LECTURE HALL TABLEAUX

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Abstract. We introduce lecture hall tableaux, which are fillings of a skew Young diagram satisfying certain conditions. Lecture hall tableaux generalize both lecture hall partitions and anti-lecture hall compositions, and also contain reverse semistandard Young tableaux as a limit case. We show that the coefficients in the Schur expansion of multivariate little q-Jacobi polynomials are generating functions for lecture hall tableaux. Using a Selberg-type integral we show that moments of multivariate little q-Jacobi polynomials, which are equal to generating functions for lecture hall tableaux of a Young diagram, have a product formula. We also explore various combinatorial properties of lecture hall tableaux.

1. Introduction

Lecture hall partitions are partitions satisfying certain conditions introduced by Bousquet-Mélou and Eriksson [2, 3]. Anti-lecture hall compositions are compositions satisfying similar conditions. Lecture hall partitions and anti-lecture hall compositions have been studied extensively in the last two decades. See the recent survey written by Savage [25]. In this paper we show that these objects are closely related to the little q-Jacobi polynomials $p_n^L(x;a,b;q)$.

For monic univariate orthogonal polynomials $p_n(x)$ with linear functional $\mathcal{L}$, the *mixed moment* $\mu_{n,k}$ and the *(normalized) moment* μ_n are defined by

$$\mu_{n,k} = \frac{\mathcal{L}(x^n p_k(x))}{\mathcal{L}(p_k(x)^2)}, \qquad \mu_n = \mu_{n,0} = \frac{\mathcal{L}(x^n)}{\mathcal{L}(1)}.$$

See for example [4, 29, 31] for surveys on moments of orthogonal polynomials. Note that $\mu_{n,k}$ is the coefficient $[p_k(x)]x^n$ of $p_k(x)$ in x^n. We define the *dual mixed moments* $\nu_{n,k}$ by the coefficient $[x^k]p_n(x)$ of x^k in $p_n(x)$. In other words, the mixed moments $\mu_{n,k}$ and the dual mixed moments $\nu_{n,k}$ satisfy

$$x^n = \sum_{k=0}^n \mu_{n,k} p_k(x), \qquad p_n(x) = \sum_{k=0}^n \nu_{n,k} x^k.$$

In this paper we show that the mixed moments and the dual mixed moments of the little q-Jacobi polynomials are generating functions for anti-lecture hall compositions and lecture hall partitions respectively. We then extend this result to the multivariate little q-Jacobi polynomials.

A *partition* is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. If $\lambda_i > 0$, we say that λ_i is a *part* of λ. The number of parts of λ is denoted by $\ell(\lambda)$. If $\ell(\lambda) = k$, we use the convention that $\lambda_i = 0$ for all $i > k$. Let $\mathcal{P}_n$ denote the set of partitions with at most n parts. For any sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of integers, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

In many cases, a family $\{p_n(x)\}_{n\ge 0}$ of univariate orthogonal polynomials generalizes naturally to a family $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ of multivariate orthogonal polynomials via

$$p_\lambda(x_1,\dots,x_n) = \frac{\det\left(p_{\lambda_j+n-j}(x_i)\right)_{i,j=1}^n}{\Delta(x)}, \tag{1}$$

where

$$\Delta(x) = \Delta(x_1,\dots,x_n) = \prod_{1\le i<j\le n} (x_i - x_j).$$

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See for example [27]. Note that the Schur function $s_\lambda(x)$ is also constructed in this way using the basis $\{x^n\}_{n\ge 0}$:

$$s_\lambda(x_1,\dots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n}{\Delta(x)}.$$

Suppose that $p_\lambda(x_1,\dots,x_n)$ are multivariate orthogonal polynomials given by (1) with orthogonality

$$\mathfrak{L}_n(p_\lambda(x_1,\dots,x_n)p_\mu(x_1,\dots,x_n)) = \delta_{\lambda\mu}K_\lambda(n),$$

where $\mathfrak{L}_n$ is a linear functional on the space of multivariate polynomials in variables $x_1,\dots,x_n$ and $K_\lambda(n)$ is some quantity depending on λ and n. Considering $s_\lambda(x_1,\dots,x_n)$ as a multivariate analog of x^i, we define the *mixed moment* $M_{\lambda,\mu}(n)$ of $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ by

$$M_{\lambda,\mu}(n) = \frac{\mathfrak{L}_n(s_\lambda(x_1,\dots,x_n)p_\mu(x_1,\dots,x_n))}{\mathfrak{L}_n(p_\mu(x_1,\dots,x_n)^2)},$$

and the *moment* $M_\lambda(n)$ of $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ by

$$M_\lambda(n) := M_{\lambda,\emptyset}(n) = \frac{\mathfrak{L}_n(s_\lambda(x_1,\dots,x_n))}{\mathfrak{L}_n(1)}.$$

When the univariate orthogonal polynomials are the Askey-Wilson polynomials, the corresponding multivariate polynomials are the Koornwinder polynomials with $q=t$. In this case the moments appear naturally in connection with exclusion processes [5, 8].

Similarly to the univariate case, the mixed moment $M_{\lambda,\mu}(n)$ is the coefficient of p_μ in the expansion of s_λ:

$$s_\lambda(x_1,\dots,x_n) = \sum_{\mu\in\mathcal{P}_n} M_{\lambda,\mu}(n)p_\mu(x_1,\dots,x_n).$$

We define the *dual mixed moment* $N_{\lambda,\mu}(n)$ by

$$p_\lambda(x_1,\dots,x_n) = \sum_{\mu\in\mathcal{P}_n} N_{\lambda,\mu}(n)s_\mu(x_1,\dots,x_n).$$

The *multivariate little q-Jacobi polynomials* $p_\lambda^L(x_1,\dots,x_n;a,b;q)$ are defined by the equation (1) using $p_n^L(x;a,b;q)$. It is known that $p_\lambda^L(x_1,\dots,x_n;a,b;q)$ are multivariate orthogonal polynomials with explicit linear functional $\mathfrak{L}^L$ related to the q-Selberg integral. Therefore, we can consider their mixed moments $M^L_{\lambda,\mu}(n;a,b)$ and the dual mixed moments $N^L_{\lambda,\mu}(n;a,b)$. In this paper we give a combinatorial interpretation for these quantities using new combinatorial objects called lecture hall tableaux.

We introduce four families of lecture hall tableaux, which are fillings of a skew Young diagram satisfying certain conditions. They generalize lecture hall partitions and anti-lecture hall compositions to a 2-dimensional array, and contain reverse semistandard Young tableaux as a limit case. Here, we define two of the four families.

For a cell (i,j) in λ, the *content* $c(i,j)$ is defined by $c(i,j)=j-i$. The notation $\mu\subseteq\lambda$ means the Young diagram containment.

Definition 1.1. For an integer n and partitions λ and μ with $\mu\subseteq\lambda$ and $\ell(\lambda)\le n$, a *lecture hall tableau* of shape λ/μ and of type $(n,\ge,>)$ is a filling T of the cells in the Young diagram λ/μ with nonnegative integers satisfying the following conditions:

$$\frac{T(i,j)}{n+c(i,j)} \ge \frac{T(i,j+1)}{n+c(i,j+1)}, \qquad \frac{T(i,j)}{n+c(i,j)} > \frac{T(i+1,j)}{n+c(i+1,j)}.$$

We denote by $\mathrm{LHT}_{(n,\ge,>)}(\lambda/\mu)$ the set of such fillings and by $\mathrm{LHT}_{(n,<,\le)}(\lambda/\mu)$ the set of fillings where the inequalities are changed to $<$ and $\le$ respectively.

See Figure 1 for an example of a lecture hall tableau. If λ/μ has only one row (resp. column), the lecture hall tableaux in $\mathrm{LHT}_{(n,\ge,>)}(\lambda/\mu)$ become anti-lecture hall compositions (resp. lecture hall partitions). Lecture hall tableaux also generalize reverse semistandard Young tableaux in the sense that if $n\to\infty$, lecture hall tableaux of type $(n,\ge,>)$ become reverse semistandard Young

$$
\begin{array}{|c|c|c|c|c|c|}
\cline{4-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & 9 & 4 & 3 \\
\cline{2-6}
\multicolumn{1}{c}{} & 5 & 6 & 4 & 3 & 1 \\
\cline{1-6}
2 & 2 & 1 & 0 & \multicolumn{2}{c}{} \\
\cline{1-4}
1 & 0 & 0 & \multicolumn{3}{c}{} \\
\cline{1-3}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|c|}
\cline{4-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & \frac{9}{8} & \frac{4}{9} & \frac{3}{10} \\
\cline{2-6}
\multicolumn{1}{c}{} & \frac{5}{5} & \frac{6}{6} & \frac{4}{7} & \frac{3}{8} & \frac{1}{9} \\
\cline{1-6}
\frac{2}{3} & \frac{2}{4} & \frac{1}{5} & \frac{0}{6} & \multicolumn{2}{c}{} \\
\cline{1-4}
\frac{1}{2} & \frac{0}{3} & \frac{0}{4} & \multicolumn{3}{c}{} \\
\cline{1-3}
\end{array}
$$

FIGURE 1. On the left is a lecture hall tableau $T \in \mathrm{LHT}_{(n,\geq,>)}(\lambda/\mu)$ for $n = 5$, $\lambda = (6, 6, 4, 3)$ and $\mu = (3, 1)$. The diagram on the right shows the number $T(i, j)/(n + c(i, j))$ for each entry $(i, j) \in \lambda/\mu$.

tableaux. Moreover, the lecture hall tableaux in $\mathrm{LHT}_{(n,\geq,>)}(\lambda)$ whose entries are at most n are exactly the reverse semistandard Young tableaux of shape λ whose entries are at most n.

Consider a sequence $\mathbf{x} = (x_0, x_1, \dots)$ of variables. For T in $LS^{(n,\geq,>)}_{\lambda/\mu}$ or $LS^{(n,<,\leq)}_{\lambda/\mu}$, the *weight* $\mathrm{wt}(T)$ is defined by

$$\mathrm{wt}(T) = \prod_{s\in\lambda/\mu} x_{T(s)} u^{\lfloor T(s)/(n+c(s))\rfloor} v^{o(\lfloor T(s)/(n+c(s))\rfloor)},$$

where $o(m)$ is 1 if m is odd and 0 otherwise. For example, if T is the lecture hall tableau in Figure 1, its weight is

$$\mathrm{wt}(T) = x_0^3 x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_9 u^3 v^3.$$

We define the *lecture hall Schur functions* of shape λ/μ and of types $(n, \geq, >)$ and $(n, <, \leq)$ by

$$LS^{(n,\geq,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = \sum_{T\in \mathrm{LHT}_{(n,\geq,>)}(\lambda/\mu)} \mathrm{wt}(T),$$

$$LS^{(n,<,\leq)}_{\lambda/\mu}(\mathbf{x}; u, v) = \sum_{T\in \mathrm{LHT}_{(n,<,\leq)}(\lambda/\mu)} \mathrm{wt}(T).$$

These lecture hall Schur functions become the usual Schur functions when $n \to \infty$:

$$\lim_{n\to\infty} LS^{(n,\geq,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = s_{\lambda/\mu}(\mathbf{x}), \qquad \lim_{n\to\infty} LS^{(n,<,\leq)}_{\lambda/\mu}(\mathbf{x}; u, v) = s_{\lambda'/\mu'}(\mathbf{x}),$$

where λ' is the conjugate of λ. We show that they also have Jacobi-Trudi type formulas, see Theorems 3.8 and 3.9.

Let $\mathbf{q} = (1, q, q^2, \dots)$ be the principal specialization of $\mathbf{x} = (x_0, x_1, x_2, \dots)$. In this paper we show that the mixed moments $M^L_{\lambda,\mu}(n; a, b; q)$ and the dual mixed moments $N^L_{\lambda,\mu}(n; a, b; q)$ for the multivariate little q-Jacobi polynomials $p^L_\lambda(x_1, \dots, x_n; a, b; q)$ are generating functions for lecture hall tableaux.

