Attractors of Hamiltonian Nonlinear Partial Differential Equations

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To the memory of Mark Vishik

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Preface

We present the theory of attractors of nonlinear Hamiltonian partial differential equations in infinite space. This is new branch of the theory of attractors of PDEs since 1990. This new theory differs significantly from the case of dissipative systems. In particular, this theory has no analogue for finite-dimensional Hamiltonian equations contrary to the case of dissipative PDEs.

This book is the first monographic publication in this direction. Included are results on global attraction, to stationary states, to solitons, and to stationary orbits, together with results on adiabatic effective dynamics of solitons and their asymptotic stability, and on the dispersion decay for linear Hamiltonian PDEs. The obtained results are generalised in the formulation of a new general conjecture on global attractors of G-invariant nonlinear Hamiltonian partial differential equations.

We also describe the results of numerical simulations.

In conclusion, we discuss possible relations of this theory with the problem of mathematical interpretation of Bohr's transitions between quantum stationary states.

The book is intended

i) to graduate and postgraduate students working with partial differential equations;

ii) to lecturers in PDEs;

iii) to mathematicians working in PDEs, Mathematical Physics, and mathematical problems of Quantum Theory.

On the required knowledge

All proofs are self-contained and their overwhelming parts rely on traditional methods of Analysis: general theory of Hilbert and Banach spaces; distributions and their Fourier transform, Sobolev spaces and embedding theorems, elementary spectral theory of the Schrödinger operators (all needed subjects are covered by [9] and Chapters 1–12 of [189]); definitions of Lie group and Lie algebra and of their representations.

The key points of the proofs rely on a novel application of subtle methods of Harmonic Analysis: the Wiener Tauberian theorem, the Titchmarsh Theorem on convolution, the theory of quasimeasures, and others. The applications are explained with all details and with exact references to the corresponding textbooks.

Keywords: nonlinear partial differential equations; Hamiltonian equations; wave equation; Maxwell equations; Klein–Gordon equation; limiting amplitude; limiting amplitude principle; limiting absorption principle; attractor; stationary state; soliton; stationary orbit; adiabatic effective dynamics; symmetry group; Lie group; Lie algebra; group representation Schrödinger equation; quantum transitions.

Contents

Introduction

This monograph presents theory of global attractors and of long time behaviour of solutions of nonlinear Hamiltonian partial differential equations in infinite space. This theory was initiated by one of the authors since 1990 and was developed in collaboration with H. Spohn since 1995, and with A. Comech, V. Imaikin, E. Kopylova, D. Stuart and B. Vainberg since 2005.

The theory of attractors for nonlinear PDEs began in Landau's 1944 famous paper [19], where he proposed the first mathematical interpretation of the onset of turbulence as the growth of the dimension of attractors of the Navier–Stokes equations when the Reynolds number increases.

The foundation for the corresponding mathematical theory was laid in 1951 by Hopf, who first established the existence of global solutions of the 3D Navier–Stokes equations [5]. He introduced the *method of compactness* which is a nonlinear version of Faedo-Galerkin approximations. This method is based on a priori estimates and Sobolev embedding theorems and has had an essential influence on the development of the theory of nonlinear PDEs (see [2, 3, 12]).

The modern development of the theory of global attractors for *dissipative PDEs*, that is, PDEs with friction, originated in 1975–1985 in publications by J. Ball, C. Foias, J.M. Ghidaglia, J.K. Hale, D. Henry, R. Temam, and was developed further by M.I. Vishik, A.V. Babin, V.V. Chepyzhov, A. Haraux, A.A. Ilyin, A. Miranville, V. Pata, E. Titi, S. Zelik, and others. An essential part of the theory up to 2000 was covered in the monographs [20]–[26].

One of the central subject of research in this theory is the global attractor of all bounded subsets of the corresponding Banach phase space. Typically this attractor is a submanifold connecting stationary states, which is an analogue of separatrices. Each single point also attracts to this submanifold, and eventually converges to one of stationary states,

$$\psi(x,t) \to S(x), \qquad t \to +\infty, \tag{0.0.1}$$

where the convergence holds in appropriate norm on the Banach phase space. In particular, the *relaxation to an equilibrium regime* in chemical and biological reactions (the 'saturation') is due to energy dissipation.

The results obtained concern a wide class of nonlinear *dissipative* PDEs, including fundamental equations of applied and mathematical physics: the Navier–Stokes equations, nonlinear parabolic equations, reaction-diffusion equations, wave equations with friction, integro-differential equations, equations with delay, with memory, and so on. The techniques of functional analysis of nonlinear PDEs were developed for the study of the structure of different types of attractors, their smoothness and their fractal and Hausdorff dimensions, dependence on parameters, on averaging, and so on.

The development of a similar theory for *Hamiltonian PDEs* seemed at first to be unmotivated and even impossible in view of energy conservation and time reversal for these equations. However, it turned out that such a theory is possible, and its development was inspired by the problem of mathematical interpretation of basic postulates of Quantum Theory. These relations to Quantum Theory are discussed in the final Chapter 8. More details can be found in [214].

Results obtained in 1990–2020 suggest that long-time global attraction to a finite-dimensional submanifold in the corresponding Hilbert phase space is in fact typical feature for nonlinear Hamiltonian PDEs in infinite space. These results are presented in our monograph.

For Hamiltonian PDEs in infinite space the theory of attractors differs significantly from the case of dissipative systems, where the global attraction to stationary states is caused by an energy dissipation which is due to friction. For Hamiltonian equations the friction and energy dissipation are absent, and the global attraction is caused by radiation which irreversibly carries energy to infinity. This peculiarity required novel tools for analysis of nonlinear Hamiltonian equations which are presented in this monograph.

Let us note however that this theory is only at an initial stage of its development and cannot be compared with the theory of attractors of dissipative PDEs with regard to richness and diversity of results.

The modern development of the theory of nonlinear Hamiltonian PDEs dates back to K. Jörgens [7], who first established the existence of global solutions for nonlinear wave equations of the form

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}^n, \tag{0.0.2}$$

by developing the Hopf method of compactness. The subsequent studies of the well-posedness for nonlinear PDEs were presented by J.-L. Lions [12], and by A. Haraux and T. Cazenave [2, 3].

The first results on *long-time asymptotics* for *linear hyperbolic PDEs* in infinite space were established in the scattering theory by P.D. Lax, C.S. Morawetz, and R.S. Phillips for the wave equation in the exterior of a star-shaped obstacle, [34]. This is the *local* *energy decay*: for any finite $R > 0$

$$\int_{|x|<R} [|\dot\psi(x,t)|^2 + |\nabla\psi(x,t)|^2 + |\psi(x,t)|^2] dx \to 0, \qquad t \to \pm\infty. \tag{0.0.3}$$

This decay means that the energy escapes each bounded region for large times. For general linear hyperbolic PDEs and systems similar local decay was established by B.R. Vainberg [40]. The extension of this decay to *nonlinear Hamiltonian PDEs* was established first by I. Segal , C.S. Morawetz and W. Strauss [35]–[39]. In these papers the local energy decay (0.0.3) was proved for solutions of equations (0.0.2) with small initial data in the case of *defocusing nonlinearities* similar to

$$f(\psi) = -m^2\psi - \varkappa|\psi|^{p-1}\psi, \tag{0.0.4}$$

where $m^2 \geq 0$, $\varkappa > 0$, and $p > 1$. Moreover, in these articles the corresponding nonlinear wave operators and scattering operators are constructed. In [80, 81] W. Strauss established the completeness of the scattering for small solutions of more general equations.

For convenience, characteristic properties of all finite-energy solutions of an equation will be referred to as *global*, in order to distinguish them from the corresponding *local* properties of the solutions with initial data sufficiently close to an attractor. Note that global attraction to a (proper) attractor is impossible for finite-dimensional Hamiltonian systems, because of energy conservation. All the above-mentioned results [35]–[39] on local energy decay (0.0.3) for nonlinear Hamiltonian PDEs mean that the corresponding *local attractor* of solutions with small initial states consists of only the zero point.

Theory of global attractors. The first results on *global attractors* for nonlinear Hamiltonian PDEs were obtained by one of the present authors in 1991–1995 for 1D equations [43, 44, 45], and were extended to nD equations in 1995–2020 in collaboration with A. Comech, V.S. Buslaev, E. Kopylova, H. Spohn, D. Stuart, B.R. Vainberg, and others.

These results were derived from an analysis of the irreversible energy radiation to infinity, which plays the role of dissipation. This progress was achieved by a novel application of subtle methods of Harmonic Analysis: the Wiener Tauberian theorem, the Titchmarsh convolution theorem, the theory of quasimeasures, the stationary scattering theory of Agmon, Jensen and Kato, the eigenfunction expansion for nonselfadjoint Hamiltonian operators based on M.G. Krein theory of J-selfadjoint operators, and others.

One of the key observations is that the results obtained so far indicate a certain dependence of long-time asymptotics of solutions on the symmetry group of the equation: for example, it may be the trivial group $G = \{e\}$, or the group of translations $G = \mathbb{R}^n$, or the unitary group $G = U(1)$, or the orthogonal group $SO(3)$. This observation suggests general conjecture for nonlinear Hamiltonian *autonomous* PDEs of type

$$\dot\Psi(t) = F(\Psi(t)), \qquad t \in \mathbb{R}, \tag{0.0.5}$$

with a Lie symmetry group G, which acts on the Hilbert or Banach phase space $\mathcal{E}$ of the equation via a representation T.

Conjecture A. (On attractors) *For* **generic** *nonlinear Hamiltonian PDEs (0.0.5) with a Lie symmetry group G, any finite-energy solution admits the asymptotics*

$$\Psi(t) \sim e^{\hat\lambda_\pm t}\Psi_\pm, \qquad t \to \pm\infty \tag{0.0.6}$$

in appropriate topology of the phase space $\mathcal{E}$.

Here $\hat{\lambda}_\pm = T'(e)\lambda_\pm$, where $\lambda_\pm$ belong to the corresponding Lie algebra $\mathfrak{g}$, while the $\Psi_\pm(x)$ are some *limiting amplitudes* depending on the trajectory $\Psi(x,t)$ considered. Both pairs $(\Psi_+, \hat{\lambda}_+)$ and $(\Psi_-, \hat{\lambda}_-)$ are solutions of the corresponding *nonlinear eigenvalue problem* (3.10.4), see more details in Section 3.10.

Let us specify the asymptotics (0.0.6) for the four symmetry groups mentioned above.

I. Equations with trivial symmetry group $G = \{e\}$. For such *generic equations* the conjecture (0.0.6) means *global attraction to stationary states*

$$\psi(x,t) \to S_\pm(x), \qquad t \to \pm\infty \tag{0.0.7}$$

as is illustrated in Fig. 1. Here the states $S_\pm(x)$ depend on the trajectory $\psi(x,t)$ under consideration, and the convergence holds in local seminorms of type $L^2(|x| < R)$ with any $R > 0$. This convergence cannot hold in global norms (that is, corresponding to $R = \infty$) due to energy conservation.

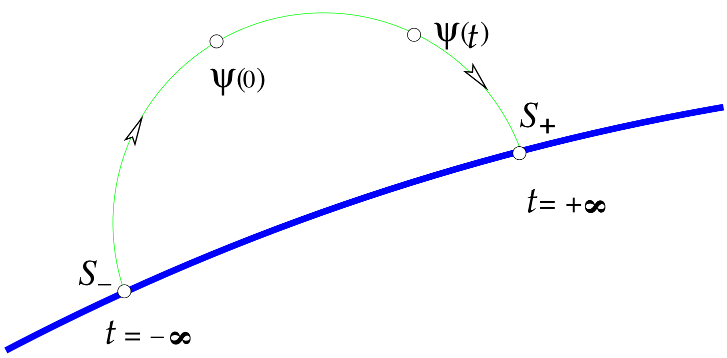

Figure 1: Convergence to stationary states

The asymptotics (0.0.7) can be symbolically written as the transitions

$$S_- \mapsto S_+. \tag{0.0.8}$$

These transitions can be considered as the mathematical model of the Bohr 'quantum jumps'.

Such attraction was proved in [43]–[55] for a variety of model equations: i) for a string coupled to nonlinear oscillators, ii) for a three-dimensional wave equation coupled to a charged particle and for the Maxwell–Lorentz equations, and also iii) for wave equation, and Dirac and Klein–Gordon equations with concentrated nonlinearities.

All proofs rely on the bounds for radiation which irreversibly carries energy to infinity. The proof of global attraction in [47, 48] rely on a novel application of the Wiener Tauberian theorem [18] which provides the relaxation of the acceleration of the particle

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty. \tag{0.0.9}$$

under the *Wiener condition* (1.5.13) on the particle charge density. These results gave the first rigorous proof of *radiation damping* (0.0.9) in Classical Electrodynamics, which has been an open problem for about 100 years.

The results of [43]–[47] and [53] are presented with details in Chapter 1.

In all problems considered here, the convergence (0.0.7) implies by the Fatou theorem the inequality

$$\mathcal{H}(S_\pm) \le \mathcal{H}(Y(t)) \equiv \text{const}, \quad t \in \mathbb{R}, \tag{0.0.10}$$

where $\mathcal{H}$ is the corresponding Hamiltonian (energy) functional. This inequality is an analog of the well known property of the weak convergence in the Hilbert and Banach spaces. Simple examples show that strong inequality in (0.0.10) is possible, that means the irreversible scattering of energy to infinity.

Example 0.0.1. The d'Alembert waves. In particular, the asymptotics (0.0.7) and the strong inequality (0.0.10) can easily be demonstrated for the d'Alembert equation with general solution $\psi(x,t) = f(x-t) + g(x+t)$. Namely, the local convergence $\psi(\cdot,t) \to 0$ in $L^2_{\rm loc}(\mathbb{R})$ obviously holds for all $f, g \in L^2(\mathbb{R})$. On the other hand, the convergence to zero in the global norm of $L^2(\mathbb{R})$ obviously fails if $f(x) \not\equiv 0$ or $g(x) \not\equiv 0$.

Example 0.0.2. Nonlinear strong Huygens principle. Similarly, a solution of the 3D wave equation with unit speed of propagation is concentrated in spherical layers $|t| - R < |x| < |t| + R$ if the initial data have support in the ball $|x| \le R$. Therefore, the solution converges to zero in $L^2_{\rm loc}(\mathbb{R}^3)$ as $t \to \pm\infty$, although its energy remains constant. This also illustrates the strong inequality in (0.0.10). This convergence corresponds to the well-known *strong Huygens principle* in Optics and Acoustics (see [1]). Thus, global attraction to stationary states (0.0.7) is a generalisation of the strong Huygens principle to non-linear equations. The difference is that for the linear wave equation the limit is always zero, while for nonlinear equations the limit can be any stationary solution.

II. Equations with the symmetry group of translations $G = \mathbb{R}^n$. Let us consider for example the case of simplest representation

$$[T(a)\psi](x) := \psi(x-a), \qquad x \in \mathbb{R}^n \tag{0.0.11}$$

for $a \in \mathbb{R}^n$. Then the asymptotics (0.0.6) means *global attraction to solitons* (traveling waves)

$$\psi(x,t) \sim \psi_\pm(x - v_\pm t), \qquad t \to \pm\infty, \tag{0.0.12}$$

where the asymptotics holds in local seminorms of type $L^2(|x - v_\pm t| < R)$ with any $R > 0$, that is *in the comoving frame of reference.*

Such soliton asymptotics was proved first for *integrable equations* (Korteweg–de Vries equation (KdV), etc), see [56, 62]. Moreover, for the Korteweg–de Vries equation more accurate soliton asymptotics in *global norms* with several solitons were first discovered by M. Kruskal and N.J. Zabuzhsky in 1965 by numerical simulation: it is the decay to solitons

$$\psi(x,t) \sim \sum_k \psi_\pm(x - v^k_\pm t) + w_\pm(x,t), \qquad t \to \pm\infty, \tag{0.0.13}$$

where $w_\pm$ are some dispersion waves. A trivial example is provided by the d'Alembert equation $\ddot\psi(x,t) = \psi''(x,t)$, for which any solution reads $\psi(x,t) = f(x-t) + g(x+t)$.

Later on, such asymptotics were proved by the method of *inverse scattering problem* for nonlinear *integrable* Hamiltonian translation-invariant equations (KdV, etc.) in the works of M.J. Ablowitz, H. Segur, W. Eckhaus, A. van Harten and others [56, 62].

For *non-integrable* equations the global attraction to solitons (0.0.12) was established for the first time in [57]–[60] for translation-invariant systems of the wave and Maxwell equations coupled to a charged relativistic particle. The result of [58] gives the first rigorous proof of the *radiation damping* for the translation-invariant system of Classical Electrodynamics.

The proofs in [57] and [58] rely on a canonical transformation to the comoving frame and variational properties of solitons, as well as on the relaxation of the acceleration (0.0.9) under the Wiener condition for the particle charge density.

The multi-soliton asymptotics (0.0.13) for *non-integrable equations* were observed numerically in [61] in the case of 1D *relativistically-invariant* nonlinear wave equations.

The results of [57] and [61] are presented with details in Chapters 2 and 6 respectively,

III. Equations with the unitary symmetry group $G = U(1)$. Let us consider for example the case of simplest representation

$$[T(e^{i\theta})\psi](x) := e^{i\theta}\psi(x), \qquad x \in \mathbb{R}^n \tag{0.0.14}$$

for $\theta \in R$. Then the asymptotics (0.0.6) means the *single-frequency asymptotics*

$$\psi(x,t) \sim \psi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{0.0.15}$$

where $\omega_\pm \in \mathbb{R}$.

Example 0.0.3. For the case of Maxwell–Schrödinger system (8.2.1) with the symmetry group $U(1)$ and its representation (8.2.6), the conjecture (0.0.6) reduces to the asymptotics (8.2.9) with

$$\hat\lambda_\pm = \begin{pmatrix} -i\omega_\pm & 0 \\ 0 & 0 \end{pmatrix}, \qquad \omega_\pm \in \mathbb{R}.$$

The asymptotics (0.0.15) also means the global attraction to the solitary manifold formed by all *stationary orbits* which are solutions of type $\psi_\omega(x)e^{-i\omega t}$. The asymptotics are expected in the local seminorms $L^2(|x| < R)$ with any $R > 0$. The global attractor is a smooth manifold formed by the circles which are the orbits of the action of the symmetry group $U(1)$ (see Fig. 2).

Such attraction *in local seminorms* $L^2(|x| < R)$ were proved i) in [64]–[70] for the Klein–Gordon and Dirac equations coupled to $U(1)$-invariant nonlinear oscillator, ii) in [63], for discrete in space and time difference approximations of such coupled systems, i.e., for the corresponding difference schemes, and iii) in [72]–[74] for the wave, Klein–Gordon, and Dirac equations with concentrated nonlinearities. More precisely, we have proved global attraction to the *solitary manifold* of all stationary orbits, though global attraction to a particular stationary orbits, with fixed $\omega_\pm$, is still an open problem.

All these results were proved under the assumption that the equations are 'strictly nonlinear'. For linear equations, the global attraction obviously fails if the discrete spectrum consists at least of two different eigenvalues.

The proofs of all these results rely on i) a nonlinear analogue of the Kato theorem on the absence of emerged eigenvalues, ii) new theory of multiplicators in the space of quasimeasures and iii) novel application of the Titchmarsh Convolution Theorem.

The results of [65]–[67] are presented with details in Chapter 3.

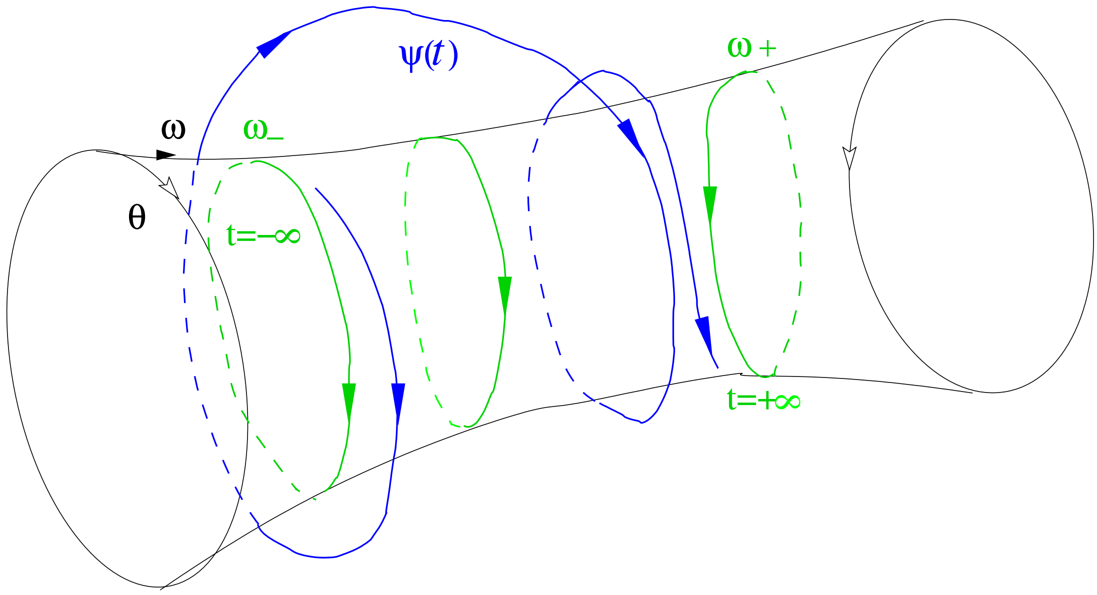

Figure 2: Convergence to stationary orbits .

Existence and orbital stability of stationary orbits. The existence of solutions $e^{\hat{\lambda}t}\Psi$ (*stationary G-orbits*) for G-invariant nonlinear wave equations (0.0.2) in the cases $G = U(1)$ and $G = \mathbb{R}^n$ was extensively studied in the 1960s–1980s. The most general results were obtained by W. Strauss, H. Berestycki and P.-L. Lions [27, 28, 33]. M. Esteban, V. Georgiev, and E. Séré constructed in [30] stationary orbits for relativistically-invariant nonlinear Maxwell–Dirac system (8.2.8) and for the Klein–Gordon–Dirac system. The key role in these papers was played by the Lusternik–Schnirelman theory of critical points [31, 32].

In [29] G. M. Coclite and V. Georgiev constructed stationary orbits for the nonlinear Maxwell–Schrödinger system with the external Coulomb potential.

General theory of *orbital stability* of stationary G-orbits was developed by M. Grillakis, J. Shatah, and W. Strauss in [103, 104].

IV. Equations with the orthogonal symmetry group $G = SO(3)$**.** For such generic equations the asymptotics (0.0.6) means that

$$\psi(x,t) \sim e^{-i\hat{\Omega}_\pm t}\psi_\pm(x), \qquad t \to \pm\infty, \tag{0.0.16}$$

where $\hat{\Omega}_\pm$ are suitable representations of real skew-symmetric 3×3 matrices $\Omega_\pm \in \mathfrak{so}(3)$. This means global attraction to 'stationary $SO(3)$-orbits'. Such asymptotics are proved in [91] for the Maxwell–Lorentz equations with rotating particle.

Generic equations. Let us emphasise that, for example, we are conjecturing asymptotics (0.0.15) for *generic $U(1)$-invariant equations*. This means that the long time behavior of solutions may be quite different for $U(1)$-invariant equations of 'positive codimension'. In particular, for solutions of the linear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t), \qquad x \in \mathbb{R}^n$$

the asymptotics (0.0.15) generally fails. Namely, any finite-energy solution admits the spectral representation

$$\psi(x,t) = \sum C_k\psi_k(x)e^{-i\omega_k t} + \int_0^\infty C(\omega)\psi(\omega,x)e^{-i\omega t}d\omega,$$

where ψ_k and $\psi(\omega, \cdot)$ are the corresponding eigenfunctions of the discrete and continuous spectrum, respectively. The last integral is a dispersion wave, which decays to zero in the norms $L^2(|x| < R)$ with any $R > 0$ (under appropriate conditions on the potential $V(x)$). Correspondingly, the attractor is the linear span of the eigenfunctions ψ_k. Thus, the long-time asymptotics does not reduce to a single term like (0.0.15), so the linear case is degenerate in this sense. Note that all our results [64]–[70] are established for a *strictly nonlinear case* (see the condition (3.1.14)), which eliminates linear equations.

Higher symmetry groups. For more sophisticated symmetry groups $G = U(N)$, the asymptotics (0.0.6) mean the global attraction to N-frequency trajectories, which can be quasi-periodic. In particular, the symmetry groups $SU(2)$, $SU(3)$ and others were suggested in 1961 by M. Gell-Mann and Y. Ne'eman for strong interaction of baryons [220, 221]. This theory provides an empirical evidence for the asymptotics (0.0.6), see Section 3.10.

On relations with Soffer's conjectures. Note that our conjecture (0.0.6) specifies the concept of *localised solution/coherent structures* from the 'Grande Conjecture' and the 'Petite Conjecture' of A. Soffer (see [161], p. 460) in the context of the Banach spaces . The Grande Conjecture was proved in [50] for the case of a 1D wave equation coupled to a nonlinear oscillator (1.2.1). Moreover, suitable versions of the Grande Conjecture were also proved in [60, 91] for the 3D wave, Klein–Gordon and Maxwell equations coupled to a relativistic particle with sufficiently small charge (2.2.1) (see Remark 2.2.1). Finally, for any matrix symmetry group G, the asymptotics (0.0.6) corresponds to the Petite Conjecture since then the localised solutions $e^{g_\pm t}\psi_\pm(x)$ are quasi-periodic.

In this book we present available results on the global attraction (0.0.7)–(0.0.16) and related numerical experiments. Moreover, we survey the results on asymptotic stability of solitons and their adiabatic effective dynamics, on the dispersion decay and relations to Quantum Mechanics.

Asymptotic stability of solitons. More precisely we should phrase 'asymptotic stability of solitary manifolds' which means a local attraction, i.e. for states sufficiently close to the manifold. There is a huge literature on this subject. In Chapter 4 we review the results on such local attraction which were developed in a series of articles [162]–[171] by V.S. Buslaev, G. Perelman, A. Soffer, D. Stuart, C. Sulem, T.P. Tsai, M. Weinstein, H.T. Yau, and others.

The crucial peculiarity of this attraction is the instability of the dynamics *along the solitary manifold*. This follows directly from the fact that solitons move with different speeds and therefore run away for large times. Analytically, this instability is caused by the presence of the eigenvalue $\lambda = 0$ in spectrum of the generator of linearised dynamics. Namely, the tangent vectors to solitary manifold are eigenvectors and associated vectors of the generator. They correspond to zero eigenvalue. Respectively, the Lyapunov theory is not applicable to this case.

This is why in the articles [162]–[170] an original strategy was developed for proving asymptotic stability of solitary manifolds. This strategy allows one to separate the unstable motion along the solitary manifold and the attraction in transversal directions to this manifold.

This approach relies on i) a special projection of a trajectory onto the solitary manifold, ii) modulation equations for parameters of the projection, and iii) time-decay of transversal component. It is a far-reaching development of the Lyapunov stability theory.

