1,\dots,d-1\} \\ 0 & \text{else} \end{cases}$$

on the basis of fundamental quasisymmetric functions and by linear extension elsewhere (here one would need to exchange $\mathbb{Q}$ depending on the choice of field, otherwise the proof works the same).
I claim that for any d-element poset P we have $\varphi(X_P) = t^m$, where m is the number of minimal elements of P.
To prove this we first need to fix some order reversing bijection $\omega : P \to [d]$. What we will do now is construct all linear extensions α of P with descent set $D = \{i+1,\dots,d-1\}$ when considered as permutations as we did above. To get this descent set we are forced to write the permutation as $(u_1,\dots,u_d)$ with $u_1 < \cdots < u_{i+1} > \cdots > u_d$ (as labels). Applying ω^{-1} pointwise we get $(v_1,\dots,v_d)$ with $v_1 \not< \cdots \not< v_{i+1} \not> \cdots \not> v_d$ (in P, with $\not<$ considered transitive), since ω is order reversing. By definition we have $v_k = \alpha^{-1}(k)$ for all $k \in [d]$ at this point, so we also get $v_1 \not> \cdots \not> v_d$ (with $\not>$ considered transitive), since α is a linear extension, hence order preserving and trivially $1 < \cdots < d$.
What we can gather from this is the following: The element v_{i+1} is minimal in P and since u_{i+1} was maximal, that means v_{i+1} is the unique minimal element of P with largest label under ω. Furthermore, v_j and v_k are incomparable for all $j,k \in [i+1]$ and if $j \in [i+1]$ and $k \in [d]\backslash[i+1]$, either $v_j < v_k$ or they are also incomparable. This leads one to the conclusion that all v_j for $j \in [i+1]$ are minimal elements of P. To summarize, when we want to construct a permutation with descent set $D = \{i+1,\dots,d-1\}$ we need an α which takes i arbitrary minimal elements which do not have the largest label under ω, listed in increasing order according to ω, then take the minimal element with largest label under ω and finally list the remaining elements in decreasing order according to ω. It should at this point be easy to see that every α constructed in this manner will satisfy our conditions, meaning that we now have an exhaustive list. We have $\binom{m-1}{i}$ such α,

where m is the number of minimal elements of P, based on the initial choices of the minimal elements. Based on Theorem 2.4.2 we may now express

$$\begin{aligned}\varphi(X_P) &= \sum_{\alpha\in\mathcal{L}(P,\omega)} \varphi\left(Q_{D(\alpha)}\right) \\ &= \sum_{i=0}^{m-1} \binom{m-1}{i} t(t-1)^i \\ &= t^m,\end{aligned}$$

finally proving the claim.
Going back to our acyclic orientations, note that sinks of $\mathcal{O}$ perfectly translate to minimal elements of $\bar{\mathcal{O}}$. This means that

$$\varphi(X_G) = \sum_{\mathcal{O}} \varphi(X_{\bar{\mathcal{O}}}) = \sum_{j\in\mathbb{N}} \mathrm{sink}(G,j)t^j.$$

Note that the sum is finite since G is finite. Now we want to compute $\varphi(e_\lambda)$ for $\lambda = (\lambda_1,\ldots,\lambda_l) \vdash d$. An easy way to do this is to find a poset P such that $X_P = e_\lambda$, one such poset being the disjoint union of chains of length $\lambda_1,\ldots,\lambda_l$. Then by what we argued earlier it follows $\varphi(e_\lambda) = t^l$. Denote $l(\lambda)$ the length of λ. Finally, we apply φ to the initial expansion of X_G to get

$$\varphi(X_G) = \sum_{\lambda\vdash d} c_\lambda \varphi(e_\lambda) = \sum_{\lambda\vdash d} c_\lambda t^{l(\lambda)}.$$

If we compare both expansions of $\varphi(X_G)$ and extract the coefficient of t^j we are left with

$$\mathrm{sink}(G,j) = \sum_{\substack{\lambda\vdash d \\ l(\lambda)=j}} c_\lambda.$$

□

Remark 2.4.5. This result once again generalizes a property of the chromatic polynomial, but this time holds significantly more information than its counterpart. Consider Theorem 1.3.2, in which we proved that $\chi(-1) = (-1)^{|V|}O(G)$, where $O(G)$ was the number of acyclic orientations of G. Notably, nowhere in this result can one extract the number of acyclic orientations with a given number j of sinks. To derive this equation from what we just proved we consider

$$\chi_G(n) = X_G(1^n) = \sum_{\lambda\vdash d} c_\lambda e_\lambda(1^n),$$

where for $\lambda = (\lambda_1,\ldots,\lambda_l)$ we have

$$e_\lambda(1^n) = \prod_{\lambda_j} \sum_{i_1<\cdots<i_l\le n} 1 = \binom{n}{\lambda_1}\cdots\binom{n}{\lambda_l}.$$

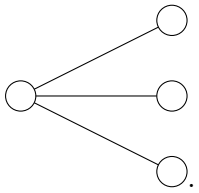
Setting $n = -1$ and using the usual definition utilizing falling factorials $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ for $k \in \mathbb{N}$ we get $\binom{-1}{\lambda_j} = (-1)^{\lambda_j}$ and so

$$\chi_G(-1) = \sum_{\lambda \vdash d} c_\lambda (-1)^{\lambda_1} \cdots (-1)^{\lambda_l} = (-1)^d \sum_{\lambda \vdash d} c_\lambda = (-1)^d O(G)$$

by Theorem 2.4.4, since we are summing over all λ irrespective of their length.

Remark 2.4.6. As of writing no way has been found to interpret the coefficients c_λ in terms of combinatorial objects similar to what we did with other bases. One problem which makes this difficult is the fact that in general we do not have $c_\lambda \geq 0$, meaning that if there is a nice interpretation it would need to be more elaborate than simple enumeration. Another issue is the fact that results about c_λ usually come in a form similar to the one in Theorem 2.4.4, that is to say sums.

Example 2.4.7. Consider the complete bipartite graph with groups of size 1 and 3, the claw graph $K_{1,3}$:



Since it is loopless every orientation is necessarily acyclic. There are 4 ways to orient it with 1 sink, 3 ways to orient it with 2 sinks, 1 way to orient it with 3 sinks and no ways to orient it with 4 sinks, so Theorem 2.4.4 gives rise to the system

$$\begin{aligned} 4 &= c_{(4)} \\ 3 &= c_{(3,1)} + c_{(2,2)} \\ 1 &= c_{(2,1,1)} \\ 0 &= c_{(1,1,1,1)}. \end{aligned}$$

Although we now know the coefficients of $e_{(4)}$, $e_{(2,1,1)}$ and $e_{(1,1,1,1)}$ we have no way to extract the other two from this. We could however express X_G in terms of p_λ using Theorem 2.3.1

$$X_G = p_{(1,1,1,1)} - 3p_{(2,1,1)} + 3p_{(3,1)} - p_{(4)}$$

and then calculate a basis change based on the Girard-Newton identities

$$\begin{aligned} p_{(4)} &= e_{(1,1,1,1)} - 4e_{(2,1,1)} + 4e_{(3,1)} + 2e_{(2,2)} - 4e_{(4)} \\ p_{(3,1)} &= e_{(1,1,1,1)} - 3e_{(2,1,1)} + 3e_{(3,1)} \\ p_{(2,1,1)} &= e_{(1,1,1,1)} - 2e_{(2,1,1)} \\ p_{(1,1,1,1)} &= e_{(1,1,1,1)}. \end{aligned}$$

Inserting into the previous equation then leaves us with

$$X_G = 4e_{(4)} + 5e_{(3,1)} - 2e_{(2,2)} + e_{(2,1,1)}.$$

A few things we can draw from this example: Although X_G is not p-positive, the absolute value of the coefficients counts combinatorial objects (for example certain edge subsets). Furthermore, X_G is not e-positive and when we tried to express it in the basis of elementary symmetric functions, we were forced to do so indirectly, losing a simple interpretation in the process. However, for some particular partitions we did find that their coefficient counts acyclic orientations with a certain number of sinks. If we could somehow refine the sums of c_λ further, then we would gain greater insight into what those coefficients might be, which we will see now.

Definition 2.4.8. Let $G = (V, E)$ be a graph with $|V| = d$. We call a partition $\lambda \vdash r \leq d$ *allowable* with respect to G if there is an induced subgraph $H \subseteq G$ with $|V(H)| = r$ and a stable partition of H of type λ. We can define a kind of maximal partition which will be useful, although this maximality is not to be understood in relation to any poset (although its connection to the dominance order will become apparent). We call a partition $\lambda \vdash r \leq d$ of length l *maximal* in the set of allowable partitions, if for every other allowable $\mu \vdash s \leq d$ we have that either $\lambda_i = \mu_i$ for all $i \leq l$ or $\lambda_1 + \cdots + \lambda_i > \mu_1 + \cdots + \mu_i$ for some $i \leq l$ (with the convention $\mu_i = 0$ if it does not exist).

Remark 2.4.9. Calling a partition maximal as above is justified, since $\lambda = (\lambda_1, \ldots, \lambda_l)$ is maximal if and only if there is an allowable partition $\mu \vdash d$ with the following properties: It extends λ in the sense that $\mu = (\lambda_1, \ldots, \lambda_l, \cdots)$ and μ is maximal in the poset of allowable partitions using dominance ordering. Note that we used $\ldots$ (on the floor) to imply the extension of a pattern, while we used $\cdots$ (centered) in place of arbitrary and unspecified elements. This might seem like an odd choice, but this convention will help formulating and proving the next Theorem with increased clarity. We will however still use centered dots for sums and other operations as is typical, since there is little chance of confusion.

Definition 2.4.10. Let G be a graph and $\mathcal{O}$ be an acyclic orientation of G. We define the *sink sequence* $\mathrm{ss}(\mathcal{O}) = (s_1, s_2, \ldots)$ of $\mathcal{O}$ in the following way: We let s_1 be the number of sinks of $\mathcal{O}$. Next, we remove those sinks from G (and consequently $\mathcal{O}$) to get a new graph. Then we let s_2 be the number of sinks of the new graph and repeat this process until there are no vertices left. Since we have already established that acyclic orientations are analogous to posets, this is equivalent to recording and removing the minimal elements of $\bar{\mathcal{O}}$ in every step. Using this alternative definition of a sink sequence we can extend the concept to finite posets P, in which case we will simply write $\mathrm{ss}(P)$.

Example 2.4.11. Let $G = K_{1,3}$, the claw graph. Since G is a tree every orientation is already acyclic. All possible sink sequences are then $(3, 1)$, $(2, 1, 1)$, $(1, 1, 2)$ and $(1, 3)$. Notice that there is a symmetry in this case, which can be accomplished by reversing the direction of every edge in a given acyclic orientation. This demonstrates that the sink sequence is in general not a partition. Note however, that a reversal of the edges does not always give the reversed sink sequence, as we will see in a later example.

Note that the following was formulated incorrectly in [6] and there has been an errata since, which was written by Timothy Chow.

Theorem 2.4.12. *Let $G = (V, E)$ be a graph and denote $d = |V|$. Let further $\mu = (\mu_1, \dots, \mu_l) \vdash r \leq d$ be a maximal allowable partition in the sense above. For $0 \leq j \leq d - r$ define $\mathrm{sink}(G, \mu, j)$ as the number of acyclic orientations $\mathcal{O}$ of G, such that the corresponding sink sequence is of the form $\mathrm{ss}(\mathcal{O}) = (\mu_1, \dots, \mu_l, j, \cdots)$. Then for the expansion $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$ we have*

$$\mathrm{sink}(G, \mu, j) = \sum_{\lambda} c_\lambda,$$

where the sum ranges over all $\lambda \vdash d$ such that $\lambda' = (\mu_1, \dots, \mu_l, j, \cdots)$ for its conjugate partition.

Proof. This proof will follow a similar reasoning to the one of Theorem 2.4.4. Fix some $0 \leq r \leq d$ and let $\mu = (\mu_1, \dots, \mu_l) \vdash r$ be a maximal allowable partition. Define the map $\varphi_\mu : \mathcal{Q}_d \to \mathbb{Q}[t]$ by

$$\varphi_\mu(Q_D) = \begin{cases} 1 & \text{if } r = d \text{ and } D = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \cdots + \mu_{l-1}\} \\ t & \text{if } r = d - 1 \text{ and } D = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \cdots + \mu_l\} \\ t(t-1)^i & \text{if } r < d - 1 \text{ and } D = \begin{smallmatrix}\{\mu_1, \mu_1+\mu_2, \dots \mu_1+\cdots+\mu_l, \\ r+i+1, r+i+2, \dots, d-1\}\end{smallmatrix} \\ 0 & \text{else} \end{cases}$$

and linear extension (once again we can simply exchange $\mathbb{Q}[t]$ to fit the field we chose). I claim that for any d-element poset P:
If $r < d - 1$ and $\mathrm{ss}(P) = (\mu_1, \dots, \mu_l, m, \cdots)$ then $\varphi_\mu(X_P) = t^m$. Also if $r = d$ and $\mathrm{ss}(P) = (\mu_1, \dots, \mu_l)$ then $\varphi_\mu(X_P) = 1 = t^m$ where $m = 0$ by convention, since we could express $\mathrm{ss}(P) = (\mu_1, \dots, \mu_l, 0, \dots)$. Furthermore, if $\mathrm{ss}(P) = (v_1, v_2, \dots)$ and for some $1 \leq i \leq l$ we have $v_1 + \cdots + v_i < \mu_1 + \cdots + \mu_i$, then $\varphi_\mu(X_P) = 0$.
To prove this claim fix an order reversing bijection $\omega : P \to [d]$. The following reasoning can be justified in the same way that we did in the proof of Theorem 2.4.4, but we will abbreviate it here. First, assume that $\mathrm{ss}(P) = (\mu_1, \dots, \mu_l, m, \cdots)$. To get the desired descent set the permutation a associated with the linear extension α must be constructed as follows: List the μ_1 elements which are P-minimal in increasing order with respect to ω. Then, do the same with the μ_2 elements which are minimal after we removed the first μ_1 from P. If $r = d$, repeat this until we have listed $\mu_1 + \cdots + \mu_{l-1}$ elements, in which case μ_l incomparable elements are left, which we must list ascending according to ω. If instead $r < d$ proceed until $\mu_1 + \cdots + \mu_l$ elements are listed. In this case let v be the remaining minimal element of largest label. If $r = d - 1$ simply append v to our list. Otherwise $r < d - 1$ and we choose any $i < d - (\mu_1 + \cdots + \mu_l) - 2$ other minimal elements $u_1, \dots, u_i$ and list them with increasing labels, then v and then the remaining elements of our poset with decreasing labels. Note that by our choice of i at least one element will be listed after v, ensuring a descent at $d - 1$. These constructions will yield the descent

sets as described above and give an exhaustive list. Now doing similar calculations as in the proof of Theorem 2.4.4, for $r = d$ we get $\varphi_\mu(X_P) = 1 = t^0$ and for $r = d-1$ we get $\varphi_\mu(X_G) = t^1$. In the last case we have $\binom{m-1}{i}$ ways to choose the remaining elements, so we get $\varphi(X_P) = t^m$ after some calculation, as claimed.
There is one part of the claim left to justify, so assume that $\mathrm{ss}(P) = (v_1, \dots)$ such that for some fixed $i \in [l]$ we have $v_1 + \cdots + v_i < \mu_1 + \cdots + \mu_i$. Let ω be a map as above. We want to show that there is no linear extension α with a descent set of a form as listed above. We will argue the cases $r < d$. For $r = d$ the reasoning will be analogous to $r = d-1$. Denote by $a = (a_1, \dots, a_d)$ the permutation resulting from the linear ordering α. Furthermore denote $(p_1, \dots, p_d) = (\omega^{-1}(a_1), \dots, \omega^{-1}(a_l)) = (\alpha^{-1}(1), \dots, \alpha^{-1}(d))$ so that $a_i = \omega(p_i)$. In all of the above forms of D the first descent happens at μ_1, so consider the first μ_1 elements. If two of them p_j and p_k, $j < k$, were P-comparable then necessarily $p_j > p_k$ to satisfy $\omega(p_j) < \omega(p_k)$, but on the other hand this leads to $j = \alpha(p_j) > \alpha(p_k) = k$, a contradiction. We conclude that the first μ_1 elements form an antichain in P. Clearly we can use the same logic for the next μ_2 elements and so on, meaning the first $\mu_1 + \cdots + \mu_l$ elements consist of l antichains of respective lengths μ_j in P, which we denote by C_j. Consider two elements p_j and p_k that are part of two distinct antichains with $j < k$. Either they are P-incomparable or they are P-comparable, in which case $p_j < p_k$ since $j = \alpha(p_j) < \alpha(p_k) = k$. This implies that for $j < k$ the antichain C_k must lie weakly above C_j, in the sense that no element of C_j is greater than any element of C_k. In the case $r = d - 1$ a single element x of P was not listed yet which we may interpret as another antichain $C_{i+1} = \{x\}$ satisfying all of the above properties. If we successively remove antichains from P starting with the largest (according to the above order), we realize that $\bigcup_{1 \le j \le k} C_j$ is an order ideal of P for all $k \le i+1$. In the other case $r < d-1$ we similarly have an additional antichain C_{k+1} as well as many consecutive descents. Similarly to what we did with x in the case $r = d - 1$, the elements associated to these descents may themselves be interpreted as 1-element antichains, so that by the same reasoning $\bigcup_{1 \le j \le k} C_j$ is an order ideal for all relevant k. Consider the set $\bigcup_{1 \le j \le i} C_j$, which is an order ideal in every case. Since it consists of i antichains it is impossible that there exists a saturated chain of length $i+1$ or larger in it. However, such an order ideal must necessarily have less than $v_1 + \cdots + v_i$ elements by the definition of the sink sequence, which would imply that $\mu_1 + \cdots + \mu_i \le v_1 + \cdots + v_i$, a contradiction. We conclude that no linear extension leads to a descent set as described above and therefore $\varphi_\mu(X_P) = 0$ in this case. With this our claim is proven.
Note that since μ is maximal there can be no acyclic orientation $\mathcal{O}$ with $\mathrm{ss}(\mathcal{O}) = (v_1, \dots)$ satisfying both $v_1 + \cdots + v_i \ge \mu_1 + \cdots + \mu_i$ for all $i \in [l]$ and also $v_1 + \cdots + v_i > \mu_1 + \cdots + \mu_i$ for some $i \in [l]$. This means that the claim above already lists all possible cases for how the sink sequence might look.
Now applying φ_μ to the equation $X_G = \sum_{\mathcal{O}} X_{\bar{\mathcal{O}}}$ (which was justified in the proof of Theorem 2.4.4) and using our claim yields

$$\varphi_\mu(X_G) = \sum_j \mathrm{sink}(G, \mu, j) t^j.$$

As in Theorem 2.4.4, the goal is now to apply φ_μ to the expansion $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$ and hence to e_λ. Recall for now that in Theorem 2.2.2 we expressed X_G in the basis of monomial symmetric functions with the use of stable partitions

$$X_G = \sum_{\lambda \vdash d} S_\lambda \tilde{m}_\lambda.$$

Denote by $\nu' = (\nu'_1, \dots)$ the conjugate partition of $\nu = (\nu_1, \dots)$. Suppose we have a partition $\nu = (\nu_1, \dots, \nu_k) \vdash d$ such that $\nu'_1 + \cdots + \nu'_i \geq \mu_1 + \cdots + \mu_i$ for all $i \in [l]$ as well as $\nu'_1 + \cdots + \nu'_i > \mu_1 + \cdots + \mu_i$ for some $i \in [l]$ (meaning that a subpartition of the conjugate of ν dominates μ in some sense). Then by the maximality of μ it follows that ν' is not an allowable partition of G and therefore $S_{\nu'} = 0$ in the expansion above. Since the basis change from e_λ to m_λ has a triangular shape (namely e_λ is a sum of m_τ where $\lambda \leq \tau'$, see [10] Theorem 7.7.4), if $c_{\nu'} \neq 0$ and ν' is maximal in the dominance order then necessarily $S_{\nu'} \neq 0$. These two facts combined mean that $c_{\nu'} = 0$ if ν is a partition satisfying the above relation to μ and such that ν' is maximal in the dominance order. Due to the triangular shape of the basis change an inductive argument shows the same is true even if ν' is not maximal, as long as the first assumption holds for all larger partitions (with respect to dominance). Therefore we may focus our attention on partitions such that $\nu' = (\mu_1, \dots, \mu_l, \cdots)$ or $\nu' = (v_1, \dots)$ with $v_1 + \cdots + v_l < \mu_1 + \cdots + \mu_l$ instead.
Choosing P_λ such that $X_{P_\lambda} = e_\lambda$ as we did in the proof of Theorem 2.4.4 and applying our newly found knowledge gives

$$\varphi_\mu(e_\lambda) = \varphi_\mu(X_{P_\lambda}) = \begin{cases} t^m & \text{if } \lambda' = (\mu_1, \dots, \mu_l, m, \cdots) \\ 0 & \text{else.} \end{cases}$$