Theorem 1.2. *We have*

$$N^L_{\lambda,\mu}(n; -uv, -u/v; q) = (-1)^{|\lambda/\mu|} LS^{(n,<,\leq)}_{\lambda/\mu}(\mathbf{q}; u, v),$$

$$M^L_{\lambda,\mu}(n; -uv, -u/v; q) = LS^{(n,\geq,>)}_{\lambda/\mu}(\mathbf{q}; u, v).$$

Equivalently,

$$p^L_\lambda(x_1, \dots, x_n; -uv, -u/v; q) = \sum_{\mu\subseteq\lambda} (-1)^{|\lambda/\mu|} LS^{(n,<,\leq)}_{\lambda/\mu}(\mathbf{q}; u, v) s_\mu(x_1, \dots, x_n),$$

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu\subseteq\lambda} LS^{(n,\geq,>)}_{\lambda/\mu}(\mathbf{q}; u, v) p^L_\mu(x_1, \dots, x_n; -uv, -u/v; q).$$

Note that the *moments* $M^L_\lambda(n; a, b; q) := M^L_{\lambda,\emptyset}(n; a, b; q)$ and the *dual moments* $N^L_\lambda(n; a, b; q) := N^L_{\lambda,\emptyset}(n; a, b; q)$ are the generating functions for lecture hall tableaux of a normal shape $\lambda = \lambda/\emptyset$.

We prove the following theorem, which shows that the moments and the dual moments have product formulas. Throughout this paper we use the standard notation for q-series:

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Theorem 1.3. *Given an integer n and a partition λ into at most n parts,*

$$LS_\lambda^{(n,\geq,>)}(\mathbf{q};u,v) = \prod_{1\le i<j\le n} \frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}} \prod_{i=1}^n \frac{(-uvq^{n-i+1};q)_{\lambda_i}}{(u^2q^{2n-i+1};q)_{\lambda_i}},$$

$$LS_\lambda^{(n,<,\leq)}(\mathbf{q};u,v) = q^{n(\lambda)-n(\lambda')} \prod_{1\le i<j\le n} \frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}} \prod_{i=1}^n \frac{(-uvq^{n-i+1};q)_{\lambda_i}}{(u^2q^{n-i+1+\lambda_i};q)_{n-i+\lambda_i}} \times \prod_{1\le i<j\le n} (1-u^2q^{2n+\lambda_i+\lambda_j-i-j+1}),$$

where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$.

The rest of this paper is organized as follows. In Section 2 we recall lecture hall partitions, anti-lecture hall compositions and their basic properties. We show that the mixed moments and the dual mixed moments of the little q-Jacobi polynomials are generating functions for lecture hall partitions and anti-lecture hall compositions respectively. We also find a connection between the mixed and dual mixed moments of univariate orthogonal polynomials and those of corresponding multivariate orthogonal polynomials. In Section 3, we introduce lecture hall tableaux and their multivariate generating functions called lecture hall Schur functions. Using lattice path models we prove Jacobi-Trudi type formulas for lecture hall Schur functions. In Section 4, we prove Theorem 1.2 using the results in Section 3. In Section 5, we prove Theorem 1.3 using a q-Selberg integral for the first identity and determinant evaluations for the second identity. In Section 6, we consider two other families of lecture hall tableaux and prove similar enumeration results. In Section 7, we propose open problems and future generalizations.

2. Definitions and basic results

In this section we define lecture hall partitions, anti-lecture hall compositions and the little q-Jacobi polynomials. We present basic results of these objects and multivariate orthogonal polynomials which are used in later sections.

2.1. **Lecture hall partitions and anti-lecture hall compositions.**

A *lecture hall partition* is a sequence $\lambda = (\lambda_1,\dots,\lambda_n)$ of integers satisfying

$$\frac{\lambda_1}{n} \ge \frac{\lambda_2}{n-1} \ge \cdots \ge \frac{\lambda_n}{1} \ge 0.$$

Let L_n denote the set of lecture hall partitions $\lambda = (\lambda_1,\dots,\lambda_n)$ satisfying the above conditions. Bousquet-Mélou and Eriksson [2] first introduced lecture hall partitions and showed that

$$\sum_{\lambda\in L_n} q^{|\lambda|} = \frac{1}{(q;q^2)_n}.$$

See [25] for the origin of lecture hall partitions and their connections with many other objects.

An *anti-lecture hall composition* (or a *planetarium composition*) is a sequence $\alpha = (\alpha_1,\dots,\alpha_n)$ of integers satisfying

$$\frac{\alpha_1}{1} \ge \frac{\alpha_2}{2} \ge \cdots \ge \frac{\alpha_n}{n} \ge 0.$$

Let AL_n denote the set of anti-lecture hall compositions $\alpha = (\alpha_1,\dots,\alpha_n)$ satisfying the above condition. Corteel and Savage [6] showed that

$$\sum_{\alpha\in AL_n} q^{|\alpha|} = \frac{(-q;q)_n}{(q^2;q)_n}.$$

Now we consider truncated versions of lecture hall partitions and anti-lecture compositions.

Definition 2.1. For integers $n \ge k \ge 0$, we define

$$L_{n,k} = \left\{ (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k : \frac{\lambda_1}{n} > \frac{\lambda_2}{n-1} > \cdots > \frac{\lambda_k}{n-k+1} \ge 0 \right\},$$
$$\overline{L}_{n,k} = \left\{ (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k : \frac{\lambda_1}{n} \ge \frac{\lambda_2}{n-1} \ge \cdots \ge \frac{\lambda_k}{n-k+1} > 0 \right\},$$
$$AL_{n,k} = \left\{ (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k : \frac{\alpha_1}{n-k+1} \ge \frac{\alpha_2}{n-k+2} \ge \cdots \ge \frac{\alpha_k}{n} \ge 0 \right\},$$
$$\overline{AL}_{n,k} = \left\{ (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k : \frac{\alpha_1}{n-k+1} > \frac{\alpha_2}{n-k+2} > \cdots > \frac{\alpha_k}{n} > 0 \right\}.$$

Note that each element in $L_{n,k}$ or $\overline{L}_{n,k}$ is obtained from a lecture hall partition in L_n by truncating the last $n-k$ integers and each element in $AL_{n,k}$ or $\overline{AL}_{n,k}$ is obtained from an anti-lecture hall composition in AL_n by truncating the first $n-k$ integers. In this paper we will mainly consider $L_{n,k}$ and $AL_{n,k}$.

For a sequence $S = (s_1, \dots, s_k)$ of positive integers and a sequence $(\beta_1, \dots, \beta_k)$ of nonnegative integers satisfying

$$\frac{\beta_1}{s_1} \ge \frac{\beta_2}{s_2} \ge \cdots \ge \frac{\beta_k}{s_k} \ge 0,$$

we define

$$\lfloor \beta \rfloor_S = \left(\left\lfloor \frac{\beta_1}{s_1} \right\rfloor, \left\lfloor \frac{\beta_2}{s_2} \right\rfloor, \dots, \left\lfloor \frac{\beta_k}{s_k} \right\rfloor \right), \qquad \lceil \beta \rceil_S = \left(\left\lceil \frac{\beta_1}{s_1} \right\rceil, \left\lceil \frac{\beta_2}{s_2} \right\rceil, \dots, \left\lceil \frac{\beta_k}{s_k} \right\rceil \right).$$

We denote by $o(\beta)$ the number of odd parts in β. If the sequence S is clear from the context, we will simply write $\lfloor \beta \rfloor$ and $\lceil \beta \rceil$. For example, if $\lambda \in L_{n,k}$ or $\lambda \in \overline{L}_{n,k}$, then $\lfloor \lambda \rfloor = \lfloor \lambda \rfloor_S$ and $\lceil \lambda \rceil = \lceil \lambda \rceil_S$ for $S = (n, n-1, \dots, n-k+1)$ and if $\alpha \in AL_{n,k}$ or $\alpha \in \overline{AL}_{n,k}$, then $\lfloor \alpha \rfloor = \lfloor \alpha \rfloor_S$ and $\lceil \alpha \rceil = \lceil \alpha \rceil_S$ for $S = (n-k+1, n-k+2, \dots, n)$.

Now we define the following generating functions:

$$L_{n,k}(u, v, q) = \sum_{\lambda \in L_{n,k}} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} q^{|\lambda|},$$
$$\overline{L}_{n,k}(u, v, q) = \sum_{\lambda \in \overline{L}_{n,k}} u^{|\lceil \lambda \rceil|} v^{o(\lceil \lambda \rceil)} q^{|\lambda|},$$
$$AL_{n,k}(u, v, q) = \sum_{\alpha \in AL_{n,k}} u^{|\lfloor \alpha \rfloor|} v^{o(\lfloor \alpha \rfloor)} q^{|\alpha|},$$
$$\overline{AL}_{n,k}(u, v, q) = \sum_{\alpha \in \overline{AL}_{n,k}} u^{|\lceil \alpha \rceil|} v^{o(\lceil \alpha \rceil)} q^{|\alpha|}.$$

Note that the floor function $\lfloor \cdot \rfloor$ is used for $L_{n,k}(u, v, q)$ and $AL_{n,k}(u, v, q)$, whereas the ceiling function $\lceil \cdot \rceil$ is used for $\overline{L}_{n,k}(u, v, q)$ and $\overline{AL}_{n,k}(u, v, q)$.

In what follows we will see that there is a simple map that gives a bijection between $L_{n,k}$ and $\overline{L}_{n,k}$ and also a bijection between $AL_{n,k}$ and $\overline{AL}_{n,k}$. First observe the following lemma whose proof is straightforward.

Lemma 2.2. *For any integers a, b and $i \ge 1$ we have*
- $\frac{a}{i+1} > \frac{b}{i}$ *if and only if* $\frac{a+1}{i+1} \ge \frac{b+1}{i}$,
- $\frac{a}{i} \ge \frac{b}{i+1}$ *if and only if* $\frac{a+1}{i} > \frac{b+1}{i+1}$,
- $\left\lceil \frac{a+1}{i} \right\rceil = \left\lfloor \frac{a}{i} \right\rfloor + 1$.

For a sequence $\beta = (\beta_1, \dots, \beta_k)$ of integers let $\beta^+ = (\beta_1 + 1, \dots, \beta_k + 1)$. Then by Lemma 2.2 the following proposition is immediate.

Proposition 2.3. *The map $\lambda \mapsto \lambda^+$ gives a bijection from $L_{n,k}$ to $\overline{L}_{n,k}$. The same map $\alpha \mapsto \alpha^+$ also gives a bijection from $AL_{n,k}$ to $\overline{AL}_{n,k}$. Moreover, $|\lambda^+| = |\lambda| + k$, $\lceil \lambda^+ \rceil = \lfloor \lambda \rfloor^+$, $|\alpha^+| = |\alpha| + k$ and $\lceil \alpha^+ \rceil = \lfloor \alpha \rfloor^+$.*

By Proposition 2.3, we have

$$\overline{L}_{n,k}(u,v,q) = (uvq)^k L_{n,k}(u,1/v,q), \tag{2}$$

$$\overline{AL}_{n,k}(u,v,q) = (uvq)^k AL_{n,k}(u,1/v,q). \tag{3}$$

Corteel and Savage [7] found product formulas for $\overline{L}_{n,k}(u,v,q)$ and $AL_{n,k}(u,v,q)$, see (5) and (6) below. Using their formulas together with (2) and (3), we can obtain product formulas for $L_{n,k}(u,v,q)$ and $\overline{AL}_{n,k}(u,v,q)$. We summarize these formulas as follows.

Proposition 2.4. *We have*

$$L_{n,k}(u,v,q) = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1})_k}{(u^2q^{2n-k+1})_k}, \tag{4}$$

$$\overline{L}_{n,k}(u,v,q) = u^k v^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uv^{-1}q^{n-k+1})_k}{(u^2q^{2n-k+1})_k}, \tag{5}$$

$$AL_{n,k}(u,v,q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1})_k}{(u^2q^{2n-2k+2})_k}, \tag{6}$$

$$\overline{AL}_{n,k}(u,v,q) = (uvq)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uv^{-1}q^{n-k+1})_k}{(u^2q^{2n-k+1})_k}. \tag{7}$$

2.2. **Mixed moments and dual mixed moments of univariate orthogonal polynomials.** Let $\{p_n(x)\}_{n\ge0}$ be a family of monic orthogonal polynomials with linear functional $\mathcal{L}$. The *(normalized) moment* μ_n of the orthogonal polynomials is defined by

$$\mu_n = \frac{\mathcal{L}(x^n)}{\mathcal{L}(1)}.$$

The *mixed moment* $\mu_{n,k}$ of the orthogonal polynomials is defined by

$$\mu_{n,k} = \frac{\mathcal{L}(x^n p_k(x))}{\mathcal{L}(p_k(x)^2)}.$$

Note that $\mu_n = \mu_{n,0}$. Moments of orthogonal polynomials give rise to interesting combinatorics [4, 29, 31]. Since $\{p_n(x)\}_{n\ge0}$ is a basis of the polynomial space, we can write

$$x^n = \sum_{k=0}^{n} c_{n,k} p_k(x).$$

Multiplying $p_k(x)$ and taking $\mathcal{L}$, we obtain

$$c_{n,k} = \frac{\mathcal{L}(x^n p_k(x))}{\mathcal{L}(p_k(x)^2)} = \mu_{n,k}.$$

Thus, the mixed moment $\mu_{n,k}$ is the coefficient $[p_k(x)]x^n$ of $p_k(x)$ when we expand x^n in terms of the basis $\{p_n(x)\}_{n\ge0}$. We define the *dual mixed moment* $\nu_{n,k}$ of $\{p_n(x)\}_{n\ge0}$ to be the coefficient $[x^k]p_n(x)$ of x^k in $p_n(x)$.