Adiabatic effective dynamics of solitons. In Chapter 5 we establish an adiabatic effective dynamics for solitons in slowly varying external potentials, when the corresponding external force is small. The existence of solitons and the global attraction to solitons (0.0.12) are typical features of translation-invariant systems. However, if the deviation of a system from translational invariance is in some sense small, the system can admit solutions which are close forever to solitons with time-dependent parameters (velocity, etc.). Moreover, in some cases it is possible to identify an 'effective dynamics' which describes the evolution of these parameters.

We present without proofs the results of [87] and [88] on *adiabatic effective dynamics* (5.1.5), (5.1.6) for the wave-particle system (1.5.1)–(1.5.2) and the Maxwell–Lorentz system (1.6.1), respectively, in the case of slowly varying external potentials. We also discuss the related *mass-energy equivalence*.

In Chapter 6 we present results of numerical simulation of soliton asymptotics and on the corresponding effective dynamics for relativistically-invariant equations.

Dispersion decay. In Chapter 7 we give i) a brief survey of basic results on the dispersion decay, and ii) new short and simplified proof of the fundamental results on the $L^1 \to L^\infty$ dispersion decay established by J.-L. Journé, A. Soffer and C.D. Sogge in [185] for the Schrödinger equation (7.1.2) with $n \geq 3$.

The dispersion decay of the corresponding linearised equations plays the key role in all results on long-time asymptotic for nonlinear Hamiltonian PDEs. One of the first fundamental results on the dispersion decay is the local energy decay (0.0.3) established in [34].

Relations to Quantum Mechanics. In the final Chapter 8 we discuss possible relationships between the theory of attractors of Hamiltonian nonlinear equations and Quantum Mechanics. The global attraction (0.0.15) was suggested by postulates of N. Bohr on transitions to *quantum stationary states* and by E. Schrödinger's definition of these quantum stationary states as solutions of type $\psi(x,t) = \psi(x)e^{-i\omega t}$ (see Chapter 8 for details). Our results confirm such attraction for *generic* $U(1)$-invariant nonlinear equations of type (3.1.1) and (3.1.18)–(3.1.20). However, for the semiclassical self-consistent Maxwell–Schrödinger system of Quantum Mechanics this attraction is still open challenging problem.

On related surveys. In conclusion let us mention the related surveys in this area [8, 11, 49]. The results on asymptotic stability of solitary manifolds were described in detail in [124] for linear equations coupled to a particle, and in [144] for the relativistically-invariant Ginzburg–Landau equations.

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Chapter 1

Global Attraction to Stationary States

In this chapter we present with details the results on global attraction to stationary states (0.0.7) for nonlinear Hamiltonian PDEs in infinite space.

In Section 1.2 we present the first result of this type obtained in [43, 44] for 1D wave equation coupled to one nonlinear oscillator ('the Lamb system'). The second result [45] for 1D wave equation coupled to several nonlinear oscillators is presented in Section 1.3, and the third result - for 1D wave equation coupled to a 'continuum of nonlinear oscilators' - is presented in Section 1.4.

In Sections 1.5 and 1.6 are presented the results [47] and [48] which concern respectively 3D wave equation and Maxwell's equations coupled to a charged relativistic particle.

Section 1.7 concerns the result [53] on three-dimensional wave equations with concentrated nonlinearity.

1.1 Free d'Alembert equation

The global attraction (0.0.7) can easily be demonstrated using the trivial (but instructive) example of the d'Alembert equation

$$\ddot{\psi}(x,t) = \psi''(x,t), \qquad x \in \mathbb{R}, \tag{1.1.1}$$

where $\dot{\psi} := \dfrac{\partial\psi}{\partial t}$, $\psi' := \dfrac{\partial\psi}{\partial x}$. All derivatives here and below are understood in the sense of distributions. This equation is formally equivalent to the Hamiltonian system

$$\dot{\psi}(t) = D_\pi \mathcal{H},\ \ \dot{\pi}(t) = -D_\psi \mathcal{H} \tag{1.1.2}$$

with Hamiltonian

$$\mathcal{H}(\psi,\pi) = \frac{1}{2}\int [|\pi(x)|^2 + |\psi'(x)|^2]\,dx, \qquad (\psi,\pi) \in \mathcal{E} := H^1_c(\mathbb{R}) \oplus [L^2(\mathbb{R}) \cap L^1(\mathbb{R})], \tag{1.1.3}$$

where $H^1_c(\mathbb{R})$ is the Hilbert space of continuous functions $\psi(x)$ with finite norm

$$\|\psi\|_{H^1_c(\mathbb{R})} := \|\psi'\|_{L^2(\mathbb{R})} + |\psi(0)|. \tag{1.1.4}$$

Let us consider solutions $(\psi(x,t),\pi(x,t))$ of (1.1.2) with initial states $(\psi(x,0),\pi(x,0)) = (\psi_0(x),\pi_0(x)) \in \mathcal{E}$. Let us assume moreover, that

$$\psi_0(x) \to C_\pm, \quad x \to \pm\infty. \tag{1.1.5}$$

For such initial data the d'Alembert formula gives

$$\begin{aligned}\psi(x,t) &= \frac{\psi_0(x+t)+\psi_0(x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\pi_0(y)dy\\ &\xrightarrow[t\to\pm\infty]{} S_\pm(x)=\frac{C_+ + C_-}{2}\pm\frac{1}{2}\int_{-\infty}^{\infty}\pi_0(y)dy,\end{aligned} \tag{1.1.6}$$

where the convergence is uniform on every finite interval $|x|<R$. Moreover,

$$\dot\psi(x,t)=\frac{\psi_0'(x+t)-\psi_0'(x-t)}{2}+\frac{\pi_0(x+t)+\pi_0(x-t)}{2}\to 0, \qquad t\to\pm\infty, \tag{1.1.7}$$

where the convergence holds in $L^2(-R,R)$ for each $R>0$. Thus, the set of stationary states $(\psi(x),\pi(x))=(C,0)$, where $C\in\mathbb{R}$ is any constant, is an attractor. Note that for positive and negative times the limits (1.1.6) may be different.

1.2 A string coupled to a nonlinear oscillator

In this section we present the first results on global attraction to stationary states (0.0.7) for nonlinear Hamiltonian PDEs obtained in [43, 44] (and developed in [50]) for the *nonlinear Lamb system* with a point nonlinearity:

$$\begin{cases} \ddot\psi(x,t) &= \psi''(x,t), \quad x\in\mathbb{R}\setminus\{0\}, \\ m\ddot y(t) &= F(y(t))+\psi'(+0,t)-\psi'(-0,t);\ y(t)\equiv\psi(0,t), \end{cases} \tag{1.2.1}$$

where $m>0$. Solutions $\psi(x,t)$ take the values in $\mathbb{R}^d$ with $d\ge 1$. This system can formally be written as the nonlinear wave equation

$$(1+m\delta(x))\ddot\psi(x,t)=\psi''(x,t)+\delta(x)F(\psi(0,t)), \qquad x\in\mathbb{R}. \tag{1.2.2}$$

The problem (1.2.1) describes small crosswise oscillations of an infinite string stretched parallel to the x-axis; a particleof mass $m>0$ is attached to the string at the point $x=0$; $F(y)$ is an external (nonlinear) force perpendicular to the string, the force subjects the particle(see Fig. 1.1)

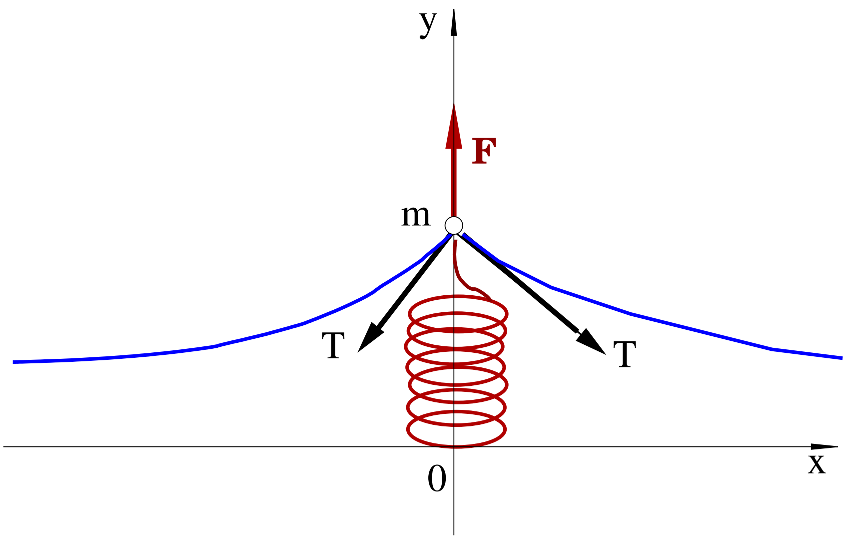

Figure 1.1: String coupled to an oscillator.

The system (1.2.1) has been introduced originally by H. Lamb [54] in the linear case when $F(y)=-\omega^2 y$. The Lamb system with nonlinear force $F(y)$ has been considered in [42] where the questions of irreversibility and nonrecurrence were discussed. The system was studied further in [43, 44, 50] where the global attraction to stationary states has been established for the first time, and in [41] where metastable regimes were studied for the stochastic Lamb system with a white noise.

The Lamb system (1.2.1) is a simplest nontrivial nonlinear time-reversible infinite-dimensional Hamiltonian system allowing an effective analysis of various questions.

Our main results for this system are as follows. Here we establish the existence of a finite-dimensional global attractor and establish the nonlinear scattering:

Each finite energy solution decays in long-time limits
to a sum of a stationary state and a dispersion wave.

The asymptotics hold in global energy norm. Moreover, in [51, 52] we have established the asymptotic completeness of the corresponding nonlinear scattering operators.

We consider the Cauchy problem for the system (1.2.1) with the initial conditions

$$\psi|_{t=0} = \psi_0(x), \qquad \dot\psi|_{t=0} = \pi_0(x), \qquad \dot y|_{t=0} = p_0. \tag{1.2.3}$$

Denote $Y(t) = (\psi(x,t), \dot\psi(x,t), \dot y(t))$. Then the Cauchy problem (1.2.1), (1.2.3) can be written as

$$\dot Y(t) = \mathbf{F}(Y(t)) \ \text{ for } \ t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{1.2.4}$$

where $Y_0 = (\psi_0, v_0, p_0)$, and

$$\mathbf{F}(Y(t)) = (\dot\psi(\cdot,t), \psi''(x,t)|_{x\neq 0}, F(\psi(0,t)) + \psi'(+0,t) - \psi'(-0,t)).$$

An exact statement of the Cauchy problem will be formulated in next section.

We will establish the scattering asymptotics

$$Y(t) \sim S_\pm + \tilde W(t)\Psi_\pm, \qquad t \to \pm\infty \tag{1.2.5}$$

where $S_\pm$ are some stationary states of the system (1.2.1), $\tilde W(t)$ is the dynamical group of the free wave equation, and $\Psi_\pm \in \mathcal{E}$ are the corresponding scattering states. The asymptotics (1.2.5) holds if the following limits exist:

$$\psi_0^+ := \lim_{x\to+\infty} \psi_0(x), \quad \psi_0^- := \lim_{x\to-\infty} \psi_0(x), \quad I_0 := \int_{-\infty}^{\infty} \pi_0(y)dy. \tag{1.2.6}$$

1.2.1 Hilbert phase space and dynamics

Let us introduce a Hilbert phase space $\mathcal{E}$ of finite energy states for the system (1.2.1). Denote by $\|\cdot\|$ resp. $\|\cdot\|_R$ the norm in the Hilbert space $L^2 := L^2(\mathbb{R},\mathbb{R}^d)$ resp. $L^2([-R,R],\mathbb{R}^d)$, and by $E_c := H_c^1(\mathbb{R}) \otimes \mathbb{R}^d$, where $H_c^1(\mathbb{R})$ is the Hilbert space with the norm (1.1.4).

Definition 1.2.1. *i) $\mathcal{E}$ is the Hilbert space of triples $(\psi(x), \pi(x), p) \in E_c \oplus L^2 \oplus \mathbb{R}^d$ with finite energy norm*

$$\|(\psi, v, p)\|_{\mathcal{E}} = \|\psi\|_{E_c} + \|\pi\| + |p| = \|\psi'\|_R + |\psi(0)| + \|\pi\| + |p|. \tag{1.2.7}$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the topology defined by the local energy seminorms

$$\|(\psi,\pi,p)\|_{\mathcal{E},R} \equiv \|\psi'\|_R + |\psi(0)| + \|\pi\|_R + |p|, \quad R > 0. \tag{1.2.8}$$

The space $\mathcal{E}_F$ is not complete, and convergence in $\mathcal{E}_F$ is equivalent to convergence in the metric

$$\text{dist}[Y_1, Y_2] = \sum_1^\infty 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathcal{E},R}}{1 + \|Y_1 - Y_2\|_{\mathcal{E},R}}, \quad Y_1, Y_2 \in \mathcal{E}. \tag{1.2.9}$$

We assume that

$$F(y) \in C^1(\mathbb{R}^d, \mathbb{R}^d), \qquad F(y) = -\nabla V(y) \tag{1.2.10}$$

$$V(y) \to +\infty, \qquad |u| \to \infty. \tag{1.2.11}$$

In this case the system (1.2.1) is formally Hamiltonian with the Hilbert phase space $\mathcal{E}$ and the Hamiltonian functional

$$\mathcal{H}(\psi,\pi,p)=\frac{1}{2}\int[|\pi(x)|^2+|\psi'(x)|^2]dx+m\frac{|p|^2}{2}+V(\psi(0)) \qquad (1.2.12)$$

for $(\psi,\pi,p)\in\mathcal{E}$. We consider solutions $\psi(x,t)$ such that $Y(t)=(\psi(\cdot,t),\dot\psi(\cdot,t),\dot y(t))\in C(\mathbb{R},\mathcal{E})$, where $y(t):=\psi(0,t)$.

Let us discuss definition of the Cauchy problem (1.2.1), (1.2.3) for the trajectories $Y(t)\in C(\mathbb{R},\mathcal{E})$. At first, $\psi\in C(\mathbb{R}^2,\mathbb{R}^d)$ for $Y(t)\in C(\mathbb{R},\mathcal{E})$. Hence, the first equation in (1.2.1) is equivalent to the d'Alembert decomposition

$$\psi(x,t)=f_\pm(x-t)+g_\pm(x+t),\quad \pm x>0, \qquad (1.2.13)$$

where

$$f_\pm,g_\pm\in C(\mathbb{R},\mathbb{R}^d).$$

Therefore,

$$\dot\psi(x,t)=-f'_\pm(x-t)+g'_\pm(x+t),\qquad \psi'(x,t)=f'_\pm(x-t)+g'_\pm(x+t)\ \text{ for }\ \pm x>0,$$

where all the derivatives are understood in the sense of distributions. The assumption $Y(t)\in C(\mathbb{R},\mathcal{E})$ implies

$$f'_\pm,g'_\pm\in L^2_{loc}(\mathbb{R},\mathbb{R}^d).$$

Now we explain the second equation of (1.2.1).

Definition 1.2.2. *In the second equation of (1.2.1) we set*

$$\psi'(0\pm,t):=f'_\pm(-t)+g'_\pm(t)\in L^2_{loc}(\mathbb{R},\mathbb{R}^d), \qquad (1.2.14)$$

while the derivative $\ddot y(t)$ of $y(t)\equiv\psi(0,t)\in C(\mathbb{R},\mathbb{R}^d)$ is understood in the sense of distributions.

Note that the functions $f_\pm$ and $g_\pm$ in (1.2.13) are unique up to an additive constant. Hence definition (1.2.14) is unambiguous.

Proposition 1.2.3. *(cf. [44]) Let the conditions (1.2.10), (1.2.11) hold, $m>0$, and $Y_0\in\mathcal{E}$. Then*
i) The Cauchy problem (1.2.4) admits a unique solution $Y(t)\in C(\mathbb{R},\mathcal{E})$.
ii) The map $U(t):Y_0\mapsto Y(t)$ is continuous in $\mathcal{E}$ and in $\mathcal{E}_F$.
iii) The energy is conserved,

$$\mathcal{H}(Y(t))=\text{const},\qquad t\in\mathbb{R}. \qquad (1.2.15)$$

iv) The a priori bounds hold,

$$\sup_{t\in\mathbb{R}}\|Y(t)\|_{\mathcal{E}}<\infty.$$

1.2.2 Main results

The stationary states $S = (s(x), 0, 0) \in \mathcal{E}$ for the system (1.2.1) are evidently determined: the set $\mathcal{S}$ of all stationary states $S \in \mathcal{E}$ is given by

$$\mathcal{S} = \{S_z = (z, 0, 0) : \quad z \in Z\}, \qquad \text{where} \quad Z := \{z \in \mathbb{R}^d : \ F(z) = 0\}. \tag{1.2.16}$$

The next theorem means that the set $\mathcal{S}$ is the global point attractor of the system (1.2.1) in the space $\mathcal{E}_F$.

Theorem 1.2.4. *(cf. [43, 44]) Let all assumptions of Proposition 1.2.3 hold and an initial state $Y_0 \in \mathcal{E}$. Then*
i) The corresponding solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem (1.2.4) attracts to the set $\mathcal{S}$,

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty \tag{1.2.17}$$

in the metric (1.2.9). This means that

$$\text{dist}[Y(t), \mathcal{S}] := \inf_{S \in \mathcal{S}} \text{dist}[Y(t), S] \to 0, \qquad t \to \pm\infty. \tag{1.2.18}$$

ii) Suppose additionally that the set Z is a discrete subset in $\mathbb{R}^d$. Then any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ attracts to some stationary states $S_\pm \in \mathcal{S}$ depending on the solution,

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \qquad t \to \pm\infty. \tag{1.2.19}$$

as is illustrated in Figure 1.

Remarks 1.2.5. i) The discreteness of the set Z is essential for the global attraction to stationary states (1.2.19). For example, let us consider the nonlinearity which vanishes on a C^1-submanifold of $\mathbb{R}^d$,

$$F(\psi) \equiv 0, \qquad \psi \in I. \tag{1.2.20}$$

Then in the case $m = 0$ any smooth function $\psi(x, t)$ with values in I is the solution to the system (1.2.1). In particular, for $d = 1$ and $I = [-1, 1]$ we can take the function

$$\psi(x, t) = \sin\log(|x - t| + 2), \qquad (x, t) \in \mathbb{R}^2. \tag{1.2.21}$$

In this case the function $(\psi(x, t), \dot{\psi}(x, t), \psi(0, t)) \in C(\mathbb{R}, \mathcal{E})$ is the solution to equation (1.2.4) with $m = 0$, and for this solution the attraction to stationary states (1.2.19) obviously breaks down. On the other hand, (1.2.17) for this solution holds. For $m > 0$ similar examples also can be easily constructed, see [44].
ii) The 'weak convergence' (1.2.19) and (1.2.11), (1.2.12) imply (0.0.10) by the Fatou lemma.

Further, let us denote $\mathcal{E}_0 = \{(\psi, v, 0) \in \mathcal{E}\}$, and $\tilde{W}(t)(\psi, v, 0) := (W(t)(\psi, v), 0)$, where $W(t)$ is the dynamical group of free wave equation (1.1.1).

Theorem 1.2.6. *([50]) Let all assumptions of Proposition 1.2.3 hold, and additionally, the finite limits (1.2.6) exist. Then the* **scattering asymptotics** *hold*

$$Y(t) = S_\pm + \tilde{W}(t)\Psi_\pm + r_\pm(t) \tag{1.2.22}$$

with $S_\pm \in \mathcal{S}$, and some asymptotic states $\Psi_\pm \in \mathcal{E}_0$; the remainder is small in the **global energy norm** *(1.2.7):*

$$\|r_\pm(t)\|_{\mathcal{E}} \to 0, \ \ t \to \pm\infty. \tag{1.2.23}$$

In [51, 52] the asymptotic completeness of the corresponding nonlinear scattering operator $S : \Psi_- \mapsto \Psi_+$ has been proved for equation (1.2.1) in the case $m = 0$.

1.2.3 Well-posedness

Recall briefly the proof of Proposition 1.2.3 from [44] since we need some constructions later, in the proofs of Theorems 1.2.4 and 1.2.6.

The construction of solutions relies on the d'Alembert representation (1.2.13). For $\pm z > 0$ the functions $f_\pm(z)$ and $g_\pm(z)$ are defined by the d'Alembert formulas

$$f_\pm(z) := \frac{\psi_0(z)}{2} - \frac{1}{2}\int_0^z v_0(y)\,dy, \quad g_\pm(z) := \frac{\psi_0(z)}{2} + \frac{1}{2}\int_0^z v_0(y)\,dy, \quad \pm z > 0. \tag{1.2.24}$$

These formulas imply that

$$f'_\pm(z), g'_\pm(z) \in L^2(\mathbb{R}^\pm, \mathbb{R}^d) \tag{1.2.25}$$

since $(\psi_0, v_0) \in \mathcal{E}$. The *reflected outgoing waves* $f_+(z)$ for $z < 0$ and $g_-(z)$ for $z > 0$ are given by

$$f_+(-t) := y(t) - g_+(t), \qquad g_-(t) := y(t) - f_-(-t), \qquad t > 0 \tag{1.2.26}$$

due to the gluing conditions $y(t) := \psi(0,t) = f_+(-t) + g_+(t) = f_-(-t) + g_-(t)$. Hence,

$$\psi(x,t) = \left\{ \begin{array}{ll} y(t-x) + g_+(x+t) - g_+(t-x), & 0 < x < t \\ y(t+x) + f_-(x-t) - f_-(-x-t), & -t < x < 0 \end{array} \right| \quad t > 0. \tag{1.2.27}$$

Substituting these representations into the second equation of (1.2.1), we immediately get the *reduced equation* for the oscillator,

$$m\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{\rm in}(t), \qquad t > 0; \quad y(0) = \psi_0(0); \; \dot{y}(0) = p_0, \tag{1.2.28}$$

where

$$w_{\rm in}(t) = g_+(t) + f_-(-t), \qquad t > 0, \tag{1.2.29}$$

is the *incident wave*. Multiplying equation (1.2.28) by $\dot{y}(t)$ and integrating, we get the energy balance

$$\frac{m\dot{y}^2(t)}{2} + U(y(t)) = \frac{m\dot{y}^2(0)}{2} + U(y(0)) - 2\int_0^t \dot{y}^2(s)ds + 2\int_0^t \dot{w}_{\rm in}(s)\dot{y}(s)ds \tag{1.2.30}$$

Note that

$$\dot{w}_{\rm in} \in L^2(\mathbb{R}^+, \mathbb{R}^d) \tag{1.2.31}$$

by (1.2.25). Hence (1.2.30) and (1.2.11) imply that the Cauchy problem (1.2.28) admits a unique solution for all $t > 0$, and the a priori bound holds:

$$\sup_{t>0}|y(t)| + \sup_{t>0}|\dot{y}(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \le B < \infty, \tag{1.2.32}$$

where B is bounded for bounded norm $\|(\psi_0, v_0, p_0)\|_{\mathcal{E}}$. These arguments imply that the Cauchy problem (1.2.4) admits a unique solution $Y(t) = (\psi(x,t), \dot{\psi}(x,t), \dot{y}(t)) \in C(\mathbb{R}, \mathcal{E})$ for any $Y_0 \in \mathcal{E}$, where $\psi(x,t)$ is defined by (1.2.13), (1.2.24), and (1.2.27) (see [44]).

The a priori bound (1.2.32) implies that $y(t) \in C(\overline{\mathbb{R}^+})$. Hence $y(0)$ exists and

$$f_+(-0) = f_+(0), \qquad g_-(-0) = g_-(+0) \tag{1.2.33}$$

since

$$f_+(-0) = y(0) - g_+(0) = \frac{\psi_0(0)}{2}, \qquad f_+(+0) = \frac{\psi_0(0)}{2} \tag{1.2.34}$$

and

$$g_-(-0) = \frac{\psi_0(0)}{2}, \qquad g_-(+0) = y(0) - f_-(-0) = \frac{\psi_0(0)}{2} \tag{1.2.35}$$

by (1.2.26) and (1.2.24).

Corollary 1.2.7. *(1.2.32) and (1.2.26) imply that*

$$f'_+ \in L^2(\mathbb{R}^-, \mathbb{R}^d), \qquad g'_- \in L^2(\mathbb{R}^+, \mathbb{R}^d) \tag{1.2.36}$$

by (1.2.25). Hence, (1.2.33) implies that

$$f'_+, g'_- \in L^2(\mathbb{R}, \mathbb{R}^d). \tag{1.2.37}$$

The formulas (1.2.24) and (1.2.27) determine the solution $\psi(x,t)$ uniquely, and $Y(t) := (\psi(x,t), \dot\psi(x,t), \psi(0,t)) \in C(\mathbb{R}, \mathcal{E})$ due to (1.2.37). Finally, the energy conservation (1.2.15) follows by differentiation, see [44]. Now Proposition 1.2.3 is proved.

Remark 1.2.8. *In the energy balance (1.2.30) the integral* $2\displaystyle\int_0^t \dot y^2(s)ds$ *is the energy radiated by the oscillator over the time interval* $[0,t]$.

1.2.4 A relaxation for reduced equation

The following lemma on relaxation for the reduced equation plays a crucial role in the proofs of Theorem 1.2.4 and Theorem 1.2.6. Let us denote $\mathcal{Z} = \{(z,0) \in \mathbb{R}^d \times \mathbb{R}^d : \ z \in Z\}$.

Lemma 1.2.9. *Let all assumptions of Theorem 1.2.4 hold. Then*
i) For every solution $y(t)$ of the equation (1.2.28)

$$(y(t), \dot y(t)) \to \mathcal{Z}, \qquad t \to \infty. \tag{1.2.38}$$

ii) Let, additionally, Z be a discrete subset in $\mathbb{R}^d$. Then there exists a point $(z,0) \in \mathcal{Z}$ such that

$$(y(t), \dot y(t)) \to (z, 0), \qquad t \to \infty.$$

Proof. Obviously, *ii)* follows from *i)*. Let us check that *i)* follows from (1.2.32). Namely, (1.2.38) is equivalent to the system

$$y(t) \ \to \ Z, \qquad t \to \infty, \tag{1.2.39}$$

$$\dot y(t) \ \to \ 0, \qquad t \to \infty. \tag{1.2.40}$$

- First, let us prove (1.2.40). Assume the contrary, that

$$|\dot y(t_k)| \ge \varepsilon > 0 \tag{1.2.41}$$

for a sequence $t_k \to \infty$. Integrating the equation (1.2.28), we get that

$$m(\dot y(t) - \dot y(s)) = \int_s^t F(y(\tau))d\tau - 2\int_s^t \dot y(\tau)d\tau + 2\int_s^t \dot w_{in}(\tau)d\tau, \qquad s,t \ge 0. \tag{1.2.42}$$

Let us estimate each of three integrals in the RHS. The first is $\mathcal{O}(|t-s|)$ since $y(\tau)$ is a bounded function by (1.2.32). The second and third integrals are $\mathcal{O}(|t-s|^{1/2})$ by (1.2.32), (1.2.31) and the Cauchy-Schwartz inequality. Hence, (1.2.42) implies that $\dot{y}(t)$ is a Hölder function of degree 1/2, i.e.

$$|\dot{y}(t)-\dot{y}(s)| \le C|t-s|^{1/2}, \qquad s,t \ge 0, \quad |t-s| \le 1.$$

Therefore, $\int_0^\infty \dot{y}^2(t)dt = \infty$ by (1.2.41) which contradicts (1.2.32).

