Determinants in Enumerative Combinatorics

Christian Krattenthaler

Universität Wien

A frequent email exchange ...

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Date: Wed, 21 Jan 2015 11:46:40 +0100 (CET)

From: Christian Krattenthaler

To: Richard Brualdi <laa_rabmath.wisc.edu>

Subject: Re: Reviewer Invitation for LAA-D-15-00074

> Ms. Ref. No.: LAA-D-15-00074
> Title: Evaluation of a Remarkable Family of
> Determinants
> Linear Algebra and its Applications

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This is not so remarkable ... The authors prove a special case of Proposition 1 in "Advanced Determinant Calculus" (the roles of i and j in the determinant have to be interchanged) in a very roundabout way.

With best wishes, Christian

ADVANCED DETERMINANT CALCULUS

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Dedicated to the pioneer of determinant evaluations (among many other things), George Andrews

Abstract. The purpose of this article is threefold. First, it provides the reader with a few useful and efficient tools which should enable her/him to evaluate nontrivial determinants for the case such a determinant should appear in her/his research. Second, it lists a number of such determinants that have been already evaluated, together with explanations which tell in which contexts they have appeared. Third, it points out references where further such determinant evaluations can be found.

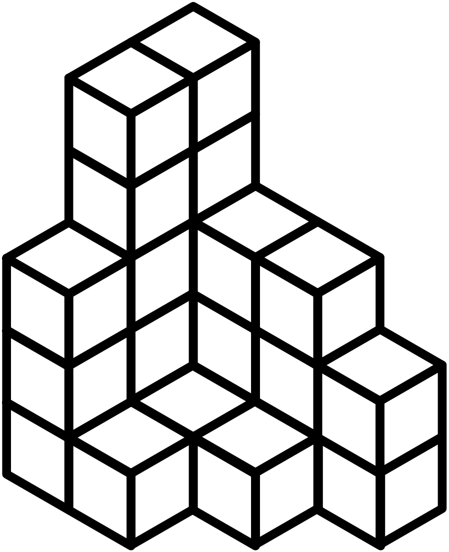
1. Introduction

Imagine, you are working on a problem. As things develop it turns out that, in order to solve your problem, you need to evaluate a certain determinant. Maybe your determinant is

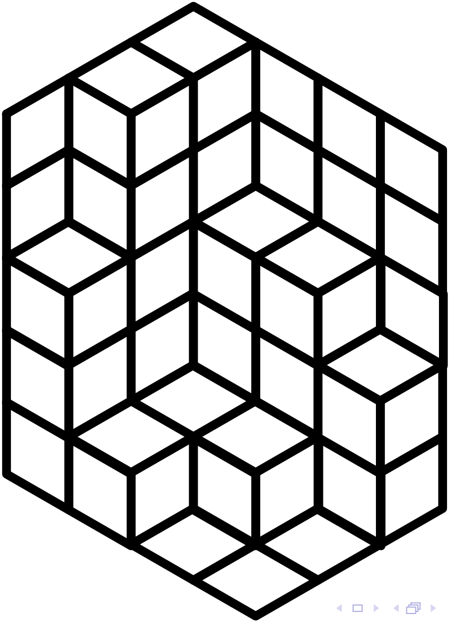
$$\det_{1\le i,j,\le n}\left(\frac{1}{i+j}\right), \tag{1.1}$$

Why determinants?

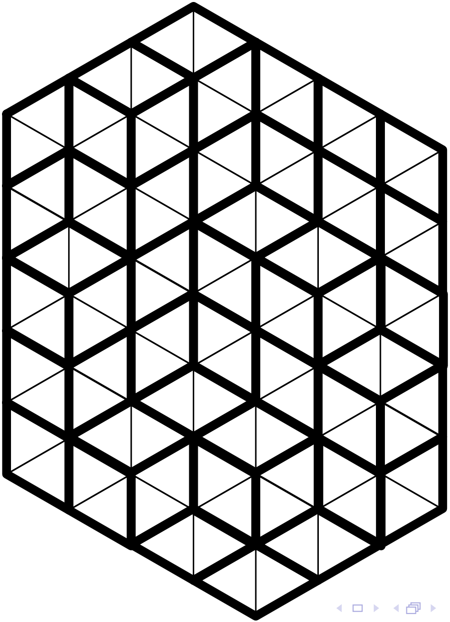
Plane Partitions



Plane Partitions ⟶ Rhombus Tilings



Plane Partitions $\longrightarrow$ Rhombus Tilings



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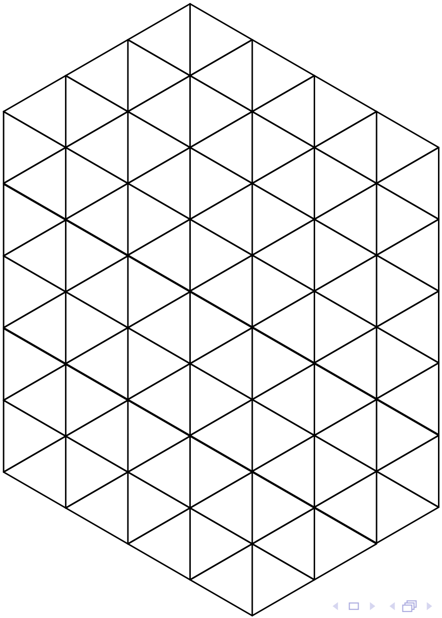

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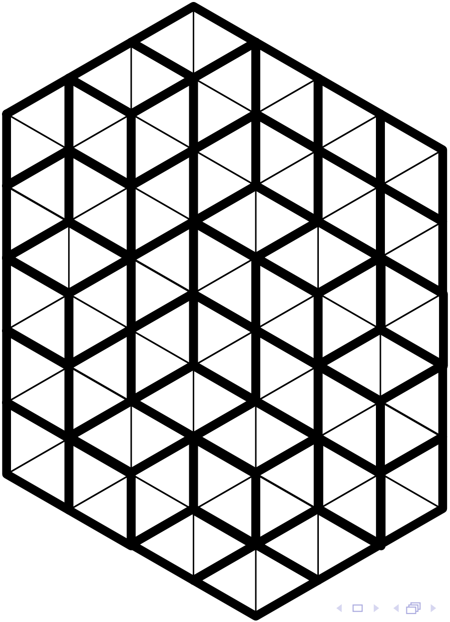

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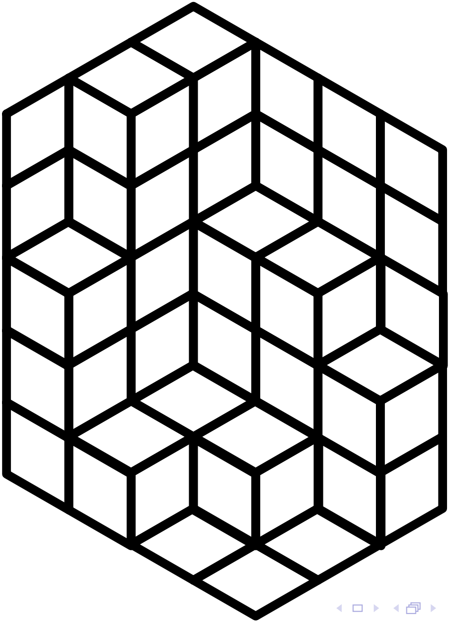
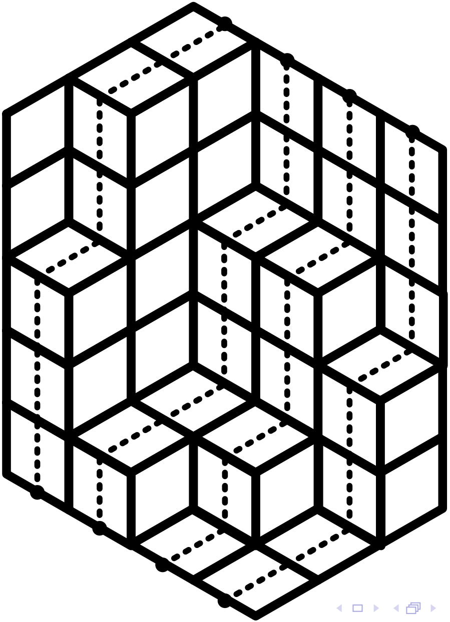