By definition, the mixed moments $\mu_{n,k}$ and the dual mixed moments $\nu_{n,k}$ satisfy

$$x^n = \sum_{k=0}^{n} \mu_{n,k} p_k(x), \qquad p_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

Therefore, we have

$$\sum_{i=0}^{m} \mu_{m,i}\nu_{i,n} = \sum_{i=0}^{m} \nu_{m,i}\mu_{i,n} = \delta_{m,n}. \tag{8}$$

Equivalently,

$$\left(\mu_{i,j}\right)_{i,j=0}^{\infty} = \left(\left(\nu_{i,j}\right)_{i,j=0}^{\infty}\right)^{-1}.$$

2.3. **The little q-Jacobi polynomials.** The *monic little q-Jacobi polynomials* $p_n^L(x;a,b;q)$ are defined by

$$p_n^L(x;a,b;q)=\frac{(aq;q)_n}{(-1)^n q^{-\binom{n}{2}}(abq^{n+1};q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},abq^{n+1}\\ aq\end{matrix};q;qx\right), \tag{9}$$

where we use the q-hypergeometric series notation

$${}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}z^n.$$

They satisfy the following three-term recurrence relation:

$$p_{n+1}^L(x;a,b;q)=(x-b_n)p_n^L(x;a,b;q)-\lambda_n p_{n-1}^L(x;a,b;q),$$

where $b_n=A_n+C_n$, $\lambda_n=A_{n-1}C_n$ for

$$A_n=\frac{q^n(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},\qquad C_n=\frac{aq^n(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \tag{10}$$

They are orthogonal with respect to the linear functional $\mathcal{L}_{a,b}^{L,q}$ given by

$$\mathcal{L}_{a,b}^{L,q}(f(x))=\sum_{k\ge 0}\frac{(bq;q)_k}{(q;q)_k}(aq)^k f(q^k).$$

See [13] for more details on the little q-Jacobi polynomials.

The (normalized) moment $\mu_n^L(a,b;q)$ and the mixed moment $\mu_{n,k}^L(a,b;q)$ of the little q-Jacobi polynomials are given by

$$\mu_n^L(a,b;q)=\frac{\mathcal{L}_{a,b}^{L,q}(x^n)}{\mathcal{L}_{a,b}^{L,q}(1)},\qquad \mu_{n,k}^L(a,b;q)=\frac{\mathcal{L}_{a,b}^{L,q}(x^n p_k^L(x;a,b;q))}{\mathcal{L}_{a,b}^{L,q}(p_k^L(x;a,b;q)^2)}.$$

Note that $\mu_{n,0}^L(a,b;q)=\mu_n^L(a,b;q)$ and $\mu_{n,k}^L(a,b;q)=0$ if $n<k$. By the q-binomial theorem,

$$\mathcal{L}_{a,b}^{L,q}(x^n)=\frac{(abq^{n+2};q)_\infty}{(aq^{n+1};q)_\infty}. \tag{11}$$

Thus, we have a product formula for the moment $\mu_n^L(a,b;q)$:

$$\mu_n^L(a,b;q)=\frac{(aq;q)_n}{(abq^2;q)_n}.$$

The mixed moments and the dual mixed moments also have product formulas.

Lemma 2.5. *We have*

$$\mu_{n,k}^L(a,b;q)=\begin{bmatrix}n\\ k\end{bmatrix}_q\frac{(aq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}}, \tag{12}$$

$$\nu_{n,k}^L(a,b;q)=(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_q\frac{(aq^{k+1};q)_{n-k}}{(abq^{n+k+1};q)_{n-k}}. \tag{13}$$

Proof. By (9) and (11), we have

$$\begin{aligned}\mathcal{L}_{a,b}^{L,q}(x^n p_k^L(x;a,b;q))&=\mathcal{L}_{a,b}^{L,q}\left(\frac{x^n(aq;q)_k}{(-1)^k q^{-\binom{k}{2}}(abq^{k+1};q)_k}\,{}_2\phi_1\left(\begin{matrix}q^{-k},abq^{k+1}\\ aq\end{matrix};q;qx\right)\right)\\
&=\frac{(aq;q)_k}{(-1)^k q^{-\binom{k}{2}}(abq^{k+1};q)_k}\sum_{i\ge 0}\frac{(q^{-k};q)_i(abq^{k+1};q)_i}{(q;q)_i(aq;q)_i}q^i\frac{(abq^{n+i+2};q)_\infty}{(aq^{n+i+1};q)_\infty}\\
&=\frac{(aq;q)_k}{(-1)^k q^{-\binom{k}{2}}(abq^{k+1};q)_k}\cdot\frac{(abq^{n+2};q)_\infty}{(aq^{n+1};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q^{-k},abq^{k+1},aq^{n+1}\\ aq,abq^{n+2}\end{matrix};q;q\right).\end{aligned}$$

By the q-Saalschütz summation formula [11, (II.12)], we obtain

$$\mathcal{L}_{a,b}^{L,q}(x^n p_k^L(x;a,b;q)) = \frac{a^k q^{k^2}(q^{n-k+1};q)_k(bq;q)_k(abq^{n+k+2};q)_\infty}{(abq^{k+1};q)_k(aq^{n+1};q)_\infty}. \tag{14}$$

Since $p_n^L(x;a,b;q)$ are monic, we have

$$\mathcal{L}_{a,b}^{L,q}(p_k^L(x;a,b;q)^2) = \mathcal{L}_{a,b}^{L,q}(x^k p_k^L(x;a,b;q)).$$

Thus, by (14), we have

$$\frac{\mathcal{L}_{a,b}^{L,q}(x^n p_k^L(x;a,b;q))}{\mathcal{L}_{a,b}^{L,q}(p_k^L(x;a,b;q)^2)} = \frac{(q^{n-k+1};q)_k(abq^{n+k+2};q)_\infty(aq^{k+1};q)_\infty}{(q;q)_k(abq^{2k+2};q)_\infty(aq^{n+1};q)_\infty},$$

which is the same (12). The second identity (13) follows from the definition (9). $\square$

The following proposition shows that the mixed moment (resp. the dual mixed moment) of the little q-Jacobi polynomials is a generating function for anti-lecture hall compositions (resp. lecture hall partitions).

Proposition 2.6. *We have*

$$\mu_{n,k}^L(-uv,-u/v;q) = AL_{n,n-k}(u,v,q), \tag{15}$$

$$\nu_{n,k}^L(-uv,-u/v;q) = (-1)^{n-k}L_{n,n-k}(u,v,q). \tag{16}$$

Proof. Equation (15) follows from (6) and (12). Equation (16) follows from (4) and (13). $\square$

Remark 2.7. It is possible to give a different proof of (12) as follows. In [29, Chapter 1, Proposition 17], Viennot showed that the moment $\mu_{n,k}^L(a,b;q)$ is the sum of weights of certain paths in the quarter plane from $(0,0)$ to (n,k). This interpretation gives the following recurrence for $\mu_{n,k}^L(a,b;q)$. If $n>k$,

$$\mu_{n,k}^L(a,b;q) = b_k\mu_{n-1,k}^L(a,b;q) + \lambda_{k+1}\mu_{n-1,k+1}^L(a,b;q) + \mu_{n-1,k-1}^L(a,b;q),$$

where b_n and λ_n are given before (10). Since $\mu_{n,n}^L(a,b;q)=1$ and $\mu_{n,k}^L(a,b;q)=0$ for $n<k$, (12) is obtained by induction. It will be interesting to find a direct combinatorial proof of (15), which is equivalent to (12).

By (8) and Proposition 2.6, we obtain the following corollary.

Corollary 2.8. *We have*

$$\sum_{i=0}^m AL_{m,m-i}(u,v,q)(-1)^{i-n}L_{i,i-n}(u,v,q) = \delta_{m,n},$$

$$\sum_{i=0}^m (-1)^{m-i}L_{m,m-i}(u,v,q)AL_{i,i-n}(u,v,q) = \delta_{m,n}.$$

There is a simple combinatorial proof of Corollary 2.8, see Proposition 3.5.

2.4. **Multivariate orthogonal polynomials.** Suppose that $p_\lambda(x_1,\dots,x_n)$ are multivariate orthogonal polynomials with linear functional $\mathfrak{L}_n$. The *mixed moment* $M_{\lambda,\mu}(n)$ and the *(normalized) moment* $M_\lambda(n)$ of $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ are defined by

$$M_{\lambda,\mu}(n) = \frac{\mathfrak{L}_n(s_\lambda(x_1,\dots,x_n)p_\mu(x_1,\dots,x_n))}{\mathfrak{L}_n(p_\mu(x_1,\dots,x_n)^2)},$$

and

$$M_\lambda(n) := M_{\lambda,\emptyset}(n) = \frac{\mathfrak{L}_n(s_\lambda(x_1,\dots,x_n))}{\mathfrak{L}_n(1)}.$$

By the orthogonality we have

$$s_\lambda(x_1,\dots,x_n) = \sum_{\mu\in\mathcal{P}_n} M_{\lambda,\mu}(n)p_\mu(x_1,\dots,x_n).$$

We define the *dual mixed moment* $N_{\lambda,\mu}(n)$ by

$$p_\lambda(x_1,\dots,x_n)=\sum_{\mu\in\mathcal{P}_n} N_{\lambda,\mu}(n)s_\mu(x_1,\dots,x_n).$$

We need the following well known lemma.

Lemma 2.9. *Let $\{p_n(x)\}_{n\ge0}$ and $\{q_n(x)\}_{n\ge0}$ be families of polynomials with $\deg p_n(x)=\deg q_n(x)=n$ and*

$$p_n(x)=\sum_{k=0}^n c_{n,k}q_k(x).$$

Then, for a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, we have

$$\det\left(p_{\lambda_j+n-j}(x_i)\right)_{i,j=1}^n=\sum_{\mu\subseteq\lambda}\det\left(c_{\lambda_i+n-i,\mu_j+n-j}\right)_{i,j=1}^n\det\left(q_{\mu_j+n-j}(x_i)\right)_{i,j=1}^n,$$

where $c_{i,j}=0$ if $i<j$.

Proof. Observe that

$$\left(p_{\lambda_i+n-i}(x_j)\right)_{i,j=1}^n=\left(c_{\lambda_i+n-i,k}\right)_{\substack{1\le i\le n\\ k\ge0}}\left(q_k(x_j)\right)_{\substack{k\ge0\\ 1\le j\le n}}.$$

By the Cauchy-Binet theorem, we have

$$\det\left(p_{\lambda_i+n-i}(x_j)\right)_{i,j=1}^n=\sum_{\mu_1\ge\cdots\ge\mu_n\ge0}\det\left(c_{\lambda_i+n-i,\mu_j+n-j}\right)_{i,j=1}^n\det\left(q_{\mu_i+n-i}(x_j)\right)_{i,j=1}^n.$$

Since $c_{i,j}=0$ for $i<j$, the summand vanishes unless $\mu\subseteq\lambda$, which finishes the proof. □

The following proposition gives a connection between the mixed and dual mixed moments of univariate orthogonal polynomials and those of corresponding multivariate orthogonal polynomials.

Proposition 2.10. *Let $\{p_i(x)\}_{i\ge0}$ be a family of univariate orthogonal polynomials. Suppose that $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ is a family of multivariate orthogonal polynomials defined by* (1). *Then the mixed moments $M_{\lambda,\mu}(n)$ and the dual mixed moments $N_{\lambda,\mu}(n)$ for $\{p_\lambda(x_1,\dots,x_n)\}_{\lambda\in\mathcal{P}_n}$ can be expressed in terms of the mixed moments $\mu_{n,k}$ and the dual mixed moments $\nu_{n,k}$ for $\{p_i(x)\}_{i\ge0}$ as follows:*

$$M_{\lambda,\mu}(n)=\det\left(\mu_{\lambda_i+n-i,\mu_j+n-j}\right)_{i,j=1}^n, \tag{17}$$

$$N_{\lambda,\mu}(n)=\det\left(\nu_{\lambda_i+n-i,\mu_j+n-j}\right)_{i,j=1}^n. \tag{18}$$

In particular, $M_{\lambda,\mu}(n)=N_{\lambda,\mu}(n)=0$ unless $\mu\subseteq\lambda$.