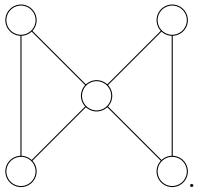
Now applying φ_μ to the expansion of X_G in terms of e_λ and comparing with what we have already proven we get

$$\sum_{\lambda, m} c_\lambda t^m = \sum_j \operatorname{sink}(G, \mu, j) t^j,$$

where the first sum ranges over all $\lambda \vdash d$ and m such that $\lambda' = (\mu_1, \dots, \mu_l, m, \cdots)$. Taking the coefficient of t^j completes the proof. □

Remark 2.4.13. At this point it is important to notice that if $r = 0$ then we have $\mu = \emptyset \vdash 0$ by convention, a trivially maximal allowable partition, and the proof still works. The reason this is interesting is that we then have $\operatorname{sink}(G, \emptyset, j) = \operatorname{sink}(G, j)$ and the relevant partitions λ are those for which $\lambda'_1 = j$, meaning $l(\lambda) = j$. This is exactly the statement of Theorem 2.4.4, showing that we indeed generalized it.

Example 2.4.14. We would like to see which new information Theorem 2.4.12 yields in regard to $K_{1,3}$. This graph is too simple however and nothing new will be gained. Instead, we consider the following graph:



Clearly the list of allowable partitions of size 5 in this case is $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$, ordered according to dominance. The single largest element among them in this case is $(2,2,1)$, but in general several maximal partitions may exist. Based on our understanding of a maximal allowable partition, we take $(2,2,1)$ and "remove the last parts" to get $\emptyset$, (2), $(2,2)$ and $(2,2,1)$. One can check that the sink sequence $(2,2,1)$ appears in 4 acyclic orientations, $(2,1,1,1)$ appears 8 times, $(1,1,1,1,1)$ appears 8 times, $(2,1,2)$ appears 4 times, $(1,1,1,2,1)$ appears 8 times and $(1,2,2)$ appears 4 times. Note that the sink sequence $(1,2,1,1,1)$ never appears in any acyclic orientation. This specifically implies that if we take an acyclic orientation with sink sequence $(1,1,1,2,1)$ and we reverse all edges the sink sequence itself does not reverse. Using this data we get the following table:

j / μ	0	1	2	3	$\cdots$	4	5
$\emptyset$	$-$	$20 = c_{(5)}$	$16 = c_{(4,1)} + c_{(3,2)}$	$0 = c_{(3,1,1)} + c_{(2,2,1)}$	$\cdots$	$0 = c_{(2,1,1,1)}$	$0 = c_{(1,1,1,1,1)}$
(2)	$-$	$12 = c_{(4,1)}$	$4 = c_{(3,2)}$	$-$	$\cdots$	$-$	$-$
$(2,2)$	$-$	$4 = c_{(3,2)}$	$-$	$-$	$\cdots$	$-$	$-$
$(2,2,1)$	$4 = c_{(3,2)}$	$-$	$-$	$-$	$\cdots$	$-$	$-$

The sections with a "$-$" also fall under our new result, but due to a mismatch in numbers we only get $0 = 0$, which would clutter the relevant information and was therefore omitted. Recalling that the first row is the information we would get from Theorem 2.4.4 we see that we gained additional information about $c_{(4,1)}$ and $c_{(3,2)}$ through this new result and we have a combinatorial interpretation for them.

We can also use this Theorem to find a class of graphs for which X_G is in fact e-positive.

Corollary 2.4.15. *Let $G = (V, E)$ be a graph and suppose that V can be partitioned into two disjoint cliques, where a clique refers to a set of vertices such that their induced subgraph is complete. Then X_G is e-positive.*

Proof. Denote as usual $d = |V|$. Since G consists of two cliques, any allowable partition consists of 1's and 2's, since if there was a 3 or higher that would imply that at least

2 vertices of a single clique are sinks simultaneously. This means that there is a unique maximal partition $\langle 1^a 2^b\rangle \vdash d$ with respect to dominance which is allowable (that is, the one with the maximal possible b). Therefor, every maximal allowable partition is either of the form $\langle 2^n\rangle$ for $0 \leq n \leq b$ or $\langle 1^n 2^b\rangle$ for $1 \leq n \leq a$. Here we use the convention m^0 to mean that the number m does not appear in the partition. Let μ be a partition as above and apply Theorem 2.4.12 to get

$$\mathrm{ss}(G, \mu, k) = \sum_\lambda c_\lambda,$$

where c_λ is the coefficient of e_λ in X_G. If we want the left hand side to not be 0 we require $k = 2$ if $l(\mu) < b$, we need $k = 1$ if $b \leq l(\mu) < a + b$ and lastly $k = 0$ is mandatory if $l(\mu) = a + b$. In those cases the sum on the right hand side ranges over all λ such that $\lambda' = (\mu_1, \ldots, \mu_l, k, \cdots)$, so they are of the form $\lambda = (i, j)$, where $i + j = d$ and $j \geq \min(l(\mu) + 1, b)$. The last restriction can be derived from considering the cases $\mu = \langle 2^n\rangle$, $0 \leq n < b$, and $\mu = \langle 1^n 2^b\rangle$, $0 \leq n \leq a$, separately.
Imagine for now what would happen if μ is not as described above. Then clearly $0 = \mathrm{ss}(G, \mu, k)$ for all k. Specifically, if $\mu \vdash d$ with $k = 0$ we get $0 = c_{\mu'}$ from Theorem 2.4.12, since μ is the only partition appearing in the sum. This shows that all $c_{\mu'}$ which are nonzero must come from a μ of the above described form. We will now show that all those coefficients are in fact nonnegative via an inductive proof.
As the base case we use $\mu = \langle 1^a 2^b\rangle$. Since $\mu \vdash d$ the only coefficient to consider is $c_{\langle 1^a 2^b\rangle'}$. Since this implies that $0 \leq \mathrm{ss}(G, \langle 1^a 2^b\rangle, 0) = c_{\langle 1^a 2^b\rangle'}$ we are done. For the induction step we successively make μ smaller.
Suppose that $\mu = \langle 1^n 2^b\rangle$ and consider $\nu = \langle 1^{n-1} 2^b\rangle$. We now compare which λ are in the respective sums of μ and ν when using k as described above. Clearly all λ appearing for μ will also appear for ν. Additionally, there is at most one new coefficient $c_{(n-1,a+b-n+1)}$ to consider which could be added based on the form of μ. Doing something similar when passing from $\mu = \langle 2^n\rangle$ to $\nu = \langle 2^{n-1}\rangle$ always adds exactly one new coefficient $c_{(a+b-n+1,n-1)}$. It is apparent that $\mathrm{ss}(G, \mu, k) \leq \mathrm{ss}(G, \nu, k')$ (with the appropriate k and k' as explained above) since all sink sequences of μ are also counted for ν. Therefore the new values c_λ added when passing from μ to ν must be nonnegative, completing our proof by induction. Since this covers all nonzero coefficients we conclude that all coefficients c_λ appearing in X_G are in fact nonnegative as was claimed. □

Remark 2.4.16. One especially easy case happens when the two cliques are not connected by edges. In fact, we may consider an arbitrary number of cliques for this. Let us first imagine how a sink sequence is constructed in the case of a complete graph: If an orientation is acyclic, it is necessarily the case that there is a unique sink and source. If we take out that sink we again have an acyclic orientation of a complete graph, so by induction the sink sequence has the form $(1, \ldots, 1)$. Note that these orientations do exist. In fact, we may take any sequence of vertices and can orient the edges such the vertices are removed in the chosen order, meaning that $\mathrm{ss}(K_n, (1, \ldots, 1), 0) = n!$. Using Theorem 2.4.12 then implies $X_{K_n} = n! e_n$. Now, if G consists of r cliques not connected to each

other, that means it is the disjoint union of complete graphs $G = K_{i_1} + \cdots + K_{i_r}$, where i_j is the size of the corresponding clique and $i_1 \geq \cdots \geq i_r$ without loss of generality. Based on our understanding of X_G it is now easy to extract

$$X_G = X_{K_{i_1}+\cdots+K_{i_r}} = X_{K_{i_1}} \cdots X_{K_{i_r}} = i_1! e_{i_1} \cdots i_r! e_{i_r} = i_1! \cdots i_r! e_{(i_1,\ldots,i_r)}.$$

Due to this one might wonder if the proof of Corollary 2.4.15 generalizes to more than two cliques. That is unfortunately not the case, since the step where we passed from μ to ν gets significantly more complicated, potentially making the induction step impossible. And indeed, such a prove cannot work as the statement itself is false. To see this consider that every graph is the union of cliques of size 1. Looking back on our claw graph $K_{1,3}$, we see that this even fails with three cliques. We could partition it into the cliques of respective sizes 2, 1 and 1, but as established $X_{K_{1,3}}$ is not e-positive.

3. More on Elementary Symmetric Functions

In this Chapter we will primarily focus on various statements one can make about the expansion of X_G in the basis of elementary symmetric functions and connected results. This will include statements regarding specific types of graphs as well as conjectures and surrounding material. Since this topic is the subject of current research it will be impossible to give an in-depth view in the scope of this work. That said, our goal will be to consider some of the more well known conjectures and results, with the implicit knowledge that we will be barely scratching the surface. On another note, since we will be talking about open conjectures it is unsurprising that we will not prove or disprove them. We will however consider properties we would expect if the conjecture were true and prove them, which in a way amounts to a (failed) attempt to disprove the original statement. Most of the below and further reading can be found in Stanley's paper [11] unless noted otherwise.

3.1. $(3+1)$-free Partially Ordered Sets

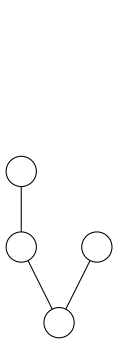
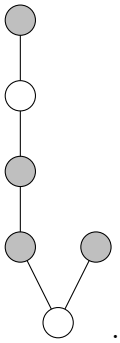
The first class of graphs we will look at are incomparability graphs of (3+1)-free partially ordered sets.

Definition 3.1.1. Let G be a graph. Then G is called *H-free*, if it does not contain the induced subgraph H. If a graph is $K_{1,3}$-free we also call it *clawfree*.

Definition 3.1.2. Let P be a partially ordered set. Define the *incomparability graph* $\mathrm{inc}(P) = (V, E)$ of P as follows: The vertices are the elements of P, meaning $V = P$. An edge $e = \{v, w\}$ is in E if and only if $v \not\leq_P w$ and $v \not\geq_P w$, meaning that v and w are P-incomparable. Note that if P is finite then the incomparability graph is finite and simple and therefore falls into our framework of the chromatic function.

Definition 3.1.3. Let P be a partially ordered set. We call it *Q-free* if P does not contain the induced subposet Q. Specifically P is $(3+1)$-free if it does not contain the disjoint union of a chain of length 3 and of length 1. Note that P is $(3+1)$-free if and only if $\mathrm{inc}(P)$ is clawfree.

Example 3.1.4. Consider the posets


and

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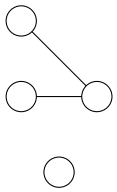
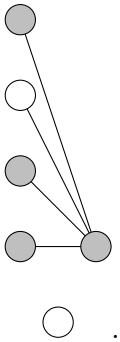
The left one is $(3+1)$-free, whereas the the right one contains several induced subposets of the form $(3+1)$, one of them being colored in gray. Their respective incomparability graphs are then


and

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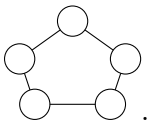
Remark 3.1.5. We already noted that the incomparability graph of a $(3+1)$-free poset is clawfree. The reverse, however, is not true: For a clawfree graph G we cannot in general find a $(3+1)$-free poset P such that $\mathrm{inc}(P) = G$. The reason this is not the case is that there are graphs which are not incomparability graphs of posets, for example the cycle on 5 vertices (which is also clawfree):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The following conjecture is due to Stanley (see [6] Conjecture 5.1), but it is worth mentioning that an equivalent formulation has been proposed before that in a joint work between Stanley and Stembridge ([12] Conjecture 5.5).

Conjecture 3.1.6. *Let P be a $(3+1)$-free finite poset. Then X_G for $G = \mathrm{inc}(P)$ is e-positive.*

Remark 3.1.7. We have already seen in example 2.4.7 that the claw graph $K_{1,3}$ is not e-positive, which makes the above plausible at first glance.
Since every e_λ is s-positive (where s refers to the Schur basis), one way the above could fail is if $\mathrm{inc}(P)$ is not s-positive, leading to our first result.

The following results on the s-positivity of X_G as well as the used definitions regarding multicolorings and P-arrays can be found in a paper by Gasharov [3].

Definition 3.1.8. Let $G = (V, E)$ be a graph and $V = \{v_1, \dots, v_d\}$. We define a *multicoloring* of G to be a function $\kappa : V \to 2^{\mathbb{N}}$, where $2^{\mathbb{N}}$ denotes the set of subsets of $\mathbb{N}$. We can easily interpret κ as assigning each vertex multiple colors and many of the concepts of regular colorings still apply. A multicoloring κ is *proper* if for all $\{v_i, v_j\} \in E$ we have $\kappa(v_i) \cap \kappa(v_j) = \emptyset$. For a composition $m = (m_1, \dots, m_d)$ (that is $m_i \in \mathbb{N}$, $m_i > 0$) we define an *m-multicoloring* to be a multicoloring κ such that $|\kappa(v_i)| = m_i$ for all i. Denote for a multicoloring κ the monomial $\mathbf{x}^\kappa = \mathbf{x}^{\kappa(v_1)} \cdots \mathbf{x}^{\kappa(v_d)}$, where for $a = \{a_1, a_2, \dots\} \subseteq \mathbb{N}$ we use $\mathbf{x}^a = x_{a_1} x_{a_2} \cdots$, a in general infinite product. Let m be a composition and define a generalization of the chromatic function for multicolorings as

$$X_G^m = \sum_{\kappa \in K^m} \mathbf{x}^\kappa = \sum_{\kappa \in K^m} \mathbf{x}^{\kappa(v_1)} \cdots \mathbf{x}^{\kappa(v_d)},$$

where K^m is the set of all proper m-multicolorings of G. Note that in this case all products are in fact finite. A similar argument to before shows that X_G^m is a symmetric function so that our theory still applies. It generalizes X_G in the sense that $X_G^{(1,\dots,1)} = X_G$.

Definition 3.1.9. Let P be any partially ordered set. A *P-array* is a finite array of elements of P with left justified rows such that $a_{i,j} < a_{i,j+1}$ if both are defined. If an element $a_{i,j}$ is not defined we will use the shorthand $a_{i,j} = \emptyset$. Note that a P-array in general does not resemble a tableau due to the fact that the row length might increase at some point. If its row length is in fact weakly decreasing and additionally $a_{i+1,j} \not< a_{i,j}$ when both exist we call it a *P-tableau*. Phrased another way, a P-tableau is a tableau of elements of P with strongly increasing rows and non decreasing columns. If P is countable then consider a sequencing $(p_1, p_2, \dots)$ of the elements of P. In this case we define the *weight* of a P-tableau T as the weak composition of infinite length $\mathrm{wt}(T) = (\#p_1 \text{ in } T, \#p_2 \text{ in } T, \dots)$.

Example 3.1.10. Take two disjoint instances $\mathbb{N} = \{1, 2, \dots\}$ and $\bar{\mathbb{N}} = \{\bar{1}, \bar{2}, \dots\}$ of the natural numbers. Define the poset $P = \mathbb{N} + \bar{\mathbb{N}}$ as their disjoint union, that is $x \leq_P y$ if either $x, y \in \mathbb{N}$ and $x \leq_{\mathbb{N}} y$ or $x, y \in \bar{\mathbb{N}}$ and $x \leq_{\bar{\mathbb{N}}} y$ with the usual order on $\mathbb{N}$ and $\bar{\mathbb{N}}$. Then the following are examples of a P-array and P-tableau respectively:

$$T = \begin{array}{lll} 1 & 2 & 3 \\ \bar{2} & & \\ \bar{2} & \bar{4} & \\ 4 & 6 & 7 \end{array} \qquad \text{and} \qquad T' = \begin{array}{lll} 1 & 2 & 3 \\ 3 & 4 & 5 \\ \bar{1} & \bar{2} & \bar{3} \\ 1 & 2 & \end{array}.$$

Note that an array does allow for an increase in row length, as well as empty rows. Also note that when we said that a tableau has non decreasing columns, that was in regards to a point wise decrease, not a global one. Given the sequencing $(1, \bar{1}, 2, \bar{2}, \dots)$ the tableau

T' has the weight $\mathrm{wt}(T') = (2,1,2,1,2,1,1,0,1,0,0,\dots)$. Since a P-array is always finite, and therefore contains a finite amount of elements, the weight of a tableau can always be associated to a finite composition, in this case $\mathrm{wt}(T') = (2,1,2,1,2,1,1,0,1)$ since elements up to 5 appear in T'. As seen in the definition, to reference the elements of an array we will use the same notation as matrices, that is $T_{1,1} = 1$, $T_{2,1} = \bar{2}$, $T_{2,2} = \emptyset$ and so on.

