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# Nonintersecting Paths, Pfaffians, and Plane Partitions

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## INTRODUCTION

Gessel and Viennot have developed a powerful technique for enumerating various classes of plane partitions [GV1, 2]. There are two fundamental ideas behind this technique. The first is the observation that most classes of plane partitions that are of interest—either by association with the representation theory of the classical groups, or for purely combinatorial reasons—can be interpreted as configurations of nonintersecting paths in a digraph (usually the lattice $\mathbf{Z}^2$). The second is the observation that the number of $r$-tuples of nonintersecting paths between two sets of $r$ vertices can (often) be expressed as a determinant.

The purpose of this article is to show by similar means that one may use pfaffians to enumerate configurations of nonintersecting paths in which the initial and/or terminal vertices of the paths are allowed to vary over specified regions of the digraph. This leads to the possibility of enumerating classes of plane partitions in which the shape is allowed to vary, whereas the previous applications of Gessel and Viennot were largely confined to plane partitions of a given shape.

We have made no attempt to catalogue all possible classes of plane partitions that one could enumerate by these techniques; rather, we have confined ourselves to providing new, simple, unified proofs of a diverse collection of known results, including identities of Gansner, Józefiak and Pragacz, Gordon, Gordon and Houten, Goulden, Lascoux and Pragacz, and Okada. In one instance, we give a new result; namely, a pfaffian for the number of totally symmetric, self-complementary plane partitions. It seems likely that the number of plane partitions belonging to the other symmetry classes for which there are only conjectured formulas (see [St3]) could also be expressed as pfaffians. We will not pursue this further here, except to note that Okada has already done this for the totally symmetric case [O].

A more detailed summary follows.

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The first half of the paper (Sects. 1–4) is concerned with the general problem of enumerating $r$-tuples of nonintersecting paths in an acyclic digraph $D$. In Section 1 we review the basic methods of Gessel and Viennot for enumerating sets of nonintersecting paths in which the initial and terminal vertices are specified. We should point out (as do Gessel and Viennot) that the fundamental idea behind this technique occurs earlier in the work of Lindström [L], although Lindström did not use it for enumerative purposes. In Section 2, we give a combinatorial definition of the pfaffian, and assemble a collection of some of its useful properties. The main results of the paper are in Sections 3 and 4. The first result (Theorem 3.1) shows that the number (or generating function) of nonintersecting $r$-tuples of paths from a set of $r$ vertices to a specified region of $D$ can, under favorable circumstances, be expressed as a pfaffian. We also give a related result, suggested by the work of Józefiak and Pragacz [JP], in which one specifies the terminal points of some of the $r$ paths. The other main result (Theorem 4.1) concerns the number (or generating function) of $r$-tuples of nonintersecting paths between two specified regions of $D$. Again assuming favorable circumstances, we show that this number can be expressed as the coefficient of $t^r$ in the pfaffian of a matrix whose entries are quadratic in $t$. This result is closely related to Okada's pfaffian for the sum of all minors of an arbitrary matrix [O], although the proof we give is independent of this.

The second half of the paper (Sects. 5–9) is concerned with applications. In our first application (Theorem 5.1), we obtain Gansner's generating function for shifted tableaux (also known as shifted reverse plane partitions) of a given shape, weighted by the sum of their entries [Ga]. A corollary of this result in the hook length formula for shifted tableaux. In Section 6, we consider Schur's $Q$-functions. These symmetric polynomials, originally defined via pfaffians, were used by Schur in his analysis of projective representations of symmetric groups [S]. More recently, it has became known that these $Q$-functions possess a natural combinatorial description, even though this description is not transparent from Schur's definition. We show (Theorem 6.1) that in fact, the methods of Section 3 do provide a simple way to reconcile Schur's definition with the combinatorial one. We also use the same methods to express the skew $Q$-functions as pfaffians, thus proving a recent result of Józefiak and Pragacz [JP]. In Section 7, we give a simple proof of a result of Gordon [Go1] which (although not originally expressed in this form) amounts to a determinant for the sum of all Schur $S$-functions with at most $k$ rows. These determinants have been used by Gessel [G], Goulden [Gl], and others to derive formulas for the number of standard tableaux with a bounded number of rows. We also show that Gordon's determinants are closely related to branching rules for the groups $SO_{2n+1}$ and $Sp_{2n}$, and this observation leads to a reconciliation between the two seemingly dissimilar proofs of the Bender–Knuth conjecture by Gordon [Go2] and Macdonald [M, p. 52]. In Section 8, we show that the number of totally symmetric, self-complementary plane partitions in a $2n \times 2n \times 2n$ prism can be expressed as a pfaffian. If this pfaffian could be evaluated explicitly, it would resolve the conjecture of Mills, *et al.* [MRR]. Finally in Section 9, we give a proof of the Giambelli determinant that expresses an arbitrary Schur function as a determinant of Schur functions indexed by hook shapes. This answers a question raised by Eğecioğlu and Remmel [ER], who gave a different proof of this determinant but noted that there did not seem to exist any known proof based on the methodology of Gessel and Viennot. The proof we give also applies to the skew version of Giambelli's determinant due to Lascoux and Pragacz [LP].

## 1. The Gessel–Viennot Method

Let $D = (V, E)$ be an acyclic directed graph. Since there will be no genuine loss of generality caused by forbidding multiple edges in what follows, we will identify the edge set $E$ as a subset of $V \times V$. The notation $u \to v$ will be used to indicate that there is an edge directed from $u$ to $v$.

For any pair of vertices $u, v \in V$, let $\mathscr{P}(u, v)$ denote the set of (directed) $D$-paths from $u$ to $v$. If $u = v$, then $\mathscr{P}(u, v)$ consists of a single path of length zero. Given any pair of $r$-tuples $\mathbf{u} = (u_1, ..., u_r)$ and $\mathbf{v} = (v_1, ..., v_r)$ of vertices, let $\mathscr{P}(\mathbf{u}, \mathbf{v})$ denote the set of $r$-tuples of paths $(P_1, ..., P_r)$ with $P_i \in \mathscr{P}(u_i, v_i)$. Two directed paths $P$ and $Q$ will be said to intersect if they share a common vertex. We will write $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$ for the subset of $\mathscr{P}(\mathbf{u}, \mathbf{v})$ consisting of nonintersecting $r$-tuples of paths.

Definition 1.1. If $I$ and $J$ are ordered sets of vertices of $D$, then $I$ is said to be $D$-compatible with $J$ if, whenever $u < u'$ in $I$ and $v > v'$ in $J$, every path $P \in \mathscr{P}(u, v)$ intersects every path $Q \in \mathscr{P}(u', v')$.

The essential point of this condition is that if $\mathbf{u}$ is $D$-compatible with $\mathbf{v}$, then the only $r$-tuples of nonintersecting paths from $\mathbf{u}$ to some permutation of $\mathbf{v}$ must connect $u_i$ to $v_i$ for $i = 1, ..., r$.

To facilitate the enumeration of various types of $D$-paths, it will be convenient to choose a weight-function $w: E \to R$ that assigns values in some commutative ring $R$ to each edge of $D$. In all applications likely to be interesting, $R$ will be a polynomial or power series ring over $\mathbf{Z}$, and the weights will be monomials. It would thus suffice to choose an indeterminate $x_e$ for each $e \in E$, and work over the formal power series ring $R = \mathbf{Z}[[x_e : e \in E]]$; all other weight-functions of interest could be obtained by specializing this one.

Once a particular weight-function $w$ has been chosen, extend $w$ multiplicatively to multisets of edges, so that $w(M) = \prod_{e \in M} w(e)$ for any such multiset $M$. (The empty set has unit weight.) The weight of a $D$-path or $r$-tuple of $D$-paths is defined as the weight of the underlying multiset of edges. Given any family $\mathscr{F}$ of edge multisets, we will write $GF[\mathscr{F}]$ for the generating function according to the weight $w$; i.e., $GF[\mathscr{F}] = \sum_{M \in \mathscr{F}} w(M)$. In particular, let us define

$$h(u, v) = GF[\mathscr{P}(u, v)] = \sum_{P \in \mathscr{P}(u,v)} w(P).$$

In order to guarantee that $h(u, v)$ is well-defined, we will always assume that the number of paths in $\mathscr{P}(u, v)$ of a given weight is finite.

A result nearly identical to the following theorem was proved by Lindström [L], although it was originally stated (incorrectly) without the assumption that $D$ is acyclic. The enumerative ramifications of Lindström's theorem have been developed to a great extent by Gessel and Viennot [GV1, 2]. It is interesting to note that Lindström used his result not for enumeration, but for studying representability of matroids.

THEOREM 1.2. *Let* $\mathbf{u} = (u_1, ..., u_r)$ *and* $\mathbf{v} = (v_1, ..., v_r)$ *be two r-tuples of vertices in an acyclic digraph D. If* $\mathbf{u}$ *is D-compatible with* $\mathbf{v}$, *then*

$$GF[\mathscr{P}_0(\mathbf{u}, \mathbf{v})] = \det[h(u_i, v_j)]_{1 \leqslant i,j \leqslant r}.$$

*Proof.* We may interpret

$$\det[h(u_i, v_j)] = \sum_{\pi \in S_r} \operatorname{sgn}(\pi)\, h(u_1, v_{\pi(1)}) \cdots h(u_r, v_{\pi(r)}) \tag{1.1}$$

as a generating function for $(r+1)$-tuples $(\pi, P_1, ..., P_r)$ with $\pi \in S_r$ and $P_i \in \mathscr{P}(u_i, v_{\pi(i)})$ in which the weight assigned to $(\pi, P_1, ..., P_r)$ is $\operatorname{sgn}(\pi)\, w(P_1) \cdots w(P_r)$. We will show that the Lindström–Gessel–Viennot involution acts on the configurations $(\pi, P_1, ..., P_r)$ with at least one pair of intersecting paths in a way that changes the sign of the weight.

To describe this involution, first choose a fixed total order of the vertices, and consider an arbitrary configuration $(\pi, P_1, ..., P_r)$ with at least one pair of intersecting paths. Among all vertices that occur as points of intersection among the paths $P_i$, let $v$ denote the least vertex with respect to the chosen total order. Such a minimum exists since the number of points of intersection must be finite. Among the paths that pass through $v$, assume that $P_i$ and $P_j$ are the two whose indices $i$ and $j$ are smallest. Using the notation $P_i(\to v)$ and $P_i(v \to)$ to denote the subpaths of $P_i$ from $u_i$ to $v$ and $v$ to $v_{\pi(i)}$

(and similarly for $P_j$), we define $(\pi', P'_1, ..., P'_r)$ to be the configuration where $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\to v)\, P_j(v\to), \qquad P'_j = P_j(\to v)\, P_i(v\to),$$

and $\pi' = \pi \circ (i, j)$.

Note that the multisets of edges appearing in $(P_1, ..., P_r)$ and $(P'_1, ..., P'_r)$ are identical. Since $D$ is assumed to be acyclic, it follows that the operation $(\pi, P_1, ..., P_r) \mapsto (\pi', P'_1, ..., P'_r)$ preserves the set of vertices of intersection, and hence is an involution. Since this involution changes the sign of the associated weight, one may cancel all of the terms appearing in (1.1), aside from those with nonintersecting paths. Since $\mathbf{u}$ is assumed to be $D$-compatible with $\mathbf{v}$, the configurations of nonintersecting paths that appear in (1.1) occur only when $\pi = \mathrm{id}$, and thus are counted with a positive weight. ∎

If we did not assume that $D$ is acyclic, then we could no longer assert that the above involution preserves the set of intersection vertices in a given configuration. For example, an intersection between a pair of paths could become part of a cycle in a single path.

Let us further consider whether the acyclic property is actually necessary for Theorem 1.2 to be valid. In a digraph with cycles there are (at least) three possible ways to interpret what a directed path should be: one interpretation would forbid repetition of vertices, another would forbid repetition of edges, and a third would offer no such restrictions. However, the digraph illustrated in Fig. 1 can be used to show that Theorem 1.2 fails for each of these interpretations. The vertex pairs $(u_1, u_2)$ and $(v_1, v_2)$ are clearly compatible in the sense of Definition 1.1, and there is exactly one pair of nonintersecting paths that connect $u_i$ to $v_i$ $(i = 1, 2)$, regardless of the interpretation of "path." If we assign unit weight to each of the edges in this example, the matrix $H = [h(u_i, v_j)]$ is either $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ (distinct vertices) or $\left[\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right]$ (distinct edges). In either case, $\det(H)$ does not yield the correct number of paths. If we impose no restrictions, then there are infinitely many paths, and so one must assign non-unit weights to the edges to

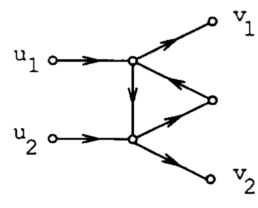


FIGURE 1

obtain well-defined generating functions. Again, one may check that $\det(H)$ is not the desired quantity.

A troublesome aspect of Theorem 1.2 is the fact that it gives no information about which sets of vertices are $D$-compatible. However, if $D$ happens to be planar (as will be the case for all the applications we have in mind), it is often possible to take advantage of the underlying topology. For example, suppose that one may pass a Jordan curve $C$ through two sets of vertices $I$ and $J$ so that all $D$-paths from $I$ to $J$ are contained in the interior of $C$. If the vertices of $I$ and $J$ are arranged along two disjoint segments of $C$, then $I$ must be $D$-compatible with $J$ (according to the order one obtains from the position of the vertices on the segments).