Proof. By Lemma 2.9 and the fact

$$x^n=\sum_{k=0}^n\mu_{n,k}p_k(x),$$

we have

$$\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n=\sum_{\mu\subseteq\lambda}\det\left(\mu_{\lambda_i+n-i,\mu_j+n-j}\right)_{i,j=1}^n\det\left(p_{\mu_j+n-j}(x_i)\right)_{i,j=1}^n.$$

Dividing both sides by $\Delta(x)$, we obtain (17). By the same arguments, we obtain (18). □

3. Lecture hall tableaux and lecture hall Schur functions

In this section we define lecture hall tableaux and their multivariate generating functions called lecture hall Schur functions. We then study their combinatorial aspects.

3.1. **Lecture hall tableaux.** For a cell (i,j) in λ, the *content* $c(i,j)$ is defined by $c(i,j)=j-i$.

Definition 3.1. Let $\prec_1$ and $\prec_2$ be inequalities in $\{<,\le,>,\ge\}$. For an integer n and partitions $\mu\subseteq\lambda$ with $\ell(\lambda)\le n$, a *lecture hall tableau of shape* λ/μ *of type* $(n,\prec_1,\prec_2)$ is a filling T of the cells in the Young diagram of λ/μ with nonnegative integers satisfying the following conditions:

$$\frac{T(i,j)}{n+c(i,j)}\prec_1\frac{T(i,j+1)}{n+c(i,j+1)},\qquad \frac{T(i,j)}{n+c(i,j)}\prec_2\frac{T(i+1,j)}{n+c(i+1,j)}.$$

We denote by $\mathrm{LHT}_{(n,\prec_1,\prec_2)}(\lambda/\mu)$ the set of such fillings.

See Figure 1 for an example of a lecture hall tableau of type $(n,\ge,>)$.

For $T\in\mathrm{LHT}_{(n,\prec_1,\prec_2)}(\lambda/\mu)$, the *weight* $\mathrm{wt}(T)$ is defined by

$$\mathrm{wt}(T)=\prod_{s\in\lambda/\mu}x_{T(s)}u^{\lfloor T(s)/(n+c(s))\rfloor}v^{o(\lfloor T(s)/(n+c(s))\rfloor)}.$$

Let $\mathbf{x}=(x_0,x_1,\dots)$. For a multivariate function $f(\mathbf{x})=f(x_0,x_1,\dots)$ we denote $f(\mathbf{q})=f(1,q,q^2,\dots)$. For a sequence $\alpha=(\alpha_1,\dots,\alpha_n)$ of nonnegative integers, we denote $\mathbf{x}_\alpha=x_{\alpha_1}\cdots x_{\alpha_n}$.

Definition 3.2. The *complete homogeneous lecture hall function* $h_k^{(n)}=h_k^{(n)}(\mathbf{x};u,v)$ is defined by

$$h_k^{(n)}(\mathbf{x};u,v)=\sum_\alpha \mathbf{x}_\alpha u^{|\lfloor\alpha\rfloor_S|}v^{o(\lfloor\alpha\rfloor_S)},$$

where $S=(n,n+1,\dots,n+k-1)$ and the sum is over all sequences $\alpha=(\alpha_1,\dots,\alpha_k)$ of integers satisfying

$$\frac{\alpha_1}{n}\ge\frac{\alpha_2}{n+1}\ge\cdots\ge\frac{\alpha_k}{n+k-1}\ge 0.$$

The *elementary lecture hall function* $e_k^{(n)}=e_k^{(n)}(\mathbf{x};u,v)$ is defined by

$$e_k^{(n)}(\mathbf{x};u,v)=\sum_\lambda \mathbf{x}_\lambda u^{|\lfloor\lambda\rfloor_S|}v^{o(\lfloor\lambda\rfloor_S)},$$

where $S=(n,n-1,\dots,n-k+1)$ and the sum is over all sequences $\lambda=(\lambda_1,\dots,\lambda_k)$ of integers satisfying

$$\frac{\lambda_1}{n}>\frac{\lambda_2}{n-1}>\cdots>\frac{\lambda_k}{n-k+1}\ge 0.$$

Note that

$$h_k^{(n)}(\mathbf{x};u,v)=\sum_{\alpha\in AL_{n+k-1,k}}\mathbf{x}_\alpha u^{|\lfloor\alpha\rfloor|}v^{o(\lfloor\alpha\rfloor)},\qquad e_k^{(n)}(\mathbf{x};u,v)=\sum_{\lambda\in L_{n,k}}\mathbf{x}_\lambda u^{|\lfloor\lambda\rfloor|}v^{o(\lfloor\lambda\rfloor)}.$$

Recall the complete homogeneous symmetric functions

$$h_k(\mathbf{x})=\sum_{i_1\ge\cdots\ge i_k\ge 0}x_{i_1}\cdots x_{i_k}$$

and the elementary symmetric functions

$$e_k(\mathbf{x})=\sum_{i_1>\cdots>i_k\ge 0}x_{i_1}\cdots x_{i_k}.$$

We have the following connections between these objects.

Proposition 3.3. *We have*

$$\lim_{n\to\infty}h_k^{(n)}(\mathbf{x};u,v)=h_k(\mathbf{x}),\tag{19}$$

$$\lim_{n\to\infty}e_k^{(n)}(\mathbf{x};u,v)=e_k(\mathbf{x}),\tag{20}$$

$$h_k^{(n)}(\mathbf{x};0,0)=h_k(x_0,x_1,\dots,x_{n-1}),\tag{21}$$

$$e_k^{(n)}(\mathbf{x};0,0)=e_k(x_0,x_1,\dots,x_{n-1}).\tag{22}$$

Proof. We will only prove (19) and (21) because (20) and (22) can be proved similarly.

First, we claim that for integers i, j, N with $0 \le i < N$, we have $\frac{i}{N} \ge \frac{j}{N+1}$ if and only if $i \ge j$. The "if part" of the claim is clear. For the "only if part", suppose that $\frac{i}{N} \ge \frac{j}{N+1}$ and $i < j$. Then $\frac{N+1}{N} \ge \frac{j}{i} \ge \frac{i+1}{i}$, which is a contradiction to the assumption $i < N$.

We now prove (19). By definition,

$$h_k^{(n)}(\mathbf{x}; u, v) = \sum_{\alpha} \mathbf{x}_\alpha u^{|\lfloor\alpha\rfloor|} v^{o(\lfloor\alpha\rfloor)},$$

where the sum is over all compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ satisfying

$$\frac{\alpha_1}{n} \ge \frac{\alpha_2}{n+1} \ge \cdots \ge \frac{\alpha_k}{n+k-1} \ge 0. \tag{23}$$

By the above claim, as $n \to \infty$, the condition (23) is equivalent to $\alpha_1 \ge \cdots \ge \alpha_k \ge 0$, which implies (19).

To prove (21), observe that since $|\lfloor\alpha\rfloor| = \lfloor\alpha_1/n\rfloor + \cdots + \lfloor\alpha_k/(n+k-1)\rfloor$, we have

$$h_k^{(n)}(\mathbf{x}; 0, 0) = \sum_{\alpha} \mathbf{x}_\alpha,$$

where the sum is over all compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ satisfying

$$1 > \frac{\alpha_1}{n} \ge \frac{\alpha_2}{n+1} \ge \cdots \ge \frac{\alpha_k}{n+k-1} \ge 0. \tag{24}$$

By the claim again, the condition (24) is equivalent to $n > \alpha_1 \ge \cdots \ge \alpha_k \ge 0$, which implies (21). □

By definition, we have $h_k^{(n)}(\mathbf{q}; u, v) = AL_{n+k-1,k}(u, v, q)$ and $e_k^{(n)}(\mathbf{q}; u, v) = L_{n,k}(u, v, q)$. Thus, we can rewrite Proposition 2.6 as follows.

Proposition 3.4. *We have*

$$\mu^L_{n,k}(-uv, -u/v; q) = AL_{n,n-k}(u, v, q) = h^{(k+1)}_{n-k}(\mathbf{q}; u, v),$$
$$\nu^L_{n,k}(-uv, -u/v; q) = (-1)^{n-k} L_{n,n-k}(u, v, q) = (-1)^{n-k} e^{(n)}_{n-k}(\mathbf{q}; u, v).$$

By (8) and Proposition 3.4,

$$\sum_{i=0}^{m} h^{(i+1)}_{m-i}(\mathbf{q}; u, v)(-1)^{i-n} e^{(i)}_{i-n}(\mathbf{q}; u, v) = \sum_{i=0}^{m} (-1)^{m-i} e^{(m)}_{m-i}(\mathbf{q}; u, v) h^{(n+1)}_{i-n}(\mathbf{q}; u, v) = \delta_{m,n}.$$

The following proposition is a multivariate analog of the above equations.

Proposition 3.5. *We have*

$$\sum_{i=0}^{m} h^{(i+1)}_{m-i}(\mathbf{x}; u, v)(-1)^{i-n} e^{(i)}_{i-n}(\mathbf{x}; u, v) = \delta_{m,n}, \tag{25}$$

$$\sum_{i=0}^{m} (-1)^{m-i} e^{(m)}_{m-i}(\mathbf{x}; u, v) h^{(n+1)}_{i-n}(\mathbf{x}; u, v) = \delta_{m,n}. \tag{26}$$

Proof. Let $f(m, n)$ be the left hand side of (25). If $m < n$, we have $f(m, n) = 0$. If $m \ge n$,

$$f(m, n) = \sum_{i=0}^{m} \sum_{\alpha,\lambda} (-1)^{i-n} \mathbf{x}_\alpha \mathbf{x}_\lambda u^{|\lfloor\alpha\rfloor| + |\lfloor\lambda\rfloor|} v^{o(\lfloor\alpha\rfloor) + o(\lfloor\lambda\rfloor)},$$

where the second sum is over all sequences α and λ of integers such that

$$\frac{\alpha_1}{i+1} \ge \frac{\alpha_2}{i+2} \ge \cdots \ge \frac{\alpha_{m-i}}{m} \ge 0,$$
$$\frac{\lambda_1}{i} > \frac{\lambda_2}{i-1} > \cdots > \frac{\lambda_{i-n}}{n+1} \ge 0.$$

Moving the first element of λ to α or vice versa gives a sign-reversing involution, which shows that $f(m, n) = \delta_{m,n}$. The second identity (26) can be proved similarly. □

Replacing n by $m-n$, we can rewrite (25) as

$$\sum_{i=0}^{n}(-1)^i e_i^{(m)}(\mathbf{x};u,v)h_{n-i}^{(m-n+1)}(\mathbf{x};u,v)=\delta_{n,0}.$$

If $m\to\infty$, the above equation becomes the well known identity

$$\sum_{i=0}^{n}(-1)^i e_i(\mathbf{x})h_{n-i}(\mathbf{x})=\delta_{n,0}.$$

3.2. **Lecture hall Schur functions.** For partitions $\mu\subseteq\lambda$ with $\ell(\lambda)\le n$ and inequalities $\prec_1,\prec_2$, we define the *lecture hall Schur function* $LS_{\lambda/\mu}^{(n,\prec_1,\prec_2)}(\mathbf{x};u,v)$ by

$$LS_{\lambda/\mu}^{(n,\prec_1,\prec_2)}(\mathbf{x};u,v)=\sum_{T\in\mathrm{LHT}_{(n,\prec_1,\prec_2)}(\lambda/\mu)}\mathrm{wt}(T).$$

By definition, we have

$$h_k^{(n)}(\mathbf{x};u,v)=LS_{(k)}^{(n,\ge,>)}(\mathbf{x};u,v),\qquad e_k^{(n)}(\mathbf{x};u,v)=LS_{(1^k)}^{(n,\ge,>)}(\mathbf{x};u,v).$$

Using the same arguments as in the proof of Proposition 3.3, we can prove the following proposition.