• Now we can prove (1.2.39). Again assume the contrary. Then

$$F(y(t_k)) \to \overline{F} \neq 0$$

for a sequence $t_k \to \infty$ since $y(t)$ is a bounded function. Moreover, (1.2.40) implies the uniform convergence

$$F(y(\tau)) \to \overline{F}, \quad |\tau - t_k| \le T$$

for any $T > 0$. Now (1.2.42) and (1.2.40), (1.2.31) imply that

$$m(\dot{y}(t_k+T) - \dot{y}(t_k-T)) = 2T\overline{F} + o(1), \qquad t_k \to \infty,$$

which contradicts (1.2.40) since $\overline{F} \neq 0$. □

1.2.5 Examples

Let us illustrate Lemma 1.2.9 by an example. For simplicity let us assume that

$$\psi_0(x) = C_\pm, \quad v_0(x) = 0, qquad \pm x > r_0$$

with some $C_\pm \in \mathbb{R}$ and $r_0 \ge 0$. Then (1.2.29) implies that $\dot{w}(t) \equiv 0$ for $t > r_0$ and (1.2.28) is an autonomous equation for $t > r_0$. In the phase plane $(y, \dot{y})$ the orbits of the reduced equation (1.2.28) are determined by the following system:

$$\dot{y}(t) = v(t), \qquad m\dot{v}(t) = F(y(t)) - 2v(t), \qquad t > r_0. \tag{1.2.43}$$

Let us compare this system with a *free* oscillator which is not coupled to the string,

$$\dot{y} = v, \qquad m\dot{v} = F(y). \tag{1.2.44}$$

There are simple relationships between phase portraits of these two systems.

A These system have the same stationary points.

B The vertical component $\dot{v}$ of the phase velocity vector of (1.2.43) is less than that of (1.2.44) if $v > 0$, and is greater if $v < 0$. The horizontal components of these vectors are equal.

C Hence the orbits of (1.2.43) intersect those of (1.2.44) from above in the halfplane $v > 0$ and from below in the halfplane $v < 0$. Let us consider for instance a nondegenerate potential of Ginzburg–Landau type

$$V(y) = \frac{1}{4}(y^2-1)^2, \quad y \in \mathbb{R} \tag{1.2.45}$$

It satisfies conditions (1.2.10) and (1.2.11). Then the system (1.2.44) has the following orbits:

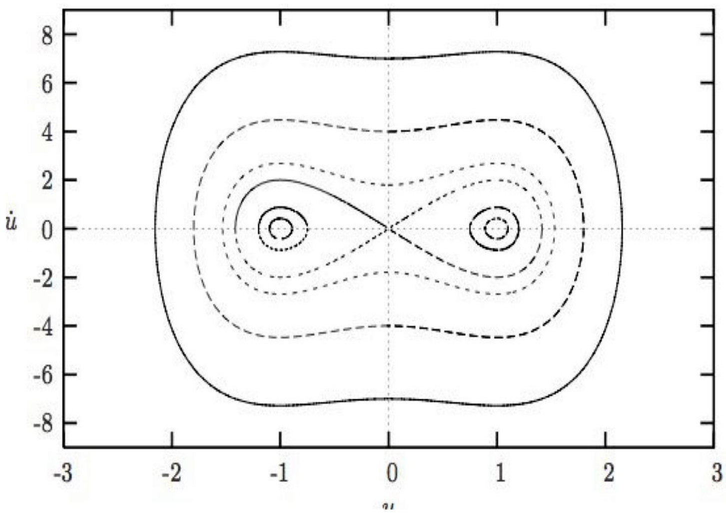

Figure 1.2: Hamiltonian system

- closed curves corresponding to periodic solutions,
- two separatrices both leaving and entering the point $(0, 0)$,
- three stationary points: a saddle at the point $(0, 0)$ and two centers at the points

$(\pm 1, 0)$, see Fig. 1.2.5. Taking into account the property **C**, we see that for the system (1.2.43) with potential (1.2.45):

- the points $(\pm 1, 0)$ are stable foci,
- the point $(0, 0)$ is a saddle, see Fig. 1.2.5.

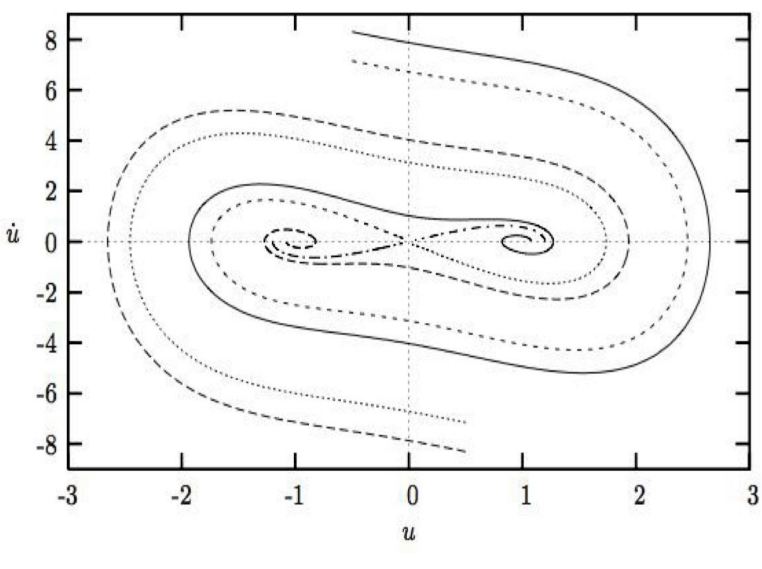

Figure 1.3: System with a friction.

1.2.6 Convergence to global attractor

Now we can prove Theorem 1.2.4. It suffices to prove it for $t \to \infty$.

Lemma 1.2.10. *Let all the assumptions of Theorem 1.2.4 hold. Then* $Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}$ *as* $t \to \infty$.

Proof. It suffices to construct $z(t) \in Z$ for $t \geq 0$ such that

$$\|Y(t) - S_{z(t)}\|_R \to 0 \text{ as } t \to \infty.$$

The convergence (1.2.39) means that there exists a function $z(t) \in Z$, $t \geq 0$, such that

$$|y(t) - z(t)| \to 0, \qquad t \to \infty. \tag{1.2.46}$$

By definitions (1.2.8) and (1.2.16),

$$\|Y(t) - S_{z(t)}\|_R = \|\psi'(\cdot,t)\|_R + |\psi(0,t) - z(t)| + \|\dot{\psi}(\cdot,t)\|_R + |\dot{y}(t)|.$$

Here both norms $\|...\|_R \to 0$ due to (1.2.13), (1.2.25), (1.2.36) and (1.2.37). Therefore, (1.2.46) and (1.2.40) complete the proof. □

Now Theorem 1.2.4 i) is proved. Then Theorem 1.2.4 ii) follows since the set $\mathcal{S}$, isomorphic to Z, is discrete.

Remark 1.2.11. *The bound (1.2.32) is provided by the friction term in the reduced equation (1.2.28) for the nonlinear oscillator. The friction means the energy radiation by the oscillator, and the integral in (1.2.32) represents the energy radiated to infinity. Thus, our proof of Theorem 1.2.4 relies on the energy radiation to infinity.*

1.2.7 The transitivity of the transitions

The next lemma shows that the transitions of type (0.0.8) exist for any two stationary states $S_\pm$.

Lemma 1.2.12. *Let conditions of Theorem 1.2.4 hold. Then for every two stationary states $S_\pm \in \mathcal{S}$ there exist solutions $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the system (1.7.13), intertwining $S_\pm$ in the sense (1.2.19).*

Proof. Let $S_\pm = (s_\pm(x), 0, 0)$ with $s_\pm(x) \equiv z_\pm \in Z$. It is possible to provide the transition $S_- \mapsto S_+$ in different ways. We choose one of them, which is possibly most obvious. Namely, we construct a solution $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), y(t)) \in C(\mathbb{R}, \mathcal{E})$ to (1.7.13) such that

$$y(t) := u(0,t) = \begin{cases} z_- & \text{for} \quad t \leq -1, \\ z_+ & \text{for} \quad t \geq 1. \end{cases} \tag{1.2.47}$$

We extend $y(t)$ for $t \in (-1, 1)$ arbitrarily so that $y \in C^2(\mathbb{R}, \mathbb{R}^d)$. Then we set $g_+(z) \equiv z_-$ and determine f_- by (1.2.28):

$$m\ddot{y}(t) = F(y(t)) + 2\big(f'_-(-t) - \dot{y}(t)\big), \quad t \in \mathbb{R}. \tag{1.2.48}$$

Then $f'_-(z) \in C(\mathbb{R}, \mathbb{R}^d)$. Since $F(z_\pm) = 0$, we have

$$f'_-(-t) \equiv 0 \quad \text{for} \quad t \leq -1 \quad \text{and for} \quad t \geq 1. \tag{1.2.49}$$

To determine f_- uniquely, we may require that

$$f_-(-t) \equiv z_- \quad \text{for} \quad t \leq -1. \tag{1.2.50}$$

Then the reflected waves g_- and f_+ are determined by (1.2.26).

Since $y(t)$, $f_-(-t)$, and $g_+(t)$ are constant for large $|t|$, $f_+(-t)$, $g_-(t)$ are also constant for large $|t|$. Then for $u(x,t)$ defined by (1.2.27), the function

$$Y(t) = (u(\cdot,t), \dot{u}(\cdot,t), \dot{u}(0,t)) \in C(\mathbb{R}, \mathcal{E})$$

is a solution to (1.2.1), and (1.2.19) holds. □

Remark 1.2.13. Physically, the inequality $z_+ \neq z_-$ means the capture of radiation by the oscillator if $V(z_+) > V(z_-)$, or the emission of radiation by the oscillator if $V(z_+) < V(z_-)$.

1.2.8 Divergent wave

Here we prove Theorem 1.2.6. First, let us construct the divergent wave

$$\tilde{W}(t)\Psi_+ = (\mathrm{w}_{\mathrm{out}}(x,t), \dot{\mathrm{w}}_{\mathrm{out}}(x,t), 0), \quad t \geq 0.$$

Here $\mathrm{w}_{\mathrm{out}}(\mathrm{x},\mathrm{t})$ is a finite energy solution to the free d'Alembert equation. Let us set

$$\mathrm{w}_{\mathrm{out}}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \tag{1.2.51}$$

where the constant C_0 will be chosen below. It remains to check (1.2.22) and (1.2.23) for $t \to \infty$ that means the representation

$$(\psi(x,t), \dot{\psi}(x,t), \dot{y}(t)) = (s_+(x), 0, 0) + (\mathrm{w}_{\mathrm{out}}(x,t), \dot{\mathrm{w}}_{\mathrm{out}}(x,t), 0) + r_+(t), \quad t > 0,$$

where

$$s_+(x) \equiv z_+ := \lim_{t\to+\infty} y(t), \tag{1.2.52}$$

and

$$\|r_+(t)\|_{\mathcal{E}} \to 0, \quad t \to +\infty. \tag{1.2.53}$$

By definition of the norm (1.2.7), (1.2.53) is equivalent to

$$\begin{aligned} \|\psi'(\cdot,t) - w'_{\mathrm{out}}(\cdot,t)\|_{L^2(\mathbb{R},\mathbb{R}^d)} + &\ |\psi(0,t) - z_+ - \mathrm{w}_{\mathrm{out}}(0,t)| \\ + &\ \|\dot{\psi}(\cdot,t) - \dot{\mathrm{w}}_{\mathrm{out}}(\cdot,t)\|_{L^2(\mathbb{R},\mathbb{R}^d)} \to 0, \quad t \to \infty \end{aligned} \tag{1.2.54}$$

since $\dot{y}(t) \to 0$ by (1.2.40).
Step i) Let us start with the second term in the LHS of (1.2.54). Since $\psi(0,t) = y(t) \to z_+$, it suffices to prove that

$$\mathrm{w}_{\mathrm{out}}(0,t) = C_0 + f_+(-t) + g_-(t) \to 0, \quad t \to +\infty. \tag{1.2.55}$$

First, (1.2.6) and (1.2.24) imply that

$$\lim_{t\to\infty} f_-(-t) = \frac{\psi_0^-}{2} - \frac{1}{2}\int_0^{-\infty} v_0(y)dy, \qquad \lim_{t\to+\infty} g_+(t) = \frac{\psi_0^+}{2} + \frac{1}{2}\int_0^{\infty} v_0(y)dy. \tag{1.2.56}$$

Second, we have by (1.2.26) and (1.2.52) that

$$\lim_{t\to\infty} f_+(-t) = z_+ - \lim_{t\to+\infty} g_+(t); \quad \lim_{t\to+\infty} g_-(t) = z_+ - \lim_{t\to\infty} f_-(-t).$$

Substituting (1.2.56), we obtain

$$\left\{ \begin{array}{rcl} \lim\limits_{t\to\infty} f_+(-t) & = & z_+ - \dfrac{\psi_0^+}{2} - \dfrac{1}{2}\displaystyle\int_0^{\infty} v_0(y)dy, \\ \lim\limits_{t\to+\infty} g_-(t) & = & z_+ - \dfrac{\psi_0^-}{2} + \dfrac{1}{2}\displaystyle\int_0^{-\infty} v_0(y)dy. \end{array} \right.$$

Hence, (1.2.55) holds if we choose

$$C_0 := \frac{\psi_0^+}{2} + \frac{\psi_0^-}{2} + \frac{I_0}{2} - 2z_+, \qquad (1.2.57)$$

where I_0 is defined in (1.2.6).

Step ii) Now, let us consider the first term in the LHS of (1.2.54). It suffices to prove for example that

$$\|\psi'(\cdot,t) - \mathrm{w}'_{\mathrm{out}}(\cdot,t)\|_{L^2(\mathbb{R}^+,\mathbb{R}^d)} \to 0, \qquad t \to \infty.$$

Using (1.2.51) and the d'Alembert representation (1.2.13) for $x > 0$, we get

$$\psi'(x,t) - \mathrm{w}'_{\mathrm{out}}(x,t) = g'_+(x+t) - g'_-(x+t), \qquad x \geq t.$$

Finally, (1.2.25) and (1.2.36) imply that

$$\begin{aligned} \|g'_+(x+t) - g'_-(x+t)\|^2_{L^2(\mathbb{R}^+,\mathbb{R}^d)} &\leq C\int_0^\infty \Big[|g'_+(x+t)|^2 + |g'_-(x+t)|^2\Big]dx \\ &= C\int_t^\infty \Big[|g'_+(z)|^2 + |g'_-(z)|^2\Big]dz \to 0, \ \ t \to \infty. \end{aligned}$$

Step iii) The third term in the LHS of (1.2.54) can be handled similarly. Theorem 1.2.6 is proved.

1.3 String coupled to several nonlinear oscillators

Here we present the results [45], which extend the results of previous section 1.2 to the case of a string with several nonlinear oscillators:

$$\ddot{\psi}(x,t) = \psi''(x,t) + \sum_1^N \delta(x - x_k) F_k(\psi(x_k,t)), \qquad x \in \mathbb{R}.$$

This equation reduces to a system of N ordinary differential equations with delay. Its study required new approach relying on a special analysis of a *relaxation* of all trajectories.

1.3.1 Introduction

Let $Q = \{x_1, \dots, x_N\}$ be a finite set of N points $x_k \in \mathbb{R}$. We establish global attraction to stationary states (0.0.7) for all finite energy solutions to the system of equation

$$\ddot{\psi}(x,t) = \psi''(x,t), \quad x \in \mathbb{R} \setminus Q \tag{1.3.1}$$

together with the gluing conditions at the points $x_k \in Q$,

$$\left\{ \begin{array}{rcl} \psi(x_k + 0, t) & = & \psi(x_k - 0, t) \\ \\ 0 & = & F_k(\psi(x_k,t)) + \psi'(x_k + 0, t) - \psi'(x_k - 0, t) \end{array} \right| . \tag{1.3.2}$$

In the case $N = 1$ this system coincides with the Lamb system (1.2.1) with $m = 0$. The solutions $\psi(x,t)$ take the values in $\mathbb{R}^d$ with $d \geq 1$. Note that the system (1.3.1) is formally equivalent to the one-dimensional nonlinear wave equation with the nonlinear term concentrated at the set Q (cr. (1.2.2)),

$$\ddot{\psi}(x,t) = \psi''(x,t) + \sum_{k=1}^N \delta(x - x_k) F_k(\psi(x_k,t)), \qquad x \in \mathbb{R}. \tag{1.3.3}$$

Physically, the system (1.3.1), (1.3.2) describes small crosswise oscillations of a string which is subject to constraint forces F_k at the points x_k, the forces are perpendicular to the string. For example, $F_k(y) = -\omega_k^2 y$ if the string is attached to a linear spring at the point x_k, see Fig. 1.4. But in general the functions $F_k(y)$ are nonlinear.

We introduce the Hilbert phase space $\mathcal{E}$ of finite energy states for the system (1.3.1), (1.3.2).

Definition 1.3.1. *i) $\mathcal{E} = E_c \oplus L^2$ is the Hilbert space of pairs $(\psi(x), \pi(x))$, with the norm*

$$\|(\psi, \pi)\|_{\mathcal{E}} = \|\psi\|_{E_c} + \|\pi\|. \tag{1.3.4}$$

iii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the topology defined by the seminorms

$$\|(\psi, \pi)\|_R \equiv \|\psi'\|_R + |\psi(0)| + \|\pi\|_R, \quad R > 0. \tag{1.3.5}$$

We assume the following conditions,

$$\left\{ \begin{array}{l} \text{all } F_k \in C^1(\mathbb{R}^d, \mathbb{R}^d), \; F_k(\psi) = -\nabla V_k(\psi) \\ \\ \inf_{y \in \mathbb{R}^d} V_k(y) > -\infty, \qquad \forall k = 1, \dots, N \\ \\ V_k(y) \to +\infty \;\text{ as }\; |y| \to \infty \;\text{ for some } k = 1, \dots, N \end{array} \right| . \tag{1.3.6}$$

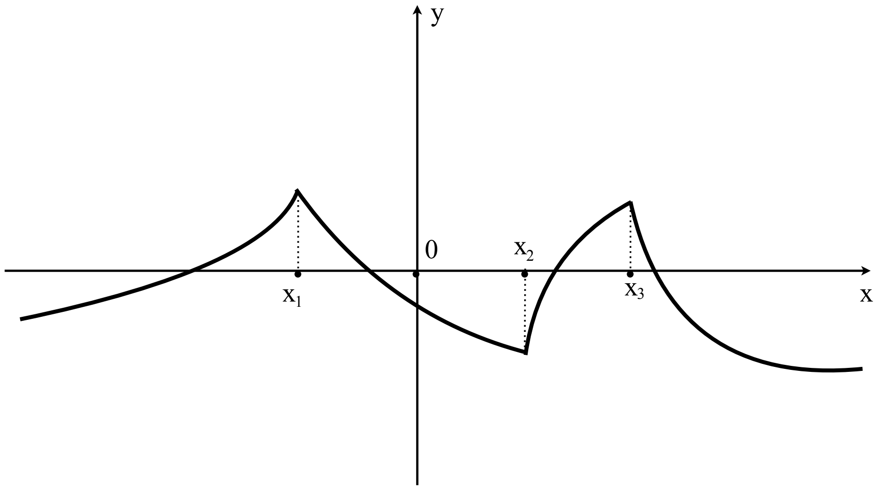

Figure 1.4: String coupled to nonlinear oscillators

Then the system (1.3.1), (1.3.2) is formally Hamiltonian with the Hilbert phase space $\mathcal{E}$ and the Hamiltonian functional

$$\mathcal{H}(\psi,\pi)=\frac{1}{2}\int_{\mathbb{R}}[|\pi(x)|^2+|\psi'(x)|^2]dx+\sum_{k=1}^{N}V_k(\psi(x_k)),\qquad(\psi,\pi)\in\mathcal{E}.\tag{1.3.7}$$

We consider solutions $Y(t)=(\psi(\cdot,t),\dot\psi(\cdot,t))\in C(\mathbb{R},\mathcal{E})$ and we write the system (1.3.1), (1.3.2) in the form

$$\dot Y(t)=\mathbf{F}(Y(t)),\qquad t\in\mathbb{R}.\tag{1.3.8}$$

Let us discuss the definition of the Cauchy problem for the functions $Y(t)\in C(\mathbb{R},\mathcal{E})$. The first equation of (1.3.2) makes sense and holds automatically because $\psi\in C(\mathbb{R}^2,\mathbb{R}^d)$ by the Sobolev embedding theorem due to $Y(t)\in C(\mathbb{R},\mathcal{E})$. The equation (1.3.1) is understood in the sense of distributions of $(x,t)\in[\mathbb{R}\setminus Q]\times\mathbb{R}$. Hence this equation is equivalent to the d'Alembert decompositions for every $k=1,\dots,N+1$,

$$\psi(x,t)=f_k(x-t)+g_k(x+t),\qquad x\in\Delta_k:=(x_{k-1},x_k),\ \ t\in\mathbb{R},\tag{1.3.9}$$

where $f_k,g_k\in C(\mathbb{R},\mathbb{R}^d)$ due to $\psi\in C(\mathbb{R}^2,\mathbb{R}^d)$, and we denote $x_0:=-\infty$ and $x_{N+1}=+\infty$. Hence, for all $k=1,\dots,N$ and $(x,t)\in\Delta_k\times\mathbb{R}$

$$\psi'(x,t)=f_k'(x-t)+g_k'(x+t),\quad \dot\psi(x,t)=-f_k'(x-t)+g_k'(x+t),\tag{1.3.10}$$

where all derivatives are understood in the sense of distributions. The assumption $Y(t)\in C(\mathbb{R},\mathcal{E})$ implies

$$f_k'(\cdot),g_k'(\cdot)\in L^2_{loc}(\mathbb{R},\mathbb{R}^d),\quad \forall k=1,\dots,N+1.\tag{1.3.11}$$

We now explain the second equation of (1.3.2).

Definition 1.3.2. *In the second equation of (1.3.2) for every $k=1,\dots,N$*

$$\left\{\begin{array}{rcl}\psi'(x_k-0,t)&:=&f_k'(x_k-t)+g_k'(x_k+t)\in L^2_{loc}(\mathbb{R},\mathbb{R}^d)\\[2ex] \psi'(x_k+0,t)&:=&f_{k+1}'(x_k-t)+g_{k+1}'(x_k+t)\in L^2_{loc}(\mathbb{R},\mathbb{R}^d)\end{array}\right| .\tag{1.3.12}$$

Note that the functions f_k and g_k in (1.3.9) are unique up to an additive constant. So the definition (1.3.12) is unambiguous.

1.3.2 Main results

We start with the existence of the dynamics.

Proposition 1.3.3. *Let $d \geq 1$ and assumptions (1.3.6) hold. Then*
i) For every initial state $Y(0) \in \mathcal{E}$ equation (1.3.8) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

ii) The mapping $W(t) : Y(0) \mapsto Y(t)$ is continuous in $\mathcal{E}$ and in $\mathcal{E}_F$ for every $t \in \mathbb{R}$.
iii) The energy (1.3.7) is conserved,

$$\mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{1.3.13}$$

This proposition will be proved in the next section.

Definition 1.3.4. *$\mathcal{S}$ denotes the set of all stationary states $S = (s(x), 0) \in \mathcal{E}$ of the system (1.3.8).*

The next proposition gives a criterion for the set $\mathcal{S}$ be a nonempty discrete subset of $\mathcal{E}_F$.

Proposition 1.3.5. *Let conditions (1.3.6) hold, $d = 1$, and all functions $F_k(y)$ with $k = 1, \ldots, N$ be real analytic on $\mathbb{R}$. Then $\mathcal{S}$ is a discrete subset of $\mathcal{E}_F$.*

The main result of this section means that the set $\mathcal{S}$ is the global attractor of the system (1.3.8) in the topology of the space $\mathcal{E}_F$.

Theorem 1.3.6. *Let $d \geq 1$, assumptions (1.3.6) hold and an initial state $Y(0) \in \mathcal{E}$. Then*
i) the corresponding solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of equation (1.3.8), attracts to the set $\mathcal{S}$ in the sense (1.2.18)

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty. \tag{1.3.14}$$

ii) Let, moreover, $d = 1$, and all functions $F_k(y_k)$ be real analytic on $\mathbb{R}$. Then any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ attracts to some stationary states $S_\pm \in \mathcal{S}$ depending on the solution,

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm, \qquad t \to \pm\infty. \tag{1.3.15}$$

Remarks 1.3.7. *i) The assertion ii) of this theorem follows from i) due to Proposition 1.3.5.*
ii) The convergence (1.3.15) and (1.3.7), (1.3.6) imply (0.0.10) by Fatou theorem.

1.3.3 Well-posedness and a priori estimates

Proof of Proposition 1.3.3. The solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to (1.3.8) can be constructed by the d'Alembert representations (1.3.9) similarly to the case $N = 1$, considered in Section 1.2. However for $N > 1$ we need to find repeatedly reflected waves from all points x_k with $k = 1, \ldots, N$. The energy conservation (1.3.13) follows by methods of [44] using the d'Alembert representations (1.3.9). □

Let us show that the energy conservation implies the following a priori estimate which we will need in the proof of Theorem 1.3.6.