Theorem 3.1.11. *Let P be a $(3+1)$-free poset with $|P| = d$. Let $G = (P,E) = \mathrm{inc}(P)$ be the incomparability graph of P and $m = (m_1,\dots,m_d)$ be a composition. Then X_G^m is s-positive.*

Proof. We can associate to each proper multicoloring κ a unique P-array T_κ in the following way. Define $\kappa^{-1}(i) = \{v_{i,1}, v_{i,2}, \dots\}$ as the set of vertices of G which contain the color i under κ. Since the coloring is proper these are stable subsets of G. A stable subset translates to pairwise comparable elements in P, that is to say a chain, so we may assume $v_{i,1} < v_{i,2} < \dots$ without loss of generality. Given this define T_κ to be

$$T_\kappa = \begin{array}{ccc} v_{1,1} & v_{1,2} & \cdots \\ v_{2,1} & v_{2,2} & \cdots \\ \vdots & \vdots & \end{array},$$

where it is clear from construction that $T_\kappa \neq T_{\kappa'}$ if $\kappa \neq \kappa'$.
On the other hand we can associate to each P-array T a unique proper multicoloring by doing this in reverse. Explicitly this means coloring all vertices in the i'th row of T with the color i. Since each row forms a chain in P it is guaranteed that it is associated to a stable partition of G, making the multicoloring proper. These two maps then form inverse bijections between the set of proper multicolorings of G and the set of P-arrays.
Consider for the space of symmetric functions the usual inner product for which $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ is the Kronecker delta. Suppose that $X_G^m = \sum_{\lambda \vdash d} b_\lambda s_\lambda$ is the expansion of our chromatic function in the Schur basis. Since the s_λ form an orthonormal basis with respect to the above inner product it follows that $\langle X_G^m, s_\lambda \rangle = b_\lambda$. To manipulate this equation we will rephrase the well known Jacobi-Trudi identity

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq l(\lambda)},$$

where $l(\lambda)$ denotes the length of λ and we use the convention $h_r = 0$ if $r \leq 0$. Given some partition $\lambda = (\lambda_1,\dots,\lambda_l)$ and permutation $\pi \in S_l$ we define the sequence of integers

$$\pi(\lambda) = (\lambda_{\pi(1)} - \pi(1) + 1, \dots, \lambda_{\pi(l)} - \pi(l) + l).$$

Now we use the Laplace expansion of the determinant so that by the definition of $\pi(\lambda)$ we get

$$s_\lambda = \sum_{\pi \in S_l} \mathrm{sgn}(\pi) h_{\pi(\lambda)},$$

where $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_n}$ for sequences of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ similar to partitions. We will also use this notation for bases other than h. Using the above expansion in the earlier inner product yields

$$b_\lambda = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) \langle X_G^m, h_{\pi(\lambda)} \rangle.$$

Based on the definition of the inner product $\operatorname{sgn}(\pi)\langle X_G^m, h_{\pi(\lambda)}\rangle$ is the coefficient of $m_{\pi(\lambda)}$ in X_G^m, which means it is also the coefficient of the monomial $\mathbf{x}^{\pi(\lambda)}$ in X_G^m. This value in turn is by definition the number of $\pi(\lambda)$-multicolorings κ, by which we mean the number of κ such that $\pi(\lambda) = (|\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots)$. Note that despite the fact that $\pi(\lambda)$ is generally not a composition this reasoning still applies, since if it contains a number $k \leq 0$ then the coefficient of $\mathbf{x}^{\pi(\lambda)}$ in X_G^m is 0. In this way the definition of an m-multicoloring as used before still suffices. Since the shape of T_κ as constructed earlier, that being the composition recording the row lengths of T_κ, is exactly $(|\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots)$, it follows that $\operatorname{sgn}(\pi)\langle X_G^m, h_{\pi(\lambda)}\rangle$ is equal to the number of P-arrays of shape $\pi(\lambda)$ and weight m. In the next step we will use convient sets and a sign reversing involution so that only P-tableau are left in the sum. Let

$$\begin{aligned} A &= \{(\pi, T) \,|\, \pi \in S_l,\ T \text{ a } P\text{-array of shape } \pi(\lambda) \text{ and weight } m\} \text{ and} \\ B &= \{(\pi, T) \in A \,|\, T \text{ is not a } P\text{-tableau}\} \end{aligned}$$

so that based on the above reasoning $b_\lambda = \sum_{(\pi,T)\in A} \operatorname{sgn}(\pi)$. Note that for a P-tableau T of shape $\pi(\lambda)$ we then have $\pi(\lambda)_1 \geq \pi(\lambda)_2 \geq \cdots \geq \pi(\lambda)_l$ since $\pi(\lambda)$ is already a partition. In this case we must also have $\pi(1) > \pi(2) > \cdots > \pi(l)$ so that $\pi = \mathrm{id}$, for if there was an inversion $i < j$ with $\pi(i) > \pi(j)$ then $\lambda_{\pi(i)} \leq \lambda_{\pi(j)}$ and so

$$\pi(\lambda)_i = \lambda_{\pi(i)} - \pi(i) + i < \lambda_{\pi(j)} - \pi(j) + j = \pi(\lambda)_j.$$

To show that $b_\lambda \geq 0$ it will therefore be enough to find a sign reversing involution on B, since only terms of the form $\operatorname{sgn}(\mathrm{id}) = 1$ will be left in the sum in that case.
Suppose we start with the P-array but not P-tableau

$$B \ni T = \begin{matrix} t_{1,1} & t_{1,2} & \cdots \\ t_{2,1} & t_{2,2} & \cdots \\ \vdots & \vdots & \end{matrix}$$

and let c be the smallest column index such that T fails to be a tableau because of it, meaning there is some i such that $t_{i,c} < t_{i+1,c}$ or $t_{i,c} = \emptyset$ despite $t_{i+1,c} \neq \emptyset$. Let furthermore r be the largest row index such that this failure occurs in the column c. Then define $\phi(\pi) = \sigma = \pi \circ (r \ \ r+1)$, where $(r \ \ r+1)$ is a transposition. Also define the construct

$$\varphi(T) = \begin{matrix} \tau_{1,1} & \tau_{1,2} & \cdots \\ \tau_{2,1} & \tau_{2,2} & \cdots \\ \vdots & \vdots & \end{matrix}$$

to be T, except that the latter parts of the rows r and $r+1$ have been exchanged, specifically $(\tau_{r,c}\,,\,\tau_{r,c+1}\,,\dots) = (t_{r+1,c+1}\,,\,t_{r+1,c+2}\,,\dots)$ and $(\tau_{r+1,c+1}\,,\,\tau_{r+1,c+2}\,,\dots) = (t_{r,c}\,,\,t_{r,c+1}\,,\dots)$. We wish to show that $\varphi(T)$ is a P-array, but not a P-tableau. The current situation is as such:

$$T = \begin{array}{ccccccc} & & \vdots & & & & \\ \cdots & t_{r,c-1} & t_{r,c} & t_{r,c+1} & t_{r,c+2} & \cdots \\ \cdots & t_{r+1,c-1} & t_{r+1,c} & t_{r+1,c+1} & t_{r+1,c+2} & \cdots \\ & & \vdots & & & & \end{array},$$

$$\varphi(T) = \begin{array}{ccccccc} & & \vdots & & & & \\ \cdots & \tau_{r,c-1} & \vert\tau_{r,c} & \tau_{r,c+1} & \tau_{r,c+2} & \cdots \\ \cdots & \tau_{r+1,c-1} & \tau_{r+1,c} & \vert\tau_{r+1,c+1} & \tau_{r+1,c+2} & \cdots \\ & & \vdots & & & & \end{array}$$

$$= \begin{array}{ccccccc} & & \vdots & & & & \\ \cdots & t_{r,c-1} & \vert t_{r+1,c+1} & t_{r+1,c+2} & t_{r+1,c+3} & \cdots \\ \cdots & t_{r+1,c-1} & t_{r+1,c} & \vert t_{r,c} & t_{r,c+1} & \cdots \\ & & \vdots & & & & \end{array}.$$

To prove $\varphi(T)$ is a P-array it suffices to show that $\tau_{r,c-1} < \tau_{r,c}$ and $\tau_{r+1,c} < \tau_{r+1,c+1}$, with those being true by convention if the respective second element does not exist. The second inequality follows immediately from the definition of c and r. For the first we rephrase the inequality to $t_{r,c-1} < t_{r+1,c+1}$ and notice that $t_{r+1,c-1} < t_{r+1,c} < t_{r+1,c+1}$ is a 3-element chain in P and $t_{r+1,c-1} \not< t_{r,c-1}$ since c was minimal, where all elements mentioned are defined. This means that if they are comparable we must have $t_{r+1,c-1} > t_{r,c-1}$. If the two are incomparable, then due to transitivity it follows that if $t_{r,c-1}$ is comparable to either $t_{r+1,c}$ or $t_{r+1,c+1}$ it has to be smaller. Now $t_{r,c-1}$ must be comparable to at least one of the three since P is $(3+1)$-free so that in every case $t_{r,c-1} < t_{r+1,c+1}$, as was claimed. It is left to show that $\varphi(T)$ is not a P-tableau. This is the case since either $\tau_{r+1,c} = t_{r+1,c} < t_{r+1,c+1} = \tau_{r,c}$ or $\tau_{r,c} = \emptyset$ despite $\tau_{r+1,c} \neq \emptyset$.
Now we have to argue that φ is a sign reversing involution on B. The rows r and $r+1$ of the array $\varphi(T)$ have lengths

$$\pi(\lambda)_{r+1} - 1 = \lambda_{\pi(r+1)} - \pi(r+1) + r = \lambda_{\sigma(r)} - \sigma(r) + r = \sigma(\lambda)_r$$

and

$$\pi(\lambda)_r + 1 = \lambda_{\pi(r)} - \pi(r) + r + 1 = \lambda_{\sigma(r+i)} - \sigma(r+1) + r + 1 = \sigma(\lambda)_{r+1}$$

respectively, meaning that $\varphi(T)$ has the shape $\sigma(\lambda)$. At this point it becomes clear that $\operatorname{sgn}(\pi) = -\operatorname{sgn}(\phi(\pi))$. Furthermore, $\varphi(T)$ will result in the same pair c, r as T, so that

$\varphi(\varphi(T)) = T$ for all $(\pi, T) \in B$. This means that this procedure is a sign reversing involution on B and we are done. □

Implicit in the proof was also the following statement, which we make explicit.

Theorem 3.1.12. *Suppose $X_G^m = \sum_{\lambda \vdash d} b_\lambda s_\lambda$ is defined as above. Then b_λ is the number of P-tableaux of shape λ and weight m.*

We can easily specialize these back to regular colorings using $m = (1, \ldots, 1)$.

Corollary 3.1.13. *Let P be a $(3+1)$-free finite poset. Let $G = (P, E) = \mathrm{inc}(P)$ be the incomparability graph of P. Then X_G is s-positive. If $X_G = \sum_\lambda b_\lambda s_\lambda$ then b_λ is equal to the number of P-tableaux of shape λ using each element of P exactly once.*

Based on the things discussed in this section it is worth asking whether or not we can generalize to s-positivity of X_G for clawfree graphs G.

3.2. Clawfree Graphs

Although difficult to find in the literature, according to Stanley [11] it seems the following conjecture was first proposed by Gasharov, despite the fact that he never published it directly.

Conjecture 3.2.1. *Let G be a clawfree graph. Then X_G is s-positive.*

Worth noting is that Theorem 3.1.11 itself is again supporting evidence for this conjecture.
The next results are connected to the fact that the claw graph $K_{1,3}$ is not nice, in a certain sense. They can be found in [11].

Definition 3.2.2. Let $G = (V, E)$ be a graph with $|V| = d$. We call G *nice*, if for every stable partition π and for every $\mu \leq \mathrm{type}(\pi)$ (in the dominance ordering) we can find a stable partition σ such that $\mu = \mathrm{type}(\sigma)$. In other words, G is nice if and only if the subposet of allowable partitions of size d is an order ideal in the poset of all partitions $\lambda \vdash d$ using dominance.

Remark 3.2.3. Let $G = (V, E)$ be a graph with $|V| = d$ and denote the set of all integer partitions of size k by A_k and the set of all allowable ones by A_k^G. If G is nice then this property not only gives insight into partitions in A_d^G as stated above, but also into partitions in A_k^G for $k < d$, since we can find an embedding of A_k into A_d which maps A_k^G into A_d^G. One way to do this is by appending the entry 1 exactly $d - k$ times to all partitions in question. To see that this procedure behaves well when considering the underlying stable partitions consider that adding a 1 to the type of a stable partition can be translated to appending a singular vertex to it. This means we take a stable partition π of an induced subgraph $H = (V_H, E_H)$ of G and map it to the stable partition $\pi \cup \{\{v\} \mid v \in V \backslash V_H\}$ of G so that everything works as described. To rephrase, every induced subgraph of a nice graph is itself nice.

Example 3.2.4. The claw graph $K_{3,1}$ is not nice, since for a stable partition of type $(3,1)$ we have $(2,2) < (3,1)$, but there is no stable partition of type $(2,2)$.

Theorem 3.2.5. *Let G be a graph. The following two statements are equivalent:*

- *All induced subgraphs $H \subseteq G$ are nice.*
- *The graph G is clawfree.*

Proof. Suppose that G is not clawfree, so it contains the induced subgraph $K_{1,3}$. Since we already established that the claw graph is not nice the first statement also fails.
Suppose on the other hand that G is clawfree. Let $H = (V, E)$ be an induced subgraph of G and π be a stable partition of H of type λ. Furthermore let μ be covered by λ in the dominance order. The only way this can be the case is if $\mu_k = \lambda_k$ for all k except for some $i < j$, in which case $\mu_i = \lambda_i - 1$ and $\mu_j = \lambda_j + 1$ (where possibly $\lambda_j = 0$). If we can show that H has a stable partition of type μ then we are done, as an inductive argument will extend our reasoning to all $\nu \leq \lambda$.
Let as above $i < j$, where $\lambda_i \neq 0$ but possibly $\lambda_j = 0$, and take respectively a block A of size λ_i and a block B of size λ_j from π and denote $W = A \dot\cup B \subseteq V$. As usual denote by H_W the induced subgraph of H with vertex set W and notice that it is bipartite by construction. Since G is clawfree it follows that H and H_W are also clawfree, meaning that every vertex in H_W has degree at most 2. The graph H_W therefore consists solely of paths and cycles. By definition we know $|A| > |B| \geq 0$, so since H_W is bipartite there must be a connected component $C = C_A \dot\cup C_B$ of H_W with $C_A \subseteq A$, $C_B \subseteq B$, $|C_A| > 1$ and $|C_A| > |C_B| \geq 0$. This means C may be understood as a path which alternates between A and B and which starts and ends in A. In π replace the block A by $(A \backslash C_A) \cup C_B$ and the block B by $(B \backslash C_B) \cup C_A$ and notice that this is a stable partition of H of type μ, so we are done. □

To understand the connection to s-positivity we need another result.

Proposition 3.2.6. *If G is such that X_G is s-positive, then G is nice.*

Proof. Based on Theorem 2.2.2 we know that G possesses a stable partition of type λ if and only if the coefficient of m_λ in X_G is nonzero (meaning it is positive). The next relevant fact is that the expansion of $s_\lambda = \sum_\mu a_\mu m_\mu$ in the basis of monomial functions is such that $a_\mu \neq 0$ if and only if $\mu \leq \lambda$ (see for example Proposition 7.10.5 and Exercise 7.12 in [10]). From the same source we know that the coefficient of m_λ in s_λ is 1.
Now let π be a stable partition of G of type λ and suppose $\mu < \lambda$. Putting the above facts together, for the expansion of X_G in the basis of the monomial symmetric functions we have $[\![m_\lambda]\!]X_G > 0$ and therefore $[\![s_\lambda]\!]X_G > 0$ since $s_\lambda = m_\lambda + \sum_{\gamma<\lambda} a_\nu m_\nu$ and X_G is s-positive. Furthermore, for those coefficients it holds $a_\nu > 0$, implying specifically $a_\mu > 0$. By the same logic the coefficient of m_μ in s_γ for $\mu < \gamma$ is also always positive. Since X_G is s-positive we conclude $[\![m_\mu]\!]X_G \geq a_\mu > 0$. By Theorem 2.2.2 this then means that there exists at least one stable partition of type μ and we are done. □

One can view the combination of these two statements as (a decidedly small bit of) supporting evidence of Conjecture 3.2.1 in the following way: If the graph G has s-positive X_G then it is nice. Therefore all of its induced subgraphs are also nice, meaning G is clawfree. This means the concepts of an s-positive graph and a clawfree graph are linked.
According to [11] there is more supporting evidence, solidifying the status of the above as a Conjecture rather than speculation.

4. Further Generalizations

One natural question to ask is whether or not the chromatic symmetric function itself may be generalized, since there are obvious analogues to a finite simple graph. Stanley describes three such generalizations in [11], the first being in the form of Tutte polynomials, which we will not discuss here. The other two follow by using different objects than graphs, namely directed graphs and hypergraphs. Stanley himself mentioned in [11] (dated at July 1995) that the related functions have not been studied thoroughly. A cursory search at this point in time seems to indicate that neither object gained significant research since, although for hypergraphs the analogue of the chromatic polynomial has been examined.
Another generalization worth mentioning has been defined by Shareshian and Wachs in [5], wherein they use regular graphs but generalize the chromatic function by introducing an additional statistic, leading to the formal power series being quasisymmetric and other different properties. This object was subject to significantly more research than the other two.
It is also possible to generalize X_G by using multicolorings as we did in Section 3.1, but we will not delve deeper into that topic.
By nature of counting objects we still require our graphs (and other objects) to be finite and for more precise restrictions we need to consider each case separately.

4.1. Directed Graphs

In this section we will deal with directed graphs $G = (V, E)$ in which both V and E are finite. Unless mentioned otherwise every directed graph here will have these restrictions. The analogue of the chromatic polynomial in this case was first introduced by Chung and Graham in [2] and its symmetric function counterpart was first introduced and studied by Chow in [1].