Proposition 3.6. *Suppose that $\succ_1$ and $\succ_2$ are any inequalities in $\{>,\ge\}$, and $\prec_1$ and $\prec_2$ are any inequalities in $\{<,\le\}$. Then for partitions λ and μ with $\mu\subseteq\lambda$ and $\ell(\lambda)\le n$,*

$$\lim_{n\to\infty}LS_{\lambda/\mu}^{(n,\succ_1,\succ_2)}(\mathbf{x};u,v)=s_{\lambda/\mu}(\mathbf{x}),\tag{27}$$

$$\lim_{n\to\infty}LS_{\lambda/\mu}^{(n,\prec_1,\prec_2)}(\mathbf{x};u,v)=s_{\lambda'/\mu'}(\mathbf{x}),\tag{28}$$

$$LS_{\lambda}^{(n,\succ_1,\succ_2)}(\mathbf{x};0,0)=s_\lambda(x_0,x_1,\dots,x_{n-1}).\tag{29}$$

Remark 3.7. Note that (29) is not true if we use a skew shape λ/μ or the other inequalities $\prec_1$ and $\prec_2$. For every lecture hall tableau contributing a nonzero weight in $LS_{\lambda}^{(n,\succ_1,\succ_2)}(\mathbf{x};0,0)$, the entry in the cell $(1,1)$ is the largest. Thus every entry is at most $n-1$. On the contrary $LS_{\lambda/\mu}^{(n,\succ_1,\succ_2)}(\mathbf{x};0,0)$ and $LS_{\lambda}^{(n,\prec_1,\prec_2)}(\mathbf{x};0,0)$ do not have such a property.

There are Jacobi-Trudi type formulas for $LS_{\lambda/\mu}^{(n,\ge,>)}(\mathbf{x};u,v)$ and $LS_{\lambda/\mu}^{(n,<,\le)}(\mathbf{x};u,v)$.

Theorem 3.8. *Let λ and μ be partitions with $\ell(\lambda)\le n$ and $\mu\subseteq\lambda$. Then*

$$LS_{\lambda/\mu}^{(n,\ge,>)}(\mathbf{x};u,v)=\det\left(h_{\lambda_i-\mu_j-i+j}^{(n-j+1+\mu_j)}(\mathbf{x};u,v)\right)_{i,j=1}^{\ell(\lambda)},\tag{30}$$

$$LS_{\lambda/\mu}^{(n,\ge,>)}(\mathbf{x};u,v)=\det\left(e_{\lambda'_i-\mu'_j-i+j}^{(n+j-1-\mu'_j)}(\mathbf{x};u,v)\right)_{i,j=1}^{\ell(\lambda')}.\tag{31}$$

Theorem 3.9. *Let λ and μ be partitions with $\ell(\lambda)\le n$ and $\mu\subseteq\lambda$. Then*

$$LS_{\lambda/\mu}^{(n,<,\le)}(\mathbf{x};u,v)=\det\left(e_{\lambda_i-\mu_j-i+j}^{(n-i+\lambda_i)}(\mathbf{x};u,v)\right)_{i,j=1}^{\ell(\lambda)},\tag{32}$$

$$LS_{\lambda/\mu}^{(n,<,\le)}(\mathbf{x};u,v)=\det\left(h_{\lambda'_i-\mu'_j-i+j}^{(n+i-\lambda'_i)}(\mathbf{x};u,v)\right)_{i,j=1}^{\ell(\lambda')}.\tag{33}$$

Note that if $n\to\infty$ in Theorems 3.8 and 3.9, we obtain the usual Jacobi-Trudi formula for $s_{\lambda/\mu}(\mathbf{x})$. In the next subsection we prove Theorem 3.8. The proof of Theorem 3.9 is similar and thus omitted.

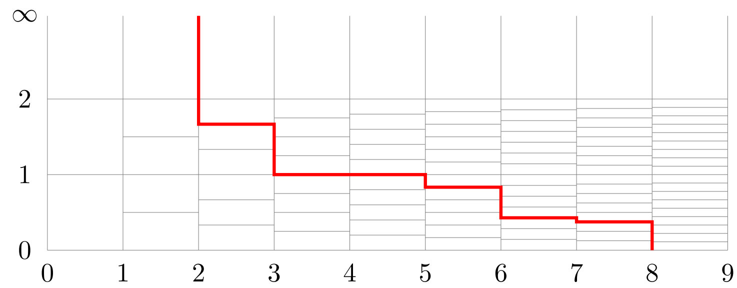

FIGURE 2. A path in $\mathrm{NW}((8,0),(2,\infty))$. This path corresponds to the anti-lecture hall composition $\alpha = (\alpha_1,\dots,\alpha_6) = (5,4,5,5,3,3)$, which satisfy $\frac{\alpha_1}{3} \ge \frac{\alpha_2}{4} \ge \frac{\alpha_3}{5} \ge \frac{\alpha_4}{6} \ge \frac{\alpha_5}{7} \ge \frac{\alpha_6}{8} \ge 0$.

3.3. **Lecture hall lattice paths.** We define the *lecture hall lattice* to be the infinite graph $G = (V, E)$, where

$$V = \bigcup_{i\ge 1, j\ge 0} \left\{ \left(i-1, \frac{j}{i}\right), \left(i, \frac{j}{i}\right) \right\}$$

and two distinct vertices $(a,b),(c,d) \in V$ with $a \le c$ are adjacent if one of the following conditions holds:

- $a = c-1$ and $b = d$,
- $a = c$ and there is no vertex $(a,e) \in V$ with $b < e < d$ or $d < e < b$.

We define the *strict lecture hall lattice* to be the infinite graph $G_s = (V_s, E_s)$, where

$$V_s = \bigcup_{i\ge 1, j\ge 0} \left\{ \left(i-1, \frac{j}{i} - \frac{1}{i^2}\right), \left(i, \frac{j}{i}\right) \right\}$$

and two distinct vertices $(a,b),(c,d) \in V_s$ with $a \le c$ are adjacent if one of the following conditions holds:

- $a = c-1$ and $b = d - 1/c^2$,
- $a = c$ and there is no vertex (a,e) with $b < e < d$ or $d < e < b$.

A *north step* is a pair (u,v) of adjacent vertices in G or G_s of the form $u = (a,b)$ and $v = (a,c)$ with $b < c$. An *east step* (resp. *west step*) is a pair (u,v) of adjacent vertices in G of the form $u = (a,b)$ and $v = (a+1,b)$ (resp. $v = (a-1,b)$). A *northeast step* (resp. *southwest step*) is a pair (u,v) of adjacent vertices in G_s of the form $u = (a,b)$ and $v = (a+1, b + \frac{1}{(a+1)^2})$ (resp. $v = (a-1, b - \frac{1}{a^2})$).

An *NW-path* (resp. *NE-path*) from A to B is a sequence $(u_1,u_2,\dots,u_k)$ of vertices in V (resp. V_s) such that $u_1 = A$, $u_k = B$ and (u_i,u_{i+1}) is a north step or a west step (resp. a north step, or a northeast step) for $1 \le i \le k-1$. For vertices A and B in V (resp. V_s), we denote by $\mathrm{NW}(A,B)$ (resp. $\mathrm{NE}(A,B)$) the set of NW-paths (resp. NE-paths) from A to B. If $B = (b,\infty)$, we define $\mathrm{NW}(A,B)$ (resp. $\mathrm{NE}(A,B)$) to be the set of infinite sequences $(u_1,u_2,\dots)$, also called NW-paths (resp. NE-paths), such that $u_1 = A$, $\lim_{k\to\infty} u_k = B$ and $(u_1,\dots,u_k)$ is an NW-path (resp. NE-path) for each k.

For $0 \le k \le n$, consider an NW-path $p \in \mathrm{NW}((n,0),(k,\infty))$. For $1 \le i \le n-k$, let $w_i = ((a_i,b_i),(a_i-1,b_i))$ be the ith leftmost west step in p. Define $\alpha = (\alpha_1,\dots,\alpha_{n-k})$ to be the composition given by $\alpha_i = a_i b_i$. Note that α_i can be considered as the number of regions in the lecture hall lattice below the step w_i. It is easy to see that the map $p \mapsto \alpha$ is a bijection from $\mathrm{NW}((n,0),(k,\infty))$ to the set $AL_{n,n-k}$ of anti-lecture hall compositions $\alpha = (\alpha_1,\dots,\alpha_{n-k})$ satisfying

$$\frac{\alpha_1}{k+1} \ge \frac{\alpha_2}{k+2} \ge \cdots \ge \frac{\alpha_{n-k}}{n} \ge 0.$$

See Figure 2 for an example of this correspondence.

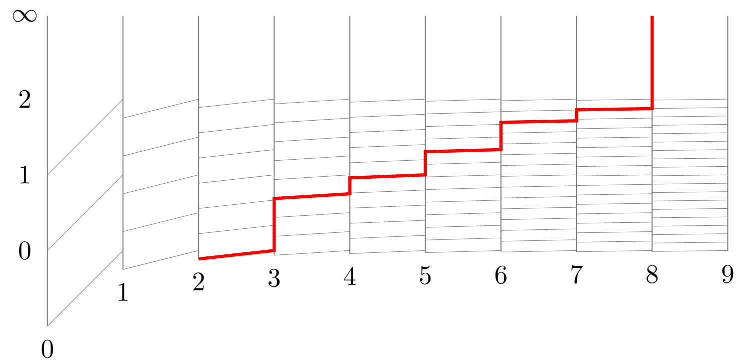

FIGURE 3. A path in $\mathrm{NE}((2,-1/3^2),(8,\infty))$. This path corresponds to the lecture hall partition $\lambda=(\lambda_1,\dots,\lambda_6)=(15,12,8,5,3,0)$ satisfying $\frac{\lambda_1}{8}>\frac{\lambda_2}{7}>\frac{\lambda_3}{6}>\frac{\lambda_4}{5}>\frac{\lambda_5}{4}>\frac{\lambda_6}{3}\ge 0$.

For $p\in \mathrm{NW}((n,0),(k,\infty))$, we define

$$\mathrm{wt}(p)=\prod_{w=((i,j),(i-1,j))} x_{ij}u^{\lfloor j\rfloor}v^{o(\lfloor j\rfloor)},$$

where the product is over all west steps w in p. Note that if α is the anti-lecture hall composition corresponding to p, we have $\mathrm{wt}(p)=x_\alpha u^{\lfloor\alpha\rfloor}v^{o(\lfloor\alpha\rfloor)}$. We also define

$$W_{i,j}=\sum_{p\in\mathrm{NW}((i,0),(j,\infty))}\mathrm{wt}(p).$$

By the correspondence between $\mathrm{NW}((i,0),(j,\infty))$ and $AL_{i,i-j}$, we have

$$W_{i,j}=h^{(j+1)}_{i-j}(\mathbf{x};u,v). \tag{34}$$

Now consider an NE-path $p\in\mathrm{NE}((k,-1/(k+1)^2),(n,\infty))$. For $1\le i\le n-k$, let $e_i=((a_i-1,b_i'),(a_i,b_i))$ be the ith rightmost step among all northeast steps in p. Define $\lambda=(\lambda_1,\dots,\lambda_{n-k})$ to be the partition given by $\lambda_i=a_ib_i$. Note that λ_i can be considered as the number of regions in the strict lecture hall lattice below the step e_i. Similarly to the NW-path case, one can check that the map $p\mapsto\lambda$ is a bijection from $\mathrm{NE}((k,-1/(k+1)^2),(n,\infty))$ to the set $L_{n,n-k}$ of lecture hall partitions $\lambda=(\lambda_1,\dots,\lambda_{n-k})$ satisfying

$$\frac{\lambda_1}{n}>\frac{\lambda_2}{n-1}>\cdots>\frac{\lambda_{n-k}}{k+1}\ge 0.$$

See Figure 3 for an example of this correspondence.