The following yields an algebraic method for identifying $D$-compatible sets of vertices.

DEFINITION 1.3. A partial order $(V, \leqslant)$ is $D$-compatible if it satisfies the following for all vertices $u, v, u', v' \in V$:

(A1) $u \to v$, $u' \to v'$, and $u < u'$ implies $u \leqslant v'$, or $v \leqslant u'$, or $v \leqslant v'$.

(A2) If $u < v' < v$ then $v'$ is incident with every $D$-path from $u$ to $v$ or $v$ to $u$.

PROPOSITION 1.4. *If $(V, \leqslant)$ is a $D$-compatible partial order, then any any two chains of this partial order are $D$-compatible.*

*Proof.* Assume to the contrary that there exists a pair of nonintersecting paths $(P, Q)$ with $P \in \mathscr{P}(u, v)$, $Q \in \mathscr{P}(u', v')$, $u < u'$, and $v > v'$. Among all such counterexamples, let us assume that we have chosen a pair for which the sum of the lengths $l(P) + l(Q)$ is minimal. If both $l(P)$ and $l(Q)$ are positive, then there exist vertices $u_0 \in P$ and $u'_0 \in Q$ such that $u \to u_0$ and $u' \to u'_0$. According to (A1), this forces $u < u'_0$, $u_0 < u'$, or $u_0 < u'_0$. (Strict inequality occurs because $P$ and $Q$ do not intersect.) We may therefore delete $u$ from $P$ and/or $u'$ from $Q$ to obtain a smaller counterexample. Hence, we must have $l(P) = 0$ or $l(Q) = 0$. In the former case, we have $u = v$ and thus $v' < u < u'$. Since $Q$ is a $D$-path from $u'$ to $v'$, (A2) implies that $u$ is incident with $Q$, a contradiction. A similar contradiction occurs when $l(Q) = 0$. ∎

## 2. PFAFFIANS

Let $\mathbf{v} = (v_1, ..., v_n)$ be an ordered list of $n$ objects, and assume that $n$ is even. By a 1-factor of $\mathbf{v}$ we mean a perfect matching; i.e., a set of (undirected) edges on the vertex set $\mathbf{v}$ with the property that each $v_i$ is incident with exactly one edge. We will write $\mathscr{F}(\mathbf{v})$ for the set of 1-factors of $\mathbf{v}$, and $\mathscr{F}_n$ for the 1-factors of $(1, 2, ..., n)$.

By convention, we will always list the edges of a 1-factor $\pi \in \mathscr{F}(\mathbf{v})$ in the form $(v_i, v_j)$ with $i<j$. Two such edges $(v_i, v_j)$ and $(v_k, v_l)$ will be said to be *crossed* if $i<k<j<l$ or $k<i<l<j$. This condition can be interpreted geometrically if we embed the vertices along the $x$-axis in the plane and represent the edges by curves in the upper half plane—crossed edges must intersect in such an embedding. Define the *sign* of $\pi$ to be $(-1)^k$, where $k$ denotes the number of crossed pairs of edges in $\pi$. Note that this quantity depends particularly on the order chosen for the elements of $\mathbf{v}$.

LEMMA 2.1. *Let $\pi \in \mathscr{F}(\mathbf{v})$ and assume that $v_i$ and $v_j$ are nonadjacent in $\pi$ and $i<j$. Let $\pi'$ denote the 1-factor obtained by interchanging $v_i$ and $v_j$. If neither $v_i$ nor $v_j$ is adjacent (in $\pi$) to any $v_k$ with $i<k<j$, then* $\operatorname{sgn}(\pi) = -\operatorname{sgn}(\pi')$.

*Proof.* Let $S$ denote the set of vertices consisting of $v_i$, $v_j$, and the two vertices to which they adjacent in $\pi$. Designate any edge whose endpoints are both in $S$ as *special.* Pairs of nonspecial edges are crossed in $\pi$ if and only if they are crossed in $\pi'$, so these pairs do not affect $\operatorname{sgn}(\pi)\operatorname{sgn}(\pi')$. A nonspecial edge $(v_k, v_l)$ will cross an odd number of special $\pi$-edges (resp., $\pi'$-edges) if and only if an odd number of vertices of $S$ lie between the endpoints $v_k$ and $v_l$. Since this quantity is the same for $\pi$ and $\pi'$, we conclude that $\operatorname{sgn}(\pi)\operatorname{sgn}(\pi') = +1$ or $-1$, according to whether the crossing status of the two special edges is or is not the same in $\pi$ and $\pi'$. In other words, we may restrict our attention to the four vertices of $S$, and regard $\pi$ and $\pi'$ as 1-factors of $S$. In that case, the assumption that no vertex $v_k$ between $v_i$ and $v_j$ belongs to $S$ implies that $v_i$ and $v_j$ are consecutive as members of $S$ (in the ordering inherited from $\mathbf{v}$). It is now easy to check that interchanging $v_i$ and $v_j$ does change the crossing status of the two special edges (cf. Fig. 2). ∎

We remark that interchanging two arbitrary (nonadjacent) vertices need not change the sign of a 1-factor. For example, consider the consequences of exchanging 1 and 3 in the 1-factor $\{(1, 2), (3, 4)\}$.

If $A = [a_{ij}]_{1 \leqslant i<j \leqslant n}$ is an upper triangular array, we define the *pfaffian* of $A$ as follows:

$$\operatorname{pf}(A) = \sum_{\pi \in \mathscr{F}_n} \operatorname{sgn}(\pi) \prod_{(i,j) \in \pi} a_{ij}.$$

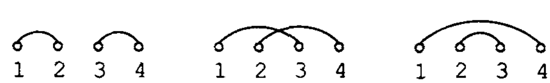


FIGURE 2

It is more conventional to regard the pfaffian as a function defined on the space of antisymmetric matrices. The following well-known result shows that the above definition does agree with the usual one. (Although the proof we give is probably *not* the usual one.)

PROPOSITION 2.2. *If $A$ is an antisymmetric matrix, then* $\mathrm{pf}(A)^2 = \det(A)$.

*Proof.* Let $E_n \subset S_n$ denote the set of permutations in which every cycle is of even length. Given $\sigma \in S_n - E_n$, let $\sigma = \sigma_1 \cdots \sigma_l$ denote the disjoint cycle decomposition of $\sigma$, labeled so that $\sigma_1$ is the odd-length cycle whose smallest element is smaller than the elements of all other odd-length cycles of $\sigma$. In these terms, the map $\sigma \mapsto \sigma' = \sigma_1^{-1}\sigma_2 \cdots \sigma_l$ defines an involution on $S_n - E_n$. If $I \subset \{1, 2, ..., n\}$ is the orbit of $\sigma_1$, then antisymmetry implies

$$\prod_{i \in I} a_{i,\sigma(i)} = (-1)^{|I|} \prod_{i \in I} a_{\sigma(i),i} = -\prod_{i \in I} a_{i,\sigma^{-1}(i)},$$

so it follows that $a_\sigma = -a_{\sigma'}$, using the notation $a_\sigma$ as an abbreviation for $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$. Since $\sigma$ and $\sigma'$ have the same sign, it follows that the net contributions of $\sigma$ and $\sigma'$ to $\det(A)$ cancel each other, and hence, $\det(A) = \sum_{\sigma \in E_n} \mathrm{sgn}(\sigma) a_\sigma$.

There is a natural bijection $(\pi, \pi') \mapsto \sigma$ from $\mathscr{F}_n \times \mathscr{F}_n$ to $E_n$ as follows. Given a pair $\pi, \pi' \in \mathscr{F}_n$, the graph formed by $\pi \cup \pi'$ is regular of degree 2 and bipartite, and hence, a disjoint union of even-length cycles. We obtain the corresponding permutation $\sigma$ by orienting each cycle in a canonical direction; namely, by starting at the smallest element of the cycle and traversing from there the edge of $\pi$ incident to this smallest element. Under these circumstances, we claim that

$$\mathrm{sgn}(\pi)\,\mathrm{sgn}(\pi') \prod_{(i,j) \in \pi \cup \pi'} a_{ij} = \mathrm{sgn}(\sigma) a_\sigma. \tag{2.1}$$

Once proved, the proposition will follow, since the terms on the left side are clearly those of $\mathrm{pf}(A)^2$, while we have already shown that those on the right yield $\det(A)$.

To prove the claim, let us define $e(\sigma) = |\{i : \sigma(i) < i\}|$ and note that (2.1) is equivalent to the assertion $\mathrm{sgn}(\pi)\,\mathrm{sgn}(\pi') = \mathrm{sgn}(\sigma)(-1)^{e(\sigma)}$. Now hypothesize that this assertion is true for some particular $\sigma \in E_n$, and consider the effect of conjugating $\sigma$ by the transposition $(i, i+1)$. Of course, this is equivalent to interchanging the positions of $i$ and $i+1$ in the orbits of $\sigma$. If $i$ and $i+1$ appear consecutively (in either order) within some orbit of $\sigma$, then this conjugation will change the parity of $e(\sigma)$. Meanwhile, it will fix one of the 1-factors $\pi$ or $\pi'$ (the one containing the edge $(i, i+1)$), while changing the sign of the other (cf. Lemma 2.1). Otherwise, if $i$ and $i+1$ are nonconsecutive or in different orbits of $\sigma$, then conjugation will preserve $e(\sigma)$ while changing the sign of both $\pi$ and $\pi'$. In either case, we conclude that the validity of (2.1) is preserved by conjugation, so we need only to consider one choice of $\sigma$ for each possible cycle-type. For example, if $\sigma$ is the $n$-cycle $(1, 2, ..., n)$, then $e(\sigma)=1$, $\pi=\{(1,2), ..., (n-1,n)\}$, and $\pi'=\{(2,3), ..., (n-2, n-1), (1,n)\}$, so the claim is clearly true in this case. Since (2.1) is multiplicative with respect to (disjoint) cycles of this type, the result follows. ∎

The practical significance of Proposition 2.2 is that it shows that the number of arithmetic operations required to compute a pfaffian is at worst a polynomial function of $n$.

We close this section with a collection of useful properties of pfaffians and related determinants. In the identities below, $A=[a_{ij}]$ denotes an $n \times n$ antisymmetric matrix.

PROPOSITION 2.3. *Assume that n is even.*

(a) $\mathrm{pf}[x_i x_j a_{ij}] = x_1 \cdots x_n \,\mathrm{pf}[a_{ij}]$.

(b) $\mathrm{pf}[X^T A X] = \det(X)\,\mathrm{pf}(A)$.

(c) $\mathrm{pf}[1]_{1 \leqslant i < j \leqslant n} = 1$, *or equivalently*, $\sum_{\pi \in \mathscr{F}_n} \mathrm{sgn}(\pi) = 1$.

(d) $\det[a_{ij} + t x_i x_j] = \det[a_{ij}]$.

(e) $\mathrm{pf}[(x_i - x_j)/(1 - x_i x_j)] = \prod_{i<j} (x_i - x_j)/(1 - x_i x_j)$.

(f) $\mathrm{pf}[x_{j-i}]_{1 \leqslant i < j \leqslant n} = \det[y_{ij}]_{1 \leqslant i,j \leqslant n/2}$, *where* $y_{ij} = x_{|i-j|+1} + x_{|i-j|+3} + \cdots + x_{i+j-1}$.

*Proof.* (a) This is a special case of (b).

(b) Proposition 2.2 implies $\mathrm{pf}[X^T A X] = \pm \det(X)\,\mathrm{pf}(A)$. Choosing $X$ to be the identity matrix shows that the positive branch must be correct.

(c) Proceed by induction on $n$. It is clear when $n=2$. For $n>2$, let $F_i \subset \mathscr{F}_n$ denote the set of 1-factors in which $i$ is adjacent to 1 $(1 < i \leqslant n)$. Given $\pi \in F_i$, the number of edges in $\pi$ that cross $(1,i)$ agrees (mod 2) with $i$, so by induction we have $\sum_{\pi \in F_i} \mathrm{sgn}(\pi) = (-1)^i$. Hence, the total contributed by each $F_i$ amounts to $\sum_{1<i\leqslant n} (-1)^i = 1$.

(d) (See also the proof of Theorem 2 in [JP].) Let $f(t) = \det[a_{ij} + t x_i x_j]$. The coefficient of $t^r$ in $f$ can be expressed as a sum of $\binom{n}{r}$ determinants corresponding to each of the ways of selecting $r$ rows from $[x_i x_j]$ and the complementary set of $n-r$ rows from $[a_{ij}]$. Since $[x_i x_j]$ is of rank one, any such matrix with $r>1$ will be singular, so we deduce that $f$ is a linear function of $t$. Since $[a_{ij}]$ is antisymmetric and $n$ is even, we obtain $f(t) = \det[a_{ji} + t x_j x_i] = \det[a_{ij} - t x_i x_j] = f(-t)$, so $f$ is an even, linear function of $t$, i.e., a constant.