Definition 4.1.1. Let $G = (V, E)$ be a directed graph. A *path cycle cover* of G is a subset $S \subseteq E$ such that every connected component of $G_S = (V, S)$ is either a directed cycle or a directed path (both of which might consist of a single vertex). Let S be a path cycle cover and interpret it as a partition of V via the connected components of G_S. Then we denote by $\pi'(S)$ the set of blocks consisting of paths and by $\sigma'(S)$ the set of blocks consisting of cycles. Denote further $\pi = \pi(S) = \text{type}(\pi'(S))$ and $\sigma = \sigma(S) = \text{type}(\sigma'(S))$, recording their respective sizes as an integer partition. Define the *cover polynomial*

$$C_G(i, j) = \sum_S i^{\underline{l(\pi)}} j^{l(\sigma)},$$

where $i^{\underline{k}} = i(i-1)\cdots(i-k+1)$ are the falling factorial and the sum ranges over all path cycle covers S. The values $l(\pi)$ and $l(\sigma)$ are easily seen to be the number of paths and cycles of S respectively. Note that adding or removing loops and multiple edges does change $C_G(i,j)$ in a non-trivial way, so we cannot assume they do not exist.

Remark 4.1.2. Let P be a finite partially ordered set. It can be shown that $C_{D(P)}(i,0) = \chi_{\mathrm{inc}(P)}(i)$, where $D(P)$ is P interpreted as a directed graph, that is (v,w) is an edge of $D(P)$ if $v >_P w$, and $\mathrm{inc}(P)$ is the incomparability graph of P. This formula gives us reason to call the cover polynomial a generalization of the chromatic polynomial. We will not prove this equality, but instead prove a variant for the symmetric function analogues, so that this fact follows by specialization.

In Section 2.3 we have seen that for undirected graphs $p_{\lambda(S)}(1^n) = n^{c(S)}$, where $\lambda(S)$ was the type of G_S considered as a connected partition and $c(S)$ was the number of connected components of G_S. Using this fact in combination with $\pi(S)$ and $\sigma(S)$, Chow proposed the following generalization of X_G on suggestion of Stanley.

Definition 4.1.3. Let $G = (V,E)$ be a directed graph and $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots)$ be commuting indeterminates. Define the *path cycle symmetric function* to be

$$\Xi_G = \Xi_G(\mathbf{x}, \mathbf{y}) = \sum_S \tilde{m}_{\pi(S)}(\mathbf{x}) p_{\sigma(S)}(\mathbf{y}),$$

where S ranges over all path cycle covers of G. It is clear that Ξ_G is symmetric in both $\mathbf{x}$ and $\mathbf{y}$, although it is not symmetric in $\mathbf{x} \cup \mathbf{y}$.

The path cycle symmetric function is linked to the cover polynomial in a similar way that the chromatic function is linked to the chromatic polynomial.

Proposition 4.1.4. *Let $G = (V,E)$ be a directed graph. Then*

$$\Xi_G(1^i, 1^j) = C_G(i,j).$$

Proof. One is easily convinced that $\tilde{m}_\lambda(1^i) = i^{\underline{l(\lambda)}}$ and $p_\lambda(1^j) = j^{l(\lambda)}$ by counting all possible monomials. Therefore we calculate

$$\Xi_G(1^i, 1^j) = \sum_S \tilde{m}_{\pi(S)}(1^i) p_{\sigma(S)}(1^j) = \sum_S i^{\underline{l(\pi)}} j^{l(\sigma)} = C_G(i,j).$$

□

To see how Ξ_G generalizes the chromatic function we consider the following.

Proposition 4.1.5. *Let P be a finite partially ordered set. Then $\Xi_{D(P)}(\mathbf{x}, \mathbf{y}) = X_{\mathrm{inc}(P)}(\mathbf{x})$, where $D(P)$ is the directed graph with vertex set P and edge set $\{(v,w) \mid v > w\} \subseteq P \times P$.*

Proof. The graph $D(P)$ is acyclic, so no path cycle cover contains any cycles. This implies that the variables $\mathbf{y}$ are obsolete and may be omitted in $\Xi_G(\mathbf{x}, \mathbf{y})$. A path in $D(P)$ corresponds to a chain in P, meaning that no elements of that path could be

connected in inc(P). Due to this a path cycle cover S of $D(P)$ will consist only of paths, which can be naturally interpreted as the blocks of a partition ρ of P. Furthermore, this partition is then stable when viewed as a partition of inc(P). By definition we have $\pi = \text{type}(\pi'(S)) = \text{type}(\rho)$, so by using Theorem 2.2.2 we get

$$\Xi_{D(P)}(\mathbf{x}, \mathbf{y}) = \sum_S \tilde{m}_\pi = \sum_{\lambda \vdash d} S_\lambda \tilde{m}_\lambda = X_{\text{inc}(P)},$$

where S_λ is the number of stable partitions (and hence path cycle covers) of type λ and $d = |P|$. □

Remark 4.1.6. Since $X_G(1^n) = \chi_G(n)$ we may now express for any poset P that $C_{D(P)}(i, 0) = \Xi_{D(P)}(1^i, 1^0) = X_{\text{inc}(P)}(1^i) = \chi_{\text{inc}(P)}(i)$, as was claimed before.

4.2. Hypergraphs

Here we will generalize some of our existing concepts to hypergraphs.

Definition 4.2.1. Let V be any set and denote by $P(V)$ its power set, that is the set of all subsets of V. Take some $E \subseteq P(V)$. Then we call the pair $H = (V, E)$ a *hypergraph* with *vertices* V and *edges* E. We will use language as if we were dealing with regular graphs, since most concepts follow naturally. The only notion necessary for us to define is that of an *induced subhypergraph* of H, by which we mean a hypergraph $G = (W, F)$ where $W \subseteq V$ and $F = \{e \cap W \mid e \in E\}$. Put simply, we take a subset of vertices and keep them connected as they would have been in H, even if that shrinks the relevant edges in size. For an induced subhypergraph we will simply call it an induced subgraph if the context is clear.

Remark 4.2.2. The connection to regular simple graphs is clear once one adopts a certain viewpoint: A simple graph $G = (V, E)$ consists of some set of vertices V and a set E of 2-element subsets of V. In this sense we simply loosened our understanding of what an edge is.

Definition 4.2.3. Let $H = (V, E)$ be a hypergraph. A *coloring* of H is a map $\kappa : V \to \mathbb{N}$. A coloring κ is *proper* if no edge is monochromatic, that is all $e \in E$ satisfy $|\{\kappa(v) \mid v \in e\}| > 1$.

Remark 4.2.4. A more natural way to define a proper coloring might be to assume that no edge contains duplicate colors. The reason this is not useful is that such colorings can easily be described by regular graphs on the same vertex set and appropriate edge sets, so there would be no need to consider hypergraphs.
On another note, if we order all elements of E by inclusion then to assume that no edge is monochromatic is to assume that no minimal edge is monochromatic. Therefore it makes sense to restrict ourselves to the case where edges form an antichain, which we will do from now on.

With these concepts being introduced, it should be easy to guess what our object of study will be.

Definition 4.2.5. Let $H = (V, E)$ be a finite hypergraph. Define

$$X_H(\mathbf{x}) = \sum_{\kappa \in K} \mathbf{x}^{\kappa},$$

where K is the set of proper colorings of H. We will call it the *chromatic function* associated to H if no confusion arises. A similar argument to before shows that X_H is in fact a symmetric function.

Remark 4.2.6. As was the case with regular graphs, it makes sense to consider a kind of simple hypergraph $H = (V, E)$. First of all, we need V to be finite for X_H to exist. Furthermore, loops, which in this case means 1-element edges, will force $X_H = 0$, so we assume they do not exist. Note that multiple edges cannot exist in the way we defined hypergraphs, so we do not need to consider them. We assume these restrictions for the rest of this section without further mention.

Definition 4.2.7. Let $H = (V, E)$ be a hypergraph. Let Π_H be the lattice of partitions of V ordered by refinement. Among all partitions in Π_H take those $\pi_1, \ldots, \pi_k$ which have a unique non-singleton block when viewed as an induced subgraph of H. Then consider the subposet L_H which is join generated by $\pi_1, \ldots, \pi_k$ in the sense that

$$\pi \in L_H \quad \text{if and only if} \quad \pi = \bigvee_{i \in I} \pi_i \quad \text{for some } I \subseteq [k],$$

where the join in Π_H is used. Notably, L_H contains $\hat{0}$ (the empty join), is finite and contains all of its joins, so it is already a lattice itself. Consider for a moment the case of a regular simple graph G: The partitions $\pi_1, \ldots, \pi_k$ are identifiable by the unique edge they still contain when viewed as induced subgraphs, meaning that L_G contains all the connected partitions. We conclude that in this case L_G is the lattice of contractions, as the notation indicates.

Remark 4.2.8. Similarly to before one can consider $X_H(1^n)$, which is easily seen to be the number of proper colorings of H. In the literature this number it is often denoted $\chi_H(n)$ and serves as an obvious analogue to the regular chromatic polynomial.

With all of these definitions one can formulate a result similar to that of regular graphs.

Theorem 4.2.9. *Let $H = (V, E)$ be a hypergraph. Then*

$$X_H = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)},$$

where $\mathrm{type}(\pi)$ *records the sizes of the blocks of π as an integer partition and p_λ is a power symmetric function.*

Proof. The proof is completely analogous to the one of Theorem 2.3.3 and will therefore be omitted. □

In Definition 1.3.6 we saw that we can link edges of regular graphs to hyperplanes. This construction naturally generalizes to the concept of subspace arrangements.

Definition 4.2.10. Let $H = (V, E)$ be a hypergraph with $|V| = d$. Order the vertices in some fixed way $V = \{v_1, \ldots, v_d\}$. Take the vector space $\mathbb{R}^d$ with the usual basis and denote by x_i the i'th component of the vector x. Consider the following: Link each edge $e = \{v_{i_1}, \ldots, v_{i_k}\}$ to a set of equalities $x_{i_1} = \cdots = x_{i_k}$. Subsequently define the $(d - k + 1)$-dimensional subspace $H_e = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_{i_1} = \cdots = x_{i_k}\}$. Similarly to a hyperplane arrangement we can now consider the subspace arrangement $\mathcal{A}_H = \{H_e \,|\, e \in E\}$. If we consider the lattice whose order relation is inclusion and is meet generated by the set $\mathcal{A}_H$ with the operation

$$A \wedge B = A \cap B = \{(x_1, \ldots, x_d) \,|\, x_{i_1} = \cdots = x_{i_k} \text{ and } x_{j_1} = \cdots = x_{j_l}\},$$

where $A = \{(x_1, \ldots, x_d) \,|\, x_{i_1} = \cdots = x_{i_k}\}$ and $B = \{(x_1, \ldots, x_d) \,|\, x_{j_1} = \cdots = x_{j_l}\}$, it is not hard to see that it is order isomorphic to the dual of L_H. This poset is called the *intersection lattice* of H and gives us another perspective on how to analyze L_H.

We can study a further generalization of X_H which is defined in a similar fashion as the symmetric function generalization of the Tutte polynomial found in [11].

Definition 4.2.11. Let $H = (V, E)$ be a hypergraph and t be an indeterminant commuting with $\mathbf{x}$. Define

$$X_H(\mathbf{x}, t) = \sum_{\kappa} (1 + t)^{m(\kappa)} \mathbf{x}^{\kappa},$$

where the sum ranges over all colorings of H (including improper ones) and $m(\kappa)$ is the number of monochromatic edges. Unlike before, loops affect X_H in a non trivial fashion, so they are once again allowed. Additionally, $X_H(\mathbf{x}, t)$ is not fully determined by the minimal edges of H, so we cannot assume that E is an antichain from this point forward. We however still require our hypergraph to be finite.

Clearly this object is also symmetric in $\mathbf{x}$ and additionally $X_H(\mathbf{x}, -1) = X_H(\mathbf{x})$ (where $0^0 = 1$), meaning it is a strict generalization of the hypergraph chromatic function.

Remark 4.2.12. In [11] Stanley generalizes the Tutte polynomial $T_G(x, y)$ for a regular graph G as $X_G(\mathbf{x}, t) = \sum_{\kappa} (1 + t)^{m(\kappa)} \mathbf{x}^{\kappa}$, so that

$$X_G(1^n, t) = n^{c(G)} t^{\rho(G)} T_G\left(\frac{t + n}{t}, t + 1\right),$$

where $c(G)$ is the number of connected components and $\rho(G)$ is the rank of the lattice L_G (and therefore $\mathcal{A}_H$).

With analogous definitions he then proposes for a hypergraph H to define a "Tutte polynomial" $T'_H(x, y)$ via the functional identity

$$X_H(1^n, t) = n^{c(H)} t^{\rho(H)} T'_H\left(\frac{t+n}{t}, t+1\right).$$

In general T'_H is not a polynomial anymore and it seems it was not subject of much research since then.

Once again we are faced with a familiar property.

Theorem 4.2.13. *Let $H = (V, E)$ be a hypergraph. Then*

$$X_H(\mathbf{x}, t) = \sum_{S \subseteq E} t^{|S|} p_{\lambda(S)}(\mathbf{x}),$$

where $\lambda(S)$ records the sizes of the connected components of the induced subhypergraph $(V, S) \subseteq H$, analogous to 2.3.1.

Proof. The proof is completely analogous to the one of Theorem 2.3.1, where the only notable differences are that E_κ is the set of edges which are monochromatic under the coloring κ and we have $t^{|S|}$ instead of $(-1)^{|S|}$. □

Remark 4.2.14. We can specialize this to gain the additional result

$$X_H(\mathbf{x}) = X_H(\mathbf{x}, -1) = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}(\mathbf{x}),$$

leading to a second expansion of the hypergraph chromatic function in the basis of power sum symmetric functions.

4.3. The Quasisymmetric Chromatic Function

In this section we will discuss definitions and results found in a paper by Shareshian and Wachs [5], which was published in 2016. We will only look at those parts which are similar to what we discussed before, meaning the original paper gives a much more in-depth view of the subject, complete with connections to other fields of mathematics and conjectures. The fundamental idea is to take the usual definition of the chromatic function and add an additional statistic. This however leads to a more involved theory, including quasisymmetric functions. We are once again dealing with regular graphs.

Definition 4.3.1. Let $G = (V, E)$ be a graph with $|V| = d$ and suppose there is a linear ordering $v_1 < \cdots < v_d$ on V. For a map $\sigma : V \to [k]$ (where not necessarily $d = k$) we say that the pair $\{i, j\}$ (where without loss of generality $i < j$) is an *ascent* of σ if $\{v_i, v_j\} \in E$ and $\sigma(v_i) > \sigma(v_j)$. We denote by $\mathrm{asc}(\sigma)$ the number of ascents of σ. Note

at this point that this also defines ascents for colorings. Let t and $\mathbf{x} = (x_1, x_2, \dots)$ be commuting indeterminates, then the *chromatic quasisymmetric function* is defined as

$$X_G(\mathbf{x}, t) = \sum_{\kappa \in K} t^{\mathrm{asc}(\kappa)} \mathbf{x}^{\kappa},$$

where K is the set of all proper colorings of G.

Remark 4.3.2. Since we are now dealing with more types of indeterminates we are no longer working in $\mathcal{Q}_{\mathbb{Q}}$ (or a similar space depending on our choice of field). Instead we are working in $\mathcal{Q}_{\mathbb{Q}}[t] \cong \mathcal{Q}_{\mathbb{Q}[t]}$ and we will switch our viewpoint depending on which is more intuitive in the moment. Specifically this means that we will sometimes regard our objects as polynomials in t, while other times we view them as formal power series in $\mathbf{x}$ or $(\mathbf{x}, t)$. Clearly there is nothing special about $\mathbb{Q}$ and one might work in $\mathbb{R}$ or $\mathbb{C}$ instead, as was mentioned for the regular chromatic function.
Once again, it is necessary that our object is finite for this function to be defined. As loops and multiple edges do not change X_G we may assume G to be simple, as before. We will therefore once again use the word graph to refer to a finite simple graph.
Since there is a linear ordering on V we may assume that the vertices are a subset of the natural numbers. Furthermore, since the distance between those numbers does not impact X_G, we may as well assume that $V = [d]$ and will do so from now on, meaning we are dealing with colorings $\kappa : [d] \to [k]$.
Ascents as defined above are a generalization of the well studied ascent statistic for permutations, however even if we were to assume that $k = d$ we still do not require the map σ to be injective, hence it is in general not a permutation. Further still if σ was a bijection, we would need to work with a line graph (meaning one with edges $\{i, i+1\}$), since we are only considering what happens on edges of G, to reproduce the classical ascents.

Proposition 4.3.3. *Let $G = (V, E)$ be a graph. Then the function $X_G(\mathbf{x}, t)$ is quasisymmetric in $\mathbf{x}$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_l)$ be some weak composition and suppose $[\![\mathbf{x}^{\alpha}]\!] X_G = a_k t^k + \cdots + a_0$. Note that a_i counts the number of colorings κ for which the associated monomial in X_G is $\mathbf{x}^{\kappa} t^i$. For any map $f : \mathbb{N} \to \mathbb{N}$ extend it to weak compositions by $f(\alpha) = (\alpha_{f(1)}, \dots, \alpha_{f(l)})$.
Let $\pi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map. In order to prove that X_G is quasisymmetric it is enough to show that $[\![\mathbf{x}^{\pi(\alpha)}]\!] X_G = b_s t^s + \cdots + b_0$ is equal to $a_k t^k + \cdots + a_0$. Let κ be some proper coloring. Since π is monotonous the coloring $\pi \circ \kappa$ has the same amount of ascents as κ, meaning that $t^{\mathrm{asc}(\kappa)} = t^{\mathrm{asc}(\pi \circ \kappa)}$. Furthermore, since π is injective we also have that the amount of colorings κ with i ascents and $\mathbf{x}^{\kappa} = \mathbf{x}^{\alpha}$ is the same as the amount of colorings κ with i ascents and $\mathbf{x}^{\pi \circ \kappa} = \mathbf{x}^{\pi(\alpha)}$. Since the former describes the number a_i and the latter the number b_i we conclude $s = k$ and $a_i = b_i$ for $1 \leq i \leq k$, proving our claim. □

Unsurprisingly the quasisymmetric chromatic function behaves well with unions.

Proposition 4.3.4. *Let G and H be graphs and denote by $G+H$ their disjoint union. Then*

$$X_{G+H}(\mathbf{x},t) = X_G(\mathbf{x},t)X_H(\mathbf{x},t).$$

Proof. The proof is the same as for the chromatic polynomial or function, that is a pair of proper colorings κ_G of G and κ_H of H naturally extend to a proper coloring κ of $G+H$ and vice versa. Additionally, the number of ascents of κ is the sum of those of κ_G and κ_H. □

For the expansion of $X_G(\mathbf{x},t)$ in bases it is helpful to use t-analogues of certain counting functions.