For $p\in\mathrm{NE}((k,-1/(k+1)^2),(n,\infty))$, we define

$$\mathrm{wt}(p)=\prod_{e=((i-1,j'),(i,j))} x_{ij}u^{\lfloor j\rfloor}v^{o(\lfloor j\rfloor)},$$

where the product is over all east or northeast steps e in p. Note that if λ is the lecture hall partition corresponding to p, we have $\mathrm{wt}(p)=\mathbf{x}_\lambda u^{\lfloor\lambda\rfloor}v^{o(\lfloor\lambda\rfloor)}$. We also define

$$E_{i,j}=\sum_{p\in\mathrm{NE}((i,-1/(i+1)^2),(j,\infty))}\mathrm{wt}(p).$$

By the correspondence between $\mathrm{NE}((i,-1/(i+1)^2),(j,\infty))$ and $L_{j,j-i}$, we have

$$E_{i,j}=e^{(j)}_{j-i}(\mathbf{x};u,v). \tag{35}$$

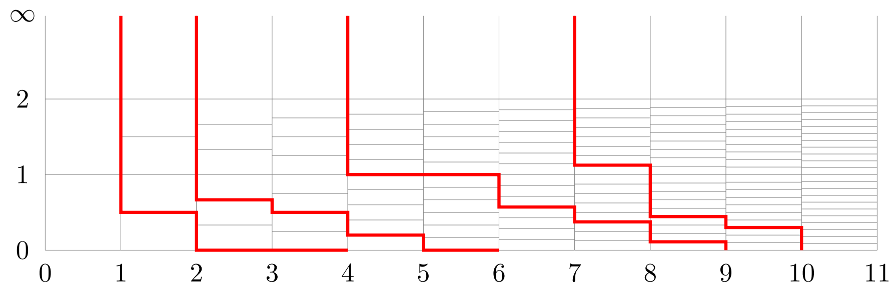

FIGURE 4. The non-intersecting NW-paths corresponding to the lecture hall tableau in Figure 1.

Lemma 3.10. *Let $\mu = (\mu_1, \dots, \mu_n) \subseteq \lambda = (\lambda_1, \dots, \lambda_n)$. For $1 \le i \le n$, let $A_i = (\lambda_i + n - i, 0)$ and $B_i = (\mu_i + n - i, \infty)$. Then*

$$LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = \sum_{(p_1,\dots,p_n)\in W} \mathrm{wt}(p_1)\cdots \mathrm{wt}(p_n),$$

where W is the set of n-tuples $(p_1, \dots, p_n)$ of non-intersecting NW-paths with $p_i \in \mathrm{NW}(A_i, B_i)$ for $1 \le i \le n$.

Proof. We construct a bijection $\phi : \mathrm{LHT}_{(n,\ge,>)}(\lambda/\mu) \to W$ as follows. Let $T \in \mathrm{LHT}_{(n,\ge,>)}(\lambda/\mu)$. The ith row of T satisfies

$$\frac{T_{i,\mu_i+1}}{\mu_i + n - i + 1} \ge \frac{T_{i,\mu_i+2}}{\mu_i + n - i + 2} \ge \cdots \ge \frac{T_{i,\lambda_i}}{\lambda_i + n - i}.$$

By the same arguments deriving (34), the ith row T corresponds to the path $p_i \in \mathrm{NW}(A_i, B_i)$ whose kth leftmost west step is $((r, s), (r - 1, s))$, where $r = \mu_i + n - i + k$ and $s = T_{i,\mu_i+k}/(\mu_i + n - i + k)$. Then we define $\phi(T) = (p_1, \dots, p_n)$, see Figure 4. Since $T \in \mathrm{LHT}_{(n,\ge,>)}(\lambda/\mu)$, we have $(p_1, \dots, p_n) \in N$. It is easy to see that the map ϕ is a bijection and $\mathrm{wt}(T) = \mathrm{wt}(p_1)\cdots \mathrm{wt}(p_n)$. This completes the proof. □

The following lemma is a dual version of Lemma 3.10.

Lemma 3.11. *Let $\mu = (\mu_1, \dots, \mu_n) \subseteq \lambda = (\lambda_1, \dots, \lambda_n)$ and $\ell' = \ell(\lambda') = \lambda_1$. For $1 \le i \le \ell'$, let $A_i = (n - \lambda'_i + i - 1, 0)$ and $B_i = (n - \mu'_i + i - 1, \infty)$. Then*

$$LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = \sum_{(p_1,\dots,p_{\ell'})\in E} \mathrm{wt}(p_1)\cdots \mathrm{wt}(p_{\ell'}),$$

where E is the set of n-tuples $(p_1, \dots, p_{\ell'})$ of non-intersecting NE-paths with $p_i \in \mathrm{NE}(A_i, B_i)$ for $1 \le i \le \ell'$.

Proof. This can be proved by the same arguments as in the proof of Lemma 3.10 except that we make the NE-path p_i from the entries of the ith column of $T \in \mathrm{LHT}_{(n\ge,>)}(\lambda/\mu)$, see Figure 5. We omit the details. □

Proof of Theorem 3.8. By Lemma 3.10 and the Lindström–Gessel–Viennot lemma, we have

$$LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = \det(W_{\lambda_i+n-i,\mu_j+n-j})^n_{i,j=1}.$$

By (34), we obtain (30). Similarly, by Lemma 3.11, we have

$$LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x}; u, v) = \det(E_{n-\lambda'_i+i-1,n-\mu'_j+j-1})^{\ell(\lambda')}_{i,j=1}.$$

By (35), we obtain (31). □

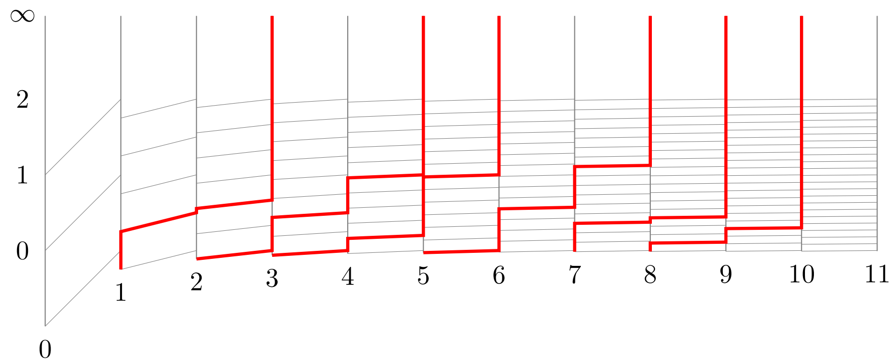

Figure 5. The non-intersecting NE-paths corresponding to the lecture hall tableau in Figure 1.

4. Multivariate little q-Jacobi polynomials

In this section we define multivariate little q-Jacobi polynomials and find a combinatorial interpretation for their mixed and dual mixed moments.

For $0<q<1$, the *q-integral* is defined by

$$\int_0^a f(x) d_q x = (1-q)\sum_{n=0}^\infty f(aq^n)aq^n,$$

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

Let $x=(x_1,\dots,x_n)$. For $f,g\in\mathbb{C}[x_1,\dots,x_n]$, $0<a<1/q$, $b<1/q$ and $0<t<1$, we define

$$\langle f,g\rangle_{L,t}^{a,b} = \int_{x_1=0}^1\int_{x_2=0}^{tx_1}\cdots\int_{x_n=0}^{tx_{n-1}} f(x)g(x)v(x;a,b,t)d_qx,$$

where

$$v(x;a,b,t)=\Delta(x)\prod_{i=1}^n x_i^\alpha \frac{(qx_i)_\infty}{(qbx_i)_\infty}\prod_{1\le i<j\le n} x_i^{2\tau-1}(q^{1-\tau}x_j/x_i)_{2\tau-1},$$

$a=q^\alpha$ and $t=q^\tau$.

The *multivariate little q-Jacobi polynomials* $\{p_\lambda^L(x;a,b;q,t):\lambda\in\mathcal{P}_n\}$ are defined by the following conditions:

(1) $p_\lambda^L(t)=m_\lambda+\sum_{\mu<\lambda} d_{\lambda,\mu}(t)m_\mu$, where $<$ is the dominance order and m_λ is the monomial symmetric function,
(2) $\langle p_\lambda^L(t),m_\mu\rangle_{L,t}^{a,b}=0$ if $\mu<\lambda$.

It is known [28, Theorem 5.1] that $p_\lambda^L(x;a,b;q,t)$ is obtained as a limit transition from the Koornwinder polynomial [14], which is the BC_n-type Macdonald polynomial generalizing the Askey-Wilson polynomial.

In this paper we consider the case $t=q$ of the multivariate little q-Jacobi polynomials, i.e.,

$$p_\lambda^L(x_1,\dots,x_n;a,b;q) := p_\lambda^L(x_1,\dots,x_n;a,b;q,q).$$

In this case $p_\lambda^L(x_1,\dots,x_n;a,b;q)$ is orthogonal with respect to the linear functional $\mathfrak{L}_{a,b}^{L,q}$ given the following formula, see [27, (5.19)]:

$$\mathfrak{L}_{a,b}^{L,q}(f(x)) = \int_{[0,1]^n} f(x)\Delta(x)^2\prod_{i=1}^n x_i^\alpha\frac{(qx_i)_\infty}{(qbx_i)_\infty}d_qx, \qquad (a=q^\alpha). \tag{36}$$

Stokman [27, Proposition 5.9] showed that if $t = q$, the multivariate little q-Jacobi polynomial can be written as a determinant of little q-Jacobi polynomials:

$$p^L_\lambda(x_1,\dots,x_n;a,b;q) = \frac{\det\left(p^L_{\lambda_j+n-j}(x_i;a,b;q)\right)_{i,j=1}^n}{\Delta(x)}.$$

Thus we can consider the mixed moment $M^L_{\lambda,\mu}(n;a,b;q)$ and the dual mixed moment $N^L_{\lambda,\mu}(n;a,b;q)$ of the multivariate little q-Jacobi polynomials $p^L_\lambda(x_1,\dots,x_n;a,b;q)$. They satisfy

$$s_\lambda(x_1,\dots,x_n) = \sum_{\mu\subseteq\lambda} M^L_{\lambda,\mu}(n;a,b;q)p^L_\mu(x_1,\dots,x_n;a,b;q), \tag{37}$$

$$p^L_\lambda(x_1,\dots,x_n;a,b;q) = \sum_{\mu\subseteq\lambda} N^L_{\lambda,\mu}(n;a,b;q)s_\mu(x_1,\dots,x_n). \tag{38}$$

The following theorem implies that the mixed moments and the dual mixed moments are generating functions for lecture hall tableaux.

Theorem 4.1. *For a partition λ with at most n parts, we have*

$$N^L_{\lambda,\mu}(n;-uv,-u/v;q) = (-1)^{|\lambda/\mu|}LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{q};u,v),$$
$$M^L_{\lambda,\mu}(n;-uv,-u/v;q) = LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{q};u,v).$$

Equivalently,

$$p^L_\lambda(x_1,\dots,x_n;-uv,-u/v;q) = \sum_{\mu\subseteq\lambda}(-1)^{|\lambda/\mu|}LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{q};u,v)s_\mu(x_1,\dots,x_n),$$
$$s_\lambda(x_1,\dots,x_n) = \sum_{\mu\subseteq\lambda} LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{q};u,v)p^L_\mu(x_1,\dots,x_n;-uv,-u/v;q).$$

Proof. By (38), in order to prove the first identity it suffices to show

$$N^L_{\lambda,\mu}(n;-uv,-u/v;q) = (-1)^{|\lambda/\mu|}LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{q};u,v).$$

By (18), Proposition 3.4 and Theorem 3.9,

$$\begin{aligned}N^L_{\lambda,\mu}(n;-uv,-u/v;q) &= \det\left(\nu_{\lambda_i+n-i,\mu_j+n-j}(-uv,-u/v;q)\right)_{i,j=1}^n\\ &= \det\left((-1)^{\lambda_i-\mu_j-i+j}e^{(\lambda_i+n-i)}_{\lambda_i-\mu_j-i+j}(\mathbf{q},u,v)\right)_{i,j=1}^n\\ &= (-1)^{|\lambda/\mu|}LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{q};u,v),\end{aligned}$$

which establishes the first identity. The second identity can be proved similarly. □

By taking the limit $n\to\infty$ in Theorem 4.1, we obtain an infinite variable symmetric function.

Corollary 4.2. *There is an infinite variable polynomial $p^L_\lambda(x_1,\dots;a,b;q)$ such that*

$$p^L_\lambda(x_1,\dots;a,b;q) = \lim_{n\to\infty} p^L_\lambda(x_1,\dots,x_n;a,b;q)$$

and

$$p^L_\lambda(x_1,\dots;a,b;q) = \sum_{\mu\subseteq\lambda}(-1)^{|\lambda/\mu|}s_{\lambda'/\mu'}(1,q,q^2,\dots)s_\mu(x_1,\dots).$$

5. Moments and dual moments of multivariate little q-Jacobi polynomials

In this section we prove product formulas for moments and dual moments of the multivariate little q-Jacobi polynomials.

The *moment* $M^L_\lambda(n;a,b;q)$ of the multivariate little q-Jacobi polynomials $p^L_\lambda(x_1,\dots,x_n;a,b;q)$ is defined by

$$M^L_\lambda(n;a,b;q) = M^L_{\lambda,\emptyset}(n;a,b;q) = \frac{\mathfrak{L}^{L,q}_{a,b}(s_\lambda(x_1,\dots,x_n))}{\mathfrak{L}^{L,q}_{a,b}(1)},$$

where $\mathfrak{L}^{L,q}_{a,b}$ is given in (36).