Rhombus Tilings $\longrightarrow$ Non-intersecting Lattice Paths



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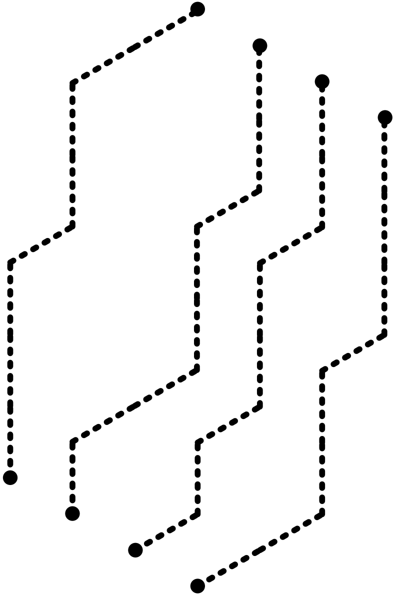

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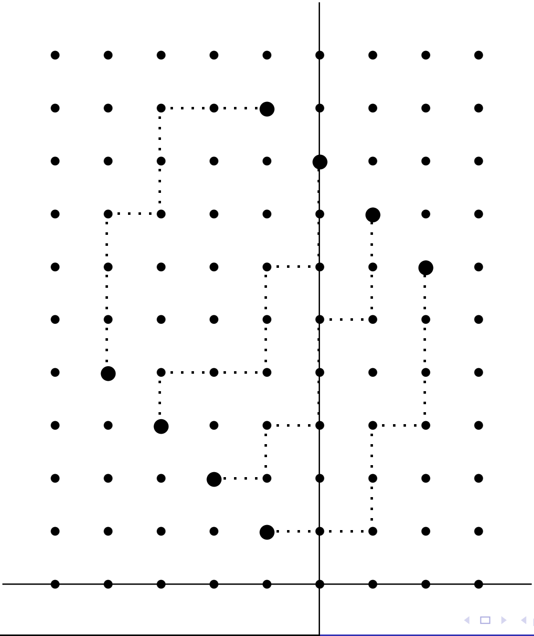

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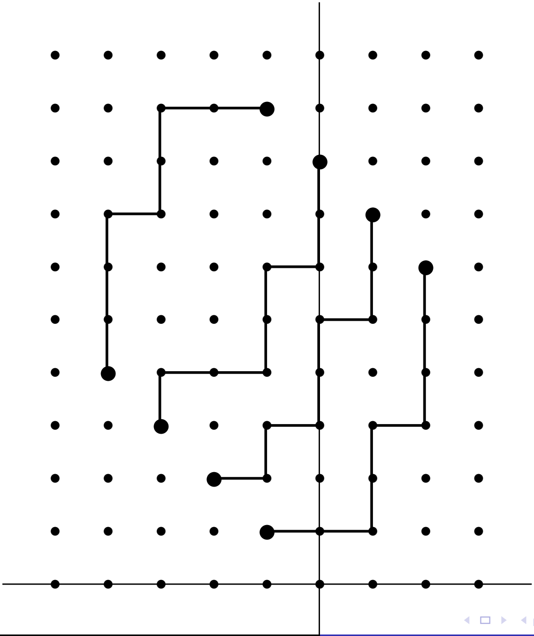
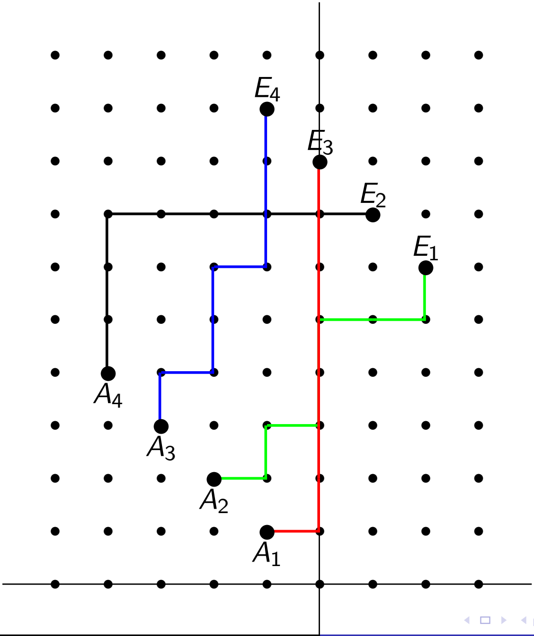
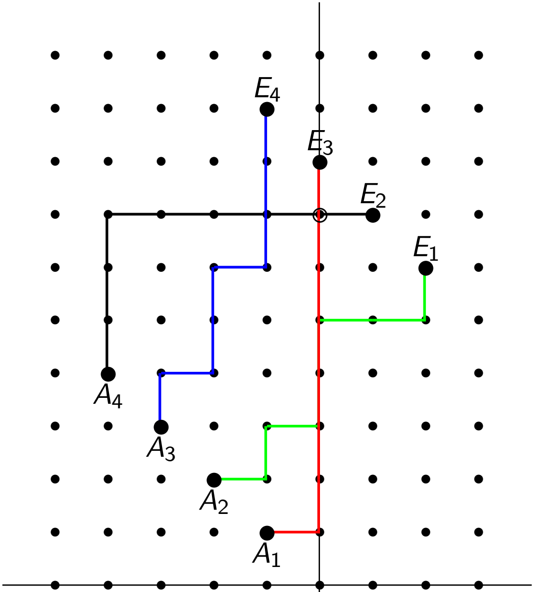


Non-intersecting Lattice Paths



$\sigma = 2413$

Non-intersecting Lattice Paths



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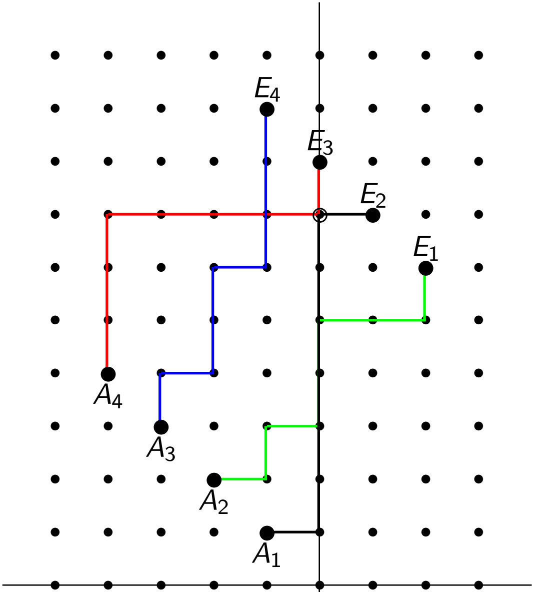
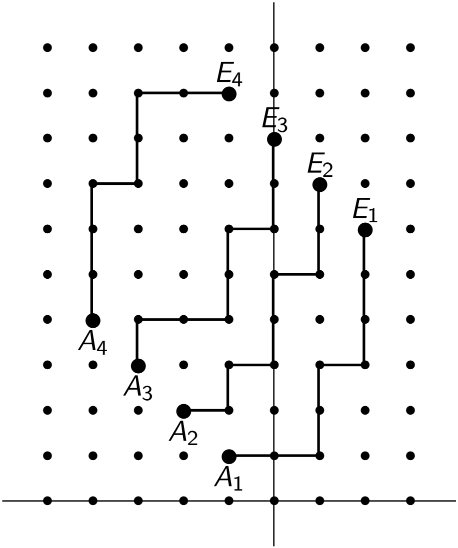
Non-intersecting Lattice Paths