Definition 4.3.5. Let $n \in \mathbb{N}$. The t-analogue of $n = 1+\cdots+1$ is $[n]_t = 1+t+\cdots+t^{n-1}$ and the the t-analogue of $n! = \prod_{i=1}^{n} i$ is $[n]_t! = \prod_{i=1}^{n}[i]_t$, where by convention $[0]_t = 0$ and $[0]_t! = 1$. There is nothing special about the letter t, apart from the fact that it is the indeterminant we chose to use in $X_G(\mathbf{x},t)$. The most common choice for other applications in the literature is q and subsequently one works with q-analogues.

Example 4.3.6. Consider the complete graph on n vertices K_n, for which we assume that the vertex set is $[n]$. A proper coloring κ is then an injective map $[n] \to \mathbb{N}$, so we note that $X_G(\mathbf{x}) = e_n n!$. For any such κ consider the corresponding proper coloring $\kappa' : [n] \to [n]$ which preserves the order of $\kappa(1),\ldots,\kappa(n)$ and notice that it is a permutation. By definition it is the case that $\mathrm{asc}(\kappa) = \mathrm{inv}(\kappa')$, where $\mathrm{inv}(\kappa')$ refers to the well known inversion statistic for permutations. Let $C_{\kappa'}$ be the set of proper colorings which reduce to κ' in the above fashion, giving us a partition of the set of all proper colorings C of K_n. This means that

$$X_{K_n}(\mathbf{x},t) = \sum_{\kappa'\in S_n}\sum_{\kappa\in C_{\kappa'}} t^{\mathrm{asc}(\kappa)}\mathbf{x}^{\kappa} = \sum_{\kappa'\in S_n} t^{\mathrm{inv}(\kappa')}\sum_{\kappa\in C_{\kappa'}}\mathbf{x}^{\kappa} = e_n\sum_{\kappa'\in S_n} t^{\mathrm{inv}(\kappa')} = e_n[n]_t!,$$

where $e_n = e_n(\mathbf{x})$ as usual and $\sum_{\sigma\in S_n} t^{\mathrm{inv}(\sigma)} = [n]_t!$ is a well known formula by MacMahon. So in this specific case the quasisymmetric chromatic function is already symmetric and we will soon see classes of graphs for which this is always the case.

For some of the following results we will make use of a particular basis for quasisymmetric functions.

Definition 4.3.7. Let $\alpha = (\alpha_1,\ldots,\alpha_l)$ be a composition and denote by $\alpha^{\mathrm{rev}} = (\alpha_l,\ldots,\alpha_1)$ its reverse composition. The *monomial quasisymmetric function* M_α is defined as

$$M_\alpha(\mathbf{x}) = \sum_{i_1<\cdots<i_l} x_{i_1}^{\alpha_1}\cdots x_{i_l}^{\alpha_l}.$$

It can be shown that the set of all M_α is a basis for the space of quasisymmetric functions. Define an involution ρ on $\mathcal{Q}_{\mathbb{Q}}$ (and subsequently $\mathcal{Q}_{\mathbb{Q}}[t]$) by $\rho(M_\alpha) = M_{\alpha^{\mathrm{rev}}}$ on the basis of monomial quasisymmetric functions and linear extension elsewhere. Note that every symmetric function is fixed by ρ, which is the case if and only if it fixes all elements

of some basis of $\Lambda_{\mathbb{Q}}$, for example $\rho(e_n) = \rho(M_{(1,\dots,1)}) = M_{(1,\dots,1)} = e_n$.
In order to prove the next result it makes sense to also define the reverse of a weak composition. The way we do this might seem slightly peculiar, however it will be convenient. Let $\alpha = (\alpha_1, \dots, \alpha_l)$ be a weak composition and let α_j be its first nonzero value. Then we define the reverse of α as $\alpha^{\text{rev}} = (\alpha_1, \dots, \alpha_{j-1}, \alpha_l, \alpha_{l-1}, \dots, \alpha_{j+1}, \alpha_j)$.

First we will consider a nice symmetry in t which is satisfied by $X_G(\mathbf{x}, t)$.

Proposition 4.3.8. *Let $G = (V, E)$ be a graph. Then*

$$\rho(X_G(\mathbf{x}, t)) = t^{|E|} X_G(\mathbf{x}, t^{-1}).$$

Proof. The goal will be to construct an involution on the set of all proper colorings which for each monomial of X_G translates t^j to $t^{|E|-j}$ while reversing the exponent of $\mathbf{x}^\kappa$ in the sense of a weak composition. Let κ be some proper coloring and let m_1 and m_2 be its smallest and largest used color respectively. Construct the coloring $\gamma(\kappa)$ such that $\gamma(\kappa)(v) = m_1 + m_2 - \kappa(v)$. Clearly $\gamma(\kappa)$ is a proper coloring if κ was one. It is also easy to see that $\gamma(\gamma(\kappa)) = \kappa$ and additionally $\text{asc}(\gamma(\kappa)) = |E| - \text{asc}(\kappa)$. Furthermore, if $\mathbf{x}^\kappa = \mathbf{x}^\alpha$ for a weak composition α then $\mathbf{x}^{\gamma(\kappa)} = \mathbf{x}^{\alpha^{\text{rev}}}$. This means γ is the involution we were looking for.
Using γ then implies that $[\![\mathbf{x}^\alpha t^j]\!] X_G = [\![\mathbf{x}^{\alpha^{\text{rev}}} t^{|E|-j}]\!] X_G$ for all weak compositions α of $|V|$ and $j \in [|E|]$, which may be rephrased as $[\![\mathbf{x}^{\alpha^{\text{rev}}} t^j]\!] X_G = [\![\mathbf{x}^\alpha t^{|E|-j}]\!] X_G$. Using a bijective argument for the $\mathbf{x}^\alpha$ we can show that for all regular compositions β if $M_\beta = \sum_\alpha \mathbf{x}^\alpha$, where α ranges over all relevant weak compositions, then $M_{\beta^{\text{rev}}} = \sum_\alpha \mathbf{x}^{\alpha^{\text{rev}}}$. This in combination with the earlier equality implies $[\![M_{\beta^{\text{rev}}} t^j]\!] X_G = [\![M_\beta t^{|E|-j}]\!] X_G$. By the definition of ρ the former coefficient is then also equal to $[\![M_\beta t^j]\!] \rho(X_G)$. Reconnecting these coefficients back to their quasisymmetric functions, we see that one of them describes $\rho(X_G(\mathbf{x}, t))$ and the other $t^{|E|} X_G(\mathbf{x}, t^{-1})$, establishing equality between the two. □

As a result of the above described symmetry of X_G one calls the quasisymmetric chromatic function *palindromic* in t.
As an immediate consequence of our reasoning we get another object which behaves in a similar fashion to the quasisymmetric chromatic function.

Corollary 4.3.9. *Let $G = (V, E)$ be a graph and ρ as above. Then*

$$\rho(X_G(\mathbf{x}, t)) = \sum_{\kappa \in K} t^{\text{des}(\kappa)} \mathbf{x}^\kappa,$$

where $\text{des}(\kappa)$ *is the number of descents defined similarly to* $\text{asc}(\kappa)$*, that is* $\text{des}(\kappa) = |\{\{i, j\} \in E \mid i < j \text{ and } \kappa(i) > \kappa(j)\}|$.

We will now expand the quasisymmetric chromatic function in the basis of fundamental quasisymmetric functions in the case of an incomparability graph. For this we need to first generalize the concepts of an inversion and the descent set of a permutation.

Definition 4.3.10. Let $G = ([d], E)$ be a graph and $\sigma \in S_d$ be a permutation. We call a pair $(i, j) \in [d]^2$ a *G-inversion* of σ if $i < j$, $\sigma(i) > \sigma(j)$ and $\{\sigma(i), \sigma(j)\} \in E$. We denote by $\mathrm{inv}_G(\sigma)$ the number of G-inversions of σ.
Let P be a partially ordered set on $[d]$ and $\sigma \in S_d$ be a permutation. We call $i \in [d-1]$ a *P-descent* if $\sigma(i) >_P \sigma(i+1)$ and denote by $\mathrm{DES}_P(\sigma)$ the set of P-descents of σ.
From now on let ω be the involution satisfying $\omega(Q_{S,d}) = Q_{[d-1]\setminus S,d}$.

Remark 4.3.11. It can easily be seen that $\mathrm{inv}_G(\sigma)$ and $\mathrm{DES}_P(\sigma)$ are generalizations of regular inversions and descent sets, by taking the complete graph K_d and the linear ordering $1 < \cdots < d$ respectively.

Theorem 4.3.12. *Let P be a partially ordered set on $[d]$ and $G = (V, E) = \mathrm{inc}(P)$ its incomparability graph. Then*

$$\omega(X_G(\mathbf{x}, t)) = \sum_{\sigma \in S_d} t^{\mathrm{inv}_G(\sigma)} Q_{\mathrm{DES}_P(\sigma),d}$$

and therefore

$$X_G(\mathbf{x}, t) = \sum_{\sigma \in S_d} t^{\mathrm{inv}_G(\sigma)} Q_{[d-1]\setminus \mathrm{DES}_P(\sigma),d}.$$

Proof. Since we will always be dealing with quasisymmetric functions of degree d we will omit this, meaning we will write for example Q_S instead of $Q_{S,d}$. Furthermore, we use the word sequencing to denote a function $\mathbb{N} \supseteq [d] \to [d] = V$, that is bringing our vertices into a sequence. We also use the word labeling to denote a function $V = [d] \to [d] \subseteq \mathbb{N}$, giving each vertex a label. This gives us a way to distinguish the two concepts despite their similarities.
Let $\mathcal{O}$ be an acyclic orientation of G and denote by $\mathrm{asc}(\mathcal{O})$ the number of edges (i, j) with $i < j$. Further denote the set of proper colorings that agree with $\mathcal{O}$ by $K(\mathcal{O})$, meaning that $\kappa \in K(\mathcal{O})$ if and only if $\kappa(i) < \kappa(j)$ for all $(i, j) \in E(\mathcal{O})$. As was argued before in this work, each proper coloring κ can be assigned a unique acyclic orientation $\mathcal{O}$ such that the two agree. Due to the definitions we also have $\mathrm{asc}(\kappa) = \mathrm{asc}(\mathcal{O})$ in that case, so that

$$X_G(\mathbf{x}, t) = \sum_{\mathcal{O}} t^{\mathrm{asc}(\mathcal{O})} \sum_{\kappa \in K(\mathcal{O})} \mathbf{x}^{\kappa}.$$

Remember that we use the notation $\bar{\mathcal{O}}$ to denote the poset whose relations are given by the transitive closure of the directed edges of $\mathcal{O}$. For any given labeling o denote by $\mathcal{L}(\bar{\mathcal{O}}, o)$ the set of linear extensions (sequencing functions) α of $\bar{\mathcal{O}}$ when viewed as permutations through o. Specifically this means that we are dealing with functions that satisfy $i < j$ if $\alpha^{-1}(i) <_{\bar{\mathcal{O}}} \alpha^{-1}(j)$ and we view them as permutations $(o(\alpha^{-1}(1)), \ldots, o(\alpha^{-1}(d)))$ (in one line notation). From now on we choose for each acyclic orientation $\mathcal{O}$ some order-reversing labeling $o_{\mathcal{O}}$.
A result by Stanley (see [10] Corollary 7.19.5) when modified slightly gives us that

$$\sum_{\kappa \in K(\mathcal{O})} \mathbf{x}^{\kappa} = \sum_{\sigma \in \mathcal{L}(\bar{\mathcal{O}}, o_{\mathcal{O}})} Q_{d-\mathrm{DES}(\sigma)},$$

where $d - S = \{i \mid d - i \in S\}$ for $S \subseteq [d]$ and $\mathrm{DES}(\sigma)$ is the classical descent set of the permutation σ.
Denote by ε the identity permutation in S_d and also consider the $o_{\mathcal{O}}$ as permutations. Then by definition $\sigma \in \mathcal{L}(\bar{\mathcal{O}}, o_{\mathcal{O}})$ if and only if $\sigma = o_{\mathcal{O}} \circ s$ for some sequencing s, where trivially $s \in \mathcal{L}(\bar{\mathcal{O}}, \varepsilon)$. This gives a bijection between $\mathcal{L}(\bar{\mathcal{O}}, o_{\mathcal{O}})$ and $\mathcal{L}(\bar{\mathcal{O}}, \varepsilon)$. Applying the bijection to the equation above yields

$$\sum_{\kappa \in K(\mathcal{O})} \mathbf{x}^{\kappa} = \sum_{s \in \mathcal{L}(\bar{\mathcal{O}}, \varepsilon)} Q_{d-\mathrm{DES}(o_{\mathcal{O}} \circ s)},$$

which can further be applied to our expansion of X_G, resulting in

$$X_G(\mathbf{x}, t) = \sum_{\mathcal{O}} t^{\mathrm{asc}(\mathcal{O})} \sum_{s \in \mathcal{L}(\bar{\mathcal{O}}, \varepsilon)} Q_{d-\mathrm{DES}(o_{\mathcal{O}} \circ s)}.$$

Denote by $\mathcal{L}(G)$ the set of all sequencing functions of G. Each $s \in \mathcal{L}(G)$ can be assigned a unique acyclic orientation $\mathcal{O}(s)$ such that $s \in \mathcal{L}(\bar{\mathcal{O}}(s), \varepsilon)$ by orienting all edges according to the labeling s^{-1}, that is s is entirely order preserving with respect to $\bar{\mathcal{O}}(s)$. This implies that $\mathcal{L}(G) = \bigcup_{\mathcal{O}} \mathcal{L}(\bar{\mathcal{O}}, \varepsilon)$ so we can use a double counting argument in the expansion of X_G which switches $\mathcal{O}$ for $\mathcal{O}(s)$. It is also the case by definition that $\mathrm{asc}(\mathcal{O}(s)) = \mathrm{inv}_G(s^{\mathrm{rev}})$ (where s^{rev} denotes the reverse permutation) so we further manipulate

$$X_G(\mathbf{x}, t) = \sum_{s \in \mathcal{L}(G)} t^{\mathrm{asc}(\mathcal{O}(s))} Q_{d-\mathrm{DES}(o_{\mathcal{O}(s)} \circ s)} = \sum_{s \in S_d} t^{\mathrm{inv}_G(s^{\mathrm{rev}})} Q_{d-\mathrm{DES}(o_{\mathcal{O}(s)} \circ s)}.$$

Everything up to this point was true for all $o_{\mathcal{O}}$ which are order reversing with respect to $\bar{\mathcal{O}}$. Now we will construct one specific such $\tilde{o}_{\mathcal{O}}$ for each $\mathcal{O}$ to work with: All $\bar{\mathcal{O}}$-maximal elements are P-comparable, so there is a unique P-maximal vertex among them. Label this element with 1 and remove it from G, $\mathcal{O}$, $\bar{\mathcal{O}}$ and P. Repeat this step with the new objects and label the resulting vertex with 2. Proceed in this manner by labeling the vertex in question with i after $i-1$ elements have been removed until all of G is labeled. By construction $\tilde{o}_{\mathcal{O}}$ is indeed an order reversing labeling.
I now claim that if x, y are $\bar{\mathcal{O}}$-incomparable and $x <_P y$, then $\tilde{o}_{\mathcal{O}}(x) > \tilde{o}_{\mathcal{O}}(y)$.
To prove this assume there are $\bar{\mathcal{O}}$-incomparable elements x, y such that $x <_P y$ and $\tilde{o}_{\mathcal{O}}(x) < \tilde{o}_{\mathcal{O}}(y)$ (note that $\tilde{o}_{\mathcal{O}}(x) \in \mathbb{N}$, so we have a linear ordering). Consider the step in the construction of $\tilde{o}_{\mathcal{O}}$ before x was labeled. Let $\bar{\mathcal{O}}_x$ and P_x be $\bar{\mathcal{O}}$ and P with all currently labeled elements removed respectively. Note that these are induced subposets and as such their order relations are inherited from the original posets. By assumption x is P_x-maximal under all $\bar{\mathcal{O}}_x$-maximal elements and since $x <_P y$ it follows that y is not $\bar{\mathcal{O}}_x$-maximal. This means that there exists some currently unlabeled z such that $y <_{\bar{\mathcal{O}}} z$ and we make a case distinction based on its relation to x. If $x \geq_{\bar{\mathcal{O}}} z$ then by transitivity $x >_{\bar{\mathcal{O}}} y$ in contradiction to their $\bar{\mathcal{O}}$-incomparability. Otherwise x and z must themselves be $\bar{\mathcal{O}}$-incomparable, so there is an $\bar{\mathcal{O}}_x$-maximal $z_0 \geq_{\bar{\mathcal{O}}} z$ which is also $\bar{\mathcal{O}}$-incomparable to x and satisfies $y <_{\bar{\mathcal{O}}} z_0$ as z did. Since x is in the process of being labeled we realize that $x >_P z_0$ and because of transitivity $y >_P z_0$. Since $z_0 >_{\bar{\mathcal{O}}} y$ the vertex z_0 will be labeled

before y, so it satisfies the same properties we assumed of x, those being $z_0 <_P y$ and $\tilde{o}_{\mathcal{O}}(z_0) > \tilde{o}_{\mathcal{O}}(y)$, while additionally being labeled after x. Using the same reasoning on the pair z_0, y implies the existence of an element z_1 which gets labeled after z_0 satisfying the same properties. By induction we can find an infinite sequence $(x, z_0, z_1, \dots)$ of such elements, in contradiction to $|V| = [d]$. We therefore conclude that the claim must be true.
Now we wish to show that

$$\mathrm{DES}_P(s) = [d-1] \setminus \mathrm{DES}(\tilde{o}_{\mathcal{O}(s)} \circ s)$$

using our new claim in order to further simplify our expansion of X_G. Let s be some sequencing and $i \in \mathrm{DES}_P(s)$. By definition we have $s(i) >_P s(i+1)$ while also $s \in \mathcal{L}(\bar{\mathcal{O}}(s), \varepsilon)$, meaning $s(i)$ and $s(i+1)$ are $\bar{\mathcal{O}}(s)$-incomparable. Hence our claim implies that $\tilde{o}_{\mathcal{O}(s)}(s(i)) < \tilde{o}_{\mathcal{O}(s)}(s(i+1))$ which is to say $i \in [d-1] \setminus \mathrm{DES}(\tilde{o}_{\mathcal{O}(s)} \circ s)$. Suppose on the other hand that $i \in [d-1] \setminus \mathrm{DES}(\tilde{o}_{\mathcal{O}(s)} \circ s)$, which means that $\tilde{o}_{\mathcal{O}(s)}(s(i)) < \tilde{o}_{\mathcal{O}(s)}(s(i+1))$. Since $\tilde{o}_{\mathcal{O}(s)}$ is order reversing we must have $s(i) \not<_{\bar{\mathcal{O}}(s)} s(i+1)$ and since $s \in \mathcal{L}(\bar{\mathcal{O}}(s), \varepsilon)$ we also have $s(i) \not>_{\bar{\mathcal{O}}(s)} s(i+1)$, meaning the two are $\bar{\mathcal{O}}(s)$-incomparable. Applying the logical negation of our claim then implies that $s(i) \not<_P s(i+1)$ and since the two are $\bar{\mathcal{O}}(s)$-incomparable it follows that there is no edge between the two in G, so by $G = \mathrm{inc}(P)$ they are P-comparable, meaning $s(i) >_P s(i+1)$, that is to say $i \in \mathrm{DES}_P(s)$. With this our equality has been established.
We now apply ω to our expansion of X_G while noting that $\omega(Q_{d-S}) = Q_{d-([d-1]\setminus S)}$ and further use the above manipulation for the descent set to get

$$\omega(X_G(\mathbf{x}, t)) = \sum_{s \in S_d} t^{\mathrm{inv}_G(s^{\mathrm{rev}})} Q_{d-\mathrm{DES}_P(s)}.$$

Finally, if we take the dual poset P^* of P we see for all permutations $\sigma \in S_d$ it holds that $d - \mathrm{DES}_P(\sigma) = \mathrm{DES}_{P^*}(\sigma^{\mathrm{rev}})$, so

$$\omega(X_G(\mathbf{x}, t)) = \sum_{\sigma \in S_d} t^{\mathrm{inv}_G(\sigma)} Q_{\mathrm{DES}_{P^*}(\sigma)}.$$

Since $G = \mathrm{inc}(P) = \mathrm{inc}(P^*)$ and $P = (P^*)^*$ we can do all of the above with P^* instead of P to arrive at our desired conclusion. □

One particular class of partially ordered sets which are of interest for the quasisymmetric chromatic function (as well as its symmetric counterpart) is that of unit interval orders. First we give the definition and a convenient classification.