(e) The following argument was suggested by I. G. Macdonald. By Proposition 2.2 and part (d), we have

$$\operatorname{pf}\left[\frac{x_i-x_j}{1-x_ix_j}\right]^2=\det\left[1+\frac{x_i-x_j}{1-x_ix_j}\right]=\det\left[\frac{(1+x_i)(1-x_j)}{1-x_ix_j}\right]$$
$$=\prod_i(1-x_i^2)\cdot\det\left[\frac{1}{1-x_ix_j}\right]=\prod_{i<j}\frac{(x_i-x_j)^2}{(1-x_ix_j)^2}.$$

The last equality is a special case of Cauchy's determinant [M, p. 38]. To complete the proof, we must show that the desired pfaffian is indeed the "positive" square root of the above expression, rather than its negative. For this, observe that the coefficient of $x^\delta=x_1^{n-1}x_2^{n-2}\cdots x_n^0$ in the claimed product (as a formal power series) is clearly 1. Meanwhile, since $(x-y)/(1-xy)=\sum_k x^{k+1}y^k-x^ky^{k+1}$, it follows that the only terms in the expansion of the pfaffian that contribute to the coefficient of $x^\delta$ must arise from the 1-factor $\{(1,2),...,(n-1,n)\}$. Furthermore, it is not hard to show that the coefficient of $x^\delta$ in this single term is also 1, as desired.

(f) This result is due to B. Gordon (see Lemma 1 of [Go1]). ∎

We remark that if $\lambda=(\lambda_1\geqslant\cdots\geqslant\lambda_n)$ is any partition with at most $n$ rows, then the coefficient of $x^{\lambda+\delta}$ in $\operatorname{pf}[(x_i-x_j)/(1-x_ix_j)]$ is either 0 or 1, the latter occurring if and only if $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$, and so on (cf. the reasoning used in part (e)). A corollary of this observation and Proposition 2.3(e) is Littlewood's well-known Schur function identity

$$\prod_{i<j}\frac{1}{1-x_ix_j}=\sum_\lambda s_\lambda,$$

in which the sum ranges over partitions $\lambda$ with columns of even length [M, p. 46].

## 3. Nonintersecting Paths and Pfaffians

Continuing the setting of Section 1, let $D=(V,E)$ be an acyclic directed graph. For any vertex $u\in V$ and subset $I\subset V$, let $\mathscr{P}(u;I)$ denote the set of $D$-paths from $u$ to any $v\in I$, and let $Q_I(u)$ denote the associated generating function; i.e.,

$$Q_I(u)=GF[\mathscr{P}(u;I)]=\sum_{P\in\mathscr{P}(u;I)}w(P)=\sum_{v\in I}h(u,v).$$

Given any $r$-tuple $\mathbf{u}=(u_1,...,u_r)$ of vertices, let $\mathscr{P}(\mathbf{u};I)$ denote the set of $r$-tuples of paths $(P_1,...,P_r)$ such that $P_i\in\mathscr{P}(u_i;I)$, and let $\mathscr{P}_0(\mathbf{u};I)$ denote the subset consisting of nonintersecting $r$-tuples. Similarly extend the $Q_I$-notation by defining

$$Q_I(\mathbf{u}) = Q_I(u_1, ..., u_r) = GF[\mathscr{P}_0(\mathbf{u}; I)].$$

Note that in the case $r = 2$, assuming $\mathbf{u}$ is $D$-compatible with some total order of $I$, Theorem 1.2 implies

$$Q_I(u_1, u_2) = \sum_{v_1 < v_2 \in I} h(u_1, v_1)\, h(u_2, v_2) - h(u_1, v_2)\, h(u_2, v_1), \tag{3.1}$$

where the sum extends to all pairs for which $v_1$ precedes $v_2$ in the ordering of $I$.

THEOREM 3.1. *Let* $\mathbf{u} = (u_1, ..., u_r)$ *be an r-tuple of vertices in an acyclic diagraph D, and assume that r is even. If* $I \subset V$ *is a totally ordered subset of the vertices such that* $\mathbf{u}$ *is D-compatible with I, then*

$$Q_I(u_1, ..., u_r) = \mathrm{pf}[Q_I(u_i, u_j)]_{1 \leqslant i < j \leqslant r}.$$

*Remark.* In case $r$ is odd, we may adjoin a phantom vertex $u_{r+1}$ to $V$, with no incident edges, and include $u_{r+1}$ in $I$. In that case, we have $Q_I(u_i, u_{r+1}) = Q_I(u_i)$. If we order all other vertices of $I$ before $u_{r+1}$, then $\mathbf{u}^* = (u_1, ..., u_{r+1})$ will be $D$-compatible with $I$, and so Theorem 3.1 can be used to provide a pfaffian of order $r + 1$ for $Q_I(\mathbf{u})$.

*Proof.* We may interpret

$$\mathrm{pf}[Q_I(u_i, u_j)] = \sum_{\pi \in \mathscr{F}(\mathbf{u})} \mathrm{sgn}(\pi) \prod_{(u_i, u_j) \in \pi} Q_I(u_i, u_j) \tag{3.2}$$

as a generating function for $(r+1)$-tuples $(\pi, P_1, ..., P_r)$ in which $P_i \in \mathscr{P}(u_i; I)$, $\pi$ is a 1-factor of $\mathbf{u}$, and for each edge $(u_i, u_j)$ of $\pi$, $P_i$ and $P_j$ must not intersect. Of course, the weight assigned to $(\pi, P_1, ..., P_r)$ shall be $\mathrm{sgn}(\pi)\, w(P_1) \cdots w(P_r)$. We will show that there is a sign-reversing involution that acts on the configurations $(\pi, P_1, ..., P_r)$ with at least one pair of intersecting paths.

To describe this involution, first choose a fixed total order of the vertices. Since $D$ is acyclic, we may also insist that this order be consistent with the edges of $D$; i.e., if $u \to v$, then $u$ must precede $v$ in the total order. Now we proceed as in the proof of Theorem 1.2. Given a configuration $(\pi, P_1, ..., P_r)$ with at least one pair of intersecting paths, let $v$ be the unique vertex of intersection that precedes all other points of intersection with respect to the chosen total order. Among the paths that pass through $v$, assume that $P_i$ and $P_j$ are the two whose indices $i$ and $j$ are smallest.

Following the notation of Theorem 1.2, define a new $r$-tuple $(P'_1, ..., P'_r)$, where $P'_i = P_i(\to v)\, P_j(v \to)$, $P'_j = P_j(\to v)\, P_i(v \to)$, and $P'_k = P_k$ for $k \neq i, j$. Also, let $\pi'$ denote the 1-factor obtained by interchanging $u_i$ and $u_j$ in $\pi$.

We claim that $(\pi', P'_1, ..., P'_r)$ is one of the configurations that appears in (3.2); i.e., we claim that for each edge $(u_k, u_l) \in \pi'$, the paths $P'_k$ and $P'_l$ do not intersect. The only cases for which this is not immediate are those involving the modified paths $P'_i$ and $P'_j$. However, the minimality of $v$ implies that there are no points of intersection (other than $v$) on the subpaths $P_i(\to v)$ and $P_j(\to v)$. Hence, the path $P_k$ will intersect $P_i$ (resp., $P_j$) if and only if $P'_k$ intersects $P'_j$ (resp., $P'_i$), and so the claim follows. Thus we conclude that the operation $(\pi, P_1, ..., P_r) \mapsto (\pi', P'_1, ..., P'_r)$ is an involution on the intersecting configurations appearing in (3.2).

To prove that this involution is sign-reversing, it is enough to show (by Lemma 2.1) that neither $u_i$ nor $u_j$ is adjacent in $\pi$ to any $u_k$ with $i<k<j$. For this, it suffices to show that if $i<k<j$, then $P_k$ must intersect $P_i$ and $P_j$. For these values of $k$, to prove that $P_k$ intersects $P_i$, let $v_i, v_k \in I$ denote the terminal vertices of $P_i$ and $P_k$. If $v_i = v_k$, there is nothing more to prove. Otherwise, recall that $I$ is totally ordered, and consider the two cases $v_i > v_k$ and $v_i < v_k$. If $v_i > v_k$, then the fact that $\mathbf{u}$ is $D$-compatible with $I$ forces $P_i$ and $P_k$ to intersect. If $v_i < v_k$, then the $D$-compatibility of $I$ forces $P'_j = P_j(\to v)\, P_i(v \to)$ to intersect $P_k$. However, as noted above, $v$ is the only intersection point in $P_j(\to v)$, so $P_k$ must intersect $P'_j$ in the subpath $P_i(v \to)$; i.e., $P_k$ intersects $P_i$. One may establish that $P_k$ must intersect $P_j$ by similar reasoning.

The existence of this sign-reversing involution shows that we may delete all of the terms of (3.2) involving intersecting configurations of paths; i.e.,

$$\mathrm{pf}[Q_I(u_i, u_j)] = GF[\mathscr{P}_0(\mathbf{u}; I)] \cdot \sum_{\pi \in \mathscr{F}_r} \mathrm{sgn}(\pi).$$

Apply Proposition 2.3(c) to complete the proof. ∎

We remark that one may derive Theorem 3.1 by applying Okada's pfaffian for the sum of the $r \times r$ minors of an $r$-rowed matrix (Theorem 3 of [O]) to Theorem 1.2. However, the above proof more clearly reveals the underlying combinatorial structure.

We next consider a result that generalizes both Theorem 1.2 and Theorem 3.1. Suppose that $\mathbf{v} = (v_1, ..., v_s)$ is an ordered list of vertices of $V$, and let $I \subset V$ be a totally ordered set that is disjoint from $\mathbf{v}$. We define $\mathbf{v} \oplus I$ to be the union of $\mathbf{v}$ and $I$, ordered so that each $v_i$ precedes each $v \in I$. Given any $r$-tuple $\mathbf{u} = (u_1, ..., u_r)$ with $r \geqslant s$, we define $\mathscr{P}_0(\mathbf{u}, \mathbf{v}; I)$ to be the set of all $r$-tuples of nonintersecting paths $(P_1, ..., P_r)$ such that $P_i \in \mathscr{P}(u_i, v_i)$ for $1 \leqslant i \leqslant s$ and $P_i \in \mathscr{P}(u_i; I)$ for $s < i \leqslant r$.

Note that there is no loss of generality in assuming that $r+s$ is even. If it is odd, merely adjoin a phantom vertex $u_{r+1}$ to $\mathbf{u}$ and $I$, with no incident edges, and order all other vertices of $I$ before $u_{r+1}$.

For convenience, we define $Q_I(u_j, u_i) = -Q_I(u_i, u_j)$ for $i \leqslant j$.

THEOREM 3.2. *Let* $\mathbf{u} = (u_1, ..., u_r)$ *and* $\mathbf{v} = (v_1, ..., v_s)$ *be sequences of vertices in an acyclic digraph D, and assume that* $r+s$ *is even. If I is a totally ordered subset of V such that* $\mathbf{u}$ *is D-compatible with* $\mathbf{v} \oplus I$ *(disjoint union), then*

$$GF[\mathscr{P}_0(\mathbf{u}, \mathbf{v}; I)] = \mathrm{pf}\begin{bmatrix} Q & H \\ -H & 0 \end{bmatrix},$$

*where* $Q = [Q_I(u_i, u_j)]$ *for* $1 \leqslant i, j \leqslant r$, *and* $H = [h(u_i, v_{s+1-j})]$ *for* $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$.

*Proof.* We have

$$\mathrm{pf}\begin{bmatrix} Q & H \\ -H & 0 \end{bmatrix} = \sum_{\pi} \mathrm{sgn}(\pi) \prod_{(u_i, u_j) \in \pi} Q_I(u_i, u_j) \prod_{(u_i, v_j) \in \pi} h(u_i, v_j), \tag{3.3}$$

summed over all 1-factors $\pi$ of $(u_1, ..., u_r, v_s, ..., v_1)$ in which there are no edges connecting any two vertices of $\mathbf{v}$. This may be interpreted as the generating function for all $(r+1)$-tuples $(\pi, P_1, ..., P_r)$ such that (1) $P_i \in \mathscr{P}(u_i, v_j)$ if there is an edge $(u_i, v_j) \in \pi$, (2) $P_i \in \mathscr{P}(u_i; I)$ if there is no such edge, and (3) for each edge $(u_i, u_j) \in \pi$, $P_i$ and $P_j$ must not intersect.

We claim that the sign-reversing involution used in the proof of Theorem 3.1 can be applied to this situation as well. To prove this, we must show that the operation $(\pi, P_1, ..., P_r) \mapsto (\pi', P'_1, ..., P'_r)$ preserves the domain of configurations described above. For this, suppose that $P_i$ and $P_j$ are the two switched paths involved in the application of the involution. Note that the interchange of $u_i$ and $u_j$ in $\pi$ will not create any edges between vertices of $\mathbf{v}$, so $\pi'$ will be a valid 1-factor for the new configuration. Now if $(u_i, v_k) \in \pi'$, then $(u_j, v_k) \in \pi$, so $P_j \in \mathscr{P}(u_j, v_k)$, which implies $P'_i \in \mathscr{P}(u_i, v_k)$ as needed. One may similarly argue that $(u_j, v_k) \in \pi'$ would imply $P'_j \in \mathscr{P}(u_j, v_k)$. Finally, one may argue that there will be no intersection of paths $P'_k$ and $P'_l$ with $(u_k, u_l) \in \pi'$ for the same reason we used in the proof of Theorem 3.1. Hence, the claim follows.