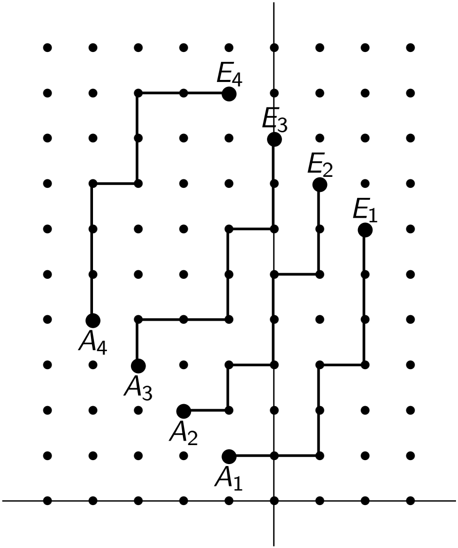


$\sigma = 2143$

Non-intersecting Lattice Paths



Non-intersecting Lattice Paths



We have $A_i = (-i, i)$ and $E_i = (a - i, c + i)$, $i = 1, 2, \ldots, b$.

How to evaluate determinants?

How to evaluate determinants?

"Method" 0. try simple-minded things; row/column operations etc.

Method 1. take out as many factors as possible until something polynomial remains; match with one of the lemmas in ADC I

Method 2. CONDENSATION

Method 3. IDENTIFICATION OF FACTORS

Method 4. LU-FACTORISATION

Method 5. THE HOLONOMIC ANSATZ

Method 6. HANKEL DETERMINANTS

“Method” 0: Simple-Minded Things

Method 1: Take factors out; match with ADC I

EXAMPLE 1. Let us consider

$$\det_{1\le i,j\le n}\left(\binom{s+i-1}{t+j-1}\right).$$

(This is from another such email exchange with Richard Brualdi.)

requires that the eigenvalues of the matrix are "nice"; see [47, 48, 84, 93, 192] for examples where that worked). Otherwise, maybe something from Sections 2.8 or 3 helps?

A final remark: It was indicated that some of the methods require that your determinant contains (more or less) parameters. Therefore it is always a good idea to:

Introduce more parameters into your determinant!

(We address this in more detail in the last paragraph of Section 2.1.) The more parameters you can play with, the more likely you will be able to carry out the determinant evaluation. (Just to mention a few examples: The condensation method needs, at least, two parameters. The "identification of factors" method needs, at least, one parameter, as well as the differential/difference equation method in Section 2.5.)

2.1. A FEW STANDARD DETERMINANTS. Let us begin with a short proof of the *Vandermonde determinant evaluation*

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\prod_{1\le i<j\le n}(X_j-X_i). \tag{2.1}$$

Although the following proof is well-known, it makes still sense to quickly go through it because, by extracting the essence of it, we will be able to build a very powerful method out of it (see Section 2.4).

If $X_{i_1}=X_{i_2}$ with $i_1\ne i_2$, then the Vandermonde determinant (2.1) certainly vanishes because in that case two rows of the determinant are identical. Hence, $(X_{i_1}-X_{i_2})$ divides the determinant as a polynomial in the X_i's. But that means that the complete product $\prod_{1\le i<j\le n}(X_j-X_i)$ (which is exactly the right-hand side of (2.1)) must divide the determinant.

On the other hand, the determinant is a polynomial in the X_i's of degree at most $\binom{n}{2}$. Combined with the previous observation, this implies that the determinant equals the right-hand side product times, possibly, some constant. To compute the constant,

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On the other hand, the determinant is a polynomial in the X_i's of degree at most $\binom{n}{2}$. Combined with the previous observation, this implies that the determinant equals the right-hand side product times, possibly, some constant. To compute the constant, compare coefficients of $X_1^0X_2^1\cdots X_n^{n-1}$ on both sides of (2.1). This completes the proof of (2.1).

At this point, let us extract the essence of this proof as we will come back to it in Section 2.4. The basic steps are:

1. *Identification of factors*
2. *Determination of degree bound*
3. *Computation of the multiplicative constant.*

An immediate generalization of the Vandermonde determinant evaluation is given by the proposition below. It can be proved in just the same way as the above proof of the Vandermonde determinant evaluation itself.

Proposition 1. *Let $X_1,X_2,\dots,X_n$ be indeterminates. If $p_1,p_2,\dots,p_n$ are polynomials of the form $p_j(x)=a_jx^{j-1}+$ lower terms, then*

$$\det_{1\le i,j\le n}\left(p_j(X_i)\right)=a_1a_2\cdots a_n\prod_{1\le i<j\le n}(X_j-X_i). \tag{2.2}$$

$\square$

Lemma

Let $X_1, X_2, \dots, X_n$ be indeterminates. If $p_1, p_2, \dots, p_n$ are polynomials of the form $p_j(x) = a_j x^{j-1}$ + lower terms, then

$$\det_{1\le i,j\le n} \left(p_j(X_i)\right) = a_1 a_2 \cdots a_n \prod_{1\le i<j\le n} (X_j - X_i).$$

(Proposition 1 from ADC I)

EXAMPLE 2. Consider

$$\det_{1\le i,j\le b}\left(\binom{a+c}{a-i+j}\right).$$

Whether or not you tried to evaluate (1.1) directly, here is an *important lesson to be learned* (it was already mentioned earlier): To evaluate (1.1) directly is quite difficult, whereas proving its generalization (2.7) is almost completely trivial. Therefore, it is always a good idea to try to *introduce more parameters into your determinant.* (That is, in a way such that the more general determinant still evaluates nicely.) More parameters mean that you have more objects at your disposal to play with.

The most stupid way to introduce parameters is to just write X_i instead of the row index i, or write Y_j instead of the column index j.[8] For the determinant (1.1) even both simultaneously was possible. For the determinant (1.2) either of the two (but not both) would work. On the contrary, there seems to be no nontrivial way to introduce more parameters in the determinant (1.4). This is an indication that the evaluation of this determinant is in a different category of difficulty of evaluation. (Also (1.3) belongs to this "different category". It is possible to introduce one more parameter, see (3.32), but it does not seem to be possible to introduce more.)

2.2. A GENERAL DETERMINANT LEMMA, PLUS VARIATIONS AND GENERALIZATIONS. In this section I present an apparently not so well-known determinant evaluation that generalizes Vandermonde's determinant, and some companions. As Lascoux pointed out to me, most of these determinant evaluations can be derived from the evaluation of a certain determinant of minors of a given matrix due to Turnbull [179, p. 505], see Appendix B. However, this (these) determinant evaluation(s) *deserve(s) to be better known.* Apart from the fact that there are numerous applications of it (them) which I am aware of, my proof is that I meet very often people who stumble across a special case of this (these) determinant evaluation(s), and then have a hard time to actually do the evaluation because, usually, their special case does not show the hidden general structure which is lurking behind. On the other hand, as I will demonstrate in a moment, if you know this (these) determinant evaluation(s) then it is a matter completely mechanical in nature to see whether it (they) is (are) applicable to your determinant or not. If one of them is applicable, you are immediately done.