The following is a Selberg-type integral due to Kadell [12], see also [30, Corollary 1.3]:

$$\begin{aligned}(39)\quad &\frac{[n]_q!}{n!}\int_{[0,1]^n} s_\lambda(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^2\prod_{i=1}^n x_i^{\alpha-1}\frac{(qx_i;q)_\infty}{(q^\beta x_i;q)_\infty}d_qx_1\dots d_qx_n\\ &= q^{\alpha\binom{n}{2}+2\binom{n}{3}}\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{\Gamma_q(\alpha+n-i+\lambda_i)\Gamma_q(\beta+i-1)\Gamma_q(i+1)}{\Gamma_q(\alpha+\beta+2n-i-1+\lambda_i)},\end{aligned}$$

where $\Gamma_q(z)=(1-q)^{1-z}(q;q)_\infty/(q^z;q)_\infty$. We refer the reader to [10] for more information on the Selberg integral.

Using (39) we obtain a product formula for the moment $M^L_\lambda(n;a,b;q)$.

Theorem 5.1. *For a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, we have*

$$M^L_\lambda(n;a,b;q)=\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{(aq^{n-i+1})_{\lambda_i}}{(abq^{2n-i+1})_{\lambda_i}}.$$

Proof. Let $a=q^\alpha$ and $b=q^\beta$. By (36) and (39),

$$\begin{aligned}M^L_\lambda(n;a,b;q)&=\frac{\mathfrak{L}^{L,q}_{a,b}(s_\lambda(x_1,\dots,x_n))}{\mathfrak{L}^{L,q}_{a,b}(s_\emptyset(x_1,\dots,x_n))}\\ &=\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{\Gamma_q(\alpha+n-i+1+\lambda_i)\Gamma_q(\alpha+\beta+2n-i+1)}{\Gamma_q(\alpha+1+n-i)\Gamma_q(\alpha+\beta+2n-i+1+\lambda_i)}.\end{aligned}$$

This is the same as the desired identity. ◻

We note that the connection between the Jacobi polynomials and the Selberg integral has been observed by Aomoto [1] and further studied by many people, see for example [16, 18, 23, 24, 27].

By Theorems 4.1 and 5.1, we obtain a product formula for $LS^{(n,\ge,>)}_\lambda(\mathbf{q};u,v)$.

Corollary 5.2. *For a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, we have*

$$LS^{(n,\ge,>)}_\lambda(\mathbf{q};u,v)=\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{(-uvq^{n-i+1})_{\lambda_i}}{(u^2q^{2n-i+1})_{\lambda_i}}.$$

Note that by Proposition 3.6, if we set $u=v=0$ in Corollary 5.2, we obtain the well known identity for the principal specialization of the Schur function:

$$s_\lambda(1,q,\dots,q^{n-1})=\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}.$$

We now consider the *dual moment* $N^L_\lambda(n;a,b;q)$ defined by

$$N^L_\lambda(n;a,b;q)=N^L_{\lambda,\emptyset}(n;a,b;q).$$

Since

$$p^L_\lambda(x_1,\dots,x_n;a,b;q)=\sum_{\mu\subseteq\lambda}N^L_{\lambda,\mu}(n;a,b;q)s_\mu(x_1,\dots,x_n),$$

the dual moment $N^L_\lambda(n;a,b;q)$ is the constant term $p^L_\lambda(0,\dots,0;a,b;q)$ of $p^L_\lambda(x_1,\dots,x_n;a,b;q)$.

We also have a product formula for the dual moment $N^L_\lambda(n;a,b;q)$.

Theorem 5.3. *For a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, we have*

$$\begin{aligned}N^L_\lambda(n;a,b;q)=(-1)^{|\lambda/\mu|}q^{n(\lambda')-n(\lambda)}&\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{(aq^{n-i+1})_{\lambda_i}}{(abq^{n-i+1+\lambda_i})_{n-i+\lambda_i}}\\ &\times\prod_{1\le i<j\le n}(1-abq^{2n+\lambda_i+\lambda_j-i-j+1}),\end{aligned}$$

where $n(\lambda)=\sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$.

Before proving the above theorem, we present its corollary. By Theorems 4.1 and 5.3, we obtain a product formula for $LS_\lambda^{(n,<,\le)}(\mathbf{q};u,v)$.

Corollary 5.4. *For a partition* $\lambda=(\lambda_1,\dots,\lambda_n)$*, we have*

$$LS_\lambda^{(n,<,\le)}(\mathbf{q};u,v)=q^{n(\lambda')-n(\lambda)}\prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}}\prod_{i=1}^n\frac{(-uvq^{n-i+1})_{\lambda_i}}{(u^2q^{n-i+1+\lambda_i})_{n-i+\lambda_i}}\times\prod_{1\le i<j\le n}(1-u^2q^{2n+\lambda_i+\lambda_j-i-j+1}).$$

In order to prove Theorem 5.3, we need the following lemma.

Lemma 5.5. *We have*

$$\det\left(\frac{1}{(ax_j)_i(b/x_j)_i}\right)_{i,j=1}^n=(-1)^nb^{-n^2}q^{-\binom{n+1}{3}}x_1\cdots x_n\frac{\prod_{1\le i<j\le n}(b-aq^{n-1}x_ix_j)(x_j-x_i)}{\prod_{j=1}^n(ax_j)_n(q^{1-n}b^{-1}x_j)_n}. \tag{40}$$

Proof. One can check that (40) is equivalent to

$$\det\left(\frac{x_j^{i-1}}{(ax_j)_i(q^{n-i}bx_j)_i}\right)_{i,j=1}^n=\frac{\prod_{1\le i<j\le n}(1-abq^{n-1}x_ix_j)(x_j-x_i)}{\prod_{j=1}^n(ax_j)_n(bx_j)_n}. \tag{41}$$

We will prove the following identity, which is equivalent to (41):

$$\det\left(x_j^{i-1}(aq^ix_j)_{n-i}(bx_j)_{n-i}\right)_{i,j=1}^n=\prod_{1\le i<j\le n}(1-abq^{n-1}x_ix_j)(x_j-x_i). \tag{42}$$

Let $f(x)$ (resp. $g(x)$) be the left (resp. right) hand side of (42). Since $f(x)=0$ whenever $x_i=x_j$ for $i\ne j$, it has $\prod_{1\le i<j\le n}(x_j-x_i)$ as a factor. Moreover, if $x_i=1/abq^{n-1}x_j$, one can check that the ith column is equal to the jth column multiplied by $(ab)^{1-n}q^{-(n-1)^2}x_j^2$. Thus $f(x)$ is a multiple of $g(x)$. Since $\deg f(x)=\deg g(x)=3\binom{n}{2}$ and their coefficients of $x_1^0x_2^1\dots x_n^{n-1}$ are equal to 1, we obtain $f(x)=g(x)$. □

We note that Lemma 5.5 is equivalent to [15, Theorem 27].

Proof of Theorem 5.3. By (18) and (13),

$$\begin{aligned}N_\lambda^L(n;-uv,-u/v;q)&=\det\left(\nu^L_{\lambda_i+n-i,\mu_j+n-j}(-uv,-u/v;q)\right)_{i,j=1}^n\\&=\det\left((-1)^{\lambda_i-\mu_j-i+j}q^{\binom{\lambda_j-j+i}{2}}\begin{bmatrix}n-j+\lambda_j\\ \lambda_j-j+i\end{bmatrix}_q\frac{(-uvq^{n-i+1})_{\lambda_j-j+1}}{(u^2q^{2n+1-i+\lambda_j-j})_{\lambda_j-j+i}}\right)_{i,j=1}^n\\&=(-1)^{|\lambda/\mu|}\prod_{i=1}^n\frac{q^{\binom{\lambda_i-i}{2}+\binom{i}{2}}(q)_{n-i+\lambda_i}(-uvq^{n-i+1})_{\lambda_i}}{(q)_{\lambda_i-i}(q)_{n-i}(u^2q^{2n+1+\lambda_i-i})_{\lambda_i-i}}\det\left(\frac{q^{i(\lambda_j-j)}}{(q^{\lambda_j-j+1})_i(u^2q^{2n+1-i+\lambda_j-j})_i}\right)_{i,j=1}^n.\end{aligned}$$

Using Lemma 5.5 with $x_j=q^{\lambda_j-j}$, we obtain the desired formula. □

Remark 5.6. Note that our proofs of Theorem 5.1 and Theorem 5.3 are different in nature. We used a Selberg-type integral to prove Theorem 5.1 and a determinant evaluation to prove Theorem 5.3. The proof technique of one identity is not obviously applicable to the other identity. It would be interesting to find another proof of each identity.

Converting $M_\lambda^L(n;a,b;q)$ as a determinant in Theorem 5.1, we obtain the following determinant evaluation.

Proposition 5.7. *We have*

$$\det\left(\frac{x_j^i(b/x_j)_i}{(ax_j)_i}\right)_{i,j=1}^n=\prod_{i=1}^n\frac{(abq^i)_{i-1}(x_i-b)}{(ax_i)_n}\prod_{1\le i<j\le n}(x_j-x_i).$$

The above identity is equivalent to

$$\det\left(x_j^i (b/x_j)_i (aq^i x_j)_{n-i}\right)_{i,j=1}^n = \prod_{i=1}^n (abq^i)_{i-1}(x_i - b) \prod_{1\le i<j\le n}(x_j - x_i). \tag{43}$$

Note that (43) cannot be shown by just identifying the roots because there are multiple roots for a and b. If we consider the both sides as polynomials in x_i's, their degrees are n^2 and $\binom{n+1}{2}$. Thus the same method does not work either. It will be interesting to find a direct proof of Proposition 5.7.

6. Enumeration of lecture hall tableaux of other types

In Sections 3 and 5, we obtained some enumeration results for lecture hall tableaux of types $(n, \ge, >)$ and $(n, <, \le)$. In this section we prove similar results for lecture hall tableaux of types $(n, >, \ge)$ and $(n, \le, <)$.

Let $T \in \mathrm{LHT}_{(n,\prec_1,\prec_2)}(\lambda/\mu)$, where $\prec_1$ and $\prec_2$ are fixed inequalities. Recall that $\mathrm{wt}(T)$ is defined by

$$\mathrm{wt}(T) = \prod_{s\in\lambda/\mu} x_{T(s)} u^{\lfloor T(s)/(n+c(s))\rfloor} v^{o(\lfloor T(s)/(n+c(s))\rfloor)}.$$

We also define $\overline{\mathrm{wt}}(T)$ by

$$\overline{\mathrm{wt}}(T) = \prod_{s\in\lambda/\mu} x_{T(s)} u^{\lceil T(s)/(n+c(s))\rceil} v^{o(\lceil T(s)/(n+c(s))\rceil)}.$$

For example, if T is the lecture hall tableau in Figure 1,

$$\mathrm{wt}(T) = x_0^3 x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_9 u^3 v^3,$$
$$\overline{\mathrm{wt}}(T) = x_0^3 x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_9 u^{13} v^{11}.$$

Recall $h_k^{(n)}(\mathbf{x}; u, v)$ and $e_k^{(n)}(\mathbf{x}; u, v)$ in Definition 3.2, where $\mathbf{x} = (x_0, x_1, \dots)$ is a sequence of variables.

Definition 6.1. We define

$$\overline{h}_k^{(n)}(\mathbf{x}; u, v) = \sum_\alpha \mathbf{x}_\alpha u^{|\lceil\alpha\rceil_S|} v^{o(\lceil\alpha\rceil_S)},$$

where $S = (n, n+1, \dots, n+k-1)$ and the sum is over all sequences $\alpha = (\alpha_1, \dots, \alpha_k)$ of integers satisfying

$$\frac{\alpha_1}{n} > \frac{\alpha_2}{n+1} > \cdots > \frac{\alpha_k}{n+k-1} > 0.$$

We define

$$\overline{e}_k^{(n)}(\mathbf{x}; u, v) = \sum_\lambda \mathbf{x}_\lambda u^{|\lceil\lambda\rceil_S|} v^{o(\lceil\lambda\rceil_S)},$$

where $S = (n, n-1, \dots, n-k+1)$ and the sum is over all sequences $\lambda = (\lambda_1, \dots, \lambda_k)$ of integers satisfying

$$\frac{\lambda_1}{n} \ge \frac{\lambda_2}{n-1} \ge \cdots \ge \frac{\lambda_k}{n-k+1} > 0.$$

By definition, we have

$$\overline{h}_k^{(n)}(\mathbf{x}; u, v) = \sum_{\alpha\in\overline{AL}_{n+k-1,k}} \mathbf{x}_\alpha u^{|\lceil\alpha\rceil|} v^{o(\lceil\alpha\rceil)}, \qquad \overline{e}_k^{(n)}(\mathbf{x}; u, v) = \sum_{\lambda\in\overline{L}_{n,k}} \mathbf{x}_\lambda u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)}.$$

Observe that the variable x_0 in $\mathbf{x}$ is never used in $\overline{h}_k^{(n)}(\mathbf{x}; u, v)$ and $\overline{e}_k^{(n)}(\mathbf{x}; u, v)$.