Proposition 1.3.8. *Let the conditions (1.3.6) hold. Then for every solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of (1.3.8) all functions $y_k(t) := \psi(x_k, t)$, are bounded:*

$$\sup_{t\in\mathbb{R}} |y_k(t)| < \infty, \qquad k = 1, \dots, N. \tag{1.3.16}$$

Proof. We prove in fact a slightly stronger statement. Namely, denote $y_k = y_k(\psi) = \psi(x_k)$ and $\overline{y} = \overline{y}(\psi) = (y_1, \dots, y_N)$ for $\psi \in E_c$. Denote by $\mathcal{U}$ the potential energy functional:

$$\mathcal{U}(\psi) \equiv \mathcal{H}(\psi, 0) = \frac{1}{2}\int_{-\infty}^{\infty} |\psi'(x)|^2\, dx + \sum_{k=1}^{N} V_k(y_k), \qquad \psi \in E_c. \tag{1.3.17}$$

Then (1.3.16) follows from

$$\mathcal{U}(\psi) \to \infty \quad as \quad |\overline{y}(\psi)| \to \infty. \tag{1.3.18}$$

To prove this it suffices to show that

$$\sup_{\mathcal{U}(\psi)\le E} |\overline{y}(\psi)| < \infty \tag{1.3.19}$$

for every $E \in \mathbb{R}$. First, all potentials V_k are bounded below by (1.3.6). Hence,

$$\sup_{\mathcal{U}(\psi)\le E} \int_{-\infty}^{\infty} |\psi'(x)|^2 dx = D < \infty. \tag{1.3.20}$$

Second, the Cauchy-Schwartz inequality gives for every $k, j = 1, \dots, N$,

$$\sup_{\mathcal{U}(\psi)\le E} |y_k - y_j| = \sup_{\mathcal{U}(\psi,0)\le E} |\int_{x_k}^{x_j} \psi'(x)\, dx| \le |x_k - x_j|^{1/2} D^{1/2}. \tag{1.3.21}$$

Therefore, (1.3.19) follows from the last condition of (1.3.6). □

1.3.4 Stationary states

In this section we prove Proposition 1.3.5. Substituting $\psi(x,t) = s(x)$ to (1.3.1) we obtain that $s''(x) = 0$ for $x \in \mathbb{R} \setminus Q$. Hence,

$$s(x) = a_k x + b_k \quad \text{for} \quad x \in \Delta_k := (x_{k-1}, x_k), \quad k = 1, \dots, N+1, \tag{1.3.22}$$

where $x_0 := -\infty$ and $x_{N+1} := +\infty$. The condition $s' \in L^2(\mathbb{R})$ implies

$$a_1 = a_{N+1} = 0. \tag{1.3.23}$$

Substituting (1.3.22) to equations (1.3.2), we obtain that

$$\left\{ \begin{array}{rl} a_k x_k + b_k & = y_k = a_{k+1} x_k + b_{k+1} \\ 0 & = F_k(y_k) + a_{k+1} - a_k \end{array} \right| , \qquad k = 1, \dots, N. \tag{1.3.24}$$

Hence, equations (1.3.23) imply that the function (1.3.22) is uniquely defined by its values $y_k = s(x_k)$ at the points x_k, $k = 1, \dots, N$:

$$a_k = \frac{y_k - y_{k-1}}{l_k}, \quad b_k = y_k - a_k x_k, \quad k = 1, \dots, N. \tag{1.3.25}$$

Here $y_0 := y_1$, $l_k := x_k - x_{k-1}$ for $k = 2, \dots, N$, and $l_1 := 1$ (for instance). For unknown y_k, $k = 1, \dots, N$, the system (1.3.24) is equivalent to

$$F_k(y_k) + \frac{y_{k+1} - y_k}{l_{k+1}} - \frac{y_k - y_{k-1}}{l_k} = 0, \quad k = 1, \dots, N. \tag{1.3.26}$$

Since $s(x) \in C^2(\overline{\Delta}_k)$, the variation $D\mathcal{U}(s)$ exists and

$$-D\mathcal{U}(s) = s''(x) + \sum_{k=1}^{N} (s'(x_k + 0) - s'(x_k - 0) - \nabla V_k(y_k))\delta(x - x_k).$$

Therefore, the system (1.3.1), (1.3.2) for stationary states implies the variation equation

$$D\mathcal{U}(s) = 0. \tag{1.3.27}$$

Proof of Proposition 1.3.5 Let us define the function in $\mathbb{R}^N$

$$\mathcal{U}_N(y_1, \dots, y_N) = \mathcal{U}(s), \tag{1.3.28}$$

where $s = s(x)$ is the stationary solution (1.3.22) with a_k and b_k defined by (1.3.25). Then (1.3.17) implies

$$\mathcal{U}_N(y_1, \dots, y_N) = \frac{1}{2} \sum_{k=2}^{N} \Big| \frac{y_k - y_{k-1}}{l_k} \Big|^2 l_k + \sum_{k=1}^{N} V_k(y_k). \tag{1.3.29}$$

Now (1.3.27) gives for stationary solutions

$$\frac{\partial \mathcal{U}_N}{\partial y_k}(y_1, \dots, y_N) = 0, \quad k = 1, \dots, N. \tag{1.3.30}$$

On the other hand, (1.3.18) implies,

$$\mathcal{U}_N(y_1, \dots, y_N) \to \infty \quad \text{as} \quad |(y_1, \dots, y_N)| \to \infty. \tag{1.3.31}$$

Hence, $\mathcal{U}_N$ gets a minimal value at a certain point $(y_1, \dots, y_N) \in R^N$, so $\mathcal{S} \neq \emptyset$.

Take $y_0(\lambda) = y_1(\lambda) = \lambda \in \mathbb{R}$. Then we can define uniquely $y_2(\lambda), \dots, y_N(\lambda)$ in a sequel according to formulas (1.3.26) with $k = 1, \dots, N-1$. Therefore the continuous map $I_1 : \ \mathcal{E}_F \to \mathbb{R}^d$ defined by

$$I_1(\psi(x), \pi(x)) = \psi(x_1)$$

is an isomorphism on $\mathcal{S}$. Hence, Proposition 1.3.5 obviously follows from the next lemma.

Lemma 1.3.9. *$Z_1 := I_1\mathcal{S}$ is a discrete subset of $\mathbb{R}$.*

Proof. All functions $y_k(\lambda)$ are real analytic on $\mathbb{R}$ for $k = 2, \dots, N$. The last equation of (1.3.24) with $k = N$ gives

$$a_{N+1} = a_N - F_N(y_N) = \frac{y_N - y_{N-1}}{l_N} - F_N(y_N). \tag{1.3.32}$$

The vector $\{y_k(\lambda) : k = 1, \dots, N\}$ defines the stationary solution $s_\lambda(x)$ via (1.3.25), (1.3.22) if and only if $a_{N+1} = 0$. Thus we get the following equation for $\lambda \in Z_1$

$$T(\lambda) := \frac{y_N(\lambda) - y_{N-1}(\lambda)}{l_N} - F_N(y_N(\lambda)) = 0. \tag{1.3.33}$$

The map $\lambda \mapsto T(\lambda)$ is real analytic on $\lambda \in \mathbb{R}$. Hence, the set Z_1 of all solutions to (1.3.33) is either discrete set in $\mathbb{R}$ or $Z_1 = \mathbb{R}$.

Let us show that the case $Z_1 = \mathbb{R}$ is impossible under conditions (1.3.6) even if the functions F_k are not real analytic. Assume the converse: $Z_1 = \mathbb{R}$. Then

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \text{const}, \qquad \lambda \in \mathbb{R}. \tag{1.3.34}$$

Indeed, since $F_k \in C^1(\mathbb{R})$, we have $y_k(\lambda) \in C^1(\mathbb{R})$ for all $k = 1, \dots, N$. Then by (1.3.30) we obtain:

$$\partial_\lambda \mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \sum_{k=1}^{N} \frac{\partial \mathcal{U}_N}{\partial y_k} y_k'(\lambda) = 0, \quad \lambda \in \mathbb{R}. \tag{1.3.35}$$

On the other hand, (1.3.29) implies

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) = \frac{1}{2} \sum_{k=2}^{N} \Big| \frac{y_k(\lambda) - y_{k-1}(\lambda)}{l_k} \Big|^2 l_k + \sum_{k=1}^{N} V_k(y_k(\lambda)). \tag{1.3.36}$$

Therefore, (1.3.34) and the middle condition (1.3.6) imply that the first sum on the right hand side of (1.3.36) is bounded for $\lambda \in \mathbb{R}$. Hence,

$$y_k(\lambda) \to \infty \quad \text{as} \quad |y_1(\lambda)| = |\lambda| \to \infty, \quad \forall k = 2, \dots, N. \tag{1.3.37}$$

However, then the second sum in the right hand side of (1.3.29) tends to infinity as $|\lambda| \to \infty$ due to the last condition of (1.3.6). Hence,

$$\mathcal{U}_N(y_1(\lambda), \dots, y_N(\lambda)) \to \infty \quad \text{as } |\lambda| \to \infty, \tag{1.3.38}$$

that contradicts to (1.3.34). □

1.3.5 Examples

In this section we consider examples of systems (1.3.1) with $d = 1$.

Example 1.3.10. Let each potential $V_k(y)$ be a polynomial of an even degree $p_k + 1 \geq 2$ with positive leading coefficient. Then all functions $F_k(y) = -\nabla V_k(y)$ are polynomials of degrees $p_k \geq 1$ and all conditions of Proposition 1.3.5 hold. By (1.3.26) each function $y_k(\lambda)$, $i \geq 2$ is a polynomial of degrees less or equals to the product $p_1 \dots p_{k-1}$. Hence the equation (1.3.33) has no more than $\overline{p} := p_1 \dots p_N$ roots $\lambda \in \mathbb{R}$, and the set $\mathcal{S}$ has no more than $\overline{p}$ points.

Next examples show that if the potentials V_k do not satisfy either some of conditions (1.3.6) or the analyticity condition, then the set $\mathcal{S}$ can be non-discrete.

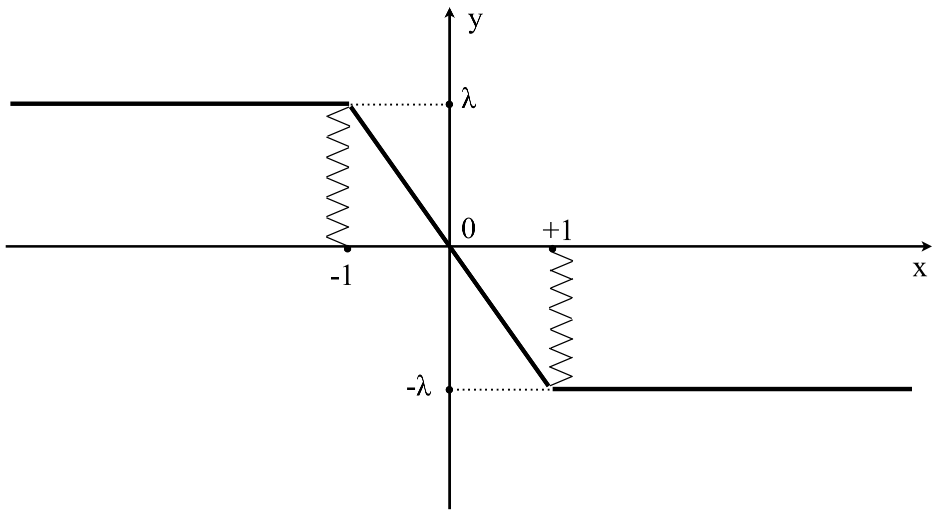

Figure 1.5: Stationary states

Example 1.3.11. The middle and the last conditions of (1.3.6) break down for the system (1.3.1) with $N = 2$, $x_1 = -1$, $x_2 = 1$, and

$$V_k(y) = -\frac{y^2}{2}, \quad k = 1, 2. \tag{1.3.39}$$

Then $F_k(y) = y$ is the force repulsing from the equilibrium position $y = 0$. In this case the system (1.3.1) has a continuum of solutions of the type (see Fig. 1.5)

$$s_\lambda(x) = \begin{cases} \lambda\ , & x \le -1, \\ -\lambda x, & -1 \le x \le 1, \\ -\lambda\ , & x \ge 1. \end{cases} \tag{1.3.40}$$

Here $y_1 = s_\lambda(-1) = \lambda$ is an arbitrary real number, so $Y_1 = \mathbb{R}$. The potentials $V_k(y)$ are real-analytic.

The last condition of (1.3.6) can be formally provided by introduction of the elastic force $F_3(y) = -y$ with the potential $V_3(y) = y^2/2$ at the point $x_3 = 0$. Then the functions (1.3.40) remain stationary solutions to the new system involving the three forces since $s_\lambda(0) = 0$ for all $\lambda \in \mathbb{R}$. So the first and last conditions of (1.3.6) and the analyticity condition hold, but the middle condition of (1.3.6) breaks down and the set $\mathcal{S}$ is not discrete.

Example 1.3.12. The last condition (1.3.6) breaks down for the system with $V_k(y) \equiv C_k$ for all k. In this case

$$F_k(y) \equiv 0, \qquad y \in \mathbb{R}.$$

Then $s_\lambda(x) \equiv \lambda$ for $x \in \mathbb{R}$ is the stationary solution to the system (1.3.1) for any $\lambda \in \mathbb{R}$. Thus, $Y_1 = \mathbb{R}$ as in previous example. The first and the middle conditions of (1.3.6) and the analyticity condition hold, but the last condition of (1.3.6) breaks down and the set $\mathcal{S}$ is not discrete.

Example 1.3.13. Now let us neglect the analyticity condition. Consider potentials $V_k(y)$ such that:

i) $V_k(y) \in C^2(\mathbb{R})$ satisfy all conditions (1.3.6).

ii) $V_k(y) \to \infty$ as $|y| \to \infty$ for every $k = 1, \dots, N$.

iii) $V_k(y) \equiv C_k$ for $y \in [a, b]$ where $a < b$. Then

$$F_k(y) \equiv 0, \qquad y \in [a, b], \quad \forall k = 1, \dots, N. \tag{1.3.41}$$

It is clear that such functions V_k exist and are not analytic. Hence, the functions $s_\lambda(x) \equiv \lambda$ are stationary solutions to the system (1.3.1), if $\lambda \in [a, b]$. Thus, the set $\mathcal{S}$ is not discrete, though all conditions (1.3.6) hold. Let us note however, that $Y_1 \neq \mathbb{R}$ here in accordance with Lemma 1.3.9.

Remark 1.3.14. In Examples 1.3.12 and 1.3.13 the global attraction (1.3.14) holds while (1.3.15) breaks down. Namely, each function $\psi(x, t)$ with values in the interval $[a, b]$ is a solution to the system (1.3.1). It is easy to construct such solution with $(\psi, \dot{\psi}) \in C(\mathbb{R}, \mathcal{E})$. For example, in the case $a = -1$ and $b = 1$, we can take the function (1.2.21).

1.3.6 Long-time asymptotics

In this section we prove Theorem 1.3.6.

Compact attracting set and global attraction

First, we construct a finite-dimensional attracting set $\mathcal{A}$. The set consists of piece-wise linear functions (1.3.22). Namely, for any $\alpha = \{(a_k, b_k) \in \mathbb{R}^{2d} : \ k = 1, \dots, N+1\} \in (\mathbb{R}^{2d})^{N+1}$ let us denote

$$\psi_\alpha(x) = a_k x + b_k, \qquad x \in \Delta_k, \quad k = 1 \dots, N+1, \tag{1.3.42}$$

and

$$A_{\mathcal{E}} = \{\alpha \in (\mathbb{R}^{2d})^{N+1} : \ \psi_\alpha(x_k - 0) = \psi_\alpha(x_k + 0), \ k = 1, \dots, N; a_1 = a_{N+1} = 0\}.$$

Then $(\psi_\alpha(x), 0) \in \mathcal{E}$ for every $\alpha \in A_{\mathcal{E}}$.

Definition 1.3.15. $\mathcal{A} = \{S_\alpha = (\psi_\alpha(x), 0) : \ \alpha \in A_{\mathcal{E}}\}$.

Obviously, $\mathcal{A}$ is a locally compact subset in $\mathcal{E}_F$. We prove next lemma in the following section.

Lemma 1.3.16. *Let all assumptions of Theorem 1.3.6 hold. Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A}, \qquad t \to \infty. \tag{1.3.43}$$

Let us deduce Theorem 1.3.6 from this lemma.

Definition 1.3.17. *Denote by $\omega(Y)$ omega-set of the trajectory $Y(t)$ in the topology of the space $\mathcal{E}_F$: $\overline{Y} \in \omega(Y)$ if and only if*

$$Y(t_k) \xrightarrow{\mathcal{E}_F} \overline{Y} \tag{1.3.44}$$

for some sequence $t_k \to \infty$.

The following lemma implies (1.3.14).

Lemma 1.3.18. *i) $\omega(Y) \neq \emptyset$ and ii) $\omega(Y) \subset \mathcal{S}$.*

Proof. i) Lemma 1.3.16 means that there exists a function $\alpha(t) \in C[0, \infty; A_{\mathcal{E}})$ such that for every $R > 0$

$$\|Y(t) - S_{\alpha(t)}\|_R \to 0 \quad \text{as} \quad t \to +\infty. \tag{1.3.45}$$

Here $S_{\alpha(t)} = (\psi_{\alpha(t)}, 0)$ and $\psi_{\alpha(t)}(x)$ is defined by (1.3.42) with $\alpha = \alpha(t) = \{(a_k(t), b_k(t)) \in \mathbb{R}^{2d} : \ k = 1, \dots, N+1\}$.

The orbit $\{S_{\alpha(t)} : t > 0\}$ is precompact in $\mathcal{E}_F$ by the bounds (1.3.16). Hence, the limit (1.3.45) implies that the orbit $\{Y(t) : t > 0\}$ with $t > 0$ also is precompact in $\mathcal{E}_F$. Therefore, $\omega(Y) \neq \emptyset$.

ii) $\omega(Y) \subset \mathcal{A}$ by (1.3.43). Moreover, the set $\omega(Y)$ is invariant with respect to dynamical group $W(t)$ due to the continuity of $W(t)$ in $\mathcal{E}_F$. Hence, for every $\overline{Y} \in \omega(Y)$ there exists a C^1-curve $t \mapsto \alpha(t) \in A_E$ such that $W(t)\overline{Y} = S_{\alpha(t)}$. Then $S_{\alpha(t)}(x) = (\psi_{\alpha(t)}(x), 0)$ is a solution to the system (1.3.8). In particular, $\partial_t \psi_{\alpha(t)}(x) \equiv 0$. Therefore, $\alpha(t) \equiv \alpha$ and $\overline{Y} = S_\alpha \in \mathcal{S}$. □

1.3.7 Attraction to a compact set

It remains to prove Lemma 1.3.16. It suffices to construct a function $\alpha(t) \in C[0, \infty; A_{\mathcal{E}})$ satisfying (1.3.45).

We may assume without loss of generality that $x_1 = 0$. Then $\psi_{\alpha(t)}(0) = b_1(t)$, and (1.3.45) according to the definition of the norm (1.3.5) means that

$$\int_{-R}^{R} |\psi'(x,t) - \psi'_{\alpha(t)}(x)|^2 \, dx + \int_{-R}^{R} |\dot\psi(x,t)|^2 \, dx + |\psi(0,t) - b_1(t)| \to 0, \quad t \to \infty. \tag{1.3.46}$$

We choose $b_1(t) = y_1(t)$ for $t > 0$. Then (1.3.46) for $R > \max(|x_1|, |x_N|)$ becomes

$$\begin{aligned} &\int_{-R}^{x_1} |\psi'(x,t)|^2 \, dx + \sum_{2 \le i \le N} \int_{x_{k-1}}^{x_k} |\psi'(x,t) - a_k(t)|^2 \, dx + \int_{X_N}^{R} |\psi'(x,t)|^2 \, dx \\ + \quad &\int_{-R}^{R} |\dot\psi(x,t)|^2 \, dx \to 0 \text{ as } t \to +\infty. \end{aligned} \tag{1.3.47}$$

It remains to check this convergence with appropriate $a_k(t)$.

1.3.8 Relaxation

To prove (1.3.47), we introduce an appropriate notion of *relaxation*. We define the Sobolev norm $\| \cdot \|_R$ of the space $H^1(-R, R)$ as usual:

$$\|z\|_R^2 \equiv \|z'(x)\|_R^2 + \|z(x)\|_R^2. \tag{1.3.48}$$

Definition 1.3.19. *i) A function $z(x) \in L^2_{loc}(\mathbb{R}^+)$ is called relaxing in L^2 if there exists a function $\overline{z}(t)$ such that for every $R > 0$*

$$\|z(\cdot + t) - \overline{z}(t)\|^2_R \to 0 \quad as \quad t \to +\infty. \tag{1.3.49}$$

We denote this by $z(t) \overset{L^2}{\sim} \overline{z}(t)$ as $t \to +\infty$.

ii) A function $z(t) \in H^1_{loc}(\mathbb{R}^+)$ is called relaxing in H^1 if there exists a function $\overline{z}(t)$ such that for every $R > 0$

$$\|z(\cdot + t) - \overline{z}(t)\|_R \to 0 \quad as \quad t \to +\infty. \tag{1.3.50}$$

We denote this relation by $z(t) \overset{H^1}{\sim} \overline{z}(t)$ as $t \to +\infty$.

The following properties of the relaxation are evident.

R0. We may assume $\overline{z}(t) \equiv z(t)$ in (1.3.50) without loss of generality.

R1. If the function $z(t)$ is relaxing in H^1, then it is relaxing stabilizing in L^2 as well.

R2. For the function $z(t)$ be relaxing in L^2 it suffices that

$$\int_0^\infty |z(t)|^2 dt < \infty. \tag{1.3.51}$$

In this case we may set $\overline{z}(t) \equiv 0$, i.e. $z(t) \overset{L^2}{\sim} 0$ as $t \to +\infty$.

R3. For the function $z(t)$ be relaxing in H^1 it suffices that

$$\int_0^\infty |z'(t)|^2 dt < \infty. \tag{1.3.52}$$

Indeed, (1.3.52) implies by the Cauchy-Schwartz inequality for $|x| \le R$

$$|z(x + t) - z(t)| = \Big| \int_t^{x+t} z'(s)\, ds \Big| \le R^{1/2} \||z'(\cdot + t)\|^{1/2}_R \to 0 \text{ as } t \to +\infty. \tag{1.3.53}$$

R4. If the function $z(t)$ is relaxing in H^1, then its derivative $z'(t)$ is relaxing in L^2, and $z'(t) \overset{L^2}{\sim} 0$ as $t \to +\infty$ according to **R2**.

R5. Conversely, if $z(t)$ is relaxing in L^2, then the integral $y(t) \equiv \displaystyle\int_{t+h_-}^{t+h_+} z(s)\, ds$ is relaxing in H^1 for any $h_\pm \in \mathbb{R}$, and we may take

$$\overline{y}(t) \equiv (h_+ - h_-)\overline{z}(t). \tag{1.3.54}$$

R6. If $z(t) \sim \overline{z}(t)$ as $t \to +\infty$ in L^2 (or in H^1), then $z(t + h) \sim \overline{z}(t)$ in L^2 (or in H^1) for every $h \in \mathbb{R}$.

R7. The set of all functions $z(t)$ relaxing in L^2 (or in H^1) is a vector space, and $z_1(t) + z_2(t) \sim \overline{z}_1(t) + \overline{z}_2(t)$, if $z_j(t) \sim \overline{z}_j(t)$, $j = 1, 2$.

R8. Let $F(\cdot) \in C^1(\mathbb{R})$ and $y(t) \in C_b(\mathbb{R}^+)$. Then $y(t) \overset{L^2}{\sim} \overline{y}(t)$ implies $F(y(t)) \overset{L^2}{\sim} F(\overline{y}(t))$.

In the next section we establish the relaxation of the Cauchy data of the solution $\psi(x, t)$ on the lines $x = x_k \pm 0$,

$$y_k(t) \equiv \psi(x_k, t) \text{ and } z^\pm_k(t) \equiv \psi'(x_k \pm 0, t), \quad t \in \mathbb{R}, \quad k = 1, \dots, N. \tag{1.3.55}$$

Lemma 1.3.20. *All the functions $y_k(t)$, $k = 1, \ldots, N$, are relaxing in H^1 and all the functions $z_k^\pm(t)$, $k = 1, \ldots, N$, are relaxing in L^2. Moreover, $y_1, y_{N+1} \overset{H^1}{\sim} 0$ and $z_1^\pm, z_{N+1}^\pm \overset{L^2}{\sim} 0$ as $t \to +\infty$.*

Let us show that this Lemma, d'Alembert representation (1.3.9) and the properties **R0**–**R8** of the relaxation imply (1.3.47). We will prove (1.3.47) for $k \geq 2$ (the case $k = 1$ is quite similar). D'Alembert representation (1.3.9) leads to well known d'Alembert formula for $x_k < x < x_{k+1}$

$$\psi(x,t) = \frac{y_k(t-(x-x_k)) + y_k(t+(x-x_k))}{2} + \frac{1}{2}\int_{t-(x-x_k)}^{t+(x-x_k)} z_k^+(s)\,ds. \tag{1.3.56}$$

Therefore

$$\begin{aligned}\psi'(x,t) &= \frac{-y_k'(t-(x-x_k)) + y_k'(t+(x-x_k))}{2} \\ &\quad + \frac{z_k^+(t+(x-x_k)) + z_k^+(t-(x-x_k))}{2}.\end{aligned} \tag{1.3.57}$$

Hence, Lemma 1.3.20 and **R7**, **R6**, **R4** imply (1.3.47) with $a_k(t) = -\overline{z_k^+}(t)$.

1.3.9 Scattering of energy to infinity

Here we analyse the energy scattering to infinity which will be applied for the proof of Lemma 1.3.20 in the next section.

Lemma 1.3.21. *The following bound holds*

$$\int_0^\infty (|\dot y_1(t)|^2 + |z_1^-(t)|^2 + |\dot y_{N+1}(t)|^2 + |z_{N+1}^+(t)|^2)\,dt < \infty. \tag{1.3.58}$$

Proof. The d'Alembert representations (1.3.9) with $k = 1$ and $k = N+1$ imply that (1.3.58) is equivalent to

$$\int_0^\infty (|f_1'(x_1-t)|^2 + |g_1'(x_1+t)|^2 + |f_{N+1}'(x_N-t)|^2 + |g_{N+1}'(x_N+t)|^2)\,dt < \infty. \tag{1.3.59}$$

The integrals for the *incident waves* $f_1'(x_1-t)$ and $g_{N+1}'(x_N+t)'$ are finite due to the d'Alembert formulas (1.2.24)

$$\begin{aligned} f_1'(x) &= \frac{\psi_0'(x)}{2} - \frac{1}{2}\pi_0(x), \quad x < x_1, \\ g_{N+1}'(x) &= \frac{\psi_0'(x)}{2} + \frac{1}{2}\pi_0(x), \quad x > x_N, \end{aligned}$$

where $(\psi_0, \pi_0) := Y(0) \in \mathcal{E}$. To derive (1.3.59) for g_1', f_{N+1}' we introduce the energy functional for $Y = (\psi(x), \pi(x)) \in \mathcal{E}$ in the interval $\Delta = [x_1, x_N]$,

$$\mathcal{H}_\Delta(Y) = \frac{1}{2}\int_{x_1}^{x_N} \Big[|\pi(x)|^2 + |\psi'(x)|^2\Big]\,dx + \sum_{k=1}^N V_k(y_k), \quad \text{where} \;\; y_k = \psi(x_k). \tag{1.3.60}$$

Let us calculate the energy flow from Δ: (1.3.1) and (1.3.9) with $k = 1$ and $N + 1$ imply for initial data $(\psi_0, \pi_0) \in \mathcal{E}$,

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_\Delta(Y(t)) &= \dot\psi\psi'\Big|_{x=x_1-0}^{x=x_N+0} \\ &= |f_1'(x_1-t)|^2 - |g_1'(x_1+t)|^2 + |g_{N+1}'(x_N+t)|^2 - |f_{N+1}'(x_N-t)|^2.\end{aligned} \tag{1.3.61}$$

Integrating, we get the energy balance,

$$\begin{aligned}&\mathcal{H}_\Delta(Y(t)) + \int_0^t (|g_1'(x_1+s)|^2 + |f_{N+1}'(x_N-s)|^2)\,ds \\ =\ &\mathcal{H}_\Delta(Y(0)) + \int_0^t (|f_1'(x_1-s)|^2 + |g_{N+1}'(x_N+s)|^2)\,ds, \qquad t\in\mathbb{R}.\end{aligned} \tag{1.3.62}$$

Then the bounds (1.3.59) for g_1', f_{N+1}' follows from the same bounds (1.3.59) for f_1', g_{N+1}' because $\inf_{Y\in\mathcal{E}} \mathcal{H}_\Delta(Y) > -\infty$ due to the middle condition of (1.3.6). □

Remark 1.3.22. The integral of the right hand side of (1.3.61) over time interval $[0, t]$ is the energy radiated outside (cf. Remark 1.2.8).