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The determinant evaluation of which I am talking is the determinant lemma from [85, Lemma 2.2] given below. Here, and in the following, empty products (like $(X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})$ for $j = n$) equal 1 by convention.

Lemma 3. *Let $X_1, \dots, X_n$, $A_2, \dots, A_n$, and $B_2, \dots, B_n$ be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\Big((X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})(X_i + B_j)(X_i + B_{j-1}) \cdots (X_i + B_2)\Big) = \prod_{1\le i<j\le n} (X_i - X_j) \prod_{2\le i\le j\le n} (B_i - A_j). \quad (2.8)$$

Lemma

Let $X_1, \dots, X_n$, $A_2, \dots, A_n$, and $B_2, \dots, B_n$ be indeterminates. Then there holds

$$\det_{1\le i,j\le n}\Big((X_i+A_n)(X_i+A_{n-1})\cdots(X_i+A_{j+1})\\ \cdot(X_i+B_j)(X_i+B_{j-1})\cdots(X_i+B_2)\Big)\\ =\prod_{1\le i<j\le n}(X_i-X_j)\prod_{2\le i\le j\le n}(B_i-A_j).$$

(Lemma 3 from ADC I)

EXAMPLE 3. Consider

$$\det_{1\le i,j\le n}\left(\binom{x+m}{j-i+m}-\binom{x+m}{m-i-j+1}\right).$$

(This is from *"Yay for determinants"* by Tewodros Amdeberhan, Christoph Koutschan and Doron Zeilberger.)

Again, Lemma 5 is tailored for applications in q-enumeration. So, also here, it may be convenient to state the according limit case that is suitable for plain enumeration (and perhaps other applications).

Lemma 7. *Let* $X_1, X_2, \dots, X_n, A_2, A_3, \dots, A_n, C$ *be indeterminates. If* $p_0, p_1, \dots, p_{n-1}$ *are polynomials with* $\deg p_j \le 2j$ *and* $p_j(C - X) = p_j(X)$ *for* $j = 0, 1, \dots, n-1$*, then*

$$\det_{1\le i,j\le n}\Big((X_i + A_n)(X_i + A_{n-1})\cdots(X_i + A_{j+1}) \cdot (X_i - A_n - C)(X_i - A_{n-1} - C)\cdots(X_i - A_{j+1} - C)\cdot p_{j-1}(X_i)\Big) = \prod_{1\le i<j\le n}(X_j - X_i)(C - X_i - X_j)\prod_{i=1}^{n} p_{i-1}(-A_i)\ . \tag{2.13}$$

$\square$

In concluding, I want to mention that, now since more than ten years, I have a different common generalization of Lemmas 3 and 4 (with some overlap with Lemma 5) in my drawer, without ever having found use for it. Let us nevertheless state it here; maybe it is exactly the key to the solution of a problem of yours.

Lemma 8. *Let* $X_1, \dots, X_n$*,* $A_2, \dots, A_n$*,* $B_2, \dots B_n$*,* $a_2, \dots, a_n$*,* $b_2, \dots b_n$*, and* C *be indeterminates. Then there holds*

$$\left(\begin{cases}(X_i + A_n)\cdots(X_i + A_{j+1})(C/X_i + A_n)\cdots(C/X_i + A_{j+1}) & j < m\\ \dots\end{cases}\right)$$

Lemma

Let $X_1, X_2, \dots, X_n, A_2, A_3, \dots, A_n, C$ *be indeterminates. If* $p_0, p_1, \dots,$ p_{n-1} *are polynomials with* $\deg p_j \le 2j$ *and* $p_j(C - X) = p_j(X)$ *for* $j = 0, 1, \dots, n-1$, *then*

$$\det_{1\le i,j\le n} \Big((X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})$$
$$\cdot (X_i - A_n - C)(X_i - A_{n-1} - C) \cdots (X_i - A_{j+1} - C) \cdot p_{j-1}(X_i)\Big)$$
$$= \prod_{1\le i<j\le n} (X_j - X_i)(C - X_i - X_j) \prod_{i=1}^{n} p_{i-1}(-A_i) .$$

(Lemma 7 from ADC I)

EXAMPLE 3. Consider

$$\det_{1\le i,j\le n}\left(\binom{x+m}{j-i+m}-\binom{x+m}{m-i-j+1}\right).$$

(This is from *"Yay for determinants"* by Tewodros Amdeberhan, Christoph Koutschan and Doron Zeilberger.)
The result is:

$$\det_{1\le i,j\le n}\left(\binom{x+m}{j-i+m}-\binom{x+m}{m-i-j+1}\right)
= \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{(x+i-j)(x+2i+j-2)}{(x+2i-j)(i+j-1)}.$$

Clearly, equation (3.9) results immediately from (3.10) by setting $q = 1$. Roughly, Kuperberg's solution [97] of the enumeration of alternating sign matrices consisted of suitably specializing the x_i's, y_i's and q in (3.10), so that each summand on the right-hand side would reduce to the same quantity, and, thus, the sum would basically count $n \times n$ alternating sign matrices, and in evaluating the left-hand side determinant for that special choice of the x_i's, y_i's and q. The resulting number of $n \times n$ alternating sign matrices is given in (A.1) in the Appendix. (The first, very different, solution is due to Zeilberger [198].) Subsequently, Zeilberger [199] improved on Kuperberg's approach and succeeded in proving the refined alternating sign matrix conjecture from [111, Conj. 2]. For a different expansion of the determinant of Izergin, in terms of *Schur functions*, and a variation, see [101, Theorem q, Theorem γ].

Next we turn to typical applications of Lemma 3. They are listed in the following theorem.

Theorem 26. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A, B be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(\begin{bmatrix} L_i+A+j \\ L_i+j \end{bmatrix}_q\right) = q^{\sum_{i=1}^n (i-1)(L_i+i)} \frac{\prod_{1\le i<j\le n}[L_i-L_j]_q}{\prod_{i=1}^n [L_i+n]_q!} \frac{\prod_{i=1}^n [L_i+A+1]_q!}{\prod_{i=1}^n [A+1-i]_q!}, \tag{3.11}$$

and

$$\det_{1\le i,j\le n}\left(q^{jL_i}\begin{bmatrix} A \\ L_i+j \end{bmatrix}_q\right) = q^{\sum_{i=1}^n iL_i} \frac{\prod_{1\le i<j\le n}[L_i-L_j]_q}{\prod_{i=1}^n [L_i+n]_q!} \frac{\prod_{i=1}^n [A+i-1]_q!}{\prod_{i=1}^n [A-L_i-1]_q!}, \tag{3.12}$$

and

$$\det_{1\le i,j\le n}\left(\binom{BL_i+A}{L_i+j}\right) = \frac{\prod_{1\le i<j\le n}(L_i-L_j)}{\prod_{i=1}^n (L_i+n)!} \prod_{i=1}^n \frac{(BL_i+A)!}{((B-1)L_i+A-1)!} \prod_{i=1}^n (A-Bi+1)_{i-1}\ , \tag{3.13}$$

and

$$\det_{1\le i,j\le n}\left(\frac{(A+BL_i)^{j-1}}{(j-L_i)!}\right) = \prod_{i=1}^n \frac{(A+Bi)^{i-1}}{(n-L_i)!} \prod_{1\le i<j\le n}(L_j-L_i). \tag{3.14}$$

$$\times \prod_{i=0} (q^{X_i-A-B})_{n-1}\,(q^{X_i+B-n+2})_{n-1} . \quad (3.17)$$

□

As another application of Lemma 5 we list two evaluations of determinants (see below) where the entries are, up to some powers of q, a difference of two q-binomial coefficients. A proof of the first evaluation which uses Lemma 5 can be found in [88, proof of Theorem 7], a proof of the second evaluation using Lemma 5 can be found in [155, Ch. VI, §3]. Once more, the second evaluation was always (implicitly) known to people in group representation theory, as it also results from a principal specialization (set $x_i = q^{i-1/2}$, $i = 1, 2, \dots$) of a symplectic character of arbitrary shape, by comparing the symplectic dual Jacobi–Trudi identity with the bideterminantal form (Weyl character formula) of the symplectic character (cf. [52, Cor. 24.24 and (24.18)]; the determinants arising in the bideterminantal form are easily evaluated by means of (2.4)).