We may now conclude that (3.3) may be restricted to nonintersecting configurations. Since $\mathbf{u}$ is $D$-compatible with $\mathbf{v} \oplus I$, such configurations arise only when $P_i \in \mathscr{P}(u_i, v_i)$ for $i \leqslant s$ and $P_i \in \mathscr{P}(u_i; I)$ for $i > s$. In particular, this forces $(u_i, v_i) \in \pi$ for $i \leqslant s$, but $\pi$ may otherwise be arbitrary. Since none of the edges $(u_i, v_i)$ can cross any edge of $\pi$ (recall the ordering chosen for the vertices of the 1-factors), it follows that the sign of $\pi$ is the same as the sign of its restriction to $\mathbf{u}' = (u_{s+1}, ..., u_r)$. Hence, the right side of (3.3) is equal to $GF[\mathscr{P}_0(\mathbf{u}, \mathbf{v}; I)] \sum_{\pi \in \mathscr{F}_{r-s}} \mathrm{sgn}(\pi)$. Apply Proposition 2.3(c). ∎

To recover Theorem 1.2 as a special case of Theorem 3.2, take $I=\varnothing$, $r=s$, and apply the fact that $\mathrm{pf}[\begin{smallmatrix} 0 & H \\ -H & 0 \end{smallmatrix}]=(-1)^{\binom{r}{2}}\det(H)$ for any $r\times r$ matrix $H$.

## 4. Nonintersecting Paths between Two Regions

The main result of the previous section (Theorem 3.1) provides a generating function for the set of $r$-tuples of nonintersecting paths from some specified set of $r$ vertices to some region $I$. In this section, we consider the more general problem of enumerating sets of nonintersecting paths in which both the initial and terminal vertices are allowed to vary over specified regions of $D$.

As in the previous section, we assume that $\mathbf{u}=(u_1, ..., u_r)$ is a sequence of vertices in an acyclic digraph $D=(V, E)$ and $I\subset V$. For $s\leqslant r$, we define $\mathscr{P}_0^{(s)}(\mathbf{u}; I)$ to be the set of $s$-tuples $(P_1, ..., P_s)$ of nonintersecting $D$-paths in $\mathscr{P}_0(\mathbf{u}'; I)$, where $\mathbf{u}'$ ranges over all $s$-subsets of $\mathbf{u}$. Let

$$Q_I^{(s)}(\mathbf{u})=GF[\mathscr{P}_0^{(s)}(\mathbf{u}; I)]=\sum_{i_1<\cdots<i_s} Q_I(u_{i_1}, ..., u_{i_s})$$

denote the corresponding generating function.

The following result is closely related to Okada's pfaffian for the sum of all minors of an arbitrary matrix (Theorem 4 of [O]); in fact, one could derive this result by means of Theorem 1.2 and Okada's pfaffian. Conversely, the techniques we give here (particularly Lemma 4.2) can be used to derive Okada's pfaffian.

THEOREM 4.1. *Let $\mathbf{u}=(u_1, ..., u_r)$ be an $r$-tuple of vertices in an acyclic digraph $D$, and assume that $I\subset V$ is an ordered subset of the vertices that is $D$-compatible with $\mathbf{u}$.*

(a) *If $r$ is even, then*

$$\sum_{s=0}^{r/2} t^s Q_I^{(2s)}(\mathbf{u})=\mathrm{pf}[(-1)^{i+j-1}+tQ_I(u_i, u_j)]_{1\leqslant i<j\leqslant r}.$$

(b) *If $r$ is odd, then*

$$\sum_{s=0}^{r} t^s Q_I^{(s)}(\mathbf{u})=\mathrm{pf}[(-1)^{i+j-1}+t^2Q_I^*(u_i, u_j)]_{1\leqslant i<j\leqslant r+1},$$

*where $Q_I^*(u_i, u_j) = Q_I(u_i, u_j)$ for $j\leqslant r$, and $Q_I^*(u_i, u_{r+1})=t^{-1}Q_I(u_i)$.*

(c) *If $r$ is even, then*

$$\sum_{s=0}^{r} t^s Q_I^{(s)}(\mathbf{u})=\mathrm{pf}[(-1)^{i+j-1}+t^2Q_I^*(u_i, u_j)]_{1\leqslant i<j\leqslant r+2},$$

*where $Q_I^*(u_i, u_j)$ is the same as in* (b) *for $j\leqslant r+1$, and $Q_I^*(u_i, u_{r+2})=0$.*

Before giving the proof, we first consider two illustrative examples. In the first example, suppose $r=3$. In that case, Theorem 4.1(b) asserts that $Q_I^{(s)}(\mathbf{u})$ is the coefficient of $t^s$ in the pfaffian of the upper triangular array

$$\begin{bmatrix} 1+t^2q_{12} & -1+t^2q_{13} & 1+tq_1 \\ & 1+t^2q_{23} & -1+tq_2 \\ & & 1+tq_3 \end{bmatrix},$$

where $q_{ij}=Q_I(u_i,u_j)$ and $q_i=Q_I(u_i)$. In the second example, suppose $r=4$. In that case, Theorem 4.1(c) asserts that $Q_I^{(s)}(\mathbf{u})$ is the coefficient of $t^s$ in the pfaffian of

$$\begin{bmatrix} 1+t^2q_{12} & -1+t^2q_{13} & 1+t^2q_{14} & -1+tq_1 & 1 \\ & 1+t^2q_{23} & -1+t^2q_{24} & 1+tq_2 & -1 \\ & & 1+t^2q_{34} & -1+tq_3 & 1 \\ & & & 1+tq_4 & -1 \\ & & & & 1 \end{bmatrix}.$$

*Proof.* First we derive parts (b) and (c) from (a).

For (b), we adjoin two phantom vertices $u_{r+1}$ and $v$ to $V$, together with a phantom edge $u_{r+1}\to v$ of weight $t^{-1}$. Note that $\mathbf{u}^*=(u_1,...,u_{r+1})$ is clearly $D$-compatible with $I^*=I\oplus\{v\}$. Furthermore, the $2s$-tuples of paths in $\mathscr{P}_0^{(2s)}(\mathbf{u}^*;I^*)$ that involve the path $u_{r+1}\to v$ can be identified with the $2s-1$-tuples in $\mathscr{P}_0^{(2s-1)}(\mathbf{u};I)$, and so we may obtain the generating function $\sum_s t^sQ_I^{(s)}(\mathbf{u})$ by applying part (a) (with $t\to t^2$) to the pair $\mathbf{u}^*$, $I^*$. Since $Q_{I^*}(u_i,u_{r+1})=t^{-1}Q_I(u_i)$, we obtain the claimed result.

For (c), we may add another phantom vertex $u_{r+2}$, with no incident edges, to the construction of part (b). Since there will not exist any $D$-paths from $u_{r+2}$ to $I$, this addition will not affect the structure of $\mathscr{P}_0^{(s)}(\mathbf{u}^*;I^*)$, and so we may apply part (a) to the pair $\mathbf{u}^{**}=(u_1,...,u_{r+2})$ and $I^*$. Since $Q_{I^*}(u_i,u_{r+2})=0$, we obtain the claimed result.

Before giving the proof of (a), we first require a lemma regarding pfaffians of sums. For any array $A$ of order $n$ and $J\subset\{1,...,n\}$, let $A_J$ denote the subarray obtained by selecting the rows and columns indexed by $J$. We will write $\sigma(J)$ for $\sum_{j\in J}j$.

LEMMA 4.2. *Assume that n is even.*

(a) *If A and B are of order n, then*

$$\mathrm{pf}[A+B]=\sum_J(-1)^{\sigma(J)-|J|/2}\,\mathrm{pf}[A_J]\,\mathrm{pf}[B_{J^c}],$$

*summed over all $J\subset\{1,...,n\}$ with $|J|$ even.*

(b) *If $A=[(-1)^{i+j-1}]_{1\leqslant i<j\leqslant n}$ and $|J|$ is even, then* $\mathrm{pf}[A_J]=(-1)^{\sigma(J)-|J|/2}$.

*Proof.* For (a), we have

$$\mathrm{pf}[A+B] = \sum_{\pi \cup \rho \in \mathscr{F}_n} \mathrm{sgn}(\pi \cup \rho) \prod_{(i,j) \in \pi} a_{ij} \prod_{(i,j) \in \rho} b_{ij},$$

summed over all ways to partition each 1-factor in $\mathscr{F}_n$ into two 1-factors $\pi$, $\rho$ on complementary subsets of $\{1, ..., n\}$. Suppose that $(\pi, \rho)$ is one of these pairs and that $J = \{j_1 < \cdots < j_k\}$ is the vertex set of $\pi$. If $(j_r, j_s) \in \pi$, then the number of edges of $\rho$ that cross $(j_r, j_s)$ will agree (mod 2) with the number of points of $\rho$ between $j_r$ and $j_s$; namely, $j_s - j_r - s + r$. The number of crossed edges between $\pi$ and $\rho$ is therefore

$$\sum_{(j_r, j_s) \in \pi} j_s - j_r - s + r = \sigma(J) + \binom{k+1}{2} = \sigma(J) - |J|/2 \pmod 2,$$

and so we have $\mathrm{sgn}(\pi \cup \rho) = (-1)^{\sigma(J) - |J|/2} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)$. It follows that

$$\mathrm{pf}[A+B] = \sum_J (-1)^{\sigma(J) - |J|/2} \sum_{\pi \in \mathscr{F}(J)} \mathrm{sgn}(\pi) \prod_{(i,j) \in \pi} a_{ij}$$

$$\cdot \sum_{\rho \in \mathscr{F}(J^c)} \mathrm{sgn}(\rho) \prod_{(i,j) \in \rho} b_{ij},$$

summed over all $J$ of even cardinality. This is clearly equivalent to the claimed formula.

For (b), note that $\mathrm{pf}[C] = (-1)^{k/2}\,\mathrm{pf}[-C]$ for any $C$ of order $k$. Therefore, it suffices to show that $\mathrm{pf}[-A_J] = (-1)^{\sigma(J)}$. However, this follows directly from Propositions 2.3(a) and 2.3(c). ∎

To complete the proof of Theorem 4.1(a), we may apply Lemmma 4.2 to the pair of arrays $A = [(-1)^{i+j-1}]$ and $B = [tQ_I(u_i, u_j)]$ to obtain

$$\mathrm{pf}[(-1)^{i+j-1} + tQ_I(u_i, u_j)] = \sum_J t^{|J|/2} \mathrm{pf}[Q_I(u_i, u_j)]_{i<j \in J},$$

summed over all $J \subset \{1, ..., r\}$ of even cardinality. By Theorem 3.1, the $J$th term $\mathrm{pf}[Q_I(u_i, u_j)]_{i<j \in J}$ may be identified as the generating function for nonintersecting paths from $\{u_j : j \in J\}$ to $I$, and so the result follows. ∎

## APPLICATIONS

### 5. Shifted Tableaux

Let $\lambda = (\lambda_1 > \cdots > \lambda_l)$ be a partition of $n$ into $l$ distinct, positive parts. Associated with $\lambda$ is the *shifted diagram*

$$D'_\lambda = \{(i,j) \in \mathbf{Z}^2 : i \leqslant j \leqslant \lambda_i + i - 1,\ 1 \leqslant i \leqslant l\},$$

which we regard as an array of cells in the plane with matrix-style coordinates. A *shifted tableau* of shape $\lambda$ is an assignment $T: D'_\lambda \to \mathbf{N}$ of nonnegative integers to the cells of $D'_\lambda$ so that the rows and columns are weakly increasing; i.e.,

$$T(i,j) \leqslant T(i,j+1), \qquad T(i,j) \leqslant T(i+1,j),$$

wherever $T$ is defined. If the tableau also satisfies $T(i,j) < T(i+1,j)$, then it is said to be *column-strict*. It is more conventional to require these tableaux to have positive entries, but we will find that the results described below are slightly more elegant when zeros are permitted. In Fig. 3b is an example of a column-strict shifted tableau of shape (6, 5, 3).

To each shifted tableau $T$, we assign the weight $|T| = \sum_{x \in D'_\lambda} T(x)$, and let $G_\lambda(q)$ denote the associated generating function; i.e.,

$$G_\lambda(q) = \sum_{T: D'_\lambda \to \mathbf{N}} q^{|T|},$$

where the sum ranges over all shifted tableaux of shape $\lambda$. Similarly, let $G'_\lambda(q)$ denote the analogous generating function for column-strict tableaux. Note that if $T$ is any shifted tableau, then the tableau $T'(i,j) = T(i,j) + i - 1$ is column-strict, and conversely. Thus we have $G'_\lambda(q) = q^{n(\lambda)} G_\lambda(q)$, where $n(\lambda) = \sum (i-1)\lambda_i$.

For any nonnegative integer $r$, let $(q)_r = (1-q)(1-q^2)\cdots(1-q^r)$. A result essentially equivalent to the following was conjectured by Stanley [St1, p. 85] and first proved by Gansner [Ga]. Another proof was later given by Sagan [Sa1].

THEOREM 5.1. *We have*

$$G_\lambda(q) = \frac{1}{(q)_{\lambda_1} \cdots (q)_{\lambda_l}} \prod_{i<j} \frac{1-q^{\lambda_i - \lambda_j}}{1-q^{\lambda_i + \lambda_j}}.$$

*Proof.* Define a directed graph $D$ on the vertex set $\mathbf{N}^2$ with an edge directed from $u$ to $v$ whenever $u - v = (1, 0)$ or $(0, 1)$, unless the first coordinates of both $u$ and $v$ are 0 (see Fig. 3a). For $u = (i, j)$, we assign the weight $q^j$ to the edge $u \to v$ if $u - v = (1, 0)$; similarly, assign the weight 1 if $u - v = (0, 1)$.