1.3.10 Proof of relaxation

We prove Lemma 1.3.20 by induction in k.

ad $k = 1$ *and* $k = N + 1$. (1.3.58) implies the needed relaxation of $y_1(t)$, $y_{N+1}(t)$ and of $z_1^-(t)$, $z_{N+1}^+(t)$ according to **R3** and to **R2** respectively. Then the relaxation of $z_1^+(t)$ and $z_{N+1}^-(t)$ follows by **R7** and **R8** from the third equation of (1.3.1) with $k = 1, N$, that is

$$z_k^+(t) - z_k^-(t) = -F_k(y_k(t)), \quad t\in\mathbb{R}, \tag{1.3.63}$$

taking into account the estimates (1.3.16).

ad $k = 2$ Let us prove the relaxation of $y_2(t)$ and $z_2^-(t)$. First, (1.3.56) with $k = 2$ and $x = x_2$ implies

$$y_2(t) = \psi(x_2, t) = \frac{y_1(t-l_2) + y_1(t+l_2)}{2} + \frac{1}{2}\int_{t-l_2}^{t+l_2} z_1^+(s)\,ds, \quad l_2 := |x_2 - x_1|. \tag{1.3.64}$$

Therefore **R5** and **R6** imply the relaxation of $y_2(t)$ in H^1. At last we take derivatives in (1.3.56) and get

$$z_2^-(t) \equiv \psi'(x_2-0, t) = \frac{-\dot y_1(t-l_2) + \dot y_1(t+l_2)}{2} + \frac{z_1^+(t+l_2) + z_1^+(t-l_2)}{2}. \tag{1.3.65}$$

Therefore **R2**, **R6** and **R7** imply the relaxation of $z_2^-(t)$ in L^2. The proof of Lemma 1.3.20 can be completed by induction.

1.4 Space-localised nonlinearity

In this section we present the result [46] on global attraction to stationary states for nonlinear wave equations with general nonlinearity

$$\ddot{\psi}(x,t) = \psi''(x,t) + f(x,\psi(x,t)), \qquad x \in \mathbb{R}. \tag{1.4.1}$$

where $f(x,\psi) = \chi(x)F(\psi)$, $F(\psi) = -\nabla U(\psi)$ for $\psi \in \mathbb{R}^d$, and

$$U(\psi) \in C^2(\mathbb{R}^d), \qquad \chi \in C_0^\infty(\mathbb{R}), \tag{1.4.2}$$

$$\chi(x) \geq 0, \qquad \chi(x) \not\equiv 0. \tag{1.4.3}$$

We will consider the Cauchy problem for equation (1.4.1) with initial conditions

$$\psi(x,0) = \psi_0(x), \quad \dot{\psi}(x,0) = \pi_0(x), \qquad x \in \mathbb{R}. \tag{1.4.4}$$

The equation (1.4.1) can be written as the dynamical system

$$\dot{Y}(t) = \mathbf{F}(Y(t)), \qquad t \in \mathbb{R}. \tag{1.4.5}$$

with $Y(t) = (\psi(t), \dot{\psi}(t))$. This equation also can be written as the Hamiltonian system (1.1.2) with Hamiltonian functional

$$\mathcal{H}(\psi,\pi) = \frac{1}{2}\int [|\pi(x)|^2 + |\psi'(x)|^2 + \chi(x)U(\psi(x,t))]\, dx, \qquad (\psi,\pi) \in \mathcal{E}, \tag{1.4.6}$$

where the Hilbert phase space $\mathcal{E}$ is defined in Definition 1.3.1. We assume that the potential U is confining, i.e.

$$U(\psi) \to \infty, \qquad |\psi| \to \infty. \tag{1.4.7}$$

Denote by $\mathcal{E}_F$ the space $\mathcal{E}$ endowed with seminorms (1.3.5).

Proposition 1.4.1. *Let $d \geq 1$ and assumptions (1.4.2), (1.4.3) and (1.4.7) hold. Then*
i) For every initial state $Y(0) \in \mathcal{E}$ equation (1.4.5) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

ii) The mapping $W(t) : Y(0) \mapsto Y(t)$ is continuous in $\mathcal{E}$ and in $\mathcal{E}_F$ for every $t \in \mathbb{R}$.
iii) The energy (1.4.6) is conserved,

$$\mathcal{H}(Y(t)) = \text{const}, \qquad t \in \mathbb{R}. \tag{1.4.8}$$

Definition 1.4.2. *$\mathcal{S}$ denotes the set of all stationary states $S = (s(x), 0) \in \mathcal{E}$ for the equation (1.4.1).*

The functions $s(x)$ satisfy the stationary equation

$$s''(x) + f(x, s(x)) = 0, \qquad x \in \mathbb{R}. \tag{1.4.9}$$

The next proposition gives a criterion for the set $\mathcal{S}$ be a nonempty discrete subset of the space $\mathcal{E}_F$. Denote by $\mathcal{U}$ the potential energy functional:

$$\mathcal{U}(\psi) := \mathcal{H}(\psi, 0) = \int_{\mathbb{R}} [\frac{1}{2}|\psi'(x)|^2 + U(\psi(x))]\, dx, \qquad \psi \in E_c. \tag{1.4.10}$$

Proposition 1.4.3. *Let conditions (1.4.2), (1.4.3) and (1.4.7) hold, and moreover, let $d = 1$ and the function $F(y)$ be real analytic on $\mathbb{R}$. Then $\mathcal{S}$ is a discrete subset of $\mathcal{E}_F$.*

The main result of [46] is the following theorem, which is illustrated by the Figure 1.

Theorem 1.4.4. i) *Let conditions* (1.4.2), (1.4.3) *and* (1.4.7) *hold and $Y(0) \in \mathcal{E}$. Then the corresponding solution $Y(t) = (\psi(t), \pi(t)) \in C(\mathbb{R}, \mathcal{E})$ to equation (1.4.5) attracts to $\mathcal{S}$ in the sense (1.2.18),*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty. \tag{1.4.11}$$

ii) *Suppose additionally that $d = 1$ and that the function $F(\psi)$ is real-analytic for $\psi \in \mathbb{R}$. Then for any solution $Y(t) = (\psi(t), \pi(t)) \in C(\mathbb{R}, \mathcal{E})$ to equation (1.4.5)*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \in \mathcal{S}, \qquad t \to \pm\infty. \tag{1.4.12}$$

Remarks 1.4.5. *i) The assertion ii) of this theorem follows from i) due to Proposition 1.4.3.*
ii) The convergence (1.4.11) and (1.4.6), (1.4.2), (1.4.3), (1.4.7) imply (0.0.10) by Fatou theorem.

1.4.1 Plan of the proof

It suffices to consider only the case $t \to \infty$. Our proofs of global attraction (1.4.11) and (1.4.12) rely on a novel method of *omega-limit trajectories* which is a development of the method of omega-limit points used in [45], see previous section 1.3. Later on this method played a central role in the theory of global attractors for $U(1)$-invariant PDEs [63]–[74].

By (1.4.2) we have

$$\operatorname{supp} \chi \subset \Delta := [-a, a] \tag{1.4.13}$$

for some $a > 0$. Conditions (1.4.3) and (1.4.7) imply the finiteness of the energy radiated from the segment Δ. Hence, similarly to (1.3.58),

$$\int_0^\infty [|\dot\psi(-a,t)|^2 + |\psi'(-a,t)|^2 + |\dot\psi(a,t)|^2 + |\psi'(a,t)|^2] dt < \infty. \tag{1.4.14}$$

This means, roughly, that

$$\psi(\pm a, t) \sim C_\pm, \qquad \psi'(\pm a, t) \sim 0, \qquad t \to \infty. \tag{1.4.15}$$

More precisely, the functions $\psi(\pm a, t)$ and $\psi'(\pm a, t)$ are slowly varying for large times, so their shifts form compact families. Namely, from an arbitrary sequence $s_k \to \infty$, one can choose a subsequence $s_{k'} \to \infty$ such that *for any* $T > 0$ the following *uniform* convergence holds,

$$\psi(\pm a, t + s_{k'}) \to C_\pm \quad \text{for} \quad t \in [0, T], \qquad k' \to \infty, \tag{1.4.16}$$

where the constants $C_\pm$ depend on the subsequence. It remains to prove that *for any* $T > 0$

$$\psi(x, t + s_{k'}) \to S_+(x) \in \mathcal{S} \quad \text{for} \quad t \in [0, T] \text{ and } x \in [-a, a], \qquad k' \to \infty, \tag{1.4.17}$$

where the convergence holds in $C([0, T]; H^1[-a, a])$. In other words, *each omega-limit trajectory is a stationary state.*

To deduce (1.4.17) from (1.4.16), we need, roughly speaking, to justify the well-posedness of the boundary value problem for a nonlinear differential equation (1.4.1) in the half-strip $-a \le x \le a$, $t > 0$, with the Cauchy boundary conditions (1.4.15) on the sides $x = \pm a$. Then the convergence (1.4.16) of boundary values implies the convergence (1.4.17) of the solution inside the strip.

Our main idea is to use evident symmetry of the wave equation with respect to interchange of variables x and t with a simultaneous change of the sign of the potential U, that is (1.4.1) can be written as

$$\psi''(x,t) = \ddot{\psi}(x,t) - f(x, \psi(x,t)). \tag{1.4.18}$$

However, in this equation with the 'time' x the condition (1.4.7) makes new potential $-U$ unbounded from below! Consequently, this dynamics with x as the time variable is not correct on the interval $|x| \le a$.

For example, in the case $U(\psi) = \psi^4$, the equation (1.4.1) for solutions of type $\psi(x,t) = \psi(x)$ is $\psi''(x) - 4\psi^3(x) = 0$. Solutions of this ordinary differential equation with finite Cauchy initial data at $x = -a$ can become infinite at any point $x \in (-a, a)$. However, in our situation local well-posedness is sufficient due to *a priori bounds*, which follow from the energy conservation (1.4.8) in view of the conditions (1.4.2), (1.4.3) and (1.4.7).

Remark 1.4.6. The discreteness of the set $\mathcal{S}$ is essential for the asymptotics (1.4.12). For example, convergence (1.4.12) fails for the solution $\psi(x,t) = \sin[\log(|x - t| + 2)]$ in the case when $d = 1$ and $F(\psi) = 0$ for $|\psi| \le 1$.

1.4.2 Well-posedness and a priori estimates

Proposition 1.4.1 follows by classical technique [12]. The energy conservation (1.3.13) implies a priori estimates

$$\sup_{t\in\mathbb{R}} \|Y(t)\|_{\mathcal{E}} < \infty \tag{1.4.19}$$

due to the conditions (1.4.3) and (1.4.7). We need however a finer characterization of the properties of the solutions.

Proposition 1.4.7. *Let the assumptions (1.4.2), (1.4.3), and (1.4.7) hold. Then*

i) The mapping $W(t)$ is Lipshitz-continuous in $\mathcal{E}_F$, and for every $R, T > 0$

$$\|W(t)Y_1 - W(t)Y_2\|_R \le L_T \|Y_1 - Y_2\|_{R+T} \quad \text{for} \quad |t| \le T, \tag{1.4.20}$$

where L_T is bounded for bounded norms $\|Y_1\|_{R+T}, \|Y_2\|_{R+T}$.

ii) For solutions $Y(t) = (\psi(t), \dot{\psi}(t)) \in C(\mathbb{R}, \mathcal{E})$ the a priori estimate holds

$$|\psi(x,t)| \le b(x) := \alpha + \beta\sqrt{|x|}, \qquad (x,t) \in \mathbb{R}^2, \tag{1.4.21}$$

where α and β are bounded for bounded energy $\mathcal{H}(Y(0))$;

iii) $\psi(x,\cdot)$ is a continuous function of $x \in \mathbb{R}$ with values in $H^1_{loc}(\mathbb{R})$), and $\psi'(x,\cdot)$ is a continuous function of $x \in \mathbb{R}$ with values in $L^2_{loc}(\mathbb{R})$);

iv) For a.a. $x \in \mathbb{R}$ and any $t \in \mathbb{R}$

$$\int_t^{t+1} (|\dot{\psi}(x,\tau)|^2 + |\psi'(x,\tau)|^2 + |\psi(x,\tau)|^2) d\tau \le e < \infty. \tag{1.4.22}$$

Proof. ad i) For solutions $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the nonlinear equation (1.4.1) the Duhamel representation holds (see [3])

$$W(t)Y(0) = W_0(t)Y(0) + \int_0^t W_0(t-s)f_*(\cdot, s)ds, \quad f_*(x,s) := (0, f(x, \psi(x,s))). \quad (1.4.23)$$

Here $W_0(t)$ denotes the dynamical group corresponding to the linear equation (1.4.1) with $f(x,u) \equiv 0$, i.e.

$$W_0(t)(\psi(0), \pi(0)) = (\psi(t), \dot{\psi}(t)),$$

where $\psi(x,t)$ is given by the d'Alembert formula (1.1.6). This formula implies the Lipschitz continuity (1.4.20) for $W_0(t)$. Then for $W(t)$ the same continuity follows from (1.4.23) by (1.4.2) and (1.4.19).

ad ii) The bound (1.4.19) implies that

$$D := \sup_{t\in\mathbb{R}} \int |\psi'(x,t)|^2 dx < \infty, \quad (1.4.24)$$

and D is bounded for bounded energy $\mathcal{H}(Y(0))$. Therefore, by the Cauchy-Schwartz inequality

$$|\psi(x,t) - \psi(0,t)| = \Big| \int_0^x \psi'(y,t)dy \Big| \le \sqrt{D}\sqrt{|x|}, \qquad (x,t) \in \mathbb{R}^2. \quad (1.4.25)$$

At last, $\sup_{t\in\mathbb{R}} |u(0,t)| < \infty$ by the bound (1.4.19). Now (1.4.25) implies (1.4.21).

ad iii) and iv) For $|x| > a$ the claimed properties follow similarly to (1.4.14). To prove them for $|x| < a$ rewrite the equation (1.4.1) as (1.4.18) and apply the integral representation of the type (1.4.23),

$$Z(x) = W_0(x+a)Z(-a) - \int_{-a}^{x} W_0(x-y)f_*(y,\cdot)dy, \qquad Z(x) := (\psi(x,\cdot), \psi'(x,\cdot)). \quad (1.4.26)$$

The claimed properties for the first term on the right hand side follow from (1.4.14), and for the integral term these properties follow from (1.4.2) and estimates (1.4.21). □

1.4.3 Stationary states

We prove Proposition 1.4.3 by a suitable modification of the arguments from the proof of Proposition 1.3.5. The stationary equation (1.4.9) and conditions (1.4.2) imply that all stationary solutions $s(x)$ are smooth and

$$s(x) = s(\pm a), \qquad \pm x \ge a \quad (1.4.27)$$

since $s'(x) \in L^2(\mathbb{R})$. Hence, the variation $D\mathcal{U}(s)$ exists and

$$D\mathcal{U}(s) = -s''(x) + f(x, s(x)).$$

Therefore, equation (1.4.1) for stationary states implies the variational equation

$$D\mathcal{U}(s) = 0. \quad (1.4.28)$$

The identities (1.4.27) imply that the continuous map $I: \mathcal{E}_F \to \mathbb{R}$ defined by

$$I(\psi(x), \pi(x)) := \psi(-a)$$

is a homeomorphism on $\mathcal{S}$. Hence, Proposition 1.4.3 obviously follows from the next lemma.

Lemma 1.4.8. *$Z := I\mathcal{S}$ is a discrete subset of $\mathbb{R}$.*

Proof. We should prove that Z has no limit points. Let us assume contrary, that there exists an infinite subsequence

$$z_k \in Z, \qquad z_k \to \overline{z} \in Z, \qquad k \to \infty. \tag{1.4.29}$$

All stationary states satisfy the boundary value problem

$$\left\{ \begin{array}{ll} s_\lambda''(x) + f(x, s_\lambda(x)) = 0 & x \in [-a, a] \\ \\ s_\lambda(-a) = \lambda, \qquad s_\lambda'(-a) = 0 & \end{array} \right| . \tag{1.4.30}$$

Let us denote by Λ the set of all $\lambda \in \mathbb{R}$ such that the solution to (1.4.30) exists. We extend $s_\lambda(x)$ to $|x| > a$ by constants,

$$s_\lambda = s_\lambda(\pm a) \text{ for } \pm x > a. \tag{1.4.31}$$

Then $S_\lambda = (s_\lambda, 0) \in \mathcal{E}$ for every $\lambda \in \Lambda$, though generally $S_\lambda \notin \mathcal{S}$.

Let us define the map $T: \Lambda \to \mathbb{R}$ by

$$T(\lambda) := s_\lambda'(a - 0). \tag{1.4.32}$$

Then

$$Z = \{\lambda \in \Lambda : \ T(\lambda) = 0\}, \tag{1.4.33}$$

and (1.4.29) implies that

$$T(\overline{z}) = T(z_k) = 0, \qquad k = 1, \ldots. \tag{1.4.34}$$

The set Λ is an open subset of $\mathbb{R}$, hence

$$\Lambda = \overset{\infty}{\underset{1}{\cup}} \Lambda_j, \tag{1.4.35}$$

where Λ_j are open intervals. We have $\overline{z} \in \Lambda_* = \Lambda_l$ with some l. Let us show that

$$\Lambda_* = \mathbb{R}. \tag{1.4.36}$$

Namely, the map T is real analytic on Λ_*, and hence, (1.4.29) and (1.4.34) imply that

$$T(\lambda) \equiv 0, \qquad \lambda \in \Lambda_* \tag{1.4.37}$$

since Λ_* is open and connected subset of $\mathbb{R}$. Hence, definition (1.4.33) implies that

$$\Lambda_* \subset Z. \tag{1.4.38}$$

Now (1.4.28) implies that

$$\partial_\lambda \mathcal{U}(s_\lambda) = \langle D\mathcal{U}(s_\lambda), \partial_\lambda s_\lambda \rangle = 0, \qquad \lambda \in \Lambda_*. \tag{1.4.39}$$

Hence,

$$\mathcal{U}(s_\lambda) \equiv \mathcal{U}(s_{\overline{z}}), \qquad \lambda \in \Lambda_*. \tag{1.4.40}$$

However, this identity implies that the set $\mathcal{S}_* := \{S_\lambda : \lambda \in \Lambda_*\}$ is bounded in $\mathcal{E}$ by conditions (1.4.2) and (1.4.3). Hence, $\mathcal{S}_*$ is precompact in $C(\mathbb{R}, \mathbb{R} \times \mathbb{R})$. Its closure in $C(\mathbb{R}, \mathbb{R} \times \mathbb{R})$ obviously belongs to $\mathcal{S}$, and hence,

$$\overline{\Lambda_*} \subset \Lambda_*. \tag{1.4.41}$$

Now (1.4.36) follows. Moreover, now (1.4.38) implies that $Z = \mathbb{R}$ which contradicts to the boundedness of $\mathcal{S}_*$ in $\mathcal{E}$. This contradiction completes the proof of Lemma 1.4.8. $\square$

1.4.4 Long-time asymptotics

We prove the Theorem 1.4.4.

Compact attracting set

Let us construct a compact attracting set $\mathcal{A}$ for the considered trajectory $Y(t)$. Let $b>0$ denote some constant to be chosen later.

Definition 1.4.9. $\mathcal{A}_b := \{S_\lambda = (s_\lambda(x),0) \in \mathcal{E} : \lambda \in \Lambda, |s_\lambda(x)| \le b \text{ for } |x| \le a\}$.

$\mathcal{A}_b$ is a compact set in $\mathcal{E}_F$ due to the equation (1.4.9). We prove the next lemma in the following section.

Lemma 1.4.10. *Let assumptions of Theorem 1.4.4 hold. Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A} = \mathcal{A}_b, \qquad t \to \pm\infty, \tag{1.4.42}$$

if the constant b is sufficiently large.

Proof of Theorem 1.3 i)

The next lemma implies the attraction to stationary states (1.4.11).

Lemma 1.4.11. *i) $\omega(Y) \neq \emptyset$ and ii) $\omega(Y) \subset \mathcal{S}$.*

Proof. i) Lemma 1.4.10 means that there exists a function $\lambda(t) \in C[0,\infty)$ such that for every $R>0$

$$\|Y(t) - S_{\lambda(t)}\|_R \to 0 \quad \text{as} \quad t \to +\infty. \tag{1.4.43}$$

Here $S_{\lambda(t)} = (s_{\lambda(t)}, 0)$ where $s_{\lambda(t)}(x)$ is defined by (1.4.30) and (1.4.31).

The orbit $\{S_{\lambda(t)} : t>0\}$ is precompact in $\mathcal{E}_F$. Hence, the limit (1.4.43) implies that the orbit $\{Y(t) : t>0\}$ with $t>0$ also is precompact in $\mathcal{E}_F$. Therefore, $\omega(Y) \neq \emptyset$.

ii) $\omega(Y) \subset \mathcal{A}$ by (1.4.42). Moreover, the set $\omega(Y)$ is invariant with respect to $W(t)$ due to the continuity of $W(t)$ in $\mathcal{E}_F$. Hence, for every $\overline{Y} \in \Omega(Y)$ there exists a C^1-curve $t \mapsto \lambda(t) \in \mathbb{R}$ such that $W(t)\overline{Y} = S_{\lambda(t)}$. Then $S_{\lambda(t)}$ is the solution to (1.4.5). In particular, $\partial_t S_{\lambda(t)} \equiv 0$. Therefore, $\lambda(t) \equiv \lambda$ and $\overline{Y} = S_\lambda \in \mathcal{S}$. □

1.4.5 Attraction to a compact set

We deduce Lemma 1.4.10 from the following lemma on 'attraction in the mean', which we prove in the next section. Let us denote for $b, R > 0$

$$\rho_{b,R}(t) = \inf_{S \in \mathcal{A}_b} \|Y(t) - S\|_R \ \text{ for } t \in \mathbb{R}. \tag{1.4.44}$$

Lemma 1.4.12. *For sufficiently large $b>0$ and every $R>0$*

$$\int_0^\infty \rho_{b,R}^2(t)dt < \infty. \tag{1.4.45}$$

Proof of Lemma 1.4.10. Let us fix a metric $\rho(\cdot,\cdot)$ on $\mathcal{E}$, defining the topology of $\mathcal{E}_F$. We prove (1.4.42) ad absurdum: let us us assume that there exist $\varepsilon > 0$ and a sequence $t_k \to \infty$, such that

$$\rho(Y(t_k), \mathcal{A}) \geq \varepsilon \text{ for all } k = 1, 2, \dots. \tag{1.4.46}$$

We will show that this is impossible and this completes the proof of Lemma 1.4.10. We may assume that $t_k + 1 < t_{k+1}$ for every k. Now (1.4.45) implies by Fatou theorem,

$$\int_0^1 \sigma_R(\theta)d\theta < \infty, \text{ where } \sigma_R(\theta) = \sum_1^\infty \rho_R^2(t_k + \theta). \tag{1.4.47}$$

Therefore, $\sigma_R(\theta) < \infty$ for every θ in a subset $\Theta(R) \subset [0,1]$ with $\displaystyle\int_{\Theta(R)} dx = 1$. Then for every $R > 0$

$$\rho_R(t_k + \theta) \to 0 \text{ as } k \to \infty \text{ for } \theta \in \Theta := \cap_{R\in\mathbb{N}}^\infty \Theta(R). \tag{1.4.48}$$

Hence $Y(t_k + \theta) \xrightarrow{\mathcal{E}_F} \mathcal{A}$ as $k \to \infty$ for every $\theta \in \Theta \subset [0,1]$, and $\displaystyle\int_\Theta dx = 1$. Hence for every $\theta \in \Theta$ the compactness of $\mathcal{A}$ in $\mathcal{E}_F$ implies that for some sequence $k(\theta) \to \infty$,

$$Y(t_{k(\theta)} + \theta) \xrightarrow{\mathcal{E}_F} \overline{Y}(\theta) \in \mathcal{A} \text{ as } k(\theta) \to \infty, \qquad \theta \in \Theta. \tag{1.4.49}$$

Then the continuity of maps $W(-\theta)$ in $\mathcal{E}_F$ implies also

$$Y(t_{k(\theta)}) \xrightarrow{\mathcal{E}_F} W(-\theta)\overline{Y}(\theta) \text{ as } k(\theta) \to \infty, \qquad \theta \in \Theta. \tag{1.4.50}$$

On the other hand, the compactness of $\mathcal{A}$ in $\mathcal{E}_F$ implies that there exists a sequence $\theta_j \in \Theta$ such that $\theta_j \to 0$ as $j \to \infty$ and

$$\overline{Y}(\theta_j) \xrightarrow{\mathcal{E}_F} Y^* \in \mathcal{A} \text{ as } j \to \infty. \tag{1.4.51}$$

Now the uniform Lipshitz continuity (1.4.20) of $W(-\theta)$ with $\theta \in [0,1]$ and the convergence $W(-\theta_j)Y^* \xrightarrow{\mathcal{E}_F} Y^*$ as $j \to \infty$ imply,

$$W(-\theta_j)\overline{Y}(\theta_j) \xrightarrow{\mathcal{E}_F} Y^* \text{ as } j \to \infty. \tag{1.4.52}$$

However this convergence together with (1.4.50) for $\theta = \theta_j$ contradict (1.4.46). □

1.4.6 Attraction in the mean

We prove Lemma 1.4.12. It suffices to construct for sufficiently large $b > 0$ a function $S_{\mu(t)} = (s_{\mu(t)}, 0) \in \mathcal{A}_b$ defined for $t \geq T$ with sufficiently large $T > 0$ such that for every $R > 0$,

$$\int_T^\infty \|Y(t) - S_{\mu(t)}\|_R^2 dt < \infty. \tag{1.4.53}$$

We will establish this inequality with

$$\mu(t) = y_-(n) := \psi(-a, n), \qquad n \leq t < n + 1, \tag{1.4.54}$$

where $n = 0, 1, \dots$ and $n \geq T$. We may change the seminorm $\|\cdot\|_R$ from (1.3.5) by an equivalent seminorm with $|\psi(-a)|$ instead of $|\psi(0)|$. Then (1.4.53) means for $R > a$ that

$$\int_T^\infty \Big(\int\limits_{|x|<a} (|\psi'(x,t) - s'_{\mu(t)}(x)|^2 + |\psi(x,t) - s_{\mu(t)}(x)|^2 + |\dot\psi(x,t)|^2) dx$$

$$+ |\psi(-a,t) - \mu(t)|^2 + \int\limits_{a<|x|<R} (|\psi'(x,t)|^2 + |\dot\psi(x,t)|^2) dx \Big) dt < \infty. \tag{1.4.55}$$

Energy scattering to infinity

The bound (1.4.14) can be written as

$$\int_0^\infty (|\dot y_-(t)|^2 + |z_-(t)|^2 + |\dot y_+(t)|^2 + |z_+(t)|^2)\, dt < \infty, \qquad (1.4.56)$$

where $y_\pm(t) = \psi(\pm a, t)$ and $z_\pm(t) = \psi'(\pm a, t)$. This follows similarly to (1.3.58) from d'Alembert representation

$$\psi(x,t) = f_\pm(t-x) + g_\pm(t+x), \quad \pm x > a, \quad t \in \mathbb{R}, \qquad (1.4.57)$$

and from finiteness of the energy flow from the segment $\Delta := [-a,a]$, differentiating the energy functional

$$\mathcal{H}_\Delta(Y) = \int_\Delta \left[\frac{|v(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + V(x,\psi(x))\right] dx, \qquad Y = (\psi(x), v(x)) \in \mathcal{E}. \quad (1.4.58)$$

Now (1.4.56) implies that

$$\int_0^\infty |\psi(-a,t) - \mu(t)|^2 dt = \int_0^\infty |y_-(t) - y_-([t])|^2 dt \le \sum_{n=0}^\infty \int_0^1 |\dot y_-(n+s)|^2 ds < \infty. \quad (1.4.59)$$

Furthermore, similarly to (1.4.56)

$$\int_0^\infty (|\dot\psi(x,t)|^2 + |\psi'(x,t)|^2\, dt \le C < \infty, \qquad a < |x| < R. \qquad (1.4.60)$$

Hence, the last integral of (1.4.55) is finite. It remains to prove the finiteness of the first integral of (1.4.55):

$$\int_T^\infty \Big(\int_{|x|<a} (|\psi'(x,t) - s'_{\mu(t)}(x)|^2 + |\psi(x,t) - s_{\mu(t)}(x)|^2 + |\dot\psi(x,t)|^2) dx\Big) dt < \infty \quad (1.4.61)$$

for sufficiently large $T > 0$. We will deduce (1.4.61) from (1.4.56) in the next section.