Theorem 30. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(q^{j(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q - q^{j(2L_i+A-1)}\begin{bmatrix}A\\ -j-L_i+1\end{bmatrix}_q\right)\right)$$
$$= \prod_{i=1}^{n} \frac{[A+2i-2]_q!}{[n-L_i]_q!\,[A+n-1+L_i]_q!} \prod_{1\le i<j\le n} [L_j-L_i]_q \prod_{1\le i\le j\le n} [L_i+L_j+A-1]_q \quad (3.18)$$

and

$$\det_{1\le i,j\le n}\left(q^{j(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q - q^{j(2L_i+A)}\begin{bmatrix}A\\ -j-L_i\end{bmatrix}_q\right)\right)$$
$$= \prod_{i=1}^{n} \frac{[A+2i-1]_q!}{[n-L_i]_q!\,[A+n+L_i]_q!} \prod_{1\le i<j\le n} [L_j-L_i]_q \prod_{1\le i\le j\le n} [L_i+L_j+A]_q . \quad (3.19)$$

□

A special case of (3.19) was the second determinant evaluation which Andrews needed in [4, (1.4)] in order to prove the *MacMahon Conjecture* (since then, ex-Conjecture) about the q-enumeration of *symmetric plane partitions*. Of course, Andrews' evaluation proceeded by LU-factorization, while Schlosser [155, Ch. VI, §3] simplified Andrews' proof significantly by making use of Lemma 5. The determinant evaluation (3.18)

Method 2: Condensation

Method 2: Condensation

This is based on a determinant formula due to Jacobi:

Proposition

Let A be an $n \times n$ matrix. Denote the submatrix of A in which rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ are omitted by $A_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}$. Then

$$\det A \cdot \det A_{1,n}^{1,n} = \det A_1^1 \cdot \det A_n^n - \det A_1^n \cdot \det A_n^1.$$

Method 3: Identification of factors

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An evaluation of the Vandermonde determinant

This is a three-step procedure:

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EXAMPLE 4. Let us consider

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right)=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n-1}{2}}$$
$$\times\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor (i+1)/2\rfloor}\left(-\mu-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i},$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$.

For proving that $(\mu + n)^E$ divides the determinant, we find E linear independent vectors in the kernel of the matrix with $\mu = -n$.

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One finds that

$$\left(0, \binom{n-2}{0}, \binom{n-2}{1}, \binom{n-2}{2}, \dots, \binom{n-2}{n-2}\right)$$

is in the kernel (as well as other similar vectors, which yield $\lfloor (n+1)/3 \rfloor$ linearly independent vectors in the kernel).

For proving that $(\mu+n)^E$ divides the determinant, we find E linear independent vectors in the kernel of the matrix with $\mu=-n$.

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is in the kernel (as well as other similar vectors, which yield $\lfloor (n+1)/3 \rfloor$ linearly independent vectors in the kernel).

For proving that this vector is indeed in the kernel, we must prove

$$\sum_{j=1}^{n-1} \binom{n-2}{j-1} \binom{-n+i+j}{2i-j} = 0$$

for $i=0,1,\dots,n-1$.

Method 4: LU-Factorisation

Method 5: The Holonomic Ansatz

This is an idea due to Doron Zeilberger (made effective by Manuel Kauers and particularly Christoph Koutschan)

Method 6: Hankel Determinants and Orthogonal Polynomials

Theorem

Let $(\mu_k)_{k\ge0}$ be a sequence with generating function $\sum_{k=0}^\infty \mu_k x^k$ written in the form

$$\sum_{k=0}^{\infty}\mu_k x^k=\cfrac{\mu_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\cdots}}}\quad .$$

Then

$$\det_{0\le i,j\le n-1}\left(\mu_{i+j}\right)=\mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}.$$

Theorem

Let $(p_n(x))_{n\ge0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree n, which is orthogonal with respect to some functional L, that is, $L(p_m(x)p_n(x)) = \delta_{m,n}c_n$, where the c_n's are some non-zero constants and $\delta_{m,n}$ is the Kronecker delta. Let

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x)$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function $\sum_{k=0}^{\infty} \mu_k x^k$ for the moments $\mu_k = L(x^k)$ can be written as continued fraction as before with the a_i's and b_i's being the coefficients in the above three-term recurrence. In particular, the previous Hankel determinant evaluation holds, with the b_i's from the above three-term recurrence.

EXAMPLE 5. In a certain problem of rhombus tiling enumeration, Markus Fulmek and myself needed to compute the determinant (among others)

$$\det_{0\le i,j\le n-1}\left(B_{i+j+2}\right),$$

where B_k denotes the k-th *Bernoulli number*. (The Bernoulli numbers are defined via their generating function, $\sum_{k=0}^{\infty} B_k z^k/k! = z/(e^z - 1)$.)

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Apparently:

$$\det_{0\le i,j,\le n-1}(B_{i+j+2}) = (-1)^{\binom n2}\left(\frac{1}{6}\right)^n \prod_{i=1}^{n-1}\left(\frac{i(i+1)^2(i+2)}{4(2i+1)(2i+3)}\right)^{n-i}$$

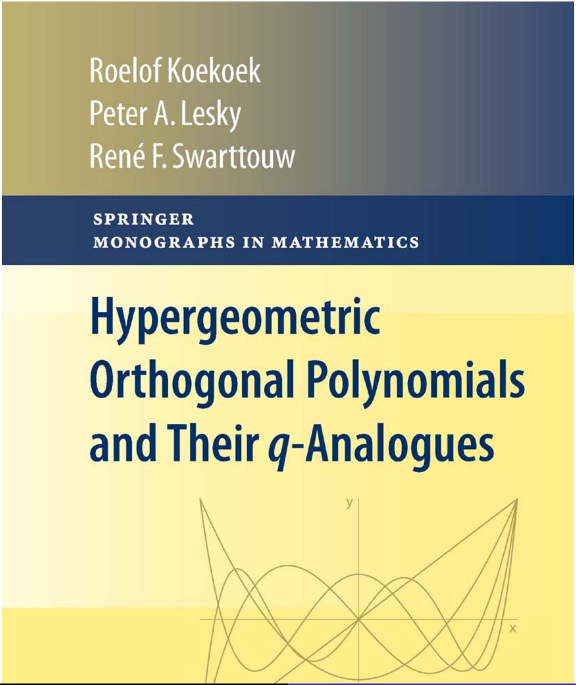
$$= (-1)^{\binom n2}\frac{1}{6}\prod_{i=1}^{n-1}\frac{i!\,(i+1)!^4\,(i+2)!}{(2i+2)!\,(2i+3)!}.$$