Choose an integer $m \geqslant 0$ and let $\mathbf{u} = (u_1, ..., u_l)$ be the $l$-tuple of vertices in which $u_i = (\lambda_i, m)$. If $l$ is odd, it will be convenient to define $\lambda_{l+1} = 0$, treat $u_{l+1} = (0, m+1)$ as a phantom vertex, and replace $l$ by $l+1$. Note

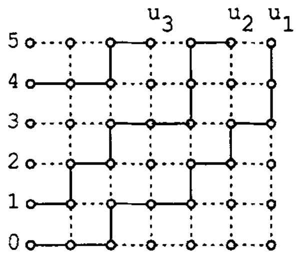


FIGURE 3a

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 2 | 3 |
| | 1 | 2 | 3 | 3 | 5 |
| | | 4 | 4 | 5 | |

FIGURE 3b

that this has no effect on the formula claimed for $G_\lambda(q)$, so we may assume for the remainder of the proof that $l$ is even.

We claim that the column-strict shifted tableaux of shape $\lambda$ with parts $\leqslant m$ can be identified as the $l$-tuples of nonintersecting $D$-paths in $\mathscr{P}_0(\mathbf{u}; I)$, where $I = \{(0, i) : i \geqslant 0\}$ denotes the set of vertices in the first column of $D$. To see this, consider the correspondence between tableaux and paths illustrated in Fig. 3. The $i$th path $P_i \in \mathscr{P}(u_i; I)$ corresponds to the $i$th row of a tableau $T$. The entries in this row can be obtained by reading the second coordinates of the horizontal steps of $P_i$ from left to right. The fact that $T(i, j) < T(i+1, j)$ corresponds to the fact that $j-i+1$st horizontal step of $P_i$ (from left to right) must be at a level strictly below the $j-i$th horizontal step of $P_{i+1}$.

Thus we have $G'_\lambda(q) = \lim_{m \to \infty} GF[\mathscr{P}_0(\mathbf{u}; I)]$.

It is clear from the geometry of $D$ that $\mathbf{u}$ is $D$-compatible with $I$, provided that $I$ is ordered so that $(0, i) < (0, j)$ iff $i < j$. Alternatively, one may check that the partial order on $\mathbf{N}^2$, defined so that $(i, j) \leqslant (i', j')$ whenever $i \geqslant i'$ and $j \leqslant j'$, is $D$-compatible. Thus, we may use Theorem 3.1 to determine $GF[\mathscr{P}_0(\mathbf{u}; I)]$. For this, note that the generating function $h(u, v)$ for paths from $u = (r, m)$ to $v = (0, i)$ is essentially a $q$-binomial coefficient. To be precise, we have

$$h(u, v) = q^{ri}(q)_{m+r-i-1}/(q)_{r-1}(q)_{m-i},$$

and so $h(u, v) \to q^{ri}/(q)_{r-1}$ in the limit $m \to \infty$. We may therefore apply (3.1) to conclude that if $r > s$, then

$$\lim_{m \to \infty} Q_I((r, m), (s, m)) = \frac{1}{(q)_{r-1}\,(q)_{s-1}} \sum_{i<j} q^{ri+sj} - q^{si+rj}$$

$$= \frac{q^s - q^r}{(q)_r\,(q)_s\,(1 - q^{r+s})}.$$

By successive applications of Theorem 3.1 and Propositions 2.3(a) and (e), we thus obtain

$$G'_\lambda(q) = \mathrm{pf}\left[\frac{q^{\lambda_j} - q^{\lambda_i}}{(q)_{\lambda_i}(q)_{\lambda_j}(1 - q^{\lambda_i + \lambda_j})}\right]$$
$$= \frac{1}{(q)_{\lambda_1} \cdots (q)_{\lambda_l}} \prod_{i<j} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_i + \lambda_j}}.$$

The claimed result now follows from the fact that $G'_\lambda(q) = q^{n(\lambda)} G_\lambda(q)$. ∎

We remark that it is an elementary exercise to verify that

$$\frac{1}{(q)_{\lambda_1} \cdots (q)_{\lambda_l}} \prod_{i<j} \frac{1 - q^{\lambda_i - \lambda_j}}{1 - q^{\lambda_i + \lambda_j}} = \prod_{x \in D'_\lambda} \frac{1}{1 - q^{h'(x)}},$$

where $h'(x)$ denotes the shifted hook length of $D'_\lambda$ at the cell $x$ [**M**, p. 135].

A (shifted) tableau with $n$ cells is said to be *standard* if the entries are $1, 2, ..., n$, in some order. If $T$ is such a tableau, we define the *descent set* $D(T)$ to consist of those integers $k$ $(1 \leqslant k < n)$ with the property that $k$ appears in a higher row of $T$ than $k+1$. The *index* of $T$ is defined to be $\sum_{k \in D(T)} n - k$, and denoted $\mathrm{ind}(T)$. For example, the following is a standard tableau with descent set $\{2, 5, 7, 10\}$ and index 32:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 7 | 10 |
| | 3 | 6 | 8 | 9 | 14 |
| | | 11 | 12 | 13 | |

Using Stanley's theory of $P$-partitions (cf. Theorem 4.5.8 of [St2]), one may deduce

COROLLARY 5.2. *If $\lambda$ is a strict partition of $n$, then*

$$\sum_T q^{\mathrm{ind}(T)} = \frac{(q)_n}{(q)_{\lambda_1} \cdots (q)_{\lambda_l}} \prod_{i<j} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_i + \lambda_j}},$$

*where $T$ ranges over the standard shifted tableaux of shape $\lambda$.*

*Proof.* Given any column-strict tableau $T$ of shape $\lambda$, we may totally order the cells of $D'_\lambda$ as follows:

$$(i,j) < (i',j') \quad \text{iff} \quad \begin{cases} T(i,j) < T(i',j'), \text{ or} \\ T(i,j) = T(i',j') \text{ and } j < j'. \end{cases}$$

If we number the cells of $D'_\lambda$ from 1 to $n$ according to this order, we obtain a standard tableau $S$. For example, the standard tableau depicted above is obtained via this process from the tableau in Fig. 3b.

Let $\mu$ be the sequence whose $i$th term is the entry of $T$ in the cell numbered $i$ by $S$. Note that the map $T \mapsto (S, \mu)$ is injective. For fixed $S$, the possible sequences $\mu$ that arise from this correspondence are characterized by the constraints $\mu_k < \mu_{k+1}$ for $k \in D(S)$, along with $\mu_n \geqslant \cdots \geqslant \mu_1 \geqslant 0$. For such $\mu$, subtract 1 from each of $\mu_n, ..., \mu_{k+1}$ for each $k \in D(S)$. The result is a sequence $\mu^*$ that differs in sum from $\mu$ by the amount $\mathrm{ind}(S)$. Since the terms of $\mu^*$ need only be weakly increasing, it follows that the generating function for the tableaux $T$ corresponding to a fixed $S$ is of the form $q^{\mathrm{ind}(S)}/(q)_n$. By adding the various choices for $S$, we obtain $G'_\lambda(q)$. Apply Theorem 5.1. ▮

If we let $q \to 1$ in the above identity we obtain a well-known formula for $g^\lambda$, the number of standard shifted tableaux of shape $\lambda$ [M, p. 135].

COROLLARY 5.3. $g^\lambda = (n!/\lambda_1! \cdots \lambda_l!) \prod_{i<j} (\lambda_i - \lambda_j)/(\lambda_i + \lambda_j)$.

## 6. SCHUR'S $Q$-FUNCTIONS

In this section we consider a variation of the shifted tableaux of Section 5 in which the entries are chosen from the ordered alphabet $\mathbf{P}' = \{1' < 1 < 3' < 2 < 3' < \cdots\}$. We define these *shifted* $\mathbf{P}'$-*tableaux* to be assignments $T: D'_\lambda \to \mathbf{P}'$ of elements of $\mathbf{P}'$ to the cells of some shifted diagram $D'_\lambda$ so that the rows and columns are weakly increasing, and

(T1) For each $k = 1, 2, 3, ...$, there is at most one $k$ per column.

(T2) For each $k = 1, 2, 3, ...$, there is at most one $k'$ per row.

An example appears in Fig. 4b.

For any such tableau $T$, we define the *content* of $T$ to be the vector $\gamma(T) = (\gamma_1, \gamma_2, ...,)$, where $\gamma_k$ denotes the number of cells $x \in D'_\lambda$ such that $T(x) = k$ or $k'$. Assign the weight $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$ to $T$, and let $Q_\lambda$ denote the associated generating function; i.e.,

$$Q_\lambda = Q_\lambda(x_1, x_2, ...,) = \sum_{T : D'_\lambda \to \mathbf{P}'} x^{\gamma(T)},$$

where the sum ranges over all shifted $\mathbf{P}'$-tableaux of shape $\lambda$.

In the course of deriving the irreducible projective characters of the symmetric groups, Schur defined a family of symmetric polynomials by means of certain pfaffians [S]. These polynomials, now known as "Schur's $Q$-functions," coincide with the generating functions $Q_\lambda$ we defined above, although the chain of deductions which led to this discovery is quite lenthy. It began with an alternative description by Schur of his $Q$-functions in terms of raising operators. This description led D. E. Littlewood [Lw] to define a more general class of symmetric polynomials (the so-called "Hall–Littlewood functions"), and these were subsequently found to have numerous applications. An explicit description of the Hall–Littlewood functions as sums of monomials was later obtained by Macdonald [M, (5.11)]. By specializing Macdonald's description to the case of Schur's $Q$-functions, it can be shown that we obtain the generating functions for $\mathbf{P}'$-tableaux defined above.

The tableau description of these $Q$-functions has been used to redevelop Schur's construction of the irreducible projective characters of $S_n$, and thus reveal their underlying combinatorial structure [Ste1]. Under these circumstances, it it therefore natural to look for a more direct proof that the tableau definition agrees with Schur's definition. The following result forms the basis for such a proof.

THEOREM 6.1. *If $\lambda$ is a partition consisting of $l$ distinct parts, then*

$$Q_\lambda = \begin{cases} \mathrm{pf}[Q_{(\lambda_i,\lambda_j)}]_{1 \leqslant i<j<l} & \text{if } l \text{ is even} \\ \mathrm{pf}[Q_{(\lambda_i,\lambda_j)}]_{1 \leqslant i<j \leqslant l+1} & \text{if } l \text{ is odd,} \end{cases}$$

*where $\lambda_{l+1}=0$ in case $l$ is odd.*

*Proof.* Define a directed graph $D$ as follows. We begin with the vertex set $\mathbf{N}^2$, and direct an edge from $u$ to $v$ whenever $u-v=(1,0)$, $(0,1)$, or $(1,1)$. Subsequently, we delete the edges $u \to v$ involving points whose first coordinates are both zero, as well as those whose second coordinates are both zero. For $u=(i,j)$, we assign the weight $x_j$ to the edges $u \to v$ with $u-v=(1,0)$ and $(1,1)$, and we assign the weight 1 to the edge with $u-v=(0,1)$. Finally, we split each of the vertices $(0,j)$ with $j>0$ into two vertices, say $(0,j)$ and $(0,j)'$, so that the edge $(1,j+1) \to (0,j)$ is redirected to $(0,j)'$, while the edge $(1,j) \to (0,j)$ remains untouched. See Fig. 4a.

Fix an integer $m>0$, and let $\mathbf{u}=(u_1, ..., u_l)$ be the $l$-tuple of vertices with $u_i=(\lambda_i, m)$. Without loss of generality, we may assume that $l$ is even (if $l$ is odd, set $\lambda_{l+1}=0$, use $u_{l+1}=(0,m)'$ as a phantom vertex, and replace $l$ by $l+1$). We claim that the shifted $\mathbf{P}'$-tableaux of shape $\lambda$ with parts $\leqslant m$

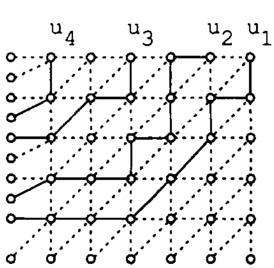


FIGURE 4a

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2′ | 3′ | 4 | | | |
| | 2′ | 2 | 2 | 3 | 5 | | | |
| | | 3 | 4′ | 4 | | | | |
| | | | 4′ | | | | | |

FIGURE 4b

can be identified with the nonintersecting paths in $\mathscr{P}_0(\mathbf{u}; I)$, where $I = \{(0,0), (0,1), (0,1)', ...\}$ denotes the set of vertices in the first column of $D$. To see this, first consider the example in Fig. 4.

The $i$th path $P_i \in \mathscr{P}(u_i; I)$ in a given configuration corresponds to the $i$th row of a tableau $T$. The entries in this row are obtained by reading the second coordinates of the horizontal and diagonal steps of $P_i$ from left to right: for each horizontal step with second coordinate $j$ we assign a $j$ to $T$, and for each diagonal step from levels $j-1$ to $j$ we assign a $j'$ to $T$. Clearly, this coding of digraph paths creates tableau rows that are weakly increasing with respect to $\mathbf{P}'$, and since there can be at most one diagonal step at given level, then there will be at most one $j'$ per row, for each $j$ (cf. property (T2)). We further claim that the conditions $T(i,j) \leqslant T(i+1,j)$ and property (T1) are equivalent to the fact that the paths are nonintersecting. To see this, suppose that the $j-i+1$st nonvertical step (from the left) in $P_i$ is at level $k$. If this step is horizontal (so that $T(i,j)=k$), then the $j-i$th step in $P_{i+1}$ must occur at level $k+1$ or higher to avoid intersection. Otherwise, if this step is diagonal (so that $T(i,j)=k'$), then the $j-i$th step in $P_{i+1}$ must either occur at a level higher than $k$, or else be a diagonal step at the same level (i.e., $T(i+1,j)=k'$).