Nonlinear Goursat problem

We consider the Goursat problem for the nonlinear wave equation (1.4.18) with the Cauchy data on the lines $x = \text{const}$:

$$\left\{\begin{array}{l} \phi''(x,t) = \ddot\phi(x,t) - f(x,\phi(x,t)), \\ \\ \phi|_{x=r} = u(t), \;\; \phi'|_{x=r} = v(t) \end{array}\right| , \; t \in \mathbb{R}, \quad x \in [r, r+\varepsilon], \qquad (1.4.62)$$

where $\varepsilon > 0$. Our assumptions (1.4.2), (1.4.3) provide that the Cauchy problem (1.4.1), (1.4.4) is well posed globally in t. On the other hand, the nonlinear Goursat problem (1.4.62) generally is not well posed globally in $x \in \mathbb{R}$.

We will establish a Lipschitz continuity of the maps

$$G(r,x) : (u(\cdot), v(\cdot)) \mapsto (\phi(x,\cdot), \phi'(x,\cdot)), \qquad x \in [r, r+\varepsilon]$$

in suitable norms for initial data $(u(\cdot), v(\cdot))$ close to $(\psi(r,\cdot), \psi'(r,\cdot))$, where $\varepsilon > 0$ *does not depend on* $r \in [-a,a]$. This continuity holds "along" the considered global solution $\psi(x,t)$ due to the a priori bounds (1.4.19). Using this continuity, we will deduce (1.4.61) from (1.4.56).

Let σ denote an arbitrary segment in $\mathbb{R}$ of the length $|\sigma|$.

Definition 1.4.13. *$\mathcal{E}(\sigma) := H^1(\sigma) \oplus L^2(\sigma)$, is the Hilbert space of functions $(u(t), v(t))$ with the norm*

$$\|(u, v)\|_{\mathcal{E}(\sigma)} = \|\dot{u}\|_{L^2(\sigma)} + \|u\|_{L^2(\sigma)} + \|v\|_{L^2(\sigma)} < \infty. \tag{1.4.63}$$

Now proposition 1.4.7 iii) and iv) imply that for any segment $\sigma \subset \mathbb{R}$

$$Z_\sigma(r) := (\psi(r, \cdot), \psi'(r, \cdot))|_\sigma \in \mathcal{E}(\sigma) \qquad \text{for a.a. } r \in \mathbb{R}$$

and

$$\|Z_\sigma(r)\|^2_{\mathcal{E}(\sigma)} \le Ce|\sigma| \qquad \text{for a.a. } r \in \mathbb{R}. \tag{1.4.64}$$

Let $\phi_j(x, t)$ with $j = 1, 2$ be two solutions of the nonlinear Goursat problem (1.4.62) for $x \in [r, r+\varepsilon)$ where $\varepsilon > 0$, such that $X_j(x) := (\phi(x, \cdot), \phi'(x, \cdot)) \in C(r, r+\varepsilon; H^1_{\rm loc}(\mathbb{R}) \oplus L^2_{\rm loc}(\mathbb{R}))$. For such solutions the Goursat problem (1.4.62) is equivalent to the integral identity of type (1.4.26),

$$X_j(x) = W_0(x - r)X_j(r) - \int_r^x W_0(x - y)(0, f(y, \phi_j(y, \cdot))dy, \qquad x \in [r, r+\varepsilon]. \tag{1.4.65}$$

For any segment $\sigma = [t_1, t_2]$ and small $\varepsilon > 0$ denote $\sigma_\varepsilon := [t_1 + \varepsilon, t_2 - \varepsilon]$.

Lemma 1.4.14. *Let assumptions (1.4.2), (1.4.3) hold, and*

$$\max_{x \in [r, r+\varepsilon], t \in \mathbb{R}} |\phi_j(x, t)| \le B < \infty, \qquad j = 1, 2. \tag{1.4.66}$$

Then for any segment $\sigma \subset \mathbb{R}$ with $|\sigma| > 2\varepsilon$

$$\|X_1(x) - X_2(x)\|_{\mathcal{E}(\sigma_{|x-r|})} \le L(B)\|X_1(r) - X_2(r)\|_{\mathcal{E}(\sigma)}, \qquad x \in [r, r+\varepsilon), \tag{1.4.67}$$

where the Lipshitz constant $L(B)$ does not depend on the segment σ.

Proof. By conditions (1.4.2)

$$M(B) := \max_{x \in \mathbb{R}, |\psi| \le B} |\nabla_\psi f(x, \psi)| < \infty. \tag{1.4.68}$$

Hence,

$$\|f(y, \phi_1(y, \cdot)) - f(y, \phi_2(y, \cdot))\|_{L^2(\sigma_{y-r})} \le M(B)\|X_1(y) - X_2(y)\|_{\mathcal{E}(\sigma)},\ y \in [r, r+\varepsilon). \tag{1.4.69}$$

Moreover, the dynamical group $W_0(y)$ admits classical estimate

$$\|W_0(z)X\|_{\mathcal{E}(\sigma_z)} \le \|X\|_{\mathcal{E}(\sigma)}, \qquad z \in [0, |\sigma|/2), \quad X \in \mathcal{E}(\sigma).$$

Now the integral equation (1.4.65) implies the integral inequality

$$m(x) \le m(r) + M(B)\int_r^x m(y)dy, \qquad x \in [r, r+\varepsilon), \tag{1.4.70}$$

where

$$m(x) := \|X_1(x) - X_2(x)\|_{\mathcal{E}(\sigma_{|x-r|})}.$$

Hence, the bounds (1.4.67) follow by the Gronwall inequality. □

Now we can prove the existence of solutions of (1.4.30).

Lemma 1.4.15. *For sufficiently large $t > 0$ the problem (1.4.30) with $\lambda = y_-(t)$ admits a unique solution $s_\lambda(x)$.*

Proof. First, the solution exists for $x \in [-a, -a+\varepsilon]$ with sufficiently small $\varepsilon > 0$ which depends on t. This local solution can be extended to all $x \in [-a, a]$ if the a priori bounds hold

$$|s_\lambda(x)| \le C, \qquad x \in [-a, a]. \tag{1.4.71}$$

This bounds follow for $\lambda = \lambda(t) := y_-(t)$ with large t by application of Lemma 1.4.14 to the following two solutions $\phi_j(x, t)$ of the nonlinear Goursat problem (1.4.62) with $r = -a$:

$$\phi_1(x, t) := \psi(x, t), \qquad \phi_2(x, t) := s_\lambda(x).$$

We will prove the bounds (1.4.71) with any $C > B_0$, where

$$B_0 := \max_{x \in [-a,a], t \in \mathbb{R}} |\psi(x, t)| \tag{1.4.72}$$

The key fact is the following convergence of the Cauchy data of these two solutions

$$\|(\psi(-a, t+\cdot), \psi'(-a, t+\cdot)) - (s_{\lambda(t)}(-a), 0)\|_{\mathcal{E}(\sigma)} \to 0, \qquad t \to \infty. \tag{1.4.73}$$

for any segment $\sigma \subset \mathbb{R}$. This convergence follows from (1.4.56). Now Lemma 1.4.14 implies that

$$\|(\psi(-a+\varepsilon, t+\cdot), \psi'(-a, t+\cdot)) - (s_{\lambda(t)}(-a+\varepsilon), s'_{\lambda(t)}(-a+\varepsilon))\|_{\mathcal{E}(\sigma_\varepsilon)} \to 0, \qquad t \to \infty$$

if we take $|\sigma| > 2\varepsilon$. Hence, by the Sobolev embedding theorem,

$$\|\psi(-a+\varepsilon, t+\cdot) - s_{\lambda(t)}(-a+\varepsilon)\|_{C(\sigma_\varepsilon)} \to 0, \qquad t \to \infty. \tag{1.4.74}$$

Therefore, the a priori bounds (1.4.71) hold. □

Proof of the attraction in the mean

Now (1.4.61) follows by the same arguments. Namely, Lemma 1.4.14 implies that for any $\varepsilon \in [0, 2a]$

$$\|(\psi(-a+\varepsilon, t+\cdot), \psi'(-a+\varepsilon, t+\cdot)) - (s_{\mu(t)}(-a+\varepsilon), s'_{\mu(t)}(-a+\varepsilon))\|_{\mathcal{E}(\sigma_\varepsilon)}$$

$$\le \quad L(C)\|(\psi(-a, t+\cdot), \psi'(-a, t+\cdot)) - (s_{\mu(t)}(-a), 0)\|_{\mathcal{E}(\sigma)}, \qquad t \ge T$$

for sufficiently large $T > 0$ and $\mu(t)$ defined by (1.4.54). Hence,

$$\|(\psi(-a+\varepsilon, t+\cdot) - s_{\mu(t)}(-a+\varepsilon)\|^2_{H^1(\sigma_\varepsilon)} + \|\psi'(-a+\varepsilon, t+\cdot) - s'_{\mu(t)}(-a+\varepsilon)\|^2_{L^2(\sigma_\varepsilon)}$$

$$\le L(C)[\|\psi(-a, t+\cdot) - s_{\mu(t)}(-a)\|^2_{H^1(\sigma)} + \|\psi'(-a, t+\cdot)\|^2_{L^2(\sigma)}], \qquad t \ge T \tag{1.4.75}$$

Choosing here $t = n$, $\sigma_\varepsilon = [0, 1]$ and summing up over $n \ge N \ge T$, we obtain

$$\int_N^\infty (|\dot\psi(x, t)|^2 + |\psi(x, t) - s_{\mu(t)}(x)|^2 + |\psi'(x, t) - s'_{\mu(t)}(x)|^2) dt < \infty, \qquad x \in [-a, a] \tag{1.4.76}$$

since the sum of the right hand sides is finite by (1.4.56). Moreover, this last sum is bounded, and hence, integrating over $x \in [-a, a]$, we obtain (1.4.61).

Now Lemma 1.4.12 is proved.

1.5 Wave-particle system

In [47], the first result on global attraction to stationary states (0.0.7) is obtained for three-dimensional real scalar wave field coupled to a relativistic particle. The scalar field satisfies 3D wave equation

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - \rho(x - q(t)), \qquad x \in \mathbb{R}^3, \tag{1.5.1}$$

where $\rho \in C_0^\infty(\mathbb{R}^3)$ is a fixed function representing the charge density of the particle, and $q(t) \in \mathbb{R}^3$ is the particleposition. The particle motion obeys the Hamiltonian equation with relativistic kinetic energy $\sqrt{1+p^2}$:

$$\dot{q}(t) = \frac{p(t)}{\sqrt{1+p^2(t)}}, \qquad \dot{p}(t) = -\nabla V(q(t)) - \int \nabla\psi(x,t)\rho(x - q(t))\, dx. \tag{1.5.2}$$

Here $-\nabla V(q)$ is external force corresponding to real potential $V(q)$, and the integral term is a self-force. Thus, wave function ψ is generated by charged particle, and plays the role of a potential acting on the particle, along with the external potential $V(q)$.

The system (1.5.1)–(1.5.2) was introduced by H. Spohn, see [55] for discussion of physical relevance of this model. This system can formally be represented in Hamiltonian form

$$\dot{\psi} = D_\pi \mathcal{H}, \quad \dot{\pi} = -D_\psi \mathcal{H}, \quad \dot{q}(t) = D_p \mathcal{H}, \quad \dot{p} = -D_q \mathcal{H} \tag{1.5.3}$$

with Hamiltonian (energy)

$$\mathcal{H}(\psi,\pi,q,p) = \frac{1}{2}\int [|\pi(x)|^2 + |\nabla\psi(x)|^2]\, dx + \int \psi(x)\rho(x-q)\, dx + \sqrt{1+p^2} + V(q). \tag{1.5.4}$$

By $\|\cdot\|$ we denote the norm in the Hilbert space $L^2 := L^2(\mathbb{R}^3)$, and $\|\cdot\|_R$ denotes the norm in $L^2(B_R)$, where B_R being the ball $|x| \le R$. Let $\mathring{H}^1 := \mathring{H}^1(\mathbb{R}^3)$ be the completion of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla\psi(x)\|$.

Definition 1.5.1. *i)* $\mathcal{E} := \mathring{H}^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ *is the Hilbert phase space of tetrads* (ψ,π,q,p) *with finite norm*

$$\|(\psi,\pi,q,p)\|_{\mathcal{E}} = \|\nabla\psi\| + \|\pi\| + |q| + |p|.$$

ii) $\mathcal{E}_\sigma$ *for* $\sigma \in \mathbb{R}$ *is the space of* $Y = (\psi,\pi,q,p) \in \mathcal{E}$ *with* $\psi \in C^2(\mathbb{R}^3)$ *and* $\pi \in C^1(\mathbb{R}^3)$ *satisfying the estimate*

$$|\nabla\psi(x)| + |\pi(x)| + |x|(|\nabla\nabla\psi(x)| + |\nabla\pi(x)|) = \mathcal{O}(|x|^{-\sigma}), \quad |x| \to \infty. \tag{1.5.5}$$

iii) $\mathcal{E}_F$ *is the space* $\mathcal{E}$ *with metric of type (1.2.9), where the corresponding seminorms are defined as*

$$\|(\psi,\pi,q,p)\|_{\mathcal{E},R} = \|\nabla\psi\|_R + \|\psi\|_R + \|\pi\|_R + |q| + |p|. \tag{1.5.6}$$

Obviously, the energy (1.5.4) is a continuous functional on $\mathcal{E}$, and $\mathcal{E}_\sigma \subset \mathcal{E}$ for $\sigma > 3/2$. The convergence in $\mathcal{E}_F$ is equivalent to the convergence in every seminorm (1.5.6). We assume the external potential be confining:

$$V(q) \to \infty, \qquad |q| \to \infty. \tag{1.5.7}$$

In this case the Hamiltonian (1.5.4) is bounded below:

$$\inf_{Y \in \mathcal{E}} \mathcal{H}(Y) = V_0 + \frac{1}{2}(\rho, \Delta^{-1}\rho), \tag{1.5.8}$$

where

$$V_0 := \inf_{q\in\mathbb{R}^3} V(q) > -\infty. \tag{1.5.9}$$

The following lemma is proved in [47, Lemma 2.1].

Lemma 1.5.2. *Let $V(q) \in C^2(\mathbb{R}^3)$ satisfies the condition (1.5.9). Then for any initial state $Y(0) \in \mathcal{E}$ there exists a unique finite energy solution $Y(t) = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$, and*
i) for every $t \in \mathbb{R}$ the map $W(t) : Y_0 \mapsto Y(t)$ is continuous both on $\mathcal{E}$ and on $\mathcal{E}_F$;
ii) the energy $\mathcal{H}(Y(t))$ is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0) \quad for \quad t \in \mathbb{R}; \tag{1.5.10}$$

iii) a priori estimates hold

$$\sup_{t\in\mathbb{R}}[\|\nabla\psi(t)\| + \|\pi(t)\|] < \infty, \qquad \sup_{t\in\mathbb{R}} |\dot{q}(t)| = \overline{v} < 1; \tag{1.5.11}$$

iv) if (1.5.7) holds, then also

$$\sup_{t\in\mathbb{R}} |q(t)| = \overline{q}_0 < \infty. \tag{1.5.12}$$

Remark 1.5.3. In the case of point particle$\rho(x) = \delta(x)$, the system (1.5.1)–(1.5.2) is incorrect, since in this case any solution of the wave equation (1.5.1) is singular at the point $x = q(t)$, and, accordingly, the integral in (1.5.2) is not defined. Energy functional (1.5.4) in this case is not bounded from below, because the last term in (1.5.8) equals $-\infty$. Indeed, in the Fourier transform, this term has the form

$$(\rho, \Delta^{-1}\rho) = -\int \frac{|\hat{\rho}(k)|^2}{k^2} dk,$$

where $\hat{\rho}(k) \equiv 1$. This is the famous 'ultraviolet divergence.' Thus, the self-energy of point charge is infinite, that suggested Abraham to introduce the model of an 'extended electron' with a continuous charge density $\rho(x)$ [203, 204].

Denote $Z = \{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$. It is easy to verify that stationary states of the system (1.5.1)–(1.5.2) have the form $S_q = (\psi_q, 0, q, 0)$, where $q \in Z$ and $\Delta\psi_q(x) = \rho(x-q)$. Therefore, $\psi_q(x)$ is the Coulomb potential

$$\psi_q(x) := -\frac{1}{4\pi}\int \frac{\rho(y-q)\,dy}{|x-y|}$$

Respectively, the set of all stationary states of this system is

$$\mathcal{S} := \{S_q : q \in Z\}.$$

If the set Z is discrete in $\mathbb{R}^3$, then the set $\mathcal{S}$ is also discrete in $\mathcal{E}$ and in $\mathcal{E}_F$. Finally, assume that the "form-factor" ρ satisfies the *Wiener condition*

$$\hat{\rho}(k) := \int e^{ikx}\rho(x)\,dx \neq 0, \qquad k \in \mathbb{R}^3. \tag{1.5.13}$$

Remark 1.5.4. The Wiener condition means a strong coupling of scalar wave field $\psi(x)$ to the particle. It is a suitable version of the 'Fermi Golden Rule" for the system (1.5.1)–(1.5.2): the perturbation $\rho(x-q)$ is not orthogonal to all eigenfunctions of continuous spectrum of the Laplacian Δ.

For simplicity of the exposition we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3), \qquad \rho(x) = 0 \ \text{ for } \ |x| \geq R_\rho, \qquad \rho(x) = \rho_r(|x|). \tag{1.5.14}$$

The main result of [47] is as follows.

Theorem 1.5.5. i) *Let the conditions* (1.5.7) *and* (1.5.13) *hold, and* $\sigma > 3/2$. *Then for any initial state* $Y(0) = (\psi_0, \pi_0, q_0, p_0) \in \mathcal{E}_\sigma$ *the corresponding solution* $Y(t) = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ *to the system* (1.5.1)–(1.5.2) *attracts to the set of stationary states:*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty, \tag{1.5.15}$$

where attraction holds in the metric (1.2.9) *defined with the seminorms*(1.5.6).

ii) *Let, additionally, the set* Z *be discrete in* $\mathbb{R}^3$. *Then*

$$Y(t) \xrightarrow{\mathcal{E}_F} S_\pm \in \mathcal{S}, \qquad t \to \pm\infty. \tag{1.5.16}$$

The key point in the proof of this theorem is the relaxation of the acceleration

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty. \tag{1.5.17}$$

This relaxation has long been known in Classical Electrodynamics as 'radiation damping'. Namely, the Liénard–Wiechert formulas for retarded potentials suggest that a particlewith a non-zero acceleration radiates energy to infinity. This radiation cannot last forever, because the total energy of the solution is finite. These arguments result in the conclusion (1.5.17) that can be found in any textbook on Classical Electrodynamics.

However, rigorous proof is not so obvious and it was done for the first time in [47]. The proof relies on calculation of total energy amount radiated to infinity using the Liénard–Wiechert formulas. The central point is the representation of this amount in the form of a convolution and subsequent application of the Wiener Tauberian theorem.

Below we give a streamlined version of this proof.

Remark 1.5.6. i) The condition (1.5.7) is not necessary for relaxation (1.5.17). The relaxation also takes place under the condition (1.5.9) (see Remark 1.5.9).

ii) The Wiener condition (1.5.13) also is not necessary for relaxation (1.5.17). For example, (1.5.17) obviously holds in the case when $V(x) \equiv 0$ and $\rho(x) \equiv 0$. More generally, such relaxation also holds when $V(x) \equiv 0$ and the norm $\|\rho\|$ is sufficiently small, see (2.2.1).

1.5.1 Liénard–Wiechert asymptotics

Let us recall long range asymptotics of the Liénard–Wiechert potentials established in [47, 48]. Denote by $\psi_r(x,t)$ the retarded potential

$$\psi_r(x,t) = -\frac{1}{4\pi}\int \frac{d^3y\ \theta(t-|x-y|)}{|x-y|}\rho(y-q(t-|x-y|)), \tag{1.5.18}$$

and set $\pi_r(x,t) = \dot{\psi}_r(x,t)$. Denote $T_r := \overline{q}_0 + R_\rho$.

Lemma 1.5.7. *The following asymptotics hold*

$$\left\{\begin{array}{l}\pi_r(x,|x|+t)=\overline{\pi}(\omega(x),t)|x|^{-1}+\mathcal{O}(|x|^{-2})\\ \\ \nabla\psi_r(x,|x|+t)=-\omega(x)\overline{\pi}(\omega(x),t)|x|^{-1}+\mathcal{O}(|x|^{-2})\end{array}\right| ,\quad |x|\to\infty \tag{1.5.19}$$

uniformly in $t\in[T_r,T]$ for any $T>T_r$. Here $\omega(x)=x/|x|$, and $\overline{\pi}(\omega(x),t)$ is given in (1.5.21).

Proof. The integrand of (1.5.18) vanishes for $|y|>T_r$. Then $|x-y|\le t$ for $t-|x|>T_r$, and (1.5.18) implies

$$\begin{aligned}\nabla\psi_r(x,t) &= \int\frac{d^3y}{4\pi|x-y|}n\nabla\rho(y-q(t-|x-y|))\cdot\dot q(t-|x-y|)+\mathcal{O}(|x|^{-2})\\ &= -\omega(x)\pi_r(x,t)+\mathcal{O}(|x|^{-2}),\qquad t-|x|>T_r,\end{aligned}$$

because $n=\dfrac{x-y}{|x-y|}=\omega(x)+\mathcal{O}(|x|^{-1})$ for bounded $|y|$. Hence, it suffices to prove asymptotics (1.5.19) for π_r only. We have

$$\pi_r(x,t)=-\int d^3y\;\frac{1}{4\pi|x-y|}\nabla\rho(y-q(\tau))\cdot\dot q(\tau),\qquad \tau:=t-|x-y|. \tag{1.5.20}$$

Replacing t by $|x|+t$ in definition of τ, we obtain

$$\tau=|x|+t-|x-y|=t+\omega(x)\cdot y+\mathcal{O}(|x|^{-1})=\overline{\tau}+\mathcal{O}(|x|^{-1}),\qquad \overline{\tau}=t+\omega\cdot y,$$

since

$$|x|-|x-y|=|x|-\sqrt{|x|^2-2x\cdot y+|y|^2}\sim|x|\Big(\frac{x\cdot y}{|x|^2}-\frac{|y|^2}{2|x|^2}\Big)=\omega(x)\cdot y+\mathcal{O}(|x|^{-1}).$$

Hence (1.5.20) implies (1.5.19) with

$$\overline{\pi}(\omega,t):=-\frac{1}{4\pi}\int d^3y\;\nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau}). \tag{1.5.21}$$

□

1.5.2 Free wave equation

Consider now the solution $\psi_K(x,t)$ of free wave equation with initial conditions

$$\psi_K(x,0)=\psi_0(x),\quad \dot\psi_K(x,0)=\pi_0(x),\qquad x\in\mathbb{R}^3. \tag{1.5.22}$$

The Kirchhoff formula gives

$$\psi_K(x,t)=\frac{1}{4\pi t}\int_{S_t(x)}d^2y\;\pi_0(y)+\frac{\partial}{\partial t}\Big(\frac{1}{4\pi t}\int_{S_t(x)}d^2y\;\psi_0(y)\Big). \tag{1.5.23}$$

Here $S_t(x)$ is the sphere $\{y:\ |y-x|=t\}$. Denote $\pi_K(x,t)=\dot\psi_K(x,t)$.

Lemma 1.5.8. *Let $Y_0\in\mathcal{E}_\sigma$. Then for any $R>0$ and any $T_2>T_1\ge 0$*

$$\int_{R+T_1}^{R+T_2}dt\int_{\partial B_R}d^2x\;\Big(|\pi_K(x,t)|^2+|\nabla\psi_K(x,t)|^2\Big)\le I_0<\infty. \tag{1.5.24}$$

Proof. Formula (1.5.23) implies

$$\nabla\psi_K(x,t)=\frac{t}{4\pi}\int_{S_1} d^2z\ \nabla\pi_0(x+tz)+\frac{1}{4\pi}\int_{S_1} d^2z\ \nabla\psi_0(x+tz)+\frac{t}{4\pi}\int_{S_1} d^2z\ \nabla_x(\nabla\psi_0(x+tz)\cdot z).$$

Here $S_1 := S_1(0)$. From (1.5.5) it follows that

$$\begin{aligned}|\nabla\psi_K(x,t)| &\le C\sum_{s=0}^{1} t^s\int_{S_1} d^2z\ |x+tz|^{-\sigma-1-s}\\ &= C\sum_{s=0}^{1}\frac{2\pi t^{s-1}}{(\sigma+s-1)|x|}\Big((t-|x|)^{-\sigma-s+1}-(t+|x|)^{-\sigma-s+1}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\int\limits_{R+T_1}^{R+T_2} dt\int\limits_{\partial B_R} d^2x|\nabla\psi_k(x,t)|^2 &\le C\int\limits_{R+T_1}^{R+T_2}\Big[\frac{(t+R)^{2-2\sigma}+(t-R)^{2-2\sigma}}{t^2}+(t-R)^{-2\sigma}\Big]dt\\ &\le C_1\int\limits_{R+T_1}^{R+T_2} dt\Big[\Big(1+\frac{R}{t}\Big)^2+\Big(1-\frac{R}{t}\Big)^2+1\Big](t-R)^{-2\sigma}\le I_0<\infty.\end{aligned}$$

The integral with $\nabla\pi_K(x,t)$ can be estimated similarly. □

1.5.3 Scattering of energy to infinity

Now we obtain a bound on the total energy radiated to infinity which we will represent as a 'radiation integral'.