Solution:

Solution:

ASKEY SCHEME OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

${}_4F_3$		Wilson	Racah	
${}_3F_2$	Continuous dual Hahn	Continuous Hahn	Hahn	dual Hahn
${}_2F_1$	Meixner – Pollaczek	Jacobi	Meixner	Kravtchouk
${}_2F_1/{}_2F_0$		Laguerre	Charlier	


Roelof Koekoek
Peter A. Lesky
René F. Swarttouw
SPRINGER
MONOGRAPHS IN MATHEMATICS
Hypergeometric
Orthogonal Polynomials
and Their q-Analogues
y
x

We need the *continuous Hahn polynomials*. They are defined by

$$p_n(a,b,c,d;x) = \frac{(\sqrt{-1})^n\,(a+c)_n\,(a+d)_n}{(a+b+c+d+n-1)_n} \times {}_3F_2\left[\begin{matrix}-n, n+a+b+c+d-1, a+x\sqrt{-1}\\ a+c, a+d\end{matrix}; 1\right].$$

They satisfy the recurrence equation

$$p_{n+1}(a,b,c,d;x) = (x - A_n(a,b,c,d))p_n(a,b,c,d;x) - B_n(a,b,c,d)p_{n-1}(a,b,c,d;x),$$

where

$$A_n(a,b,c,d) = \sqrt{-1}\left(a + \frac{n\,(b+c+n-1)\,(b+d+n-1)}{(a+b+c+d+2n-2)\,(a+b+c+d+2n-1)} + \frac{(1-a-b-c-d-n)\,(a+c+n)\,(a+d+n)}{(-1+a+b+c+d+2n)\,(a+b+c+d+2n)}\right),$$

and

$$
\begin{aligned}
B_n(a,b,c,d) \\
&= -\frac{n\,(a+c+n-1)\,(b+c+n-1)\,(a+d+n-1)}{(a+b+c+d+2n-3)\,(a+b+c+d+2n-2)^2} \\
&\qquad \times \frac{(b+d+n-1)\,(a+b+c+d+n-2)}{(a+b+c+d+2n-1)}.
\end{aligned}
$$

They are orthogonal with respect to the measure

$$
\begin{aligned}
L(p(x)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} &\Gamma(a+x\sqrt{-1})\,\Gamma(b+x\sqrt{-1}) \\
&\cdot \Gamma(c-x\sqrt{-1})\,\Gamma(d-x\sqrt{-1})\,p(x)\,dx\,.
\end{aligned}
$$

and

$$
\begin{aligned}
&B_n(a,b,c,d)\\
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\end{aligned}
$$

It turns out that for $a=b=c=d=1$ we get

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L(x^n) = (\sqrt{-1})^{-n} B_{n+2}.
$$

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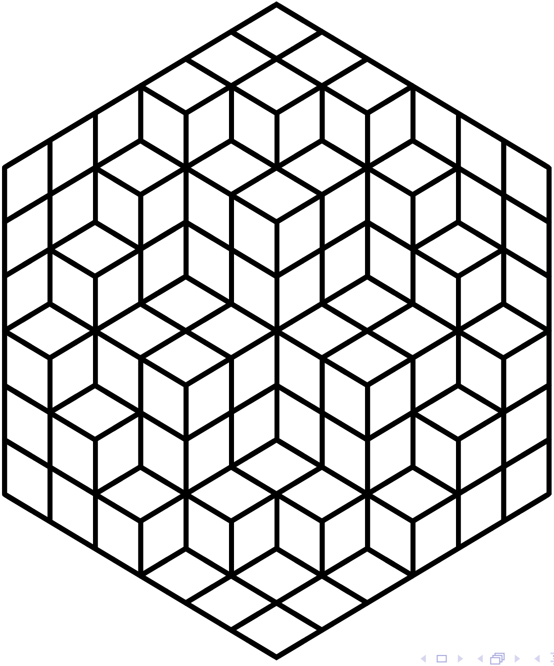
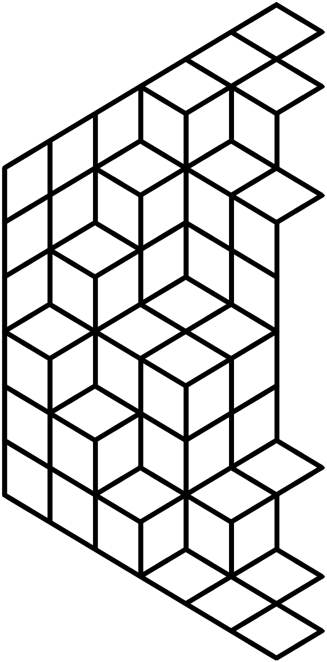
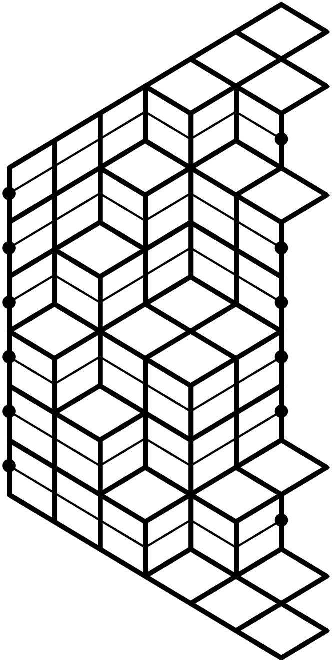
If one works everything out, then, according to Heilermann's theorem, we obtain indeed:

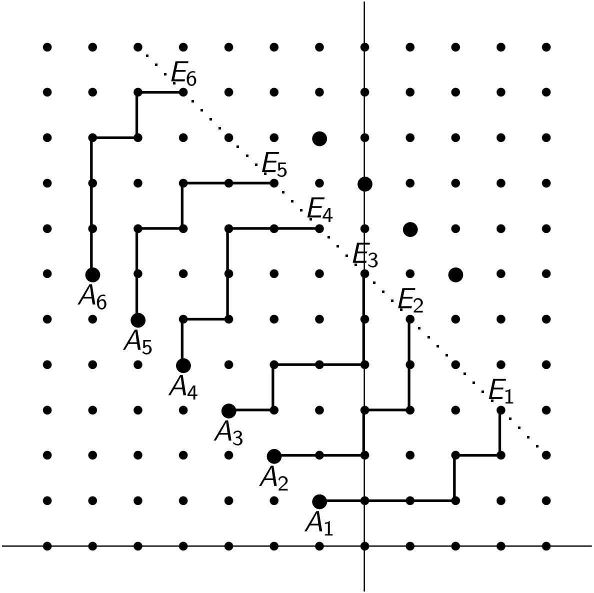
$$\begin{aligned}\det_{0 \le i,j, \le n-1} (B_{i+j+2}) &= (-1)^{\binom{n}{2}} \left(\frac{1}{6}\right)^n \prod_{i=1}^{n-1} \left(\frac{i(i+1)^2(i+2)}{4(2i+1)(2i+3)}\right)^{n-i} \\ &= (-1)^{\binom{n}{2}} \frac{1}{6} \prod_{i=1}^{n-1} \frac{i!\,(i+1)!^4\,(i+2)!}{(2i+2)!\,(2i+3)!}.\end{aligned}$$

