1. Show that, if $p_1, p_2, \dots, p_n$ are polynomials of the form $p_j(x) = a_j x^{j-1} +$ lower terms, then

$$\det_{1\le i,j\le n} (p_j(X_i)) = a_1 a_2 \cdots a_n \prod_{1\le i<j\le n} (X_j - X_i).$$

2. Evaluate

$$\det_{1\le i,j\le n} \left(\binom{i+j}{i} \right).$$

Afterthought: Can you find a combinatorial proof of the determinant evaluation?

3. Show that

$$\det_{1\le i,j\le n} \Big((X_i + A_n)(X_i + A_{n-1}) \cdots (X_i + A_{j+1})(X_i + B_j)(X_i + B_{j-1}) \cdots (X_i + B_2) \Big)$$
$$= \prod_{1\le i<j\le n} (X_i - X_j) \prod_{2\le i\le j\le n} (B_i - A_j).$$

4. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number. Show that

$$\det_{0\le i,j\le n-1} (C_{m+i+j}) = \prod_{1\le i\le j\le m-1} \frac{2n+i+j}{i+j}.$$

(*Hint*: Write $C_n = (-1)^n 2^{2n+1} \binom{1/2}{n+1}$.)

5. Show that

$$\det_{1\le i,j\le n} \left(\frac{x_j^i \, (b/x_j; q)_i}{(ax_j; q)_i} \right) = \prod_{i=1}^{n} \frac{(abq^i; q)_{i-1} \, (x_i - b)}{(ax_i; q)_n} \prod_{1\le i<j\le n} (x_j - x_i),$$

where the q-shifted factorials are defined by

$$(\alpha; q)_k := (1-\alpha)(1-\alpha q) \cdots (1-\alpha q^{k-1})$$

if k is a positive integer, and $(\alpha; q)_0 := 1$. (This is from *"Lecture Hall Tableaux"* by Sylvie Corteel and Jang Soo Kim.)

6. Define the q-binomial coefficient $\begin{bmatrix} N \\ M \end{bmatrix}_q$ by

$$\begin{bmatrix} N \\ M \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_M \, (q;q)_{N-M}}.$$

For a lattice path P above the x-axis, let $A(P)$ denote the area between P and the x-axis. Show that

$$\sum_P q^{A(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where the sum is over all lattice paths P from $(0,0)$ to (a,b).

7. Show that

$$\sum_{\pi} q^{|\pi|} = q^{-a\binom{b+1}{2}} \det_{1\le i,j\le b}\left(q^{j(a-i+j)}\begin{bmatrix} a+c \\ a-i+j \end{bmatrix}_q\right),$$

where the sum is over all plane partitions contained in the $a\times b\times c$ box.

8. Show that

$$q^{-a\binom{b+1}{2}} \det_{1\le i,j\le b}\left(q^{j(a-i+j)}\begin{bmatrix} a+c \\ a-i+j \end{bmatrix}_q\right) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

9. Show that

$$\det_{1\le i,j\le n}\left(\frac{1}{X_i+Y_j}\right) = \frac{\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\le i,j\le n}(X_i+Y_j)}.$$

Some exercises on Pfaffians

The *Pfaffian* of a skew-symmetric matrix $A=(A_{i,j})_{1\le i,j\le 2n}$ is defined as

$$\operatorname{Pf} A = \sum_{m \text{ perfect matching of } \{1,\dots,2n\}} \operatorname{sgn} m \prod_{\{i,j\}\in m} A_{i,j},$$

where the product is over all pairs $\{i,j\}$ with $i<j$ in the matching m, and $\operatorname{sgn} m = (-1)^{\#\text{crossings of } m}$, with a *crossing* being a quadruple (i,j,k,l) with $i<j<k<l$ and $\{i,k\}$ and $\{j,l\}$) being pairs in the matching m. For example,

$$\operatorname{Pf}\begin{pmatrix} 0 & A_{1,2} & A_{1,3} & A_{1,4} \\ -A_{1,2} & 0 & A_{2,3} & A_{2,4} \\ -A_{1,3} & -A_{2,3} & 0 & A_{3,4} \\ -A_{1,4} & -A_{2,4} & -A_{3,4} & 0 \end{pmatrix} = A_{1,2}A_{3,4} - A_{1,3}A_{2,4} + A_{1,4}A_{2,3}.$$

It is a fact that

$$(\operatorname{Pf} A)^2 = \det A,$$

so that computing a Pfaffian is, up to sign, equivalent to a determinant evaluation. (See the article of Stembridge listed on the website

`http://www.mat.univie.ac.at/~kratt/bedlewo`

of the course.)

The significance of Pfaffians in enumeration is (at least) two-fold:

(1) The number (generating function) of non-intersecting lattice paths with fixed starting points but end points varying in some given set can be expressed in terms of a Pfaffian. (One may even fix *some* of the end points and let the others be chosen

from the set.) This follows from the minor summation formula of Ishikawa and Wakayama in Exercise 11.

(2) The number of perfect matchings of a planar graph can be expressed as the Pfaffian of the adjacency matrix in which the sign of some entries has been changed according to a certain rule. This is a theorem due to Kasteleyn, and is explained in Section 2 of the article by Tesler on the website of the course.

10. Show that $\mathrm{Pf}(E) = 1$, where E is the $2n \times 2n$ skew-symmetric matrix with all 1s above the main diagonal.

11. Let m, n, p be integers such that $n+m$ is even and $0 \le n-m \le p$. Let M be an $n \times p$ matrix, H an $n \times m$ matrix, and $A = (a_{ij})_{1\le i,j\le p}$ a skew-symmetric matrix. Show that

$$\sum_K \mathrm{Pf}\left(A_K^K\right) \det\left(M_K \vdots H\right) = (-1)^{\binom{m}{2}}\, \mathrm{Pf}\begin{pmatrix} M\,A\,M^t & H \\ -H^t & 0 \end{pmatrix}.$$

where the sum is over all subsets K of $\{1,2,\dots,p\}$ of cardinality $n-m$, A_K^K is the skew-symmetric matrix obtained from A by selecting the rows and columns indexed by K, and M_K is the submatrix of M which consists of the columns indexed by K.
(*Hint*: Show that left-hand and right-hand side of the identity are linear in the columns of H. Use this observation to reduce the claim to $m = 0$. Then show that the two sides of the remaining identity are both linear in the rows of M. Use this second observation to conclude the proof.)

12. Let A be a $2n \times 2n$ skew-symmetric matrix. Show that

$$\mathrm{Pf}(A) = \frac{1}{2^n\, n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}\sigma \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}.$$

13. Let A and B be $n \times n$ matrices. Show that

$$\mathrm{Pf}\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} = (-1)^{\binom{n}{2}} \det B.$$