This integral has to be bounded a priori by (1.5.11). Indeed, the energy $\mathcal{H}_R(t)$ at time $t\in\mathbb{R}$ in the ball B_R is defined by

$$\mathcal{H}_R(t)=\frac{1}{2}\int_{B_R} d^3x\ \Big(|\pi(x,t)|^2+|\nabla\psi(x,t)|^2\Big)+\sqrt{1+p^2(t)}+V(q(t))+\int d^3x\ \psi(x,t)\rho(x-q(t)).$$

Consider the energy $I_R(T_1,T_2)$ radiated from the ball B_R during the time interval $[T_1,T_2]$ with $T_2>T_1>0$:

$$I_R(T_1,T_2)=\mathcal{H}_R(T_1)-\mathcal{H}_R(T_2).$$

This energy is bounded a priori, because by (1.5.11) the energy $\mathcal{H}_R(T_1)$ is bounded from above, while $\mathcal{H}_R(T_2)$ is bounded from below. Thus,

$$I_R(T_1,T_2)\le I<\infty, \tag{1.5.25}$$

where I does not depend on T_1, T_2 and R. Further, one has

$$\frac{d}{dt}\mathcal{H}_R(t)=\int_{\partial B_R} d^2x\ \omega(x)\cdot\pi(x,t)\nabla\psi(x,t),\quad t>R.$$

Hence, (1.5.25) implies

$$\int_{R+T_1}^{R+T_2} dt\int_{\partial B_R} d^2x\ \omega(x)\cdot\pi(x,t)\nabla\psi(x,t)\le I.$$

The solution admits the splitting $\pi = \pi_r + \pi_K$, $\psi = \psi_r + \psi_K$, and hence,

$$\int_{R+T_1}^{R+T_2} dt \int_{\partial B_R} d^2x\ \omega(x)\cdot(\pi_r\nabla\psi_r + \pi_K\nabla\psi_r + \pi_r\nabla\psi_K + \pi_K\nabla\psi_K) \le I.$$

Lemmas 1.5.7 and 1.5.8 together with the Cauchy-Schwarz inequality imply

$$\int_{T_r}^{T} dt \int_{S_1} d^2\omega\, |\overline{\pi}(\omega,t)|^2 \le I_1 + T\mathcal{O}(R^{-1}), \qquad T > T_r,$$

where $I_1 < \infty$ does not depend on T and R. Taking the limit $R\to\infty$ and then $T\to\infty$ we obtain the finiteness of the energy radiated to infinity:

$$\int_0^\infty dt \int_{S_1} d^2\omega |\overline{\pi}(\omega,t)|^2 < \infty. \tag{1.5.26}$$

1.5.4 Convolution representation and relaxation of acceleration

Applying a partial integration in (1.5.21), we obtain

$$\begin{aligned}\overline{\pi}(\omega,t) &= \int d^3y\ \nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau}) = \int d^3y\ \nabla_y\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau})\frac{1}{1-\omega\cdot\dot q(\overline{\tau})}\\ &= -\int d^3y\ \rho(y-q(\overline{\tau}))\frac{\partial}{\partial y_\alpha}\frac{\dot q_\alpha(\overline{\tau})}{1-\omega\cdot\dot q(\overline{\tau})} = \frac{1}{4\pi}\int d^3y\ \rho(y-q(\overline{\tau}))\frac{\omega\cdot\ddot q(\tau)}{(1-\omega\cdot\dot q(\tau))^2}\end{aligned} \tag{1.5.27}$$

The function $\overline{\pi}(\omega,t)$ is globally Lipschitz continuous in ω and t due to (1.5.11) Hence, (1.5.26) implies

$$\lim_{t\to\infty}\overline{\pi}(\omega,t) = 0 \tag{1.5.28}$$

uniformly in $\omega\in S_1$. Denote $r(t) = \omega\cdot q(t)$, $s = \omega\cdot y$, $\tilde\rho(q_3) = \int dq_1dq_2\rho(q_1,q_2,q_3)$ and decompose the y-integration in (1.5.27) along and transversal to ω. Then we obtain the convolution

$$\begin{aligned}\overline{\pi}(\omega,t) &= \int ds\,\tilde\rho(s-r(t+s))\,\frac{\ddot r(t+s)}{(1-\dot r(t+s))^2}\\ &= \int d\tau\,\tilde\rho(t-(\tau-r(\tau)))\,\frac{\ddot r(\tau)}{(1-\dot r(\tau))^2} = \int d\theta\,\tilde\rho(t-\theta)g_\omega(\theta) = \tilde\rho * g_\omega(t).\end{aligned}$$

Here $\theta = \theta(\tau) = \tau - r(\tau)$ is a nondegenerate diffeomorphism of $\mathbb{R}$ since $\dot r \le \overline{r} < 1$ due to (1.5.11), and

$$g_\omega(\theta) = \frac{\ddot r(\tau(\theta))}{(1-\dot r(\tau(\theta)))^3}. \tag{1.5.29}$$

Let us extend $q(t) = 0$ for $t < 0$. Then $\tilde\rho * g_\omega(t)$ is defined for all t, and coincides with $\overline{\pi}(\omega,t)$ for sufficiently large t. Hence, (1.5.28) reads as a convolution limit

$$\lim_{t\to\infty}\tilde\rho * g_\omega(t) = 0. \tag{1.5.30}$$

Moreover, $g'_\omega(\theta)$ is bounded by (1.5.11). Therefore, (1.5.30) and the Wiener condition (1.5.13) imply

$$\lim_{\theta\to\infty} g_\omega(\theta) = 0, \qquad \omega\in S_1 \tag{1.5.31}$$

by Pitt's extension of the Wiener Tauberian theorem, cf. [18, Thm. 9.7(b)]. Hence, (1.5.29) implies

$$\lim_{t\to\infty} \ddot q(t) = 0. \tag{1.5.32}$$

since $\theta(t) \to \infty$ as $t \to \infty$. Finally,

$$\lim_{t\to\infty} \dot q(t) = 0, \tag{1.5.33}$$

since $|q(t)| \le \overline{q}_0$ due to (1.5.11).

Remark 1.5.9. (i) We have used condition (1.5.7) in the proof of (1.5.25). However, (1.5.9) at this point is also sufficient. Hence, the relaxation (1.5.32) holds also under condition (1.5.9).
(ii) For point charge $\rho(x) = \delta(x)$, (1.5.30) implies (1.5.31) directly.
(iii) Condition (1.5.13) is necessary for the implication (1.5.31)⇒(1.5.32). Indeed, if (1.5.13) is violated, then $\hat\rho_a(\xi) = 0$ for some $\xi \in \mathbb{R}$, and with the choice $g(\theta) = \exp(i\xi\theta)$ we have $\rho_a * g(t) \equiv 0$ whereas g does not decay to zero.

1.5.5 A compact attracting set

Here we show that the set

$$\mathcal{A} = \{S_q :\ q \in \mathbb{R}^3,\ |q| \le \overline{q}_0\} \tag{1.5.34}$$

is an attracting subset. It is compact in $\mathcal{E}_F$ since $\mathcal{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$.

Lemma 1.5.10. *The following attraction holds,*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{A}, \quad t \longrightarrow \pm\infty. \tag{1.5.35}$$

Proof. We need to check that for every $R > 0$

$$\operatorname{dist}_R(Y(t), \mathcal{A}) = |p(t)| + \|\pi(t)\|_R + \inf_{S_q\in\mathcal{A}} \Big(|q(t) - q| + \|\psi(t) - \psi_q\|_R + \|\nabla(\psi(t) - \psi_q)\|_R\Big) \to 0 \tag{1.5.36}$$

as $t \to +\infty$ We estimate each summand separately.
i) $|p(t)| \to 0$ as $t \to \infty$ by (1.5.32).
ii) $\inf\limits_{|q|\le\overline{q}_0} |q(t) - q| = 0$ for any $t \in \mathbb{R}$ by (1.5.11).
iii) (1.5.18) implies for $t > R + T_r$ and $|x| < R$

$$|\pi_r(x,t)| \le\ C \max_{t-R-T_r\le\tau\le t} |\dot q(\tau)| \int_{|y|<T_r} d^3y\, \frac{1}{|x-y|} |\nabla\rho(y - q(t - |x-y|))|\,.$$

The integral in the RHS is bounded uniformly in $t > R + T_r$ and $x \in B_R$. Hence, $\|\pi_r(t)\|_R \to 0$ as $t \to \infty$ by (1.5.33). Then also $\|\pi(t)\|_R \to 0$.
iv) We can replace q with $q(t)$ in the last line of (1.5.36). Then for $t > R+T_r$ and $|x| < R$, one has

$$\psi_r(x,t) - \psi_{q(t)}(x) = -\int_{|y|<T_r} d^3y \frac{1}{4\pi|x-y|} \Big(\rho(y - q(t - |x-y|)) - \rho(y - q(t))\Big)$$

by (1.5.18). Moreover, $\rho(y-q(t-|x-y|))-\rho(y-q(t)) \to 0$ as $t \to \infty$ uniformly in $x \in B_R$ due to (1.5.33). Hence, $\|\psi_r(t) - \psi_{q(t)}\|_R \to 0$ as $t \to \infty$. Then also $\|\psi(t) - \psi_{q(t)}\|_R \to 0$. Finally, $\|\nabla(\psi(t) - \psi_{q(t)})\|_R$ can be estimated in a similar way. □

1.5.6 Global attraction to stationary states

Now we complete the proof of Theorem 1.5.5.
i) Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be any finite energy solution to the system (1.5.1)–(1.5.2). If the attraction (1.5.15) does not hold, there is a sequence $t_k \to \infty$ for which

$$\operatorname{dist}(Y(t_k), \mathcal{S}) \geq \delta > 0, \qquad k = 1, 2, \dots \tag{1.5.37}$$

Since $\mathcal{A}$ is a compact set in $\mathcal{E}_F$, (1.5.35) implies that

$$Y(t_{k'}) \xrightarrow{\mathcal{E}_F} \overline{Y} \in \mathcal{A}, \qquad k' \to \infty \tag{1.5.38}$$

for some subsequence $k' \to \infty$. It remains to check that $\overline{Y} = S_{q_*} \in \mathcal{S}$ with some $q_* \in Z$, since this contradicts (1.5.37).

First, $\overline{Y} = S_q$ with some $|q| \leq \overline{q}_0$ by the definition (1.5.34). Similarly, by the continuity of the map $W(t)$ in $\mathcal{E}_F$,

$$W(t)Y(t_{k'}) = Y(t_{k'} + t) \xrightarrow{\mathcal{E}_F} W(t)\overline{Y} = S_{Q(t)}, \qquad k' \to \infty, \tag{1.5.39}$$

where $Q(\cdot) \in C^2(\mathbb{R}, \mathcal{E})$, since $W(t)\overline{Y} \in C(\mathbb{R}, \mathcal{E})$ is a solution to the system (1.5.1)–(1.5.2). Finally, for $S_{Q(t)}$ to be a solution to the system (1.5.1)–(1.5.2), there must be $\dot{Q}(t) \equiv 0$. Therefore, $Q(t) \equiv q_* \in Z$ and $\overline{Y} = S_{q_*} \in \mathcal{S}$.

ii) If the set Z is discrete in $\mathbb{R}^3$, then solitary manifold $\mathcal{S}$ is discrete in $\mathcal{E}_F$. □

1.6 Maxwell–Lorentz equations: radiation damping

In [48] global attraction to stationary states similar to (1.5.15), (1.5.16) was established for the Maxwell–Lorentz equations with charged relativistic particle:

$$\left\{\begin{array}{l}\dot E(x,t)=\operatorname{rot}B(x,t)-\dot q\rho(x-q),\qquad \dot B(x,t)=-\operatorname{rot}E(x,t)\\ \\ \operatorname{div}E(x,t)=\rho(x-q),\qquad \operatorname{div}B(x,t)=0,\qquad \dot q(t)=\dfrac{p(t)}{\sqrt{1+p^2(t)}}\\ \\ \dot p(t)=\displaystyle\int[E(x,t)+E^{\rm ext}(x,t)+\dot q(t)\wedge(B(x,t)+B^{\rm ext}(x,t))]\rho(x-q(t))\,dx\end{array}\right| . \tag{1.6.1}$$

Here $\rho(x-q)$ is the particle charge density, $\dot q\rho(x-q)$ is the corresponding current density, and $E^{\rm ext}=-\nabla\phi^{\rm ext}(x)$ and $B^{\rm ext}=-\operatorname{rot}A^{\rm ext}(x)$ are external static Maxwell fields. Similarly to (1.5.7), we assume that *effective scalar potential* is confining:

$$V(q):=\int\phi^{\rm ext}(x)\rho(x-q)\,dx\to\infty,\qquad |q|\to\infty. \tag{1.6.2}$$

This system describes Classical Electrodynamics with‘extended electron’ introduced by M. Abraham [203, 204]. In the case of a point electron, when $\rho(x)=\delta(x)$, such system is not well defined. Indeed, in this case, any solutions $E(x,t)$ and $B(x,t)$ of the Maxwell equations (the first line of (1.6.1)) are singular for $x=q(t)$, and, accordingly, the integral in the last equation (1.6.1) does not exist.

This system may be formally presented in Hamiltonian form, if the fields are expressed in terms of potentials $E(x,t)=-\nabla\phi(x,t)-\dot A(x,t)$, $B(x,t)=-\operatorname{rot}A(x,t)$. The corresponding Hamiltonian functional reads

$$\begin{aligned}\mathcal H &= \frac12[\langle E,E\rangle+\langle B,B\rangle]+V(q)+\sqrt{1+p^2}\\ &= \frac12\int[E^2(x)+B^2(x)]\,dx+V(q)+\sqrt{1+p^2}.\end{aligned} \tag{1.6.3}$$

The Hilbert phase space of finite energy states is defined as $\mathcal E:=L^2\oplus L^2\oplus\mathbb R^3\oplus\mathbb R^3$. Under the condition (1.6.2) a solution $Y(t)=(E(x,t),B(x,t),q(t),p(t))\in C(\mathbb R,\mathcal E)$ of finite energy exists and is unique for any initial state $Y(0)\in\mathcal E$.

The Hamiltonian (1.6.3) is conserved along solutions, what provides *a priori estimates*, which play an important role in proving global attraction of the type (1.5.15), (1.5.16) in [48]. The key role in the proof is played again by relaxation of acceleration (1.5.17), which is derived by a suitable generalisation of our methods [47]: the expression of energy radiated to infinity via Liénard–Wiechert retarded potentials, its representation in the form of a convolution and the use of the Wiener Tauberian theorem.

In Classical Electrodynamics the **radiation damping** (1.5.17) is traditionally derived from the Larmor and Liénard formulas for radiation power of a point particle (see formulas (14.22) and (14.24) of [212]), but this approach ignores field feedback although it plays the key role in the relaxation of the acceleration. The main problem is that this reverse field reaction for point particles is infinite. A rigorous sense of these classical calculations was first found in [47, 48] for the Abraham model of ‘extended electron’ under the Wiener condition (1.5.13). A detailed discussion can be found in [55].

1.7 Wave equations with concentrated nonlinearities

Here we prove the result of [53] on global attraction to to stationary states for 3D wave equation with point coupling to an $\mathbf{U}(1)$-invariant nonlinear oscillator. This goal is inspired by fundamental mathematical problem of an interaction of point particles with the fields.

Point interaction models were considered since 1930 in the papers of E. Wigner, H. Bethe and R. Peierls, E. Fermiand others (see [93] for a detailed survey) and of Dirac [96]. Rigorous mathematical results were obtained since 1960 by Ya. B. Zeldovich, F. Berezin, L. Faddeev, F.H.J. Cornish, D. Yafaev, E. Zeidler and others [94, 95, 97, 99, 101], and since 2000 by D. Noja, A. Posilicano, and others [98, 100, 92].

We consider real wave field $\psi(x,t)$ coupled to a nonlinear oscillator

$$\left\{\begin{array}{c} \ddot\psi(x,t)=\Delta\psi(x,t)+\zeta(t)\delta(x) \\ \\ \lim\limits_{x\to 0}(\psi(x,t)-\zeta(t)G(x))=F(\zeta(t)) \end{array}\right| \quad x\in\mathbb{R}^3,\quad t\in\mathbb{R}, \tag{1.7.1}$$

where $G(x)=\dfrac{1}{4\pi|x|}$ is the Green function of the operator $-\Delta$ in $\mathbb{R}^3$. Nonlinear function $F(\zeta)$ admits a potential:

$$F(\zeta)=U'(\zeta),\quad \zeta\in\mathbb{R},\quad U\in C^2(\mathbb{R}). \tag{1.7.2}$$

We assume that the potential is confining, i.e.,

$$U(\zeta)\to\infty,\quad \zeta\to\pm\infty. \tag{1.7.3}$$

The system (1.7.1) admits stationary solutions $\psi_q=qG(x)\in L^2_{loc}(\mathbb{R}^3)$, where $q\in Q:=\{q\in\mathbb{R}:F(q)=0\}$. We assume that the set Q is nonempty and does not contain intervals, i.e.,

$$[a,b]\not\subset Q \tag{1.7.4}$$

for any $a<b$.

As before, $\|\cdot\|$ and $\|\cdot\|_R$ denote the norms in $L^2=L^2(\mathbb{R}^3)$ and in $L^2(B_R)$ respectively, and $\mathring{H}^1=\mathring{H}^1(\mathbb{R}^3)$ is the completion of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla\psi(x)\|$. Denote

$$\mathring{H}^2=\mathring{H}^2(\mathbb{R}^3):=\{f\in\mathring{H}^1,\quad \Delta f\in L^2\},\quad t\in\mathbb{R}.$$

We define the function sets

$$D=\{\psi\in L^2:\psi(x)=\psi_{reg}(x)+\zeta G(x),\quad \psi_{reg}\in\mathring{H}^2,\quad \zeta\in\mathbb{R},\quad \lim_{x\to 0}\psi_{reg}(x)=F(\zeta)\}$$

and

$$\dot D=\{\pi\in L^2(\mathbb{R}^3):\pi(x)=\pi_{reg}(x)+\eta G(x),\quad \pi_{reg}\in\mathring{H}^1,\quad \eta\in\mathbb{R}\}.$$

Obviously, $D\subset\dot D$.

Definition 1.7.1. *$\mathcal{D}$ is the Hilbert manifold of states $\Psi=(\psi,\pi)\in D\times\dot D$.*

First, we prove global well-posedness for the system (1.7.1) established in [98].

Theorem 1.7.2. *Let conditions (1.7.2) and (1.7.3) hold. Then*

(i) *For every initial data* $\Psi_0 = (\psi_0, \pi_0) \in \mathcal{D}$ *the system (1.7.1) has a unique solution* $\Psi(t) = (\psi(t), \dot\psi(t)) \in C(\mathbb{R}, \mathcal{D})$.

(ii) *The energy is conserved:*

$$\mathcal{H}(\Psi(t)) := \frac{1}{2}\Big(\|\dot\psi(t)\|^2 + \|\nabla\psi_{reg}(t)\|^2\Big) + U(\zeta(t)) = \text{const}, \quad t \in \mathbb{R}. \tag{1.7.5}$$

(iii) *The following a priori bound holds*

$$|\zeta(t)| \le C(\Psi_0), \quad t \in \mathbb{R}. \tag{1.7.6}$$

Proof. It suffices to prove the theorem for $t \ge 0$.
Step i) First we consider free wave equation with initial data from $\mathcal{D}$:

$$\ddot\psi_f(x,t) = \Delta\psi_f(x,t), \quad (\psi_f(0), \dot\psi_f(0)) = (\psi_0, \pi_0) = (\psi_{0,reg}, \pi_{0,reg}) + (\zeta_0 G, \eta_0 G) \in \mathcal{D}, \tag{1.7.7}$$

where $(\psi_{0,reg}, \pi_{0,reg}) \in \mathring{H}^2 \oplus \mathring{H}^1$.

Lemma 1.7.3. *There exists a unique solution* $\psi_f(t) \in C([0;\infty), L^2_{loc}(\mathbb{R}^3))$ *to (1.7.7). Moreover, for any* $t > 0$ *there exists the limit*

$$\lambda(t) := \lim_{x\to 0} \psi_f(x,t) \in C[0,\infty),$$

and

$$\dot\lambda(t) \in L^2_{loc}[0,\infty). \tag{1.7.8}$$

Proof. We split $\psi_f(x,t)$ as

$$\psi_f(x,t) = \psi_{f,reg}(x,t) + g(x,t),$$

where $\psi_{f,reg}$ and g are solutions to free wave equation with initial data $(\psi_{0,reg}, \pi_{0,reg})$ and $(\zeta_0 G, \eta_0 G)$, respectively. First, $\psi_{f,reg} \in C([0,\infty), \mathring{H}^2)$ by the energy conservation. Hence, $\lim\limits_{x\to 0} \psi_{f,reg}(x,t)$ exists for any $t \ge 0$ since $\mathring{H}^2(\mathbb{R}^3) \subset C(\mathbb{R}^3)$.

Let us obtain an explicit formula for g. Note, that the function $h(x,t) = g(x,t) - (\zeta_0 + \eta_0 t)G(x)$ satisfies

$$\ddot h(x,t) = \Delta h(x,t) - (\zeta_0 + \eta_0 t)\delta(x), \quad h(x,0) = 0, \ \ \dot h(x,0) = 0. \tag{1.7.9}$$

The unique solution to (1.7.9) is spherical wave :

$$h(x,t) = -\frac{\theta(t-|x|)}{4\pi|x|}(\zeta_0 + \eta_0(t-|x|)), \quad t \ge 0. \tag{1.7.10}$$

Here θ is the Heaviside function. Hence,

$$\begin{aligned} g(x,t) &= h(x,t) + (\zeta_0 + \eta_0 t)G(x) \\ &= -\frac{\theta(t-|x|)(\zeta_0 + \eta_0(t-|x|))}{4\pi|x|} + \frac{\zeta_0 + \eta_0 t}{4\pi|x|} \in C([0,\infty), L^2_{loc}(\mathbb{R}^3)), \end{aligned}$$

and then

$$\lim_{x\to 0} g(x,t) = \frac{\eta_0}{4\pi}, \quad t > 0.$$

Finally, $\dot\psi_{f,reg}(0,t) \in L^2_{loc}([0,\infty))$ by [53, Lemma 3.4]. Hence, (1.7.8) follows. □

Step ii) Now we prove local well-posedness. We modify the nonlinearity F so that it becomes Lipschitz-continuous. Define

$$\Lambda(\Psi_0) = \sup\{|\zeta| : \zeta \in \mathbb{R},\, U(\zeta) \le \mathcal{H}(\Psi_0)\}.$$

We may pick a modified potential function $\tilde{U}(\zeta) \in C^2(\mathbb{R})$, so that

$$\begin{cases} \tilde{U}(\zeta) = U(\zeta), & |\zeta| \le \Lambda(\Psi_0), \\ \tilde{U}(\zeta) > \mathcal{H}(\Psi_0), & |\zeta| > \Lambda(\Psi_0), \end{cases} \tag{1.7.11}$$

and the function $\tilde{F}(\zeta) = \tilde{U}'(\zeta)$ is Lipschitz-continuous:

$$|\tilde{F}(\zeta_1) - \tilde{F}(\zeta_2)| \le C|\zeta_1 - \zeta_2|, \quad \zeta_1, \zeta_2 \in \mathbb{R}.$$

The following lemma is trivial.

Lemma 1.7.4. *For small* $\tau > 0$ *the Cauchy problem*

$$\frac{1}{4\pi}\dot{\zeta}(t) + \tilde{F}(\zeta(t)) = \lambda(t), \quad \zeta(0) = \zeta_0 \tag{1.7.12}$$

has a unique solution $\zeta \in C^1[0,\tau]$.

Denote

$$\psi_S(t,x) := \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|), \quad t \in [0,\tau],$$

with ζ from Lemma 1.7.4.

Lemma 1.7.5. *The function* $\psi(x,t) := \psi_f(x,t) + \psi_S(x,t)$ *is a unique solution to the system*

$$\left\{\begin{array}{c} \ddot{\psi}(x,t) = \Delta\psi(x,t) + \zeta(t)\delta(x) \\ \lim\limits_{x\to 0}(\psi(x,t) - \zeta(t)G(x)) = \tilde{F}(\zeta(t)) \\ \psi(x,0) = \psi_0(x), \quad \dot{\psi}(x,0) = \pi_0(x) \end{array}\right| \quad x \in \mathbb{R}^3, \quad t \in [0,\tau], \tag{1.7.13}$$

satisfying the condition

$$(\psi(t), \dot{\psi}(t)) \in \mathcal{D}, \quad t \in [0,\tau]. \tag{1.7.14}$$

Proof. Initial conditions of (1.7.13) follow from (1.7.7). Further,

$$\lim_{x\to 0}(\psi(t,x) - \zeta(t)G(x)) = \lambda(t) + \lim_{x\to 0}\Big(\frac{\theta(t-|x|)\zeta(t-|x|)}{4\pi|x|} - \frac{\zeta(t)}{4\pi|x|}\Big) = \lambda(t) - \frac{1}{4\pi}\dot{\zeta}(t) = \tilde{F}(\zeta(t)).$$

Thus, the second equation of (1.7.13) is satisfied. At last,

$$\ddot{\psi} = \ddot{\psi}_f + \ddot{\psi}_S = \Delta\psi_f + \Delta\psi_S + \zeta\delta = \Delta\psi + \zeta\delta$$

and ψ solves the first equation of (1.7.13) then.