Definition 4.3.13. Let $\mathcal{I}$ be a finite collection of intervals of the form $[x, x+1]$ in $\mathbb{R}$ and define an order relation on them by $[x, x+1] <_{\mathcal{I}} [y, y+1]$ if $x+1 < y$, that is to say the interval $[x, x+1]$ lies completely to the left of $[y, y+1]$. We will identify the interval $[x, x+1]$ with the number x and subsequently only write $x <_{\mathcal{I}} y$. We call any partially ordered set P isomorphic to such an $\mathcal{I}$ a *unit interval order*.
Let P be a partially ordered set on $[d]$ satisfying:

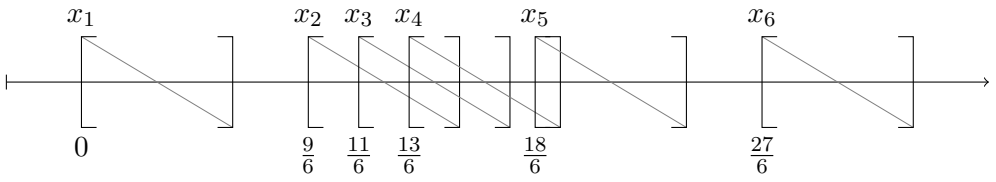
- If $x <_P y$ then $x < y$ and
- if $\{x, z\}_P + \{y\}$ with $x <_P z$ is an induced subposet of P, then $x < y < z$. Another way to phrase this condition is to say that y is pairwise P-incomparable to both x and z.

Here by $\{x, z\}_P$ we mean the induced subposet of P on the elements x, z and $+$ is the disjoint union. In this case we call P a *natural unit interval order*.
Let $m = (m_1, \ldots, m_{d-1})$ be a sequence of natural numbers such that

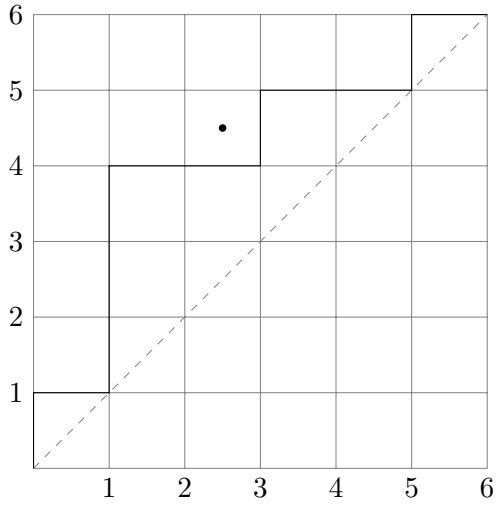
- $m_1 \leq \cdots \leq m_{d-1} \leq d$ and
- $m_i \geq i$ for all i.

We then denote by $P(m)$ the partially ordered set on $[d]$ with order relation $i <_{P(m)} j$ if $j \in [d]\backslash[m_i]$ for $i \in [d-1]$. One might visualize $P(m)$ as a lattice path from $(0, 0)$ to (d, d) with steps either $(0, 1)$ or $(1, 0)$ staying weakly above the line $x = y$. To do this, for each $i \in [d-1]$ draw a horizontal line from $(i-1, m_i)$ to (i, m_i), then add a line from $(d-1, d)$ to (d, d) and vertical lines as necessary to connect everything. Given this view the order relations $i <_{P(m)} j$ are then the points $(i - \frac{1}{2}, j - \frac{1}{2})$ which lie strictly above our lattice path.

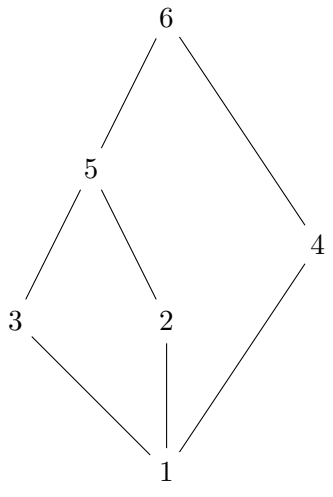
Example 4.3.14. Consider the sequences $\mathcal{I} = (x_1, \ldots, x_6) = (0, \frac{9}{6}, \frac{11}{6}, \frac{13}{6}, \frac{18}{6}, \frac{27}{6})$ and $m = (1, 4, 4, 5, 5)$. To better understand how the posets $\mathcal{I}$ and $P(m)$ behave in this case we visualize them, starting with $\mathcal{I}$.



Each pair of brackets represents a unit interval on the real number line (with diagonals to make reading easier). By definition, we have $x_i <_{\mathcal{I}} x_j$ if the unit intervals of x_i and x_j are disjoint and x_j is to the right of x_i. Conversely, if two intervals intersect then their respective numbers are $\mathcal{I}$-incomparable. In order to later make a comparison with $P(m)$ we will use the notation $i <_{\mathcal{I}} j$ to mean $x_i <_{\mathcal{I}} x_j$. Looking at the diagram above, we see for example that $3 <_{\mathcal{I}} 5$.
For the visualization of $P(m)$ we go with the above explained lattice path model.



To read off the order relations from this diagram take $i < j$ and consider the point $(i - \frac{1}{2}, j - \frac{1}{2})$ which lies in the center of some square. If the lattice path passes under this point then $i <_{P(m)} j$, otherwise the two are $P(m)$-incomparable. One example of comparable elements is $3 <_{P(m)} 5$ (denoted by a dot in the above). After noting all relations one can draw $P(m)$ which has the following form:



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It turns out that the poset $\mathcal{I}$ also has this form. Furthermore, one can check that the above also defines a natural unit interval order. We will now see that these facts are not mere coincidence.

Proposition 4.3.15. *Let P be a partially ordered set on $[d]$. The following statements are equivalent:*

1. *P is a natural unit interval order.*

2. *$P = P(m)$ for some $m = (m_1, \ldots, m_{d-1})$ as described above.*
3. *There exists a unit interval order $\mathcal{I} = \{x_1, \ldots, x_d\}$, $x_1 < \cdots < x_d$, such that $x_i <_{\mathcal{I}} x_j$ if and only if $i <_P j$.*

Proof. We will prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.
$1 \Rightarrow 2$: Assume that P is a natural unit interval order. For $i \in [d-1]$ let $m_i = \max\{j \in [d] \,|\, j \not>_P i\}$. We wish to show that those m_i satisfy the above mentioned conditions, where trivially $m_i \leq d$ and since $i \not>_P i$ it follows that $m_i \geq i$. We still need to show that $m_i \leq m_j$ if $i < j$ in order for the m_i to satisfy all needed properties.
Suppose we had such a pair $i < j$ where this was not the case, so $m_i > m_j$. It must be the case that $m_i >_P j$ since m_j is the largest element for which $m_j \not>_P j$.
I claim that i is P-incomparable to m_i and j in this case. Suppose i is comparable to m_i. By the definition of m_i it must then be the case that $m_i \leq_P i$ and therefore $m_i \leq i$, so we conclude $m_i = i$ since $m_i \geq i$. As a result we have $i = m_i >_P j$, meaning that i and j are P-comparable, leading to the second case of our claim. Now suppose i is P-comparable to j. Since $j > i$ we need $j >_P i$ by the definition of a natural unit interval order, so that $m_i >_P j >_P i$, a contradiction to the definition of m_i. We conclude that i is in fact P-incomparable to both m_i and j.
Because of this we get that $\{j, m_i\}_P + \{i\}$ is an induced subposet of P, implying that $j < i < m_i$, a contradiction to the fact $i < j$. We conclude that in fact $m_i \leq m_j$, meaning $m = (m_1, \ldots, m_{d-1})$ defines a poset $P(m)$ as above.
It is left to show that $P = P(m)$. Assume that $i \not<_{P(m)} j$. Either $i >_{P(m)} j$ in which case $i > m_j$ by the definition of $P(m)$, so i is not part of $\{k \in [d] \,|\, k \not>_P j\}$, meaning $i >_P j$. Otherwise they are $P(m)$-incomparable, meaning that $j \leq m_i$. Consider the case that $i <_P j$. Then m_i needs to be P-comparable to either i or j, since otherwise we would have the induced subposet $\{i, j\}_P + \{m_i\}$ and it would follow $i < m_i < j$. By definition m_i is P-incomparable to i, so it is P-comparable to j, meaning in our case $m_i \geq_P j$ since $m_i \geq j$. As a result of this we have that $m_i \geq_P j >_P i$, a contradiction. This shows that $i \not<_P j$ if $i \not<_{P(m)} j$.
For the converse assume $i <_{P(m)} j$. Then $j > m_i$ by definition, meaning j is not element of $\{k \in [d] \,|\, k \not>_P i\}$. We therefore conclude that $j >_P i$. With this we have proven that $i <_P j$ if and only if $i <_{P(m)} j$, which shows $P = P(m)$.
$2 \Rightarrow 3$: Now assume that we have $P = P(m)$ for some $m = (m_1, \ldots, m_{d-1})$. We will prove this claim by induction on d, where the case $d = 1$ is trivial. From now on assume that $d > 1$. If $m_1 = d$ then P is an antichain, so we can choose $x_i = \frac{i}{d}$ and are done. As such we now also suppose $m_1 < d$. Define $m' = (m'_1, \ldots, m'_{d-2})$ by $m'_i = \min\{d-1, m_i\}$. Note that the subposet of P induced by the vertex set $[d-1]$ is $P(m')$. By our induction hypothesis we find $\mathcal{I}' = \{x_1, \ldots, x_{d-1}\}$ such that $x_i <_{\mathcal{I}'} x_j$ if and only if $i <_{P(m')} j$. Assume without loss of generality that $x_i - x_j \neq 1$ for all i, j, which we can ensure by progressively adding small numbers to each x_i such that the interval order is still intact. If $m_{d-1} = d-1$ then $i <_P d$ for all $i \in [d-1]$. We can therefore choose $\mathcal{I} = \mathcal{I}' \cup \{x_{d-1}+1\}$ and are done. If $m_1 < m_{d-1} = d$ take the smallest i such that $m_i = d$. In that case j and d are P-incomparable if and only if $i \leq j < d$, for all $j < i$ we instead have $j <_P d$. In order to construct a unit interval order $\mathcal{I}$ as planned we would need $x_j + 1 > x_d$ for

all such incomparable j. To make this work pick any $x_d \in (\max\{x_{i-1}+1, x_{d-1}\}, x_i+1)$ and use $\mathcal{I} = \mathcal{I}' \cup \{x_d\}$. The interval for x_d is necessarily nonempty since otherwise either $x_{i-1} > x_i$, in contradiction to the definition of $\mathcal{I}'$, or $x_{d-1} > x_i + 1$, which would imply that $i <_{P(m')} d-1$ in contradiction to our choice of i.
$3 \Rightarrow 1$: Finally assume that we have such a unit interval order $\mathcal{I}$. If $i <_P j$ then it follows $x_i < x_i + 1 < x_j$ and therefore $i < j$. Suppose that $i <_P k$ where i, k are pairwise P-incomparable to j. Consider the induced subposets $\{i,k\}_P + \{j\}$ of P. Then $x_i + 1 < x_k$ and $x_j \in [x_i - 1, x_i + 1] \cap [x_k - 1, x_k + 1]$ by definition. It follows that $x_i < x_j < x_k$ and by extension $i < j < k$, showing that P is a natural unit interval order. □

Remark 4.3.16. Consider the isomorphism class of a unit interval order and call its elements $x_1, \ldots, x_d$, where $x_1 < \cdots < x_d$. One way to characterize this class is by the following construction: Start with x_1 and for x_2 decide whether or not it is inside or outside of $[x_1, x_1 + 1]$. Similarly, for x_3 decide whether or not it is inside or outside of $[x_1, x_1 + 1]$ and $[x_2, x_2 + 1]$ respectively (in accordance with the rules of a unit interval order). Proceed in this way until all order relations are defined.
Based on this construction consider the following: If we start at x_1 and move along the real line until $x_d + 1$ we encounter all intervals. Every time we enter an interval we record "(" and every time we leave one we record ")". Clearly we recorded each symbol d times and if we list them in order we get a string of parenthesis. We might imagine that each pair of parenthesis corresponds to the application of a binary operation on two elements, for example in a groupoid. Such strings of parenthesis are well known to be enumerated by the Catalan numbers $C_d = \frac{1}{d+1}\binom{2d}{d}$. It should also be clear that each list of parentheses of this form corresponds to a unique isomorphism class of unit interval orders, hence this is a bijection and we conclude that there are C_d isomorphism classes. Another object which is enumerated by C_d is the number of lattice paths from $(0,0)$ to (d,d) with steps $(1,0)$ or $(0,1)$ staying weakly above the line $x = y$, which is to say that the number of posets $P(m)$ with $m = (m_1, \ldots, m_{d-1})$ as described above is also C_d. Given the fact that $P(m) = P(n)$ implies $m = n$ we see that the number of posets $P(m)$ is the same as the number of natural unit interval orders of size d. This implies that each natural unit interval order can be associated to exactly one class of unit interval orders. If we include our knowledge from before to do a natural association, we get the following result.

Proposition 4.3.17. *Each natural unit interval order is order isomorphic to exactly one class of unit interval orders.*

Now we finally prove that the quasisymmetric chromatic function associated to the incomparability graph of a natural unit interval order is symmetric. To do this we first need a technical Lemma.

Lemma 4.3.18. *Let P be a natural unit interval order and $G = (V, E) = \mathrm{inc}(P)$. Let further κ be a proper coloring of G and $a \in \mathbb{N}$. Denote by $G_{\kappa,a}$ the induced subgraph of G consisting of vertices that are colored either a or $a+1$ under κ. Then each connected component of $G_{\kappa,a}$ is some path $i_1 - \cdots - i_k$ for which $i_1 < \cdots < i_k$.*

Proof. Clearly $G_{\kappa,a}$ is bipartite and therefore contains no cycles of odd length, specifically none of length three.
Consider the case when $\{x,y\},\{y,z\} \in E$ but $\{x,z\} \notin E$. By definition x and z are P-comparable and are pairwise P-incomparable to y. By the properties of a natural unit interval order this implies that $x < y < z$ or $z < y < x$. Clearly this property also extends to $G_{\kappa,a}$, since it is an induced subgraph. If we now take any path $i_1 - \cdots - i_k$ in $G_{\kappa,a}$ we see that there cannot be an edge of the form $\{i_j, i_{j+2}\}$ in G, since this would introduce a cycle of length 3 to $G_{\kappa,a}$. We can therefore apply the above reasoning $k-2$ times, since $\{i_j,i_{j+1}\},\{i_{j+1},i_{j+2}\} \in E$ but $\{i_j,i_{j+2}\} \notin E$ for all $1 \leq j \leq k-2$, to arrive at the conclusion that either $i_1 < \cdots < i_k$ or $i_1 > \cdots > i_k$. This is still true in the case that the path already forms a cycle, which would mean $i_1 = i_k$, but $i_1 < i_k$ or $i_1 > i_k$. By this we conclude that no cycles can exist in $G_{\kappa,a}$. By definition this makes $G_{\kappa,a}$ a forest, so if we can also show that each vertex has a degree of at most 2 the claim follows.
Consider on the contrary a vertex v with neighbors x, y and z in $G_{\kappa,a}$. We may assume that $x < v$, since at least one such vertex connected to v must exist. Since we have the paths $x - v - y$ and $x - v - z$ we infer that $x < v < y$ and $x < v < z$. But there is also the path $y - v - z$, so either $y < v < z$ or $z < v < y$, a contradiction. This shows that no vertex of degree 3 or higher exists in $G_{\kappa,a}$ and we are done. □

Theorem 4.3.19. *Let P be a natural unit interval order and $G = (V,E) = \mathrm{inc}(P)$. Then $X_G(\mathbf{x},t)$ is a symmetric function with respect to $\mathbf{x}$.*

Proof. We will construct for each $a \in \mathbb{N}$ an involution φ_a on the set of proper colorings which will exchange the amount of occurences of the colors a and $a+1$ while keeping $t^{\mathrm{asc}(\kappa)}$ constant. Since the transpositions $(a, a{+}1)$ generate $S_{\mathbb{N}}$ this will show that $X_G(\mathbf{x},t)$ is symmetric in $\mathbf{x}$.
Let κ be some proper coloring. By Lemma 4.3.18 we can consider paths in $G_{\kappa,a}$, say $i_1 - \cdots - i_k$, which are colored only with a and $a+1$ so that $i_1 < \cdots < i_k$. Define $\varphi_a(\kappa)$ by keeping all colors intact, except in the case of the above mentioned paths with odd k. In their case, swap around the occurrences of a and $a+1$. It should be clear that $\varphi_a(\kappa)$ is a proper coloring, that the amount of occurrences of a and $a+1$ is swapped and that φ_a is an involution on the set of proper colorings. The fact that $\mathrm{asc}(\kappa) = \mathrm{asc}(\varphi_a(\kappa))$ follows since for odd k we have an ascent in every second step of the path $i_1 - \cdots - i_k$ by $i_1 < \cdots < i_k$ and there are an even number of edges. □

A. Refresher on Discrete Mathematics

This part serves as a refresher or brief introduction to topics from discrete mathematics which we use in this work. It is assumed that the reader had at least some exposure to the most fundamental concepts and as such they will only be briefly mentioned. Those unfamiliar with the basics of partially ordered sets and especially graphs are encouraged to read up on those topics elsewhere.