Note that if we did not split vertex $(0,i)$ into two vertices, then a pair of $D$-paths whose last steps were, e.g., $(1,2) \to (0,1)$ and $(1,1) \to (0,1)$, would necessarily intersect. However, this corresponds to a legitimate tableau configuration of the form $\left[\begin{smallmatrix} 1 & x \\ * & 2' \end{smallmatrix}\right]$, where $*$ denotes a "cell" that is outside of the shifted diagram of $\lambda$. This configuration would be illegitimate if $*$ were part of the interior of the diagram since no element of $\mathbf{P}'$ could be assigned to $*$ without violating one or more of the rules for $\mathbf{P}'$-tableaux.

It is clear from the geometry of $D$ that $\mathbf{u}$ is $D$-compatible with $I$, so we may apply Theorem 3.1 and pass to the limit $m \to \infty$ to complete the proof. ∎

Schur formulated his original definition of the $Q$-functions [S, p. 224] by introducing the formal power series

$$Q(z) = \prod_i \frac{1 + x_i z}{1 - x_i z} = \sum_{k \geqslant 0} Q_k(x_1, x_2, ...,) \, z^k$$

whose coefficients $Q_k$ were defined to be the $Q$-functions indexed by single-rowed partitions. It is easy to verify that this agrees with the tableau definition we gave earlier. Subsequently, one notes that $Q(z)\,Q(-z) = 1$, and so the expression $(Q(z_1)\,Q(z_2) - 1)/(z_1 + z_2)$ is a well-defined formal power series. Schur then used an expansion equivalent to

$$(Q(z_1)\,Q(z_2) - 1)\frac{z_1 - z_2}{z_1 + z_2} = \sum_{m,n \geqslant 0} Q_{(m,n)}(x_1, x_2, ...,) \, z_1^m z_2^m \tag{6.1}$$

to define the $Q$-functions indexed by two-rowed partitions. For partitions $\lambda$ of length $>2$, Schur defined $Q_\lambda$ to be the pfaffian of $[Q_{(\lambda_i,\lambda_j)}]$, in agreement with Theorem 6.1. Hence, to completely verify the equivalence of the two definitions, it remains only to address the two-rowed case. We will return to this question after the following digression.

A *skew* shifted diagram is a set of cells in the plane of the form $D'_{\lambda/\mu} := D'_\lambda - D'_\mu$, where $D'_\lambda$ and $D'_\mu$ are any pair of shifted diagrams such that $D'_\mu \subset D'_\lambda$. A *skew shifted* $\mathbf{P}'$*-tableau* of shape $\lambda/\mu$ is defined to be a map $T: D'_{\lambda/\mu} \to \mathbf{P}'$ satisfying the same rules as nonskew $\mathbf{P}'$-tableaux, and we let

$$Q_{\lambda/\mu} = Q_{\lambda/\mu}(x_1, x_2, ...,) = \sum_{T: D'_{\lambda/\mu} \to \mathbf{P}'} x^{\gamma(T)}$$

denote the associated generating function. These skew $Q$-functions, like the ordinary $Q$-functions, arise naturally in the study of projective representations of symmetric groups.

Recently, Józefiak and Pragacz have shown that the skew $Q$-functions can be expressed as pfaffians [JP]. The following proof of their result shows that it can also be derived by combinatorial methods. We should point out that in their formulation of the identity, an algebraic definition of the skew function $Q_{\lambda/\mu}$ was used, rather than the tableau definition we gave above. A proof of the equivalence of the two definitions can be found in Proposition 8.2 of [Ste1], for example.

In the following, we set $Q_{-r} = 0$ for $r > 0$ and $Q_{(r,s)} = -Q_{(s,r)}$ for $r \leqslant s$.

THEOREM 6.2. *Let $\lambda$ and $\mu$ be strict partitions of lengths $l$ and $m$, respectively. If $l+m$ is odd, then define $\lambda_{l+1} = 0$ and replace $l$ by $l+1$, so that $l+m$ is even. We have*

$$Q_{\lambda/\mu} = \mathrm{pf}\begin{bmatrix} Q & H \\ -H & 0 \end{bmatrix},$$

*where $Q = [Q_{(\lambda_i,\lambda_j)}]$ for $1 \leqslant i, j \leqslant l$, and $H = [Q_{\lambda_i - \mu_{m-j+1}}]$ for $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant m$.*

*Proof.* Let $D$ be the digraph defined above, and let $I$, as before, denote the set of vertices with first coordinate zero. Fix an integer $n \geqslant 0$, and define $\mathbf{u} = (u_1, ..., u_l)$ and $\mathbf{v} = (v_1, ..., v_m)$ to be the vertex sequences consisting of $u_i = (\lambda_i, n)$ and $v_j = (\mu_j, 0)$. By reasoning analogous to the proof of Theorem 6.1, the skew shifted $\mathbf{P}'$-tableaux of shape $\lambda/\mu$ can be identified with $\mathscr{P}_0(\mathbf{u}, \mathbf{v}; I)$, i.e., with the $l$-tuples $(P_1, ..., P_l)$ of nonintersecting paths such that $P_i \in \mathscr{P}(u_i, v_i)$ for $1 \leqslant i \leqslant m$ and $P_i \in \mathscr{P}(u_i; I)$ for $i > m$. Since $\mathbf{u}$ is clearly $D$-compatible with $\mathbf{v} \oplus I$, we may apply Theorem 3.2 to obtain the desired result. ∎

Now consider the question of equivalence between (6.1) and the tableau definition of the two-rowed $Q$-functions. By transferring the factor $z_1 + z_2$ to the right side of (6.1) and extracting the coefficient of $z_1^{m+1} z_2^n$, we obtain the identity

$$Q_{m,n} + Q_{m+1,n-1} = Q_m Q_n - Q_{m+1} Q_{n-1} = Q_{(m+1,n)/1},$$

the latter equality being a special case of Theorem 6.2. Since this recursively determines the two-rowed $Q$-functions, it suffices to verify that the corresponding tableau generating functions satisfy the same recurrence. For this, we may apply the Sagan–Worley *jeu de taquin* [Sa2, Sect. 11] to remove the "hole" in each skew $\mathbf{P}'$-tableau of shape $(m+1, n)/1$; one finds that the resulting tableaux are those of shape $(m, n)$ and $(m+1, n-1)$.

## 7. Plane Partitions and Gordon's Determinants

Let $\lambda$ be a partition of length $l = l(\lambda)$. The *diagram* of $\lambda$ consists of the array of cells

$$D_\lambda = \{(i, j) \in \mathbf{Z}^2 : 1 \leqslant j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l\}.$$

A *tableau* of shape $\lambda$ is an assignment $T : D_\lambda \to \mathbf{P}$ of positive integers to the diagram of $\lambda$ so that the row and columns are weakly increasing. A *plane partition* is an analogous map with weakly decreasing rows and columns. More generally, if $\mu$ is a partition for which $D_\mu \subset D_\lambda$, then the *skew diagram* $D_{\lambda/\mu}$ is defined to be $D_\lambda - D_\mu$, and a *skew tableau* of shape $\lambda/\mu$ is a map $T : D_{\lambda/\mu} \to \mathbf{P}$, subject to the same rules as nonskew tableaux.

The weight of a tableau or plane partition $T$ is defined to be $x_1^{\gamma_1} x_2^{\gamma_2} \cdots$, where $\gamma_k$ denotes the number of $k$'s in $T$. We recall that the generating function one obtains for the (skew) column-strict tableaux of shape $\lambda/\mu$ with respect to this weight is the Schur function $s_{\lambda/\mu}$ [M]. Since Schur functions are known to be symmetric functions of $x_1, x_2, \ldots$, we may also regard $s_{\lambda/\mu}$ as a generating function for (skew) column-strict plane partitions.

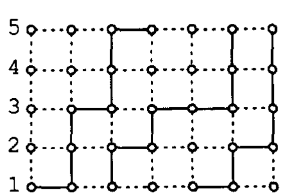


FIGURE 5a

| | | |
|---|---|---|
| | 1 | 2 |
| 2 | 3 | 3 |
| 1 3 | 5 | |

FIGURE 5b

One of the primary applications of the Gessel–Viennot technique is the interpretation of column-strict tableaux as configurations of nonintersecting paths [GV2]. For this, one defines a digraph $D$ with vertex set $\mathbf{N}^2$ and edges directed from $u$ to $v$ whenever $v-u=(0,1)$ or $(1,0)$. Note that this orientation is in a direction opposite to our previous digraphs; it is simply a matter of convenience. For $u=(i,j)$, one assigns the weight $x_j$ (resp. 1) to the edge with $v-u=(1,0)$ (resp., $(0,1)$). Under these circumstances, one may identify the column-strict tableaux of shape $\lambda/\mu$ having parts $\leqslant n$ with the configurations of nonintersecting paths in $\mathscr{P}_0(\mathbf{u},\mathbf{v})$, where $u_i=(\mu_i+l-i,1)$ and $v_i=(\lambda_i+l-i,n)$ for $i=1,...,l$. An example of this correspondence appears in Fig. 5. As in all of the previous applications, the $i$th path encodes the $i$th row of the tableaux.

Note that if $u=(j,1)$ and $v=(i,n)$, then the generating function $h(u,v)$ for paths from $u$ to $v$ is the coefficient of $t^{i-j}$ in $\prod_{k=1}^{n}(1-x_k t)^{-1}$ (or zero, if $j>i$). Therefore, in the limit $n\to\infty$, Theorem 1.2 implies the well-known determinantal formula

$$s_{\lambda/\mu}=\det[h_{\lambda_i-\mu_j-i+j}]_{1\leqslant i,j\leqslant l},$$

where $h_r$ denotes the coefficient of $t^r$ in $\prod_{k\geqslant 1}(1-x_k t)^{-1}$ [M, (5.4)].

In an analysis of column-strict plane partitions with at most $k$ rows, Gordon and Houten [GH] derived pfaffians for the associated generating functions, and Gordon [Go1] later showed that these pfaffians could be rewritten as determinants. (See also [BK].) In the following, we show that their pfaffians can be easily obtained by combinatorial methods. For completeness, we include Gordon's determinantal reductions.

THEOREM 7.1. *Let* $h=\sum_i h_i$ *and* $g_r=g_{-r}=\sum_i h_i h_{i+r}$. *We have*

(a) $\sum_{l(\lambda)\leqslant 2k} s_\lambda=\det[g_{i-j}+g_{i+j-1}]_{1\leqslant i,j\leqslant k}$.

(b) $\sum_{l(\lambda)\leqslant 2k+1} s_\lambda=h\cdot\det[g_{i-j}-g_{i+j}]_{i\leqslant i,j\leqslant k}$.

*Proof.* Using the digraph $D$ defined above, let $\mathbf{u}$ denote the vertex sequence consisting of $u_i=(i,1)$ for $i=1,...,l$. We may identify the column-strict tableaux with parts $\leqslant n$ and at most $l$ rows as the configurations in $\mathscr{P}_0(\mathbf{u};I)$, where $I=\{(i,n):i\geqslant 1\}$. Apply Theorem 3.1. In the case $l=2k$, we obtain

$$\sum_{l(\lambda)\leqslant 2k} s_\lambda=\lim_{n\to\infty}\operatorname{pf}[Q_I(u_i,u_j)]_{1\leqslant i<j\leqslant 2k}.$$

To evaluate this limit, note that (3.1) implies $Q_I(u_i,u_j)\to c_{j-i}$ as $n\to\infty$, where

$$c_r=\sum_{j>i\geqslant 0} h_i h_{j-r}-h_j h_{i-r}=\sum_{i>-r} g_i-\sum_{i>r} g_i=g_r+g_{r-1}+\cdots+g_{-(r-1)}.$$

Proposition 2.3(f) shows that the pfaffian of $[c_{j-i}]$ is the determinant of the $k \times k$ matrix whose $i, j$-entry is $c_{|i-j|+1} + c_{|i-j|+3} + \cdots + c_{i+j-1}$. By subtracting the $i$th row from the $i+1$st $(i = k-1, ..., 1)$, and then performing the same operations on the columns, we obtain the determinant claimed in (a).

For the case $l = 2k+1$, add a phantom vertex $u_{2k+2}$ to $\mathbf{u}$. Theorem 3.1 implies

$$\sum_{l(\lambda) \leqslant 2k+1} s_\lambda = \mathrm{pf}[q_{ij}]_{1 \leqslant i < j \leqslant 2k+2},$$

where $q_{ij} = c_{j-i}$ (for $i < j < 2k+2$) and $q_{ij} = h$ (for $i < j = 2k+2$). Treating $[q_{ij}]$ as an antisymmetric matrix, we now subtract the $i$th row of $[q_{ij}]$ from the $i-1$st $(i = 2, ..., 2k+1)$, and then perform the same operations on the columns. This yields a matrix whose $i, j$ entry is $g_{j-i-1} - g_{j-i+1}$, except in the last row and column; the last column will be zero, except for an $h$ in the $2k+1$st row. Since these row and column operations preserve the pfaffian (cf. Proposition 2.3(b)), we may use a Laplace type expansion to conclude that

$$\sum_{l(\lambda) \leqslant 2k+1} s_\lambda = h \cdot \mathrm{pf}[g_{j-i-1} - g_{j-i+1}]_{1 \leqslant i < j \leqslant 2k}. \tag{7.1}$$

Apply Proposition 2.3(f) to obtain the determinant claimed in (b). ∎

We remark that the same argument can be applied to the nonintersecting $l$-tuples of paths from the vertices $u_i = (\mu_i + l - i, 1)$ to $I$, and thus one may obtain pfaffians for the sums $\sum_\lambda s_{\lambda/\mu}$, where $\mu$ is fixed and $\lambda$ ranges over paritions with at most $l$ rows. These pfaffians are implicit in Lemma 1 of [GH], although the proof given there is unrelated to the one given above. Gessel [G] has used these pfaffians to show that for fixed $l$ and $\mu$, the number of standard Young tableaux of size $n$ with at most $l$ rows and any shape of the form $\lambda/\mu$ is a $P$-recursive function of $n$.