Pfaffians

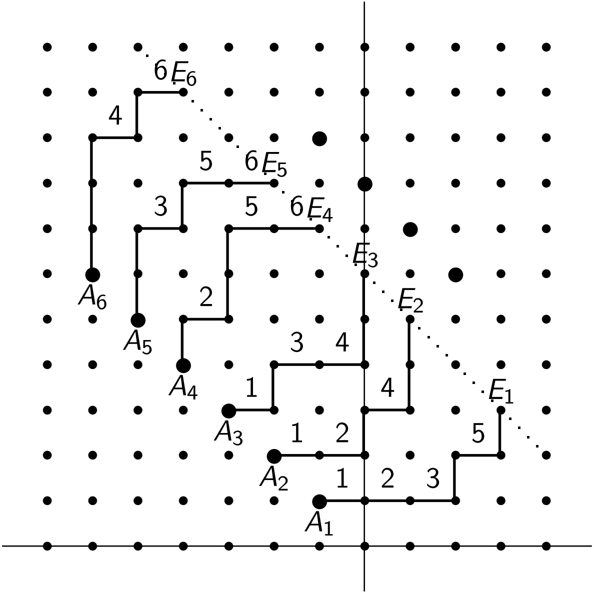








E_6
E_5
E_4
E_3
E_2
E_1
A_6
A_5
A_4
A_3
A_2
A_1


6 E_6
4
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6 E_4
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4
E_1
A_3
1
2
5
A_2
1
2
3
A_1

The minor summation formula of Ishikawa and Wakayama

Theorem

Let m and p be positive integers and $M = (M_{i,j})_{1 \le i \le m, 1 \le j \le p}$ any $m \times p$ matrix. If m is even and $A = (a_{r,s})_{1 \le r,s \le p}$ is any $p \times p$ skew-symmetric matrix, then we have

$$\sum_K \operatorname{Pf}\left(A_K^K\right) \det\left(M_K\right) = \operatorname{Pf}\left(M\, A\, M^t\right),$$

where $K = (k_1, \dots, k_m)$ runs over all increasing sequences $1 \le k_1 < \cdots < k_m \le p$ of integers, and $[m] := (1, 2, \dots, m)$.

The minor summation formula of Ishikawa and Wakayama

Corollary

Let m and p be positive integers and $M = (M_{i,j})_{1\le i\le m, 1\le j\le p}$ any $m \times p$ matrix. If m is even, then we have

$$\sum_K \det M_K = \operatorname*{Pf}_{1\le i<j\le} \left(\sum_{1\le r<s\le p} (M_{i,r}M_{j,s} - M_{i,s}M_{j,r}) \right),$$

where $K = (k_1, \dots, k_m)$ runs over all increasing sequences $1 \le k_1 < \dots < k_m \le p$ of integers.

If one applies the corollary to our situation, one obtains

$$\sum_{\lambda:\lambda_1\le 2n} s_\lambda(X_a) = \underset{1\le i<j\le 2n}{\mathrm{Pf}} \left(\sum_{m>n+i-j} e_m(X_a)e_n(X_a) \right. \\ \left. - \sum_{m<n+i-j} e_m(X_a)e_n(X_a) \right)$$

If one applies the corollary to our situation, one obtains

$$\sum_{\lambda:\lambda_1\le 2n} s_\lambda(X_a) = \operatorname*{Pf}_{1\le i<j\le 2n}\left(\sum_{m>n+i-j} e_m(X_a)e_n(X_a) - \sum_{m<n+i-j} e_m(X_a)e_n(X_a)\right)$$

By row and column operations, this can be transformed into

$$\sum_{\lambda:\lambda_1\le 2n} s_\lambda(X_a) = \operatorname*{Pf}_{1\le i<j\le 2n}\left(\sum_{r=-j+i+1}^{j-i}\sum_{\ell\ge 0} e_\ell(X_a)e_{\ell+r}(X_a)\right)$$

For the final step, one uses a Pfaffian-to-determinant reduction due to Basil Gordon.

Lemma

If the quantities z_i, $i \in \mathbb{Z}$, satisfy $z_{-i} = -z_i$, then we have

$$\operatorname*{Pf}_{1\le i,j\le 2h}(z_{j-i}) = \det_{1\le i,j\le h}\left(z_{|j-i|+1} + z_{|j-i|+3} + z_{|j-i|+5} + \cdots + z_{i+j-1}\right).$$

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This leads to the identity

$$\sum_{\lambda:\lambda_1\le 2n} s_\lambda(X_a) = \det_{1\le i,j\le n} \left(\sum_{\ell\ge 0} e_\ell(X_a) e_{\ell+j-i}(X_a) + \sum_{\ell\ge 0} e_\ell(X_a) e_{\ell+i+j-1}(X_a) \right).$$

was used in [88] in the proof of *refinements* of the *MacMahon (ex-)Conjecture* and the *Bender–Knuth (ex-)Conjecture*. (The latter makes an assertion about the generating function for tableaux with bounded entries and a bounded number of columns. The first proof is due to Gordon [59], the first published proof [3] is due to Andrews.)

Next, in the theorem below, we list two very similar determinant evaluations. This time, the entries of the determinants are, up to some powers of q, a sum of two q-binomial coefficients. A proof of the first evaluation which uses Lemma 5 can be found in [155, Ch. VI, §3]. A proof of the second evaluation can be established analogously. Again, the second evaluation was always (implicitly) known to people in group representation theory, as it also results from a principal specialization (set $x_i = q^i$, $i = 1, 2, \dots$) of an odd orthogonal character of arbitrary shape, by comparing the orthogonal dual Jacobi–Trudi identity with the bideterminantal form (Weyl character formula) of the orthogonal character (cf. [52, Cor. 24.35 and (24.28)]; the determinants arising in the bideterminantal form are easily evaluated by means of (2.3)).

Theorem 31. *Let n be a nonnegative integer, and let $L_1, L_2, \dots, L_n$ and A be indeterminates. Then there hold*

$$\det_{1\le i,j\le n}\left(q^{(j-1/2)(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q + q^{(j-1/2)(2L_i+A-1)}\begin{bmatrix}A\\ -j-L_i+1\end{bmatrix}_q\right)\right)$$
$$= \prod_{i=1}^{n}\frac{(1+q^{L_i+A/2-1/2})}{(1+q^{i+A/2-1/2})}\frac{[A+2i-1]_q!}{[n-L_i]_q!\,[A+n+L_i-1]_q!}$$
$$\times \prod_{1\le i<j\le n}[L_j-L_i]_q\,[L_i+L_j+A-1]_q \tag{3.20}$$

and

$$\det_{1\le i,j\le n}\left(q^{(j-1/2)(L_j-L_i)}\left(\begin{bmatrix}A\\ j-L_i\end{bmatrix}_q + q^{(j-1/2)(2L_i+A-2)}\begin{bmatrix}A\\ -j-L_i+2\end{bmatrix}_q\right)\right)$$

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$$\times \prod_{1\le i<j\le n}[L_j-L_i]_q\,[L_i+L_j+A-1]_q \qquad (3.20)$$

and

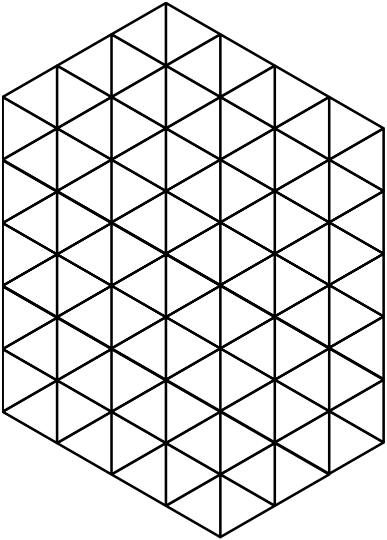
$$\det_{1\le i,j\le n}\left(q^{(j-1/2)(L_j-L_i)}\left(\begin{bmatrix} A\\ j-L_i\end{bmatrix}_q + q^{(j-1/2)(2L_i+A-2)}\begin{bmatrix} A\\ -j-L_i+2\end{bmatrix}_q\right)\right)$$
$$= \frac{\prod_{i=1}^{n}(1+q^{L_i+A/2-1})}{\prod_{i=2}^{n}(1+q^{i+A/2-1})}\prod_{i=1}^{n}\frac{[A+2i-2]_q!}{[n-L_i]_q!\,[A+n+L_i-2]_q!}$$
$$\times \prod_{1\le i<j\le n}[L_j-L_i]_q\,[L_i+L_j+A-2]_q\,. \qquad (3.21)$$

□

A special case of (3.20) was the first determinant evaluation which Andrews needed in [4, (1.3)] in order to prove the *MacMahon Conjecture* on *symmetric plane partitions*. Again, Andrews' evaluation proceeded by LU-factorization, while Schlosser [155, Ch. VI, §3] simplified Andrews' proof significantly by making use of Lemma 5.