A.1. Graphs

As graphs are one of the most fundamental parts of discrete mathematics we will simply give a demonstration on how the notations surrounding them is used in this work. Let $G = (V, E)$ be a graph with vertices V which are connected by edges E. We will also use the notation $V(G) = V$ and $E(G) = E$ in cases where the vertex and edge set are not further specified. For a simple graph we can identify each edge with a two element subset of V and will therefore simply denote them by $\{v, w\} \in E$ for $v, w \in V$. In the case of a directed graph $D = (V, E)$ we will similarly write $V = V(D)$ and $E = E(D)$. Additionally, if it simple then each edge may be identified by a two element tuple and we will therefore write $(v, w) \in E$ for $v, w \in V$ if the edge goes from v to w.

A.2. Partially Ordered Sets

Here we will show the most basic aspects of partially ordered sets needed to understand the rest of this work. Due to this it is very fast paced and key concepts which are relevant to posets on a grander scale will be completely missing in this section.

Definition A.2.1. A *partially ordered set* or *poset* is a set P with an order relation $\leq$, that is it satisfies reflexivity, antisymmetry and transitivity. Specifically this means that for all $x, y, z \in P$ we have

- $x \leq x$,
- if $x \leq y$ and $y \leq x$ then $x = y$ and
- if $x \leq y$ and $y \leq z$ then $x \leq z$.

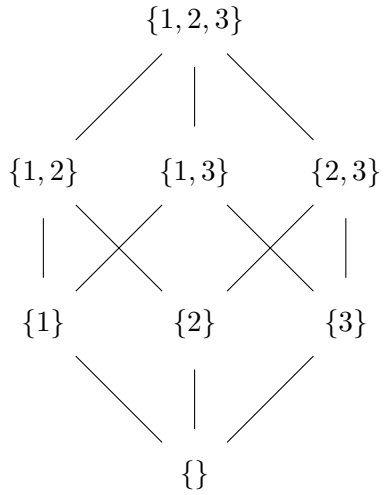
We use the notation $x < y$ to mean that $x \leq y$ and $x \neq y$. Depending on the context we will also write $y \geq x$ instead of $x \leq y$. When discussing several posets simultaneously we will use $\leq_P$ instead of $\leq$ for clarity. We will however still use the notation $\leq$ for the usual order on the real numbers in that case. If for two elements x, y neither $x \leq y$ nor

$y \leq x$ then we call them *incomparable*. If all elements of P are pairwise comparable then we call P a *linear ordering*.

Example A.2.2. One of the easiest examples of posets is the natural numbers $\mathbb{N}$ with the usual order. This is also an example of a linear ordering. For another example consider the nonnegative integers with the relation

$$x \leq y \quad \Longleftrightarrow \quad y = xz \text{ for some nonnegative integer } z.$$

This poset is usually called the *division lattice*. As a last example consider the set $A = \{1, 2, 3\}$. On the set of subsets of A let $B \leq C$ if $B \subseteq C$. In this case $\{1\} < \{1, 2, 3\}$ and $\{2, 3\} < \{1, 2, 3\}$ but $\{1\}$ and $\{2, 3\}$ are incomparable. We call this poset the *Boolean algebra* on $\{1, 2, 3\}$. A common way to visualize posets is the following:



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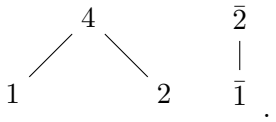
The above is a representation of the Boolean algebra as described above. The way to read it is to start at some element, for example $\{1, 2, 3\}$, and travel downwards along any lines, for example to $\{2\}$, in order to conclude that the first element is larger than the second, in this case $\{2\} < \{1, 2, 3\}$.

Definition A.2.3. Let P be a poset and $Q \subseteq P$ also be a poset. We call Q a *subposet* if $x \leq_Q y$ implies $x \leq_P y$. We say that Q is an *induced* subposet if instead $Q \subseteq P$ and $x \leq_Q y$ if and only if $x \leq_P y$ for all $x, y \in Q$.
Let P and Q be posets such that $P \cap Q = \emptyset$. Then the *disjoint union* of the two $P + Q$ is the poset on $P \cup Q$ with order relations $x \leq_{P+Q} y$ if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$.

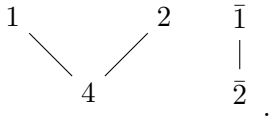
Example A.2.4. Consider the set $\{1, 2, 4\}$ with the relations $1 < 2$ and $1 < 4$. This is an example of a subposet of $\mathbb{N}$ with the usual order. It is however not an induced subposet, since $2 \not< 4$.
Call the above poset A and consider the poset $B = \{1, 2\}$ in which $1 <_B 2$. Since A

and B are not disjoint it is technically not possible to consider their disjoint union. It is however possible to instead consider an isomorphic copy of B (which will be explained later) which we call $\bar{B} = \{\bar{1}, \bar{2}\}$ with $\bar{1} <_{\bar{B}} \bar{2}$ and then consider the poset $A + \bar{B}$. Since such a step is always possible no matter the posets involved it is sometimes convenient to use the notation $A+B$ with the implicit knowledge that we are possibly talking about isomorphic copies. The poset $A + \bar{B}$ is then of the form



Definition A.2.5. Let P and Q be some posets and $f : P \to Q$ be any map. We say that f is *order preserving* or an *order homomorphism* if $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for all $x, y \in P$. If f is bijective and satisfies that $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$ then we call it an *order isomorphism*. By definition every order isomorphism has an order preserving inverse f^{-1}. Similarly we call f *order reversing* if $x \leq_P y$ implies $f(x) \geq_Q f(y)$ for all $x, y \in P$.
Let L be any linearly ordered poset. An injective order preserving map $f : P \to L$ is also called a *linear extension* of P. One also often calls the poset $f(P)$ a linear extension of P. In the case that $L = \mathbb{N}$ we may assume that the codomain of such a map is always of the form $[n]$ or $\mathbb{N}$, since we can always shift the linear ordering to comply with this. For every poset define its *dual poset* P^* to be P with the opposite order relations. In other words the map $f : P \to P^*, x \mapsto x$ is order reversing.

Example A.2.6. The map $f : \mathbb{N} \to \mathbb{N}, x \mapsto x + 1$ is order preserving, but not an order isomorphism since it is not bijective. One linear extension of the above poset $A + \bar{B}$ is defined be the relations $1 < \bar{1} < 2 < 4 < \bar{2}$. The dual $(A + \bar{B})^*$ of $A + \bar{B}$ is of the form



Definition A.2.7. An element y is said to *cover* x, if $x < y$ and there is no z such that $x < z < y$. An element x is *maximal* if there exists no z with $x < z$. Similarly, an element y is *minimal* if there exists no z such that $z < y$. A *chain* $c \subseteq P$ is a collection of elements which is linearly ordered as an induced subposet of P. Depending on the context we may interpret the chain c as a poset or set. A chain is *saturated* if the elements cover each other. An *antichain* $a \subseteq P$ is a collection of elements which are pairwise incomparable.

Example A.2.8. Let P be the division lattice as above. Then 12 covers 6 since $12 = 6 \cdot 2$, so $6 < 12$, and there is no element k such that $k = 6x$ and $12 = ky$. Clearly 1 is a minimal element, since it divides every nonnegative integer. Interestingly enough, this poset also has the maximal element 0 since $0 = x \cdot 0$ for all x. Note that 0 does not cover any element despite being maximal. As an example of a finite chain consider $\{0, 1, 2, 4, 8\}$, with the linear ordering $1 < 2 < 4 < 8 < 0$. It is not saturated, since $8 < 16 < 0$. Every

chain can however be extended to at least one saturated chain by "filling in the missing elements". In this case one possible extension to a saturated chain is by including all powers of 2, that is $\{0, 1, 2, 4, 8, 16, \dots\}$. An antichain in this case consists of pairwise coprime elements, for example $\{2, 27, 77\}$.

Definition A.2.9. An induced subposet $I \subseteq P$ is an *order ideal* if for all $x \in P$ and $y \in I$ it holds that $x \leq_P y$ implies $x \in I$. If I is an order ideal and there exists a set $A \subseteq P$ such that $I = \{x \in P \mid x \leq_P a \text{ for some } a \in A\}$, then we say that A *generates* I. In the case that I is finite it is always generated by its set of maximal elements, which necessarily form an antichain. If I is generated by a single element we call it a *principal order ideal*. If we instead had the property that for all $x \in P$ and $y \in I$ we have $x \geq y$ implies $x \in I$, then we call I a *dual order ideal*. This is due to the fact that it is an ideal in the dual poset P^*.

Example A.2.10. It is sometimes convenient to think of the visualization of a poset when considering ideals. This is due to the fact that if one element is in the ideal, then all elements below it connected by a line are also in the ideal. In the case that we have a generating set A we can simply consider the elements of A and imagine that everything below is part of the ideal.
Consider the Boolean algebra on $\{1, 2, 3\}$. Then the ideal generated by $\{1, 2\}$ and $\{2, 3\}$ is $\{\{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{\}\}$.

Remark A.2.11. Posets and simple directed graphs can be linked in a certain fashion: Take any directed graph $D = (V, E)$ and consider the "poset" P defined on the set V by $w < v$ if and only if $(v, w) \in E$. This P as defined above is in general not a partially ordered set due to a lack of transitivity. However, if we take these relations to be cover relations and extend everything transitively then P satisfies the axioms of a partially ordered set. Due to this we call P the *transitive closure* of D.
On the other hand, if we have a poset P we can define a directed graph D by using the vertex set P and edge set $\{(v, w) \in P^2 \mid w < v\}$. This will necessarily result in a simple directed graph. Depending on the context it may also be useful to define a directed graph using only the cover relations of P instead. This is not always possible, since cover relations do not fully characterize a poset. If P is finite however this is the case. One particular benefit is that the two constructions mentioned here are then inverse to each other.

Definition A.2.12. Let P be any poset and let $x, y \in P$. If there exists a $z \in P$ such that $x \leq z$ and $y \leq z$ then we call z an *upper bound* of x and y. We call it the *least* upper bound if it is smaller than every upper bound of x and y. Similarly if $z \leq x$ and $z \leq y$ we call z a *lower bound* and if it is larger than every lower bound we call it the *greatest* lower bound of x and y.
Suppose that a least upper bound exists for x and y. Then we denote it by $x \vee y$ and also call it the *join* of x and y. If the greatest lower bound exists we denote it by $x \wedge y$ and call it the *meet* of x and y.
If every pair of elements in P has both a meet and a join, then we call P a *lattice*.

Remark A.2.13. There are several additional properties which follow for lattices but not general posets. One easy example is the fact that a finite lattice must have unique minimal and maximal elements denoted $\hat{0}$ and $\hat{1}$ respectively. This can be proven by induction. We will not delve deeper into other properties due to the restraints of this work.

Example A.2.14. The easiest example of a poset in which some elements do not have a meet or join is in the case of a disjoint union. Consider the poset $A + \bar{B}$ from above. It is clear that 1 and $\bar{1}$ do not have a meet or join, meaning that $A + \bar{B}$ is not a lattice. Even in the case that a pair of elements does have an upper bound, it does not have to have a join. Take the set $[0, 1) \cup x$, where we take the usual order on $\mathbb{R}$ and in addition $x < y$ for all $y \in (0, 1)$. Then the two minimal elements 0 and x do have an infinite amount of upper bounds, however no join exists. For another example consider the set $\{a, b, c, d\}$ with the relations $a > c$, $a > d$, $b > c$ and $b > d$. Clearly both c and d are lower bounds of a and b. However, since c and d are incomparable no meet exists.
For examples of lattices consider the division lattice. In it the join of two elements is given by their smallest common multiple, while their meet is given by their largest common denominator. Another example is the Boolean algebra on $\{1, 2, 3\}$ (or any other finite or infinite set). In this case the join is given by the set union and the meet is given by the set intersection.

Definition A.2.15. Let P be a poset. We call P *graded of rank* k if every chain of maximal length in P has k elements. Let $c = \{c_0, \dots, c_{k-1}\}$ be any chain with $c_0 < \cdots < c_{k-1}$. Then we define the *rank* of c_i as $\mathrm{rk}(c_i) = i$. Due to the restrictions of a graded poset this can be extended to a well defined function $\mathrm{rk} : P \to \{0, \dots, k-1\}$.
Let L be a finite lattice. If L is graded and for all $x, y \in L$ we have

$$\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y)$$

then we call L *semimodular*. If additionally every $x \in L$ can be represented as

$$x = \bigvee_{a \in A} a,$$

where $A \subseteq \{y \in L \mid \mathrm{rk}(y) = 1\}$, then we call L a *geometric* lattice. Note that $\bigvee_{a \in A} a$ is well defined since it is a finite join and $\vee$ can be shown to be associative. Additionally by convention $\hat{0} = \bigvee_{a \in \emptyset} a$.

Example A.2.16. Consider again the Boolean algebra L on $\{1, 2, 3\}$ (or any finite set S). It admits a rank function, which assigns each set its number of elements. Due to this we can calculate

$$\mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y) = |x \cap y| + |x \cup y| = |x \cap y| + |x| + |y| - |x \cap y| = |x| + |y|,$$

meaning this is also a semimodular lattice. Additionally $\{y \in L \mid \mathrm{rk}(y) = 1\} = \{\{s\} \mid s \in S\}$ and clearly every element of L is the finite union of such elements, making it a geometric lattice.

Definition A.2.17. Let P be a poset. Denote by $[x, y]$ the set of all elements $z \in P$ such that $x \leq z \leq y$, also called the *interval* from x to y. P is called *locally finite* if all intervals $[x, y] \subseteq P$ are finite. In particular, every finite poset is already locally finite.
If we have a locally finite poset P we define its *incidence algebra* $I(P)$ as the set of functions $f : P \times P \to \mathbb{C}$ which satisfy $f(x, y) = 0$ if $x \not\leq y$. As the name implies, the incidence algebra is a $\mathbb{C}$-algebra with addition and scalar multiplaction as usual, where vector multiplication comes in the form of involution, that is

$$(f * g)(x, y) = \sum_{z \in P} f(x, z)g(z, y) = \sum_{z \in [x,y]} f(x, z)g(z, y),$$

where the latter equality holds due to the properties of an incidence algebra. Define for P its *zeta function* $\zeta \in I(P)$ by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else.} \end{cases}$$

Based on the theory of incidence algebras one knows that ζ has a multiplicative inverse μ, meaning that

$$\delta_{x,y}(x, y) = (\mu * \zeta)(x, y) = (\zeta * \mu)(x, y) = \sum_{z \in P} \zeta(x, z)\mu(z, y),$$

where $\delta_{x,y}$ is the Kronecker delta. This fact already implies that for fixed $x, y \in P$ we get

$$\begin{aligned} \sum_{x \leq z \leq y} \mu(z, y) &= \sum_{x \leq z \leq y} \zeta(x, z)\mu(z, y) = 0 \quad \text{and} \\ \sum_{x \leq z \leq y} \mu(x, z) &= \sum_{x \leq z \leq y} \mu(x, z)\zeta(z, y) = 0 \quad \text{if } x < y \text{ as well as} \\ \mu(x, x) &= 1. \end{aligned}$$

The function μ is called the *Möbius function* of P and has certain useful properties due to being the inverse of ζ.

Theorem A.2.18 (Möbius Inversion)**.** *Let P be a locally finite poset such that every principal order ideal is finite. Denote by ζ and μ the zeta function and Möbius function of P respectively. Let f and g be some functions $P \to \mathbb{C}$. Then the following two statements are equivalent:*

$$\begin{aligned} g(x) &= \sum_{y \leq x} f(y) \quad \textit{for all } x \in P \textit{ and} \\ f(x) &= \sum_{y \leq x} g(y)\mu(y, x) \quad \textit{for all } x \in P. \end{aligned}$$

Proof. The proof consists of elementary manipulations. Suppose the first equality is true and let $x \in P$ be fixed, then

$$\begin{aligned}
\sum_{y \leq x} g(y)\mu(y,x) &= \sum_{y \leq x} \mu(y,x) \sum_{z \leq y} f(z) \cdot 1 \\
&= \sum_{y \leq x} \mu(y,x) \sum_{z \leq y} f(z)\zeta(z,y) \\
&= \sum_{z \in P} f(z) \sum_{z \leq y \leq x} \zeta(z,y)\mu(y,x) \\
&= \sum_{z \in P} f(z) \sum_{y \in P} \zeta(z,y)\mu(y,x) \\
&= \sum_{z \in P} f(z)\delta_{z,x} \\
&= f(x).
\end{aligned}$$

On the other hand, if the second equality is true then

$$\begin{aligned}
\sum_{y \leq x} f(y) &= \sum_{y \leq x} 1 \sum_{z \leq y} g(z)\mu(z,y) \\
&= \sum_{y \leq x} \zeta(y,x) \sum_{z \leq y} g(z)\mu(z,y) \\
&= \sum_{z \in P} g(z) \sum_{z \leq y \leq x} \mu(z,y)\zeta(y,x) \\
&= \sum_{z \in P} g(z) \sum_{y \in P} \mu(z,y)\zeta(y,x) \\
&= \sum_{z \in P} g(z)\delta_{z,x} \\
&= g(x).
\end{aligned}$$

Note that the sums are finite or absolutely convergent at every step. This is the case since most values appearing in them are in fact 0 and we assumed principal order ideals to be finite. □

Remark A.2.19. If for a partially ordered set P we apply Möbius inversion to its dual poset and reinterpret the statements back to P we get the following (with notation as above):
Let P be a locally finite poset such that every dual principal order ideal is finite. Then the following two statements are equivalent:

$$\begin{aligned}
g(x) &= \sum_{y \geq x} f(y) \quad \text{for all } x \in P \text{ and} \\
f(x) &= \sum_{y \geq x} \mu(x,y)g(y) \quad \text{for all } x \in P.
\end{aligned}$$

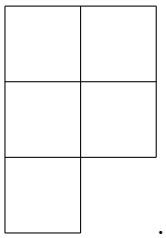
We will refer to this formulation also simply by Möbius inversion. In the case that we have a finite poset P then it is trivially locally finite and every (dual) principal order ideal is trivially finite. As a result both formulations of the Möbius inversion hold in that case.