Let $\lambda'$ denote the partition conjugate to $\lambda$; one says that $\lambda'$ is *even* if the columns of $\lambda$ (i.e., the rows of $\lambda'$) are all of even length. Recently, Goulden [Gl] has noted that standard techniques from the theory of symmetric functions imply the following corollary of (7.1).

Corollary 7.2. *We have*

$$\sum_{\substack{l(\lambda) \leqslant 2k \\ \lambda' \text{ even}}} s_\lambda = \mathrm{pf}[g_{j-i-1} - g_{j-i+1}]_{1 \leqslant i < j \leqslant 2k}.$$

In view of Proposition 2.3(f), we may also write write this in the form

$$\sum_{\substack{l(\lambda) \leqslant 2k \\ \lambda' \text{ even}}} s_\lambda = \det[g_{i-j} - g_{i+j}]_{1 \leqslant i,j \leqslant k}. \tag{7.2}$$

Goulden used this observation, together with algebraic techniques, to derive some extensions of Gessel's results on $P$-recursive sequences of standard tableaux. He also noted that the entries in the above pfaffian had a nice combinatorial interpretation, and posed the problem of using this as the basis of a combinatorial proof. The following argument provides one such proof.

*Combinatorial Proof of Corollary* 7.2. Let $D$ be the diagraph defined above, and fix an integer $n \geqslant 1$. For $1 \leqslant i < j \leqslant 2k$, define $\mathscr{P}_{ij}$ to be the set of pairs of nonintersecting paths $(P, Q)$ with $P \in \mathscr{P}(u, v)$ and $Q \in \mathscr{P}(u', v')$, where $u = (i, 1)$, $u' = (j, 1)$, $v = (m, n)$, and $v' = (m+1, n)$ for some $m \geqslant 1$. Note that

$$\lim_{n \to \infty} GF[\mathscr{P}_{ij}] = \sum_{m \geqslant 1} h_{m-j+1} h_{m-i} - h_{m-i+1} h_{m-j} = g_{j-i-1} - g_{j-i+1}.$$

We may interpret

$$\text{pf}[GF[\mathscr{P}_{ij}]] = \sum_{\pi \in \mathscr{F}_n} \text{sgn}(\pi) \prod_{(i,j) \in \pi} GF[\mathscr{P}_{ij}] \tag{7.3}$$

as a generating function for configurations $(\pi, P_1, ..., P_{2k})$ consisting of a 1-factor $\pi$ and a set of $2k$ paths $P_i$ such that $(P_i, P_j) \in \mathscr{P}_{ij}$ whenever $(i, j) \in \pi$. Furthermore, we claim that the sign-reversing involution constructed in the proof of Theorem 3.1 is stable with respect to intersecting configurations of this type. For this, one only needs to check that the involution preserves the "adjacency" of the endpoints (i.e., the property that the endpoints of a pair of paths in $\mathscr{P}_{ij}$ must be of the form $(m, n)$ and $(m+1, n)$, for some $m$). It is a straightforward exercise to verify this.

Once verified, we may delete all of the terms in (7.3) that corresond to intersecting configurations. Each of the surviving $2k$-tuples of nonintersecting paths will appear only once in (7.3), since the only possible choice for $\pi$ would be $\{(1, 2), ..., (2k-1, 2k)\}$ (cf. the property of adjacent endpoints). Hence, the surviving terms of (7.3) can be identified with the configurations in $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$, where $u_i = (i, 1)$, $v_i = (\alpha_i, n)$, and $\alpha$ is any increasing sequence with $\alpha_2 - \alpha_1 = 1$, $\alpha_4 - \alpha_3 = 1$, etc. These configurations correspond to column-strict tableaux with parts $\leqslant n$, at most $2k$ rows, and columns of even length. In the limit $n \to \infty$, the generating function for these tableaux is the Schur function sum claimed above. ∎

The two determinants that appear in Theorem 7.1 are closely related to some determinantal formulas for characters of $SO_{2n+1}(\mathbf{C})$ and $Sp_{2n}(\mathbf{C})$. To explain this connection, let us define $SO_{2n+1}(\omega)$ (resp., $Sp_{2n}(\omega)$) to be the character of the representation of $SO_{2n+1}$ (resp., $Sp_{2n}$) indexed by the highest weight vector $\omega$. We choose to regard these characters as Laurent polynomials in the variables $x_1, ..., x_n$, where $x_1^{\pm 1}, ..., x_n^{\pm 1}$ (and 1, in the case of $SO_{2n+1}$) denote the eigenvalues of a generic member of the group in question.

In the following, $e_r$ (resp., $\dot{e}_r$) denotes the $r$th elementary symmetric function of the variables $x_1, x_2, ...,$ (resp., $x_1^{\pm 1}, ..., x_n^{\pm 1}$), and we set $\Delta = \prod_{i=1}^n (x_i^{1/2} + x_i^{-1/2})$. Also, $\omega_n$ denotes the $n$th fundamental weight of either $SO_{2n+1}$ or $Sp_{2n}$, depending on the context.

Proposition 7.3. *We have*

(a) $SO_{2n+1}(2k\omega_n) = \det[\dot{e}_{n-i+j} + \dot{e}_{n-i-j+1}]_{1 \leqslant i,j \leqslant k}$.

(b) $SO_{2n+1}((2k+1)\,\omega_n) = \Delta \cdot \det[\dot{e}_{n-i+j} - \dot{e}_{n-i-j}]_{1 \leqslant i,j \leqslant k}$.

(c) $Sp_{2n}(k\omega_n) = \det[\dot{e}_{n-i+j} - \dot{e}_{n-i-j}]_{i \leqslant i,j \leqslant k}$.

*Proof.* Let $\lambda$ be a partition of length $l \leqslant n$. The $SO_{2n+1}$ character indexed by $\lambda$ is

$$\det[\ddot{e}_{\lambda'_i - i+j} + (1-\delta_{1,j})\,\ddot{e}_{\lambda'_i - i-j+2}]_{1 \leqslant i,j \leqslant l},$$

where $\ddot{e}_r = \dot{e}_r + \dot{e}_{r-1}$ (see (3.3) of [K], or Sect. A2 of [P2], or [KT]). If we subtract the $j$th column from the $j+1$st ($j = 1, ..., l-1$), we obtain $[\dot{e}_{\lambda'_i - i+j} + \dot{e}_{\lambda'_i - i-j+1}]$. For the weight vector $2k\omega_n$, the corresponding partition $\lambda$ is the $n$-rowed rectangle $(k, ..., k)$. For this $\lambda$ one has $\lambda'_i = n$, and so (a) follows.

Similarly, the $Sp_{2n}$-character indexed by $\lambda$ is

$$\det[f_{\lambda'_i - i+j} + (1-\delta_{1,j}) f_{\lambda'_i - i-j+2}]_{1 \leqslant i,j \leqslant l},$$

where $f_r = \dot{e}_r - \dot{e}_{r-2}$ (see (3.3) in [K], Sect. A2 of [P2], or [KT]). If we add the $j$th column to the $j+2$nd ($j = 1, ..., l-2$), we obtain $[\dot{e}_{\lambda'_i - i+j} - \dot{e}_{\lambda'_i - i-j}]$. For the weight vector $k\omega_n$, the corresponding partition is the same as above, and so we obtain (c).

Finally, recall that if $\omega$ and $\omega'$ denote the highest weight vectors for $Sp_{2n}$ and $SO_{2n+1}$ corresponding to some partition $\lambda$, then $\Delta \cdot Sp_{2n}(\omega) = SO_{2n+1}(\omega' + \omega_n)$. (This is a corollary of the Weyl character formula; e.g., see Proposition A.2.1 of [P2].) Thus, we may obtain (b) as a consequence of (c). ∎

There is an automorphism of the ring of symmetric functions, denoted $\omega$ in [M], with the property that $s_\lambda \mapsto s_{\lambda'}$. This automorphism also interchanges $h_r$ and $e_r$. If we apply this automorphism to $g_r$, and then specialize to the variables $\mathbf{x}=(x_1, ..., x_n)$ (i.e., set $x_{n+1}=x_{n+2}=\cdots=0$), we obtain

$$g_r \mapsto \sum_i e_i(\mathbf{x})\, e_{i+r}(\mathbf{x}) = (x_1 \cdots x_n) \sum_i e_i(\mathbf{x})\, e_{n-r-i}(\mathbf{x}^{-1}) = (x_1 \cdots x_n)\, \dot{e}_{n-r}.$$

Similarly, one finds $h \mapsto (x_1 \cdots x_n)^{1/2} \Delta$. Therefore, this operation transforms the determinants of Theorem 7.1(a), 7.1(b), and (7.2) into the determinants of Proposition 7.3(a), 7.3(b), and 7.3(c), respectively. From this we may deduce the following known but difficult-to-prove identities (e.g., see [M, p. 51] for (a) and Theorem 4.1 of [Ste2] for (b)).

COROLLARY 7.4. (a) $\sum_{\lambda_1 \leqslant k} s_\lambda(x_1, ..., x_n) = (x_1 \cdots x_n)^{k/2}\, SO_{2n+1}(k\omega_n)$.

(b) $\sum_{\lambda_1 \leqslant 2k} s_{2\lambda}(x_1, ..., x_n) = (x_1 \cdots x_n)^k\, Sp_{2n}(k\omega_n)$.

The identities of Corollary 7.4(a) and Theorem 7.1 share an interesting history related to the MacMahon and Bender–Knuth Conjectures on plane partitions. These conjectures gave explicit generating functions for certain classes of plane partitions weighted by $q^{|T|}$, where $|T|$ denotes the sum of the entries in the plane partition $T$. For MacMahon's Conjecture, the class consists of the set of symmetric plane partitions (i.e., $T(i, j) = T(j, i)$) with parts $\leqslant n$ and at most $k$ rows; in the Bender–Knuth Conjecture [BK], the class consists of shifted plane partitions with parts $\leqslant n$ and at most $k$ rows. These conjectures were first proved by Andrews [A], Gordon [Go2], and Macdonald [M, p. 52] by methods that at first seem to be completely unrelated. However, we can show that the proofs of Gordon and Macdonald are in fact very closely connected.

Without violating the spirit of either proof, one could say that both proofs begin with the observation that the generating functions for the two classes in question can be obtained by applying appropriate specializations to the Schur function sums that appear in Theorem 7.1 and/or Corollary 7.4(a). In Macdonald's proof, one then observes that when the same specializations are applied to the character $SO_{2n+1}(k\omega_n)$, one may easily obtain an explicit factorization via the Weyl denominator formula. Meanwhile, in Gordon's proof, one directly evaluates the (specialized) determinants of Theorem 7.1 via a lengthy calculation. If it had been known at the time that Gordon's determinants were essentially characters of $SO_{2n+1}$, these lengthy calculations could have been avoided.

It is interesting to note that Corollary 7.4(b) has been used to derive explicit generating functions for some other classes of plane partitions (see [P1] or Corollary 4.3 of [Ste2]). These generating functions were also obtained independently by Désarménien [D] using other methods (see also [SV] for the case $q=1$).

## 8. Totally Symmetric, Self-Complementary Plane Partitions

A plane partition $T$ can be treated as a three-dimensional array of nodes in $\mathbf{P}^3$ consisting of those points $(i, j, k)$ for which $k \leqslant T(i, j)$. From this point of view, it is easy to see that there is a 6-fold symmetry group acting on plane partitions by permutation of the three coordinates. A plane partition is thus said to be *totally symmetric* if it is invariant under all six of these symmetries.

We remark that Andrews and Robbins have conjectured an explicit generating function for totally symmetric plane partitions with parts $\leqslant n$ (e.g., see [St3]). Furthermore, there is a known bijection (e.g., [O]) between totally symmetric plane partitions with parts $\leqslant n$ and row-strict shifted plane partitions with parts $\leqslant n$. From this it is not difficult to set up a correspondence between totally symmetric plane partitions and configurations of nonintersecting paths in an appropriate digraph. This leads (via Theorem 4.1) to a matrix whose pfaffian is the generating function for totally symmetric plane partitions with parts $\leqslant n$. We will not pursue the details here since this has already been done by Okada [O], although his methods are slightly different from those we have described here.

Mills, Robbins, and Rumsey have extended the group of symmetry operations on plane partitions by introducing the notion of complementation. Given a plane partition $T$ that fits inside an $a \times b \times c$ prism in $\mathbf{P}^3$, one defines the *complement* $T^c$ (with respect to $(a, b, c)$) to be the set of nodes $(a+1-i,\ b+1-j,\ c+1-k)$, where $(i, j, k)$ ranges over the nodes of $T$. This operation, together with permutation of coordinates, generates a symmetry group $G$ of order 12. This suggested to Stanley the idea of studying, for each subgroup $H$ of $G$, the class of $H$-invariant plane partitions, and this led to several new conjectures and theorems by various people [St3].