Rhombus Tilings $\longrightarrow$ Perfect Matchings

Rhombus Tilings ⟶ Perfect Matchings



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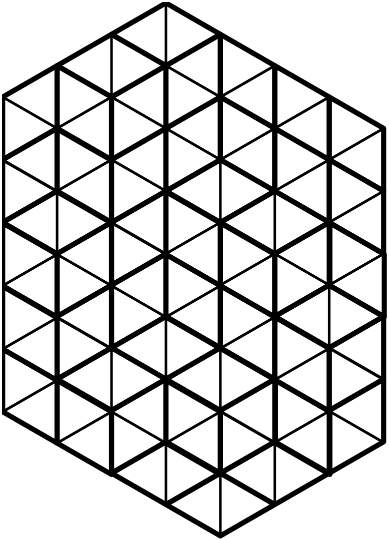

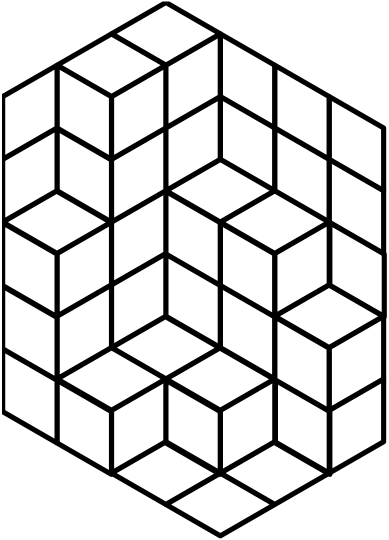

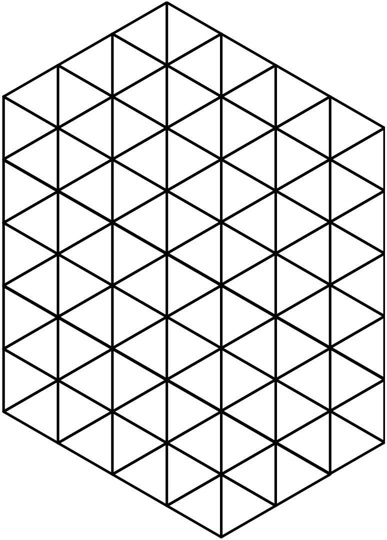

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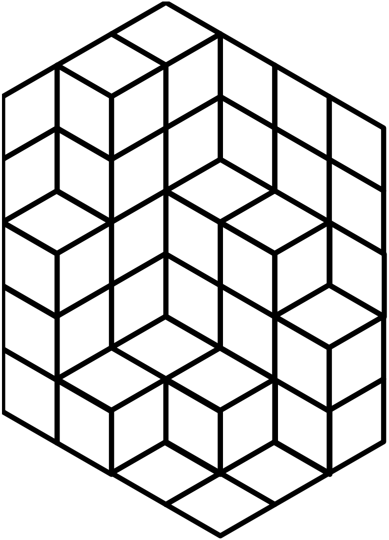

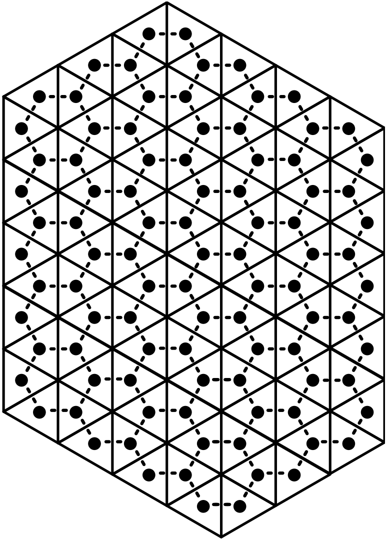

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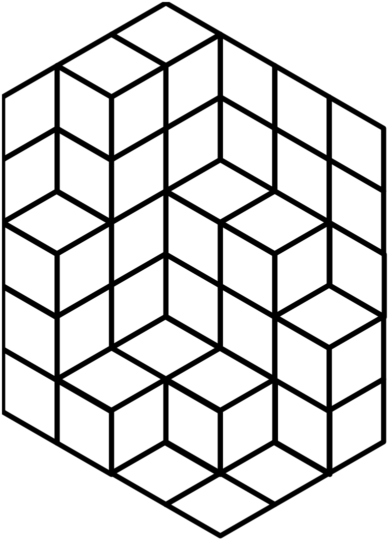

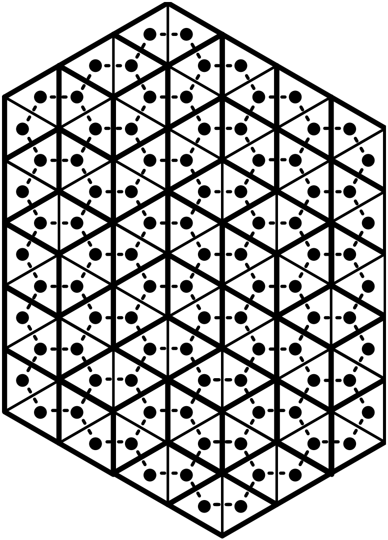

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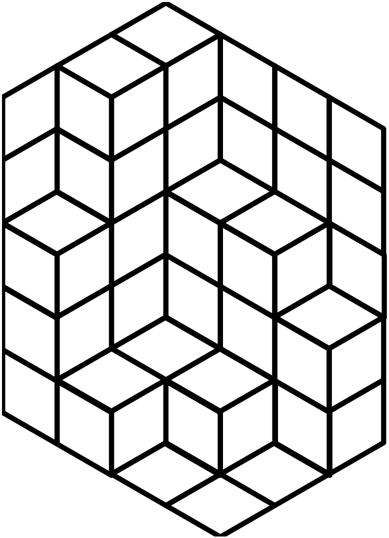

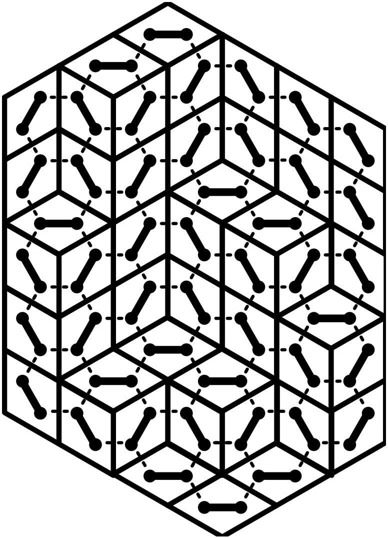

Rhombus Tilings ⟶ Perfect Matchings