A.3. Partitions

Definition A.3.1. Let $n \in \mathbb{N}$. A *partition* $\lambda = (\lambda_1, \dots, \lambda_l)$ of n is a weakly decreasing sequence of natural numbers $\lambda_1 \geq \cdots \geq \lambda_l > 0$ such that $n = \lambda_1 + \cdots + \lambda_l$. We also write $\lambda \vdash n$ and call $l = l(\lambda)$ the *length* of λ. We may also use the notation $\lambda = \langle 1^{r_1} 2^{r_2} \dots w^{r_w} \rangle$ to indicate that λ contains the number i exactly r_i times. A *composition* $a = (a_1, \dots a_l)$ is an arbitrary sequence of natural numbers such that $a_1 + \cdots + a_l = n$. We will similarly call $l = l(a)$ the length of the composition. A *weak composition* is a composition for which the entry 0 is allowed for all but the last entry. Alternatively, a weak composition of n is an infinite sequence $a = (a_1, a_2, \dots)$ such that $0 \leq a_i \in \mathbb{N}$ and $\sum_{i \in \mathbb{N}} a_i = n$.

Example A.3.2. Take $4 \in \mathbb{N}$. Then $(2, 1, 1)$, $(1, 2, 1)$ and $(0, 3, 0, 1)$ are examples of a partition, composition and weak composition of 4 respectively. Note that every partition is a composition and every composition is a weak composition. When using the alternative definition of a weak composition we instead have $(0, 3, 0, 1, 0, \dots)$.

Remark A.3.3. One can imagine a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ as squares arranged in a certain way. For example, the partition $(2, 2, 1)$ can be visualized as



.

This is also called the *Ferrers diagram* associated to λ. Transposing the Ferrers diagram results in another Ferrers diagram. This leads one to define the *conjugate* of λ as the partition which results when transposing the Ferrers diagram, also denoted λ'. In the case $\lambda = (2, 2, 1)$ this means $\lambda' = (3, 2)$.

Definition A.3.4. One can introduce several order relations for the set of partitions. We will only discuss the dominance order, as other relations are not relevant to this work. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\rho = (\rho_1, \dots, \rho_r)$ be partitions of the same number n. Use the convention $\lambda_i = 0$ and $\rho_j = 0$ if $i > l$ and $j > r$ respectively. We say that $\lambda \leq \rho$ in the *dominance order* if

$$\lambda_1 + \cdots + \lambda_i \leq \rho_1 + \cdots + \rho_i$$

for all $1 \leq i \leq \max(l, r)$.

Example A.3.5. In general broad partitions (meaning those which begin with big numbers) are larger in the dominance order than long partitions (those which have lots of small entries). Consider for example $\lambda = (3, 2, 2)$ and $\rho = (5, 2)$, then it is easy to see that $\lambda < \rho$. It should be noted that the dominance order is not linear in most cases, meaning there are incomparable elements. The smallest number for which this happens is $n = 6$, where $(4, 1, 1)$ is incomparable to $(3, 3)$ and $(3, 1, 1, 1)$ is incomparable to $(2, 2, 2)$.

Definition A.3.6. A *semistandard Young tableau* T is an array of natural numbers with weakly decreasing row length and for which entries decrease weakly along rows and strongly along columns. The *shape* of T is the partition λ consisting of the row lengths of T. Given two partitions λ, μ such that $\mu_i \leq \lambda_i$ for all i we call an array T a semistandard Young tableau of *skew shape* λ/μ, if T follows all the rules of a semistandard Young tableaux of shape λ, but an array of shape μ has been removed from it.

Example A.3.7. Consider the following semistandard Young tableaux of shape $(3, 3, 1)$ and $(3, 3, 1)/(2, 1)$ respectively:

$$\begin{array}{lll} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & & \end{array} \quad \text{and} \quad \begin{array}{lll} - & - & 4 \\ - & 1 & 5 \\ 2 & & \end{array} = \begin{array}{ll} & 4 \\ 1 & 5 \\ 2 & \end{array} .$$

B. Symmetric Functions

B.1. Basic Concepts

Here we will look at basic definitions relating to symmetric functions. This part serves mainly to introduce readers unfamiliar with the concepts used in the rest of the work and to document notation. It is however assumed that the reader is familiar with formal power series and basic operations relating to them. As symmetric functions play an essential role in algebraic combinatorics, it is strongly advised that the uninitiated seek out additional information on the topic, for example in [11].

Definition B.1.1. Let R be a commutative ring. Denote the R-module of formal power series with respect to the indeterminate x and coefficients in R by $R[[x]]$. More generally, if we have several commuting indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ we denote the R-module of formal power series with coefficients in R by $R[[\mathbf{x}]]$. For a weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$ of $n \in \mathbb{N}$ (that is $0 \leq \alpha_i \in \mathbb{N}$ and $\sum_{i=1}^{l} \alpha_i = n$) we denote $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_l^{\alpha_l}$. In this case define the *degree* of the monomial as $\deg(\mathbf{x}^\alpha) = n$. Then a *symmetric function* $f(\mathbf{x}) \in R[[\mathbf{x}]]$ of degree n is a formal power series of the form

$$f(\mathbf{x}) = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha,$$

where α ranges over all weak compositions of n and for all permutations $\pi \in S_{\mathbb{N}}$ we have $f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2),\dots})$. The set of all symmetric functions of degree n is denoted by Λ_R^n and forms an R-module. The *space of symmetric functions* is then $\Lambda_R = \bigcup_{i \in \mathbb{N}} \Lambda_R^i$ and forms an R-algebra. As is also the case with generic formal power series and polynomials, we denote the coefficient of $\mathbf{x}^\alpha$ in f by $c_\alpha = [\![\mathbf{x}^\alpha]\!] f$. Most commonly in enumeration applications one uses $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, which are all fields and thus the respective sets Λ_R^n are already vector spaces.

From now on we will be working in $\Lambda = \Lambda_{\mathbb{Q}}$, although most of the mentioned facts and definitions translate to $\Lambda_{\mathbb{R}}$ and $\Lambda_{\mathbb{C}}$ with little effort. Since it is an algebra, it would be interesting to know if there are any bases which are of importance. There are in fact several and we give a brief overview of those relevant in this work.

Example B.1.2. A common way of viewing symmetric functions are in terms of partitions. Take for example $\lambda = (2, 1) \vdash 3$ and consider which symmetric function f of degree 3 contains all monomials of the form λ, by which we mean monomials of the form $x_i^2 x_j$ where $i \neq j$. Clearly

$$f(\mathbf{x}) = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + \dots = \sum_{i<j} x_i^2 x_j + \sum_{i>j} x_i^2 x_j.$$

We consider the smallest monomial in the canonical order, that being $x_1^2 x_2$. Notice now that if we apply a permutation $\pi \in S_{\mathbb{N}}$ to $x_1^2 x_2$ the resulting monomial will still retain its form. Furthermore, all monomials of that form can be gained by starting with $x_1^2 x_2$ and applying some permutation. Hence $x_1^2 x_2$ and by extension λ generate f in the sense that if we know λ, then all of f is already fixed.

Definition B.1.3. The *monomial symmetric functions* are given by

$$m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l},$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition and the sum ranges over all pairwise distinct indices $i_1, \ldots, i_l \in \mathbb{N}$ such that the resulting monomial is unique. This restriction ensures that the coefficient of any given monomial in m_λ is in fact 1. For some uses it is convenient to add an additional weight, leading to the *augmented monomial symmetric functions* given by

$$\tilde{m}_\lambda = r_1! \cdots r_w! m_\lambda$$

where $\lambda = \langle 1^{r_1} 2^{r_2} \ldots w^{r_w} \rangle$ is again a partition, which contains the number k exactly r_k times. The set of all m_λ (and by extension $\tilde{m}_\lambda$) form a basis for the space of symmetric functions.

Example B.1.4. The easiest example arises from the partition $\lambda = (1) = \langle 1^1 \rangle$, such that

$$m_\lambda = x_1 + x_2 + \cdots = \tilde{m}_\lambda.$$

For a more instructive partition consider $\rho = (2,2) = \langle 2^2 \rangle$, meaning

$$m_\rho = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + \cdots = \sum_{i<j \in \mathbb{N}} x_i^2 x_j^2 \quad \text{and} \quad \tilde{m}_\rho = 2 m_\rho.$$

Thinking back on example B.1.2 it can be easily seen that

$$f = m_{(2,1)},$$

meaning the way of thinking we applied there corresponds to the expansion of a function in the basis of m_λ.

Definition B.1.5. The *power sum symmetric functions* are given by

$$p_n = m_{(n)} = \sum_{i \in \mathbb{N}} x_i^n$$

for all $n \in \mathbb{N}$ and are extended for partition indices by

$$p_{(\lambda_1, \ldots, \lambda_l)} = p_{\lambda_1} \cdots p_{\lambda_l}.$$

Once again, the set of all power sum symmetric functions gives a basis for the space of symmetric functions.

Example B.1.6. As an easy example consider $\lambda = (2,1)$ so that

$$p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2 + \dots)(x_1 + x_2 + \dots).$$

Notice that $p_{(2,1)}$ differs from $m_{(2,1)}$ by the existence of terms of the form x_i^3. This means that if we wish to expand $m_{(2,1)}$ in the basis of power sum symmetric functions we have to correct for that fact, so that

$$m_{(2,1)} = p_{(2,1)} - p_{(3)}.$$

In more complicated cases a similar logic can be applied, leading to several steps of inclusion exclusion in general.

Definition B.1.7. The *elementary symmetric functions* are defined by

$$e_n = m_{(1^n)} = \sum_{i_1 < \dots < i_n \in \mathbb{N}} x_{i_1} \dots x_{i_n}$$

and can be extended to partition indices by

$$e_{(\lambda_1,\dots,\lambda_l)} = e_{\lambda_1} \cdots e_{\lambda_l}.$$

The set of e_λ can again be shown to be a basis of the space of symmetric functions.

Example B.1.8. Consider the partition $\lambda = (2,2)$, so that

$$e_{(2,2)} = e_2 e_2 = (x_1x_2 + x_1x_3 + \dots + x_2x_3 + \dots)(x_1x_2 + \dots).$$

Notice that the coefficient of $x_1^2 x_2 x_3$ is 2, whereas it does not appear in $m_{(2,2)}$. Trying to expand the function $m_{(2,1)}$ in the basis of elementary symmetric functions gives us

$$m_{(2,1)} = e_{(2,1)} - 3e_{(3)},$$

since $e_{(2,1)} = (x_1x_2 + \dots)(x_1 + \dots)$ contains all monomials of the form $x_i x_j x_k$ exactly $3 = \binom{3}{1}$ times and all of the form $x_i^2 x_j$ exactly once for pairwise disjoint i, j, k.

The last basis we will discuss is that of the Schur symmetric functions. Although we will not see this here, there are several different equivalent ways to define them due to the fact that they arise in several other disciplines of mathematics. We merely picked a convenient one for the context of this work and will not motivate or discuss them further.

Definition B.1.9. Let λ, μ be partitions such that $\mu_i \leq \lambda_i$ for all i. The *skew Schur function* $s_{\lambda/\mu}$ is defined as

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum ranges over all semistandard Young tableau T of skew shape λ/μ and $\mathbf{x}^T = \mathbf{x}^\nu = x_1^{\nu_1} \cdots x_l^{\nu_l}$ with $\nu = (\#1 \text{ in } T, \dots, \#l \text{ in } T)$ being the weight of T. In the case that $\mu = \emptyset$ we call $s_\lambda = s_{\lambda/\mu}$ a *Schur function*. Schur functions are in fact symmetric and the set of (non skew) Schur functions is a basis for $\Lambda_{\mathbb{Q}}$.

Example B.1.10. Consider the partition $(2,1)$. Since the monomials occuring in $s_{(2,1)}$ are of degree 3 we will restrict ourselves to x_1, x_2 and x_3 for now. This means that our tableaux may only have entries in $\{1,2,3\}$ and after listing them one comes to the conclusion that

$$\begin{aligned} s_{(2,1)}(x_1,x_2,x_3) &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 \\ &= m_{(2,1)}(x_1,x_2,x_3) + 2m_{(1,1,1)}(x_1,x_2,x_3) \end{aligned}$$

where we used the notation $f(x_1,x_2,x_3)$ for $f(\mathbf{x})$ in the case $\mathbf{x} = (x_1,x_2,x_3,0\dots)$. This analysis holds for any selection of three distinct variables, so that

$$\begin{aligned} s_{(2,1)}(\mathbf{x}) &= s_{(2,1)}(x_1,x_2,x_3) + s_{(2,1)}(x_1,x_2,x_4) + \dots \\ &= m_{(2,1)}(\mathbf{x}) + 2m_{(1,1,1)}(\mathbf{x}). \end{aligned}$$

Definition B.1.11. Let B be a basis for Λ. We call $f \in \Lambda$ *B-positive*, if the expansion of $f = \sum_{b\in B} c_b b$ in the basis B has only nonnegative coefficients $c_b \geq 0$. For the previously discussed bases we will, for example, simply call f *e-positive* if it has nonnegative coefficients in the base of elementary symmetric functions.

Definition B.1.12. Define an inner product on $\Lambda_{\mathbb{Q}}$ by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$$

and bilinear extension, where δ is the Kronecker delta. It can be shown that the Schur functions s_λ form an orthonormal basis for $\Lambda_{\mathbb{Q}}$ with this inner product, which is to say that

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

B.2. Quasisymmetric Functions

One concept which generalizes symmetric functions is that of quasisymmetric functions. Once again, a more detailed introduction into the theory can be found in [10].

Definition B.2.1. Let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting indeterminates and consider $R[[\mathbf{x}]]$ the space of formal power series over the commutative ring R in the variables $\mathbf{x}$. We call $f(\mathbf{x}) \in R[[\mathbf{x}]]$ a *quasisymmetric function* if

$$[\![x_{i_1}^{a_1} \dots x_{i_n}^{a_n}]\!] f(\mathbf{x}) = [\![x_{j_1}^{a_1} \dots x_{j_n}^{a_n}]\!] f(\mathbf{x})$$

whenever $i_1 < \dots < i_n \in \mathbb{N}$ and $j_1 < \dots < j_n \in \mathbb{N}$. In the case that a quasisymmetric function f only contains monomials of degree d we also say that the degree of f is d. It is easily seen that the set of quasisymmetric functions of a given degree d define an R-module and we denote it by $\mathcal{Q}_{R,d}$. The algebra of all quasisymmetric functions of arbitrary degree is then denoted by $\mathcal{Q}_R = \bigcup_{d\in\mathbb{N}} \mathcal{Q}_{R,d}$. If no confusion arises we will omit the underlying ring and simply write $\mathcal{Q}_d$ and $\mathcal{Q}$ respectively.

Remark B.2.2. To gain a basic intuition about quasisymmetric functions one might want to reformulate the definition of a symmetric function in a similar way: A symmetric function $f(\mathbf{x})$ is a formal power series such that

$$[\![x_{i_1}^{a_1} \dots x_{i_n}^{a_n}]\!] f(\mathbf{x}) = [\![x_{j_1}^{a_1} \dots x_{j_n}^{a_n}]\!] f(\mathbf{x})$$

whenever $i_1, \dots, i_n$ and $j_1, \dots, j_n$ are respectively pairwise distinct.
This view also makes immediately clear that every symmetric function is quasisymmetric, since symmetric functions have a strictly more restrictive condition imposed on their monomials. As we will see shortly, the reverse is not the case.

When we worked with symmetric functions we saw that expanding a given function f in one of our bases followed a specific pattern. We will now see that the same is true for quasisymmetric functions, but first we need to find a suitable basis.

Definition B.2.3. Consider a set $S \subseteq [d-1]$ for a fixed $d \in \mathbb{N}$. Then the *fundamental quasisymmetric functions* of degree d are defined as

$$Q_{S,d}(\mathbf{x}) = \sum_{\substack{i_1 \leq \dots \leq i_d \in \mathbb{N} \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} \dots x_{i_d}.$$

We will write Q_S instead of $Q_{S,d}$ if the context is clear. The set of fundamental quasisymmetric functions of a given degree d are in fact linearly independent. Furthermore, they span $\mathcal{Q}_{R,d}$ making them a basis. This also means that the set of all fundamental quasisymmetric functions is a basis for $\mathcal{Q}_R$.

Example B.2.4. Suppose we wanted to expand the quasisymmetric function

$$f = \sum_{i_1 < i_2} x_{i_1} x_{i_2}^2 + \sum_{j_1 \leq j_2} x_{j_1} x_{j_2}$$

in the basis of fundamental quasisymmetric functions. The first type of monomial is inconvenient for our construction, so we instead consider all monomials of the form $x_{i_1} x_{i_2} x_{i_3}$ for $i_1 < i_2 \leq i_3$ and at the end substract all those for which $i_2 < i_3$. From these restrictions one can easily read off $S_1 = \{1\} \subseteq [3]$, $S_2 = \{1, 2\} \subseteq [3]$ and $S_3 = \{1\} \subseteq [2]$. This means that our desired quasisymmetric function contains $Q_{S_1,3}$ and $Q_{S_3,2}$ but not $Q_{S_2,3}$ so that

$$f(\mathbf{x}) = Q_{S_1,3} - Q_{S_2,3} + Q_{S_3,2}.$$

Notice that monomials of the form $x_{i_1}^2 x_{i_2}$ with $i_1 < i_2$ do not appear in f, meaning that this function is not symmetric. Despite that, some quasisymmetric functions are already familiar, for example $Q_{\{1\},2} = e_2$ is symmetric.

Definition B.2.5. In the case of $\mathcal{Q}_{\mathbb{C}}$, define an involution $\omega : \mathcal{Q}_{\mathbb{C},d} \to \mathcal{Q}_{\mathbb{C},d}$ by $\omega(Q_{S,d}) = Q_{[d-1]\setminus S,d}$ and linear extension, essentially switching the weak and strict inequalities in the definition of the fundamental quasisymmetric function. Clearly this map can be extended to $\mathcal{Q}_{\mathbb{C}}$ and then restricted to $\mathcal{Q}_{\mathbb{R}}$, $\mathcal{Q}_{\mathbb{Q}}$ and $\mathcal{Q}_{\mathbb{Z}}$ and we will use the notation ω in all cases if no confusion arises. It is easy to see that $\omega(h_n) = e_n$ in the symmetric case, making this a generalization of the well studied involution on symmetric functions also denoted ω.

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