In particular, Mills et al. [MRR] have studied the class of plane partitions that are invariant under the full group $G$ (i.e., totally symmetric and self-complementary), and conjectured a formula for the number of such partitions. To describe their conjecture, first note that for plane partitions of this type to exist, it is necessary that the complementing prism be of the form $2n \times 2n \times 2n$ for some $n$. Therefore, let us define $t_n$ to be the number of tsscpp's of order $n$; i.e., the number of totally symmetric plane partitions that are self-complementary with respect to a $2n \times 2n \times 2n$ prism.

Conjecture 8.1 [MRR]. $t_n = \prod_{i=0}^{n-1} (3i+1)!/(n+i)!$.

Recently, W. Doran has shown that there is a natural way to encode tsscpp's by means of nonintersecting configurations of paths [Do]. To describe this encoding, we define a digraph $D$ on the vertex set $V = \{(i, j) \in \mathbf{Z}^2 : 1 \leqslant i \leqslant j\}$, with an edge directed from $u$ to $v$ whenever $u - v = (-1, 0)$ or $(0, 1)$. Assign unit weight to each edge. We prefer to embed $D$ in a plane with matrix-style coordinates, so that the vertices occupy the positions on or above the main diagonal. Define $\mathbf{u}$ to be the vertex sequence consisting of $u_i = (i, 2i)$ for $i = 1, ..., n-1$, and let $I = \{(i, i) : i \geqslant 1\}$ be the set of vertices on the main diagonal of $D$. Doran's result can be stated as follows.

THEOREM 8.2. *We have* $t_n = |\mathscr{P}_0(\mathbf{u}; I)|$; *i.e., there is a one-to-one correspondence between the tsscpp's of order n and sequences of nonintersecting paths* $(P_1, ..., P_{n-1})$ *such that* $P_i \in \mathscr{P}(u_i; I)$.

We include a proof for the sale of completeness.

*Proof.* By a result of Mills et al. (cf. the remarks following the proof of Theorem 1 in [MRR]), one knows that a tsscpp of order $n$ is equivalent to a shifted tableau $T$ of shape $\delta = (n-1, n-2, ..., 1)$, with entires in the range $\{1, ..., n\}$, such that $T(i, j) \geqslant j$. (We should point out that this description is obtained by reflecting the arrays $b_{ij}$ in [MRR] across a line at a 45° angle.) Given one of these tableaux $T$, let $D_k$ denote the shifted partition diagram formed by the cells $\{(i, j) \in D'_\delta : T(i, j) \leqslant k\}$. The sequence $(D_1, ..., D_{n-1})$ uniquely determines $T$. Furthermore, these sequences are characterized by the fact that the first row of $D_k$ has length at most $k$, together with the property $D_1 \subset D_2 \subset \cdots \subset D_{n-1}$.

Associate with the shifted diagram $D_i$ the $D$-path $Q_i$ from $(1, i+1)$ to $I$ obtained by arranging the steps of $Q_i$ so that the geometric configuration of vertices of $D$ that appear northwest of $Q_i$ is congruent to the arrangement of cells in $D_i$. For example, if $D_5$ is the shifted diagram of (4, 3, 1), then the path $Q_5$ is the one illustrated in Fig. 6.

The fact that $D_i \subset D_{i+1}$ implies that the translated path $(1, 1) + Q_{i+1}$, obtained by adding the vector (1, 1) to each vertex of $Q_{i+1}$, will not intersect $Q_i$. The converse holds as well. Therefore, if we define $P_i = (i-1, i-1) + Q_i$ for $1 \leqslant i < n$, we obtain a configuration of nonintersecting paths from $u_i = (i, 2i)$ to $I$; conversely, any such configuration corresponds to a unique tableau of the type described above. ∎

An example consisting of a family of nonintersecting paths, together with the corresponding tableau, is illustrated in Fig. 7.

It is now a simple matter to derive a pfaffian for $t_n$.

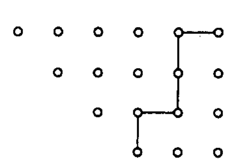

FIGURE 6

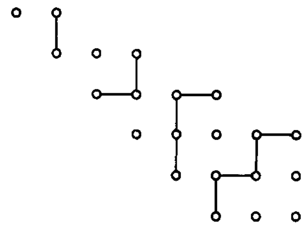

FIGURE 7a

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 5 |
| | 3 | 5 | 5 |
| | | 5 | 5 |
| | | | 5 |

FIGURE 7b

THEOREM 8.3. *Define* $a_{ij}=\sum_{2i-j<r\leqslant 2j-i}\binom{i+j}{r}$ *for* $0\leqslant i<j$. *We have*

$$t_n=\begin{cases}\operatorname{pf}[a_{ij}]_{0\leqslant i<j\leqslant n-1} & \text{if } n \text{ is even}\\ \operatorname{pf}[a_{ij}]_{1\leqslant i<j\leqslant n-1} & \text{if } n \text{ is odd.}\end{cases}$$

*Proof.* If $n$ is even, adjoin a phantom vertex $u_0=(0,0)$ to the digraph $D$, and include $u_0$ in $I$. We may then use Theorems 3.1 and 8.2 to deduce that $t_n=\operatorname{pf}[Q_I(u_i,u_j)]$, where the indices $i<j$ range from 0 to $n-1$ for even $n$, and 1 to $n-1$ for odd $n$. Therefore, we need only to prove $Q_I(u_i,u_j)=a_{ij}$. For this, we fix $i<j$ and let $b_{kl}$ denote the number of pairs of *intersecting* paths $(P,Q)$ in which $P$ (resp., $Q$) runs from $u_i=(i,2i)$ to $(k,k)$ (resp., $u_j$ to $(l,l)$). Since the total number of paths from $u_i$ to $I$ is $2^i$, it follows that $2^{i+j}-Q_I(u_i,u_j)$ is the number of pairs of intersecting paths from $u_i$ and $u_j$ to $I$. Hence,

$$2^{i+j}-Q_I(u_i,u_j)=\sum_{k,l} b_{kl}=\sum_{k<l} b_{lk}+\sum_{k\leqslant l} b_{lk},$$

the last equality being a consequence of the fact that $b_{kl}=b_{lk}$ (cf. the path-switching involution of Sect. 1). When $k\leqslant l$, every path from $u_i$ to $(l,l)$ must intersect every path from $u_j$ to $(k,k)$, so we have $b_{lk}=\binom{i}{l-i}\binom{j}{k-j}$ and thus

$$2^{i+j}-Q_I(u_i,u_j)=\sum_{0\leqslant k<l\leqslant 2i-j}\binom{i}{l+j-i}\binom{j}{k}+\sum_{0\leqslant k\leqslant l\leqslant 2i-j}\binom{i}{l+j-i}\binom{j}{k}.$$

If we apply the substitution $l\to 2i-j-l$ to the first sum, and $k\to 2i-j-k$ to the second sum, we will obtain

$$2^{i+j}-Q_I(u_i,u_j)=\sum_{k+l<2i-j}\binom{i}{i-l}\binom{j}{j-k}$$
$$+\sum_{k+l\geqslant 2i-j}\binom{i}{2i-j-l}\binom{j}{2i-j-k}.$$

For fixed values of $r=k+l$, both sums are Vandermonde convolutions, and so

$$Q_I(u_i, u_j) = 2^{i+j} - \sum_{r>2j-i} \binom{i+j}{r} - \sum_{r \leqslant 2i-j} \binom{i+j}{r}.$$

This expression clearly agrees with the definition of $a_{ij}$. ∎

We have used this result to verify Conjecture 8.1 for $n \leqslant 30$.

## 9. The Giambelli Determinant

Let $\lambda$ be a partition with (Durfee) rank $r$, and let $\lambda = (\alpha \mid \beta)$ be the Frobenius notation. We recall that this means there are $r$ cells on the main diagonal of $D_\lambda$, and that $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $1 \leqslant i \leqslant r$. In the particular case $r=1$, $(a \mid b)$ denotes the hook shaped partition with arm length $a$ and leg length $b$. We define the $k$th *principal hook* of $D_\lambda$ to be the set of cells $(i,j) \in D_\lambda$ such that $i=k \leqslant j$ or $j=k \leqslant i$. The parameters $\alpha_i$ and $\beta_i$ can be interpreted as the number of cells in the $i$th principal hook that are above or below the main diagonal, respectively.

We define a digraph $D$ on the vertex set $\mathbf{Z} \times \mathbf{P}$ as follows. Direct an edge from $u=(i,j)$ to $v$ if (1) $i>0$ and $u-v=(0,1)$ or $(1,0)$, or (2) $i<0$ and $u-v=(0,-1)$ or $(1,-1)$, or (3) $i=0$ and $u-v=(0,1)$ or $(1,-1)$. An outline of this digraph appears in Fig. 8. We attach weights to the edges $u \to v$ as follows: if $u-v=(0,\pm 1)$, assign the weight 1; otherwise, assign the weight $x_j$, assuming $u=(i,j)$.

A $D$-path $P$ from $u=(a,n)$ to $v=(-b-1, n+1)$ can be interpreted as a column-strict tableau of shape $(a \mid b)$ with parts $\leqslant n$. The entries of the tableau are obtained by reading the (indices of the) weights of the nonvertical edges. Diagonal edges correspond to the entries in the first column of the tableau, and the horizontal edges correspond to entries in the first row beyond the cell $(1,1)$. We will use this observation to give a simple proof of Giambelli's Schur function determinant [M, p. 30] (cf. also [ER]); i.e.,

THEOREM 9.1. *We have* $s_{(\alpha \mid \beta)} = \det[s_{(\alpha_i \mid \beta_j)}]_{1 \leqslant i,j \leqslant r}$.

*Proof.* Following the above correspondence between hook tableaux and paths, we claim that the column-strict tableaux $T$ of shape $(\alpha \mid \beta)$ with parts $\leqslant n$ correspond to the $r$-tuples of nonintersecting paths in $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$, where $u_i = (\alpha_i, n)$ and $v_i = (-\beta_i - 1, n+1)$. In this correspondence, the $i$th path corresponds to the restriction of $T$ to the $i$th principal hook. An example appears in Fig. 8. To prove this, observe that the subpaths from $\mathbf{u}$ to the region $I = \{(-1, i) : i \geqslant 1\}$ correspond to paths in the digraph of

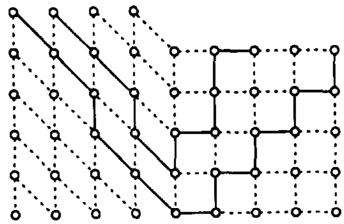

FIGURE 8a

1 1 2 3 4
2 2 3 5
4 4
5 5

FIGURE 8b

Section 5; we know that nonintersecting configurations of this type correspond to column-strict shifted tableaux. Similarly, the subpaths from the region $J=\{(0,i):i\geqslant 1\}$ to $\mathbf{v}$ correspond to paths in the digraph of Section 6 in which only diagonal edges are used. We know that these paths correspond to shifted $\mathbf{P}'$-tableaux with entries taken from $\{1',...,n'\}$, i.e., row-strict shifted tableaux. Since an unshifted column-strict tableau can be decomposed into a pair consisting of a column-strict and a row-strict shifted tableau that share the same diagonal, the claim follows. Apply Theorem 1.2 to complete the proof. ∎

For integers $r\geqslant 0$, recall that $h_r$ is the Schur function $s_r$ corresponding to the one-rowed partition $r$, and similarly, $e_r$ is the Schur function $s_{1^r}$ corresponding to a single column. By convention, we define $h_r=e_r=0$ for $r<0$.

An extension of Giambelli's determinant to skew Schur functions has been given by Lascoux and Pragacz [LP]. In the following, we show that the above argument can be used to prove their identity as well.

THEOREM 9.2. *If $\lambda=(\alpha\,|\,\beta)$ has rank $r$ and $\mu=(\gamma\,|\,\delta)$ has rank $s$, then*

$$s_{\lambda/\mu}=(-1)^s\det\begin{bmatrix}A & H\\ E & 0\end{bmatrix},$$

*where $A=[s_{(\alpha_i|\beta_j)}]$ $(r\times r)$, $H=[h_{\alpha_i-\gamma_j}](r\times s)$, and $E=[e_{\beta_j-\delta_i}](s\times r)$.*

*Proof.* An (unshifted) column-strict skew tableau can be decomposed into a pair consisting of a column-strict and a row-strict shifted skew tableau that share their main diagonals. It follows that if we define $u_i=(\alpha_i,n)$ and $v_i=(-\beta_i-1,n+1)$ for $1\leqslant i\leqslant r$, and $u_{r+i}=(-\delta_i-1,1)$ and $v_{r+i}=(\gamma_i,1)$ for $1\leqslant i\leqslant s$, then we may identify the (skew) column-strict tableaux having shape $\lambda/\mu$ and parts $\leqslant n$ with the configurations of nonintersecting paths $(P_1,...,P_{r+s})$ such that $P_i\in\mathscr{P}(u_i,v_{r+i})$ for $1\leqslant i\leqslant s$, $P_i\in\mathscr{P}(u_i,v_i)$ for $s<i\leqslant r$, and $P_{r+i}\in\mathscr{P}(u_{r+i},v_i)$ for $1\leqslant i\leqslant s$. Even though $\mathbf{u}$ need not be $D$-compatible with $\mathbf{v}$, it is still true that the only noninter-

secting configurations that connect **u** to some permutation of **v** must connect $u_i$ to $v_{\pi(i)}$, where $\pi$ denotes the involution that interchanges $i$ and $r+i$ for $i=1, ..., s$. We may therefore apply Theorem 1.2, with the conclusion modified to take into account the sign of $\pi$. ▮

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