For a sequence of variables $\mathbf{x} = (x_0, x_1, \dots)$, we define $\mathbf{x}^+ = (x_0^+, x_1^+, \dots)$, where $x_i^+ = x_{i+1}$. In fact, $\mathbf{x}^+$ is the same as $(x_1, x_2, \dots)$. However, in order to emphasis that the index of the sequence begins with 0 we use $\mathbf{x}^+ = (x_0^+, x_1^+, \dots)$. The following lemma is an immediate consequence of the map $\lambda \mapsto \lambda^+$ in Proposition 2.3.

Lemma 6.2. *Let $\mathbf{x} = (x_0, x_1, \dots)$ be a sequence of variables. Then for integers $n \ge k \ge 0$, we have*

$$\overline{h}_k^{(n)}(\mathbf{x}; u, v) = u^k v^k h_k^{(n)}(\mathbf{x}^+; u, v^{-1}),$$
$$\overline{e}_k^{(n)}(\mathbf{x}; u, v) = u^k v^k e_k^{(n)}(\mathbf{x}^+; u, v^{-1}).$$

Let $\prec_1$ and $\prec_2$ be any inequalities in $\{>, <, \ge, \le\}$. Recall that lecture hall tableaux in $\mathrm{LHT}_{\lambda/\mu}^{(n,\prec_1,\prec_2)}$ may have entries equal to 0. We define $\overline{\mathrm{LHT}}_{\lambda/\mu}^{(n,\prec_1,\prec_2)}$ to be the set of lecture hall tableaux in $\mathrm{LHT}_{\lambda/\mu}^{(n,\prec_1,\prec_2)}$ all of whose entries are positive. We also define

$$\overline{LS}_{\lambda/\mu}^{(n,\prec_1,\prec_2)}(\mathbf{x}; u, v) = \sum_{T \in \overline{\mathrm{LHT}}_{(n,\prec_1,\prec_2)}(\lambda/\mu)} \overline{\mathrm{wt}}(T).$$

Note that

$$\overline{h}_k^{(n)}(\mathbf{x}; u, v) = \overline{LS}_{(k)}^{(n,>,\ge)}(\mathbf{x}; u, v), \qquad \overline{e}_k^{(n)}(\mathbf{x}; u, v) = \overline{LS}_{(1^k)}^{(n,>,\ge)}(\mathbf{x}; u, v).$$

For any lecture hall tableau T, we define T^+ to be the tableau obtained from T by increasing every entry by 1. By Lemma 2.2, the map $T \mapsto T^+$ gives a bijection from $\mathrm{LHT}_{\lambda/\mu}^{(n,\ge,>)}$ to $\overline{\mathrm{LHT}}_{\lambda/\mu}^{(n,>,\ge)}$ and a bijection from $\mathrm{LHT}_{\lambda/\mu}^{(n,<,\le)}$ to $\overline{\mathrm{LHT}}_{\lambda/\mu}^{(n,\le,<)}$. Therefore we obtain a relation between lecture hall Schur functions as follows.

Proposition 6.3. *Let λ and μ be partitions with $\ell(\lambda) \le n$ and $\mu \subseteq \lambda$. Then*

$$\overline{LS}_{\lambda/\mu}^{(n,>,\ge)}(\mathbf{x}; u, v) = (uv)^{|\lambda/\mu|} LS_{\lambda/\mu}^{(n,\ge,>)}(\mathbf{x}^+; u, v^{-1}),$$
$$\overline{LS}_{\lambda/\mu}^{(n,\le,<)}(\mathbf{x}; u, v) = (uv)^{|\lambda/\mu|} LS_{\lambda/\mu}^{(n,<,\le)}(\mathbf{x}^+; u, v^{-1}).$$

Recall that in Theorems 3.8 and 3.9 we have Jacobi-Trudi type formulas for $LS_{\lambda/\mu}^{(n,\ge,>)}(\mathbf{x}; u, v)$ and $LS_{\lambda/\mu}^{(n,<,\le)}(\mathbf{x}; u, v)$. Combining these results with Lemma 6.2 and Proposition 6.3, we obtain Jacobi-Trudi type formulas for $\overline{LS}_{\lambda/\mu}^{(n,>,\ge)}(\mathbf{x}; u, v)$ and $\overline{LS}_{\lambda/\mu}^{(n,\le,<)}(\mathbf{x}; u, v)$.

Theorem 6.4. *Let λ and μ be partitions with $\ell(\lambda) \le n$ and $\mu \subseteq \lambda$. Then*

$$\overline{LS}_{\lambda/\mu}^{(n,>,\ge)}(\mathbf{x}; u, v) = \det\left(\overline{h}_{\lambda_i-\mu_j-i+j}^{(n-j+1+\mu_j)}(\mathbf{x}; u, v)\right)_{i,j=1}^{\ell(\lambda)} = \det\left(\overline{e}_{\lambda'_i-\mu'_j-i+j}^{(n+j-1-\mu'_j)}(\mathbf{x}; u, v)\right)_{i,j=1}^{\ell(\lambda')},$$
$$\overline{LS}_{\lambda/\mu}^{(n,\le,<)}(\mathbf{x}; u, v) = \det\left(\overline{e}_{\lambda_i-\mu_j-i+j}^{(n-i+\lambda_i)}(\mathbf{x}; u, v)\right)_{i,j=1}^{\ell(\lambda)} = \det\left(\overline{h}_{\lambda'_i-\mu'_j-i+j}^{(n+i-\lambda'_i)}(\mathbf{x}; u, v)\right)_{i,j=1}^{\ell(\lambda')}.$$

In Corollaries 5.2 and 5.4, we have product formulas for $LS_{\lambda}^{(n,\ge,>)}(\mathbf{q}; u, v)$ and $LS_{\lambda}^{(n,<,\le)}(\mathbf{q}; u, v)$. By Proposition 6.3, we obtain product formulas for $\overline{LS}_{\lambda}^{(n,>,\ge)}(\mathbf{q}; u, v)$ and $\overline{LS}_{\lambda}^{(n,\le,<)}(\mathbf{q}; u, v)$.

Theorem 6.5. *For a partition λ with $\ell(\lambda) \le n$,*

$$\overline{LS}_{\lambda}^{(n,>,\ge)}(\mathbf{q}; u, v) = (uvq)^{|\lambda|} \prod_{1 \le i < j \le n} \frac{q^{\lambda_j+n-j} - q^{\lambda_i+n-i}}{q^{i-1} - q^{j-1}} \prod_{i=1}^{n} \frac{(-uv^{-1}q^{n-i+1})_{\lambda_i}}{(u^2 q^{2n-i+1})_{\lambda_i}}, \tag{44}$$

$$\overline{LS}_{\lambda}^{(n,\le,<)}(\mathbf{q}; u, v) = (uvq)^{|\lambda|} q^{n(\lambda')-n(\lambda)} \prod_{1 \le i < j \le n} \frac{q^{\lambda_j+n-j} - q^{\lambda_i+n-i}}{q^{i-1} - q^{j-1}} \times \prod_{i=1}^{n} \frac{(-uv^{-1}q^{n-i+1})_{\lambda_i}}{(u^2 q^{n-i+1+\lambda_i})_{n-i+\lambda_i}} \prod_{1 \le i < j \le n} (1 - u^2 q^{2n+\lambda_i+\lambda_j-i-j+1}). \tag{45}$$

7. Further study

A lot of natural questions arise from these lecture hall tableaux. We list a few of them in this section.

(1) In this paper, we study the case $q = t$ of the multivariate little q-Jacobi polynomials. Stokman defined the (q,t)-analog in [27]. Do we get nice combinatorics if we set $t = q^k$? Can we use the $t = q^k$ version of Warnaar's q-Selberg integral in this setting [30]?

(2) There is a lot of recent activities around the enumeration of skew (semi)-standard Young tableaux [19]. Naruse [21] found a subtraction-free formula for the number of standard Young tableaux of shape λ/μ. Morales, Pak and Panova [19] proved the following q-analog of Naruse's result:

$$s_{\lambda/\mu}(1,q,q^2,\dots) = \sum_{S\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus S} \frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}},$$

where $s_{\lambda/\mu}(1,q,q^2,\dots)$ is the principal specialization of the skew Schur function, the elements in $\mathcal{E}_{\lambda/\mu}$ are certain subsets of λ/μ called excited diagrams and $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$. In our setting, when $n\to\infty$, both $LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x};u,v)$ and $LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{x};u,v)$ converge to the principal specialization of a skew Schur function. It is therefore natural to ask whether there exist Naruse-type formulas for $LS^{(n,\ge,>)}_{\lambda/\mu}(\mathbf{x};u,v)$ and $LS^{(n,<,\le)}_{\lambda/\mu}(\mathbf{x};u,v)$.

(3) The lecture hall tableaux define lecture hall tableau polytopes. When $n\to\infty$, these polytopes are the Gelfand-Tsetlin polytopes, which are known to have nice properties [9]. The lecture hall polytopes were also studied by different authors. For a survey of those results, see [25]. Do the lecture hall tableau polytopes inherit nice properties of these families of polytopes?

(4) Given a sequence $a = (a_1, a_2, \dots)$, an integer $n \ge 0$ and a partition λ, the a-lecture hall tableaux can be defined as fillings T of the diagram of λ such that

$$\frac{T_{i,j}}{a_{n+j-i}} \ge \frac{T_{i,j+1}}{a_{n+j+1-i}}; \quad \frac{T_{i,j}}{a_{n+j-i}} \ge \frac{T_{i+1,j}}{a_{n+j-1-i}}.$$

Here we study the case $a = (1,2,3,\dots)$. Are there natural sequences? In the case of lecture hall partitions, Bousquet-Mélou and Eriksson [3] showed that, for example, given $\ell\ge 2$, the sequence with $a_i = \ell a_{i-1} - a_{i-2}$ for all i gives beautiful generating functions. Savage and Visontai [26] studied a-Eulerian polynomials coming from a-lecture hall partitions. Can we build a tableau analog of a-Eulerian polynomials?

(5) In [29] Viennot found a combinatorial interpretation for $\mu_{n,k,\ell} = \frac{\mathcal{L}(x^n p_k p_\ell)}{\mathcal{L}(p_\ell^2)}$ in terms of Motzkin paths. Since

$$x^n = \sum_{k=0}^{n} \mu_{n,k} p_k(x), \qquad p_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k,$$

we have

$$x^n p_k = \sum_{i=0}^{k} \nu_{k,i} x^{n+i} = \sum_{i=0}^{k} \nu_{k,i} \sum_{\ell=0}^{n+i} \mu_{n+i,\ell}\cdot p_\ell.$$

Multiplying both sides by p_ℓ and taking $\mathcal{L}$, we obtain

$$\mu_{n,k,\ell} = \sum_{i=\max(n-\ell,0)}^{k} \nu_{k,i}\mu_{n+i,\ell}.$$

Note that $\mu_{0,k,\ell} = \delta_{k,\ell}$ is the orthogonality of the polynomials, whose general version for the little q-Jacobi polynomials is proved combinatorially in Proposition 3.5. Is there any nice combinatorial interpretation for $\mu_{n,k,\ell}$ for the little q-Jacobi polynomials in terms of lecture hall partitions? Is there a multivariate analog?

(6) Last but not least, this phenomena that multivariate (dual) moments give rise to interesting combinatorics seems to be a "universal" phenomena. In [5], a combinatorial interpretation for the multivariate moments of the Koornwinder polynomials at $q = t$ for a specific λ, which gives a positivity result for the Koornwinder moments. They conjectured that the positivity is true for general λ, see [8]. It is also known that Macdonald polynomials are Schur positive; but no combinatorial model is known to prove this. It would be great to have a general combinatorics theory for multivariate (dual) moments of orthogonal polynomials.

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