It remains to check (1.7.14). Note, that the function $\varphi_{reg}(x,t) = \psi(x,t) - \zeta(t)G_1(x) = \psi_{reg}(x,t) + \zeta(t)(G(x) - G_1(x))$, where $G_1(x) = G(x)e^{-|x|}$, satisfies

$$\ddot{\varphi}_{reg}(x,t) = \Delta\varphi_{reg}(x,t) + (\zeta(t) - \ddot{\zeta}(t))G_1(x)$$

with initial data from $H^2 \oplus H^1$. Moreover, (1.7.8) and (1.7.12) imply that $\ddot\zeta \in L^2([0,\tau])$. Hence,

$$(\varphi_{reg}(x,t), \dot\varphi_{reg}(x,t)) \in H^2 \oplus H^1, \quad t \in [0,\tau]$$

by [53, Lemma 3.2]. Therefore,

$$\psi_{reg}(x,t) = \psi(x,t) - \zeta(t)G(x) = \varphi_{reg}(x,t) + \zeta(t)(G_1(x) - G(x))$$

satisfies $(\psi_{reg}(t), \dot\psi_{reg}(t)) \in \mathring{H}^2 \oplus \mathring{H}^1$, $t \in [0,\tau]$, and (1.7.14) holds then.

It remains to prove the uniqueness. Suppose now that there exists another solution $\tilde\psi = \tilde\psi_{reg} + \tilde\zeta G$ to the system (1.7.13), with $(\tilde\psi, \dot{\tilde\psi}) \in \mathcal{D}$. Then, by reversing the above argument, the second equation of (1.7.13) implies that $\tilde\zeta$ solves the Cauchy problem (1.7.12). The uniqueness of the solution of (1.7.12) implies that $\tilde\zeta = \zeta$. Then, defining

$$\psi_S(t,x) := \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|), \quad t \in [0,\tau],$$

for $\tilde\psi_f = \tilde\psi - \psi_S$ one obtains

$$\ddot{\tilde\psi}_f = \ddot{\tilde\psi} - \ddot\psi_S = \Delta\tilde\psi_{reg} - (\Delta\psi_S + \zeta\delta) = \Delta(\tilde\psi_{reg} - (\psi_S - \zeta G)) = \Delta\tilde\psi_f\,,$$

i.e. $\tilde\psi_f$ solves the Cauchy problem (1.7.7). Hence, $\tilde\psi_f = \psi_f$ by the uniqueness of the solution to (1.7.7), and hence, $\tilde\psi = \psi$. □

According to [53, Lemma 3.7]

$$\mathcal{H}_{\tilde F}(\Psi(t)) = \|\dot\psi(t)\|^2 + \|\nabla\psi_{reg}(t)\|^2 + \tilde U(\zeta(t)) = \text{const}, \quad t \in [0,\tau]. \tag{1.7.15}$$

Now we are able to prove Theorem 1.7.2 on the global well-posedness. First, note that

$$\tilde U(\zeta(t)) = U(\zeta(t)), \quad t \in [0,\tau]. \tag{1.7.16}$$

Indeed, $\mathcal{H}_F(\Psi_0) \ge U(\zeta_0)$ by the definition of energy in (1.7.5). Therefore, $|\zeta_0| \le \Lambda(\Psi_0)$, and then $\tilde U(\zeta_0) = U(\zeta_0)$, $\mathcal{H}_{\tilde F}(\Psi_0) = \mathcal{H}_F(\Psi_0)$. Further,

$$\mathcal{H}_F(\Psi_0) = \mathcal{H}_{\tilde F}(\Psi(t)) \ge \tilde U(\zeta(t)), \quad t \in [0,\tau],$$

and (1.7.11) implies that

$$|\zeta(t)| \le \Lambda(\Psi_0), \quad t \in [0,\tau]. \tag{1.7.17}$$

Now we can replace $\tilde F$ by F in Lemma 1.7.5 and in (1.7.15). The solution $\Psi(t) = (\psi(t), \dot\psi(t)) \in \mathcal{D}$ constructed in Lemma 1.7.5 exists for $0 \le t \le \tau$, where the time span τ in Lemma 1.7.4 depends only on $\Lambda(\Psi_0)$. Hence, the bound (1.7.17) at $t = \tau$ allows us to extend the solution Ψ to the time interval $[\tau, 2\tau]$. We proceed by induction to obtain the solution for all $t \ge 0$. Theorem 1.7.2 is proved. □

The main result of [53] is as follows.

Theorem 1.7.6. *Let $\Psi(x,t) = (\psi(x,t), \dot\psi(x,t))$ be a solution to (1.7.1) with initial data from $\mathcal{D}$. Then*

$$\Psi(x,t) \to (\psi_{q_\pm}, 0), \quad t \to \pm\infty,$$

where $q_\pm \in Q$ and the convergence holds in $L^2_{loc}(\mathbb{R}^3) \oplus L^2_{loc}(\mathbb{R}^3)$.

Proof. It suffices to prove this theorem for $t \to +\infty$ only. By Lemma 1.7.5, the solution $\psi(x,t)$ to (1.7.1) with initial data $(\psi_0, \pi_0) \in \mathcal{D}$, can be represented as the sum

$$\psi(x,t) := \psi_f(x,t) + \psi_S(x,t), \quad t \geq 0, \tag{1.7.18}$$

where *dispersive component* $\psi_f(x,t)$ is a unique solution to (1.7.7), and *singular component* $\psi_S(x,t)$ is a unique solution to the following Cauchy problem

$$\ddot{\psi}_S(x,t) = \Delta\psi_S(x,t) + \zeta(t)\delta(x), \quad \psi_S(x,0) = 0, \quad \dot{\psi}_S(x,0) = 0. \tag{1.7.19}$$

Here $\zeta(t) \in C_b^1([0,\infty))$ is a unique solution to

$$\frac{1}{4\pi}\dot{\zeta}(t) + F(\zeta(t)) = \lambda(t), \quad \zeta(0) = \zeta_0. \tag{1.7.20}$$

Now we can prove local decay of $\psi_f(x,t)$.

Lemma 1.7.7. *For any $R > 0$, the following convergence holds*

$$\Big\|(\psi_f(t), \dot{\psi}_f(t))\Big\|_{H^2(B_R)\oplus H^1(B_R)} \to 0, \quad t \to \infty. \tag{1.7.21}$$

Here B_R is the ball of radius R.

Proof. We represent the initial data $(\psi_0, \pi_0) = (\psi_{0,reg},\ \pi_{0,reg}) + (\zeta_0 G,\ \eta_0 G) \in \mathcal{D}$ as

$$(\psi_0,\ \pi_0) = (\varphi_0,\ p_0) + (\zeta_0\chi G,\ \eta_0\chi G),$$

where a cut-of function $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfies

$$\chi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases} \tag{1.7.22}$$

Let us show that

$$(\varphi_0,\ p_0) \in H^2 \oplus H^1.$$

Indeed,

$$(\varphi_0,\ p_0) = (\psi_0 - \zeta_0\chi G,\ \pi_0 - \eta_0\chi G) \in L^2 \oplus L^2.$$

On the other hand,

$$(\varphi_0,\ p_0) = (\psi_{0,reg} + \zeta_0(1-\chi)G,\ \pi_{0,reg} + \eta_0(1-\chi)G) \in \mathring{H}^2 \oplus \mathring{H}^1.$$

Now we split the dispersion component $\psi_f(x,t)$ as

$$\psi_f(x,t) = \varphi(x,t) + \varphi_G(x,t), \quad t \geq 0,$$

where φ and φ_G are defined as solutions to the free wave equation with initial data $(\varphi_0,\ p_0)$ and $(\zeta_0\chi G, \eta_0\chi G)$, respectively, and study the decay properties of φ_G and φ.

First, by the strong Huygens principle

$$\varphi_G(x,t) = 0 \ \text{ for } \ |x| \leq t-2.$$

Indeed, $\varphi_G(x,t) = \zeta_0\dot{\psi}_G(x,t) + \eta_0\psi_G(x,t)$, where $\psi_G(x,t)$ is the solution to the free wave equation with initial data $(0, \chi G) \in H^1 \oplus L^2$, and $\psi_G(x,t)$ satisfies the strong Huygens principle by Theorem XI.87 of [16], v. III.

It remains to check that

$$\|(\varphi(t),\dot\varphi(t))\|_{H^2(B_R)\oplus H^1(B_R)}\to 0,\quad t\to\infty,\quad \forall R>0, \tag{1.7.23}$$

For $r\ge 1$ denote $\chi_r=\chi(x/r)$, where $\chi(x)$ is a cut-off function (1.7.22). Denote $\phi_0=(\varphi_0,\pi_0)$. Let $u_r(t)$ and $v_r(t)$ be solutions to free wave equations with the initial data $\chi_r\phi_0$ and $(1-\chi_r)\phi_0$, respectively, so that $\varphi(t)=u_r(t)+v_r(t)$. By the strong Huygens principle

$$u_r(x,t)=0 \ \text{ for } \ t\ge |x|+2r.$$

To conclude (1.7.23), it remains to note that

$$\begin{aligned}\|(v_r(t),\dot v_r(t))\|_{H^2(B_R)\oplus H^1(B_R)} &\le C(R)\|(v_r(t),\dot v_r(t))\|_{\mathring H^2\oplus H^1}=C(R)\|(1-\chi_r)\phi_0\|_{\mathring H^2\oplus H^1}\\ &\le C(R)\|(1-\chi_r)\phi_0\|_{H^2\oplus H^1}\end{aligned} \tag{1.7.24}$$

by the energy conservation for the free wave equation. We also use the Sobolev embedding theorem $\mathring H^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$. The right-hand side of (1.7.24) could be made arbitrarily small if $r\ge 1$ is sufficiently large. □

Due to (1.7.18) and (1.7.21), for the proof of Theorem 1.7.6 it suffices to verify the convergence of $\psi_S(x,t)$ to stationary states:

Lemma 1.7.8. *Let $\psi_S(x,t)$ and $\zeta(t)$ be solutions to (1.7.19) and (1.7.20), respectively. Then*

$$(\psi_S(t),\dot\psi_S(t))\to(\psi_{q_\pm},\ 0),\quad t\to\infty,$$

where $q_\pm\in Q$ and the convergence holds in $L^2_{loc}(\mathbb{R}^3)\oplus L^2_{loc}(\mathbb{R}^3)$.

Proof. The unique solution to (1.7.19) is the spherical wave

$$\psi_S(x,t)=\frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|),\quad t\ge 0, \tag{1.7.25}$$

cf. (1.7.9)–(1.7.10). Then a priori bound (1.7.6) and equation (1.7.20) imply that

$$(\psi_S(t),\dot\psi_S(t))\in L^2(B_R)\oplus L^2(B_R),\quad 0\le R<t.$$

First, we prove the convergence of $\zeta(t)$. From (1.7.6) it follows that $\zeta(t)$ has the upper and lower limits:

$$\underline{\lim}_{t\to\infty}\zeta(t)=a,\quad \overline{\lim}_{t\to\infty}\zeta(t)=b. \tag{1.7.26}$$

Suppose that $a<b$. Then the trajectory $\zeta(t)$ oscillates between a and b. Assumption (1.7.4) implies that $F(\zeta_0)\ne 0$ for some $\zeta_0\in(a,b)$. For the concreteness, let us assume that $F(\zeta_0)>0$. The convergence (1.7.21) implies that

$$\lambda(t)=\psi_f(0,t)\to 0,\qquad t\to\infty. \tag{1.7.27}$$

Hence, for sufficiently large T we have

$$-F(\zeta_0)+\lambda(t)<0,\quad t\ge T.$$

Then for $t\ge T$ the transition of the trajectory from left to right through the point ζ_0 is impossible by (1.7.20). Therefore, $a=b=q_+$, where $q_+\in Q$ since $F(q_+)=0$ by (1.7.20). Hence (1.7.26) implies

$$\zeta(t)\to q_+,\quad t\to\infty, \tag{1.7.28}$$

Further,

$$\theta(t-|x|) \to 1, \quad t \to \infty \tag{1.7.29}$$

uniformly in $|x| \le R$. Then (1.7.25) and (1.7.28) imply that

$$\psi_S(t) \to q_+ G, \quad t \to \infty,$$

where the convergence holds in $L^2_{loc}(\mathbb{R}^3)$. It remains to verify the convergence of $\dot\psi_S(t)$. We have

$$\dot\psi_S(x,t) = \frac{\theta(t-|x|)}{4\pi|x|}\dot\zeta(t-|x|), \quad |x| < t.$$

From (1.7.20), (1.7.27) and (1.7.28) it follows that $\dot\zeta(t) \to 0$ as $t \to \infty$. Then

$$\dot\psi_S(t) \to 0, \qquad t \to \infty$$

in $L^2_{loc}(\mathbb{R}^3)$ by (1.7.29). □

This completes the proof of Theorem 1.7.6. □

1.8 Comparison with dissipative systems

All above results on global attraction to stationary states refer to 'generic' systems with a trivial symmetry group. These systems are characterized by a suitable discreteness of attractors, by Wiener condition, etc.

Global attraction to stationary states (0.0.7) resembles similar asymptotics (0.0.1) for dissipative systems. However, there are a number of fundamental differences:

I. In dissipative systems

• the (point) global attractor always consists of *stationary states*;

• the global attraction (0.0.7) to stationary states is due to the energy absorption;

• the global attraction (0.0.1) *holds only as* $t \to +\infty$;

• this attraction can hold *in bounded and unbounded domains*;

• this attraction is due to the absorption of energy and holds mainly in suitable global norms;

• such global attraction to stationary states also *holds for all finite-dimensional dissipative systems*.

II. On the other hand, in Hamiltonian systems

• the *global attractor may differ from the set of stationary states*, as will be seen below;

• the global attraction (0.0.7) to stationary states is due to the *radiation of energy to infinity*, which plays the role of energy absorption;

• this attraction takes place both *as* $t \to \infty$*, and as* $t \to -\infty$;

• this attraction holds *only in unbounded domains*;

• the attraction holds *only in local seminorms*;

• the attraction to a *proper subset* cannot hold for finite-dimensional Hamiltonian systems due to energy conservation.

Chapter 2

Global Attraction to Solitons

In this chapter we present the first results [57, 60] on global attraction to solitons (0.0.12) for the scalar wave field coupled to the charged relativistic particle. This result was extended in [58, 59] to similar system with the Maxwell field.

2.1 Translation-invariant wave-particle system

In [57] the system (1.5.1)–(1.5.2) was considered in the case of zero potential $V(x) \equiv 0$:

$$\left\{ \begin{array}{rcl} \ddot{\psi}(x,t) & = & \Delta\psi(x,t) - \rho(x-q(t)), \qquad x \in \mathbb{R}^3 \\ \dot{q}(t) & = & \dfrac{p(t)}{\sqrt{1+p^2(t)}}, \ \dot{p}(t) = -\displaystyle\int \nabla\psi(x,t)\rho(x-q(t))\,dx \end{array} \right| , \tag{2.1.1}$$

which can be written in the Hamiltonianform (1.5.3). The Hamiltonian of this system is given by (1.5.4) with $V = 0$, and it is conserved along trajectories. By Lemma 1.5.2 with $V(x) \equiv 0$, global solutions exist for all initial data $Y(0) \in \mathcal{E}$, and a priori estimates (1.5.11) hold.

This system is translation-invariant, so the corresponding full momentum

$$P = p - \int \pi(x)\nabla\psi(x)\,dx \tag{2.1.2}$$

is also conserved. Respectively, the system (2.1.1) admits traveling-wave type solutions (solitons)

$$\psi_v(x-a-vt), \ \ q(t) = a + vt, \ \ p_v = v/\sqrt{1-v^2}, \tag{2.1.3}$$

where $v, a \in \mathbb{R}^3$, and $|v| < 1$. The solitons are easily determined: for $|v| < 1$ there is a unique function ψ_v which makes (2.1.3) a solution to (2.1.1),

$$\psi_v(x) = \ -\int d^3y(4\pi\ |(y-x)_\parallel + \lambda(y-x)_\perp|)^{-1}\rho(y), \tag{2.1.4}$$

where we set $\lambda = \sqrt{1-v^2}$ and $x = x_\parallel + x_\perp$, where $x_\parallel \| v$ and $x_\perp \perp v$ for $x \in \mathbb{R}^3$. Indeed, substituting (2.1.3) into the wave equation of (2.1.1), we get the stationary equation

$$(v\cdot\nabla)^2\psi_v(x) = \Delta\psi_v(x) - \rho(x). \tag{2.1.5}$$

Through the Fourier transform

$$\hat{\psi}_v(k) = -\hat{\rho}(k)/(k^2 - (v\cdot k)^2), \tag{2.1.6}$$

which implies (2.1.4). The set of all solitons forms 6-dimensional *solitary manifold* in the Hilbert phase space $\mathcal{E}$:

$$\mathcal{S} = \{S_{v,a} = (\psi_v(x-a),\, \pi_v(x-a),\, a,\, p_v): \quad v,\, a \in \mathbb{R}^3, \quad |v| < 1\}, \tag{2.1.7}$$

where $\pi_v := -v\nabla\psi_v$. Recall that the spaces $\mathcal{E}$ and $\mathcal{E}_\sigma$ and the corresponding norms were introduced in Definition 1.5.1. The following theorem is the main result of [57].

Theorem 2.1.1. *Let the Wiener condition* (1.5.13) *hold and* $\sigma > 3/2$. *Then for any initial state* $Y(0) \in \mathcal{E}_\sigma$, *the corresponding solution* $Y(t) = (\psi(t), \pi(t), q(t), p(t))$ *of the system* (2.1.1) *converges to the solitary manifold* $\mathcal{S}$ *in the following sense:*

$$\ddot{q}(t) \to 0, \quad \dot{q}(t) \to v_\pm, \qquad t \to \pm\infty, \tag{2.1.8}$$

$$(\psi(x,t), \dot{\psi}(x,t)) = (\psi_{v_\pm}(x-q(t)), \pi_{v_\pm}(x-q(t))) + (r_\pm(x,t), s_\pm(x,t)), \tag{2.1.9}$$

where the remainder decreases locally in the **comoving frame***: for each* $R > 0$

$$\|\nabla r_\pm(q(t)+x,t)\|_R + \|r_\pm(q(t)+x,t)\|_R + \|s_\pm(q(t)+x,t)\|_R \to 0, \qquad t \to \pm\infty. \tag{2.1.10}$$

The theorem means that, in particular,

$$\psi(x,t) \sim \psi_v(x - v_\pm t + \theta_\pm(t)), \quad \text{where} \quad \dot\theta_\pm(t) \to 0, \quad t \to \pm\infty. \tag{2.1.11}$$

The proof [57] relies on a) relaxation of acceleration (1.5.17) in the case $V = 0$ (see Remark 1.5.9 i)), and b) on the *canonical change of variables* to the comoving frame. The key role is played by the fact that the soliton $S_{v,a}$ minimises the Hamiltonian (1.5.4) (in the case $V = 0$) with a fixed total momentum (2.1.2), which implies *orbital stability of solitons*, see [103, 104]. In addition, the proof essentially relies on the *strong Huygens principle* for the three-dimensional wave equation.

Before entering into more precise and technical discussion, it may be useful to give general idea of our strategy. As was mentioned above, the total momentum (2.1.2) is conserved because of translation invariance.

We transform the system (2.1.1) to new variables $(\Psi(x), \Pi(x), Q, P) = (\psi(q+x), \pi(q+x), q, P(\psi, q, \pi, p))$. The key role in our strategy is played by the fact that this transformation is canonical, which is proved in Section 2.1.4. Through this canonical transformation one obtains the new Hamiltonian

$$\begin{aligned}&\mathcal{H}_P(\Psi,\Pi) = \mathcal{H}(\psi,\pi,q,p)\\ &= \int d^3x \left(\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Psi(x)|^2 + \Psi(x)\rho(x)\right) + \left[1 + \left(P + \int d^3x\, \Pi(x)\, \nabla\Psi(x)\right)^2\right]^{1/2}.\end{aligned}$$

Since Q is the cyclic coordinate (i.e., the Hamiltonian $\mathcal{H}_P$ does not depend on Q), we may regard P as a fixed parameter and consider the reduced system for (Ψ, Π) only. Let us define

$$\pi_v(x) = -v \cdot \nabla\psi_v(x), \quad P(v) = p_v + \int d^3x\, v \cdot \nabla\psi_v(x)\, \nabla\psi_v(x), \;\; p_v = v/(1-v^2)^{1/2}. \tag{2.1.12}$$

We will prove that (ψ_v, π_v) is the unique critical point and moreover, global minimum of $\mathcal{H}_{P(v)}$. Thus, if initial data is close to (ψ_v, π_v), then corresponding solution must remain close forever by conservation of energy, which translates into the orbital stability of the solitons. Here we follow the ideas of the D. Bambusi and L. Galgani paper [102], were the orbital stability of solitons for the Maxwell–Lorentz equations was proved for the first time. For a general class of nonlinear wave equations with symmetries such approach to orbital stability of the solitons was developed in [103, 104].

However, the orbital stability by itself is not enough. It only ensures that initial states, close to a soliton, remain so, but does not yield the convergence of $\dot q(t)$ in (2.1.8), and even less the asymptotics (2.1.9), (2.1.10). Thus we need an additional, not quite obvious argument which combines the relaxation (1.5.17) with the orbital stability in order to establish the soliton-like asymptotics (2.1.8), (2.1.9), (2.1.10). As one essential input we will use the strong Huygens principle for wave equation.

2.1.1 Canonical transformation and reduced system

Since the total momentum is conserved, it is natural to use P as a new coordinate. To maintain the symplectic structure we have to complete this coordinate to a canonical transformation of the Hilbert phase space $\mathcal{E}$.

Definition 2.1.2. *Let the transform $T:\mathcal{E}\to\mathcal{E}$ be defined by*

$$T:Y=(\psi,\pi,q,p)\mapsto Y^T=(\Psi(x),\Pi(x),Q,P)=(\psi(q+x),\pi(q+x),q,P(\psi,q,\pi,p))\ , \tag{2.1.13}$$

where $P(\psi,q,\pi,p)$ is the total momentum (2.1.2).

Remarks 2.1.3. i) The map T is continuous on $\mathcal{E}$ and Fréchet-differentiable at points $Y{=}(\psi,q,\pi,p)$ with sufficiently smooth $\psi(x),\pi(x)$, but it is not everywhere differentiable. ii) In the T-coordinates the solitons $Y_{v,a}(t)=(\psi_v(x-a-vt),\pi_v(x-a-vt),q=a+vt,p_v)$ become stationary except for the coordinate Q,

$$TY_{v,a}(t)=(\psi_v(x),\pi_v(x),a+vt,P(v)) \tag{2.1.14}$$

with the total momentum $P(v)$ of the soliton defined in (2.1.12).

Denote $\mathcal{H}^T(Y)=\mathcal{H}(T^{-1}Y)$ for $Y=(\Psi,\Pi,Q,P)\in\mathcal{E}$. Then

$$\mathcal{H}^T(\Psi,\Pi,Q,P)=\mathcal{H}_P(\Psi,\Pi)=\mathcal{H}(\Psi(x-Q),\Pi(x-Q),Q,P+\int d^3x\,\Pi(x)\,\nabla\Psi(x))$$
$$=\int d^3x\,\Big[\frac{1}{2}|\Pi(x)|^2+\frac{1}{2}|\nabla\Psi(x)|^2+\Psi(x)\rho(x)\Big]+\Big(1+\Big[P+\int d^3x\,\Pi(x)\,\nabla\Psi(x)\Big]^2\Big)^{1/2}.$$

The functionals $\mathcal{H}^T$ and $\mathcal{H}$ are Fréchet-differentiable on the Hilbert phase space $\mathcal{E}$.

Proposition 2.1.4. *Let $Y(t)\in C(\mathbb{R},\mathcal{E})$ be a solution to the system (2.1.1). Then*

$$Y^T(t):=TY(t)=(\Psi(t),\Pi(t),q(t),p(t))\in C(\mathbb{R},\mathcal{E})$$

is a solution to the Hamiltonian system

$$\left\{\begin{array}{ll}\dot\Psi=D_\Pi\mathcal{H}^T, & \dot\Pi=-D_\Psi\mathcal{H}^T\\ \dot Q=D_P\mathcal{H}^T, & \dot P=-D_Q\mathcal{H}^T\end{array}\right|. \tag{2.1.15}$$

Proof. The equations for $\dot\Psi$, $\dot\Pi$ and $\dot Q$ can be checked by direct computation, while the one for $\dot P$ follows from conservation of the total momentum (2.1.2) since the Hamiltonian $\mathcal{H}^T$ does not depend on Q. □

Remark 2.1.5. *Formally, Proposition 2.1.4 follows from the fact that T is a canonical transform, see Section 2.1.4.*

Recall that Q is a cyclic coordinate. Hence, the system (2.1.15) is equivalent to a reduced Hamiltonian system for Ψ and Π only, which can be written as

$$\dot\Psi=D_\Pi\mathcal{H}_P,\qquad \dot\Pi=-D_\Psi\mathcal{H}_P. \tag{2.1.16}$$

Due to (2.1.14), the soliton (ψ_v,π_v) is a stationary solution to (2.1.16) with $P=P(v)$. Moreover, for every fixed $P\in\mathbb{R}^3$, the functional $\mathcal{H}_P$ is Fréchet-differentiable on the Hilbert phase space $\mathcal{F}=\mathring{H}^1\oplus L^2$. Hence, (2.1.16) implies that the soliton is a critical point of $\mathcal{H}_{P(v)}$ on $\mathcal{F}$. The next lemma demonstrates that (ψ_v,π_v) is a global minimum of $\mathcal{H}_{P(v)}$ on $\mathcal{F}$.

Lemma 2.1.6. *i) For every $v\in\mathbb{R}^3$ with $|v|<1$ the functional $\mathcal{H}_{P(v)}$ has the lower bound*

$$\mathcal{H}_{P(v)}(\Psi,\Pi)-\mathcal{H}_{P(v)}(\psi_v,\pi_v)\geq\frac{1-|v|}{2}\Big(\|\nabla(\Psi-\psi_v)\|^2+\|\Pi-\pi_v\|^2\Big),\quad(\Psi,\Pi)\in\mathcal{F}.\quad(2.1.17)$$

ii) $\mathcal{H}_{P(v)}$ has no other critical points on $\mathcal{F}$ except the point (ψ_v,π_v).

Proof. i) Denoting $\Psi-\psi_v=\psi$ and $\Pi-\pi_v=\pi$, we have

$$\mathcal{H}_{P(v)}(\psi_v+\psi,\pi_v+\pi)-\mathcal{H}_{P(v)}(\psi_v,\pi_v)=\int d^3x(\pi_v(x)\pi(x)+\nabla\psi_v(x)\cdot\nabla\psi(x)+\rho(x)\psi(x))$$

$$+\frac{1}{2}\int d^3x\,(|\nabla\psi(x)|^2+|\pi(x)|^2)+(1+(p_v+m)^2)^{1/2}-(1+p_v^2)^{1/2},\qquad(2.1.18)$$

where $p_v=P(v)+\displaystyle\int d^3x\,\pi_v(x)\,\nabla\psi_v(x)$, and

$$m=\int d^3x\,(\pi(x)\,\nabla\psi_v(x)+\pi_v(x)\,\nabla\psi(x)+\pi(x)\,\nabla\psi(x)).$$

Taking into account that $v=(1+p_v^2)^{-1/2}p_v$, we obtain

$$\begin{aligned}&\mathcal{H}_{P(v)}(\psi_v+\psi,\pi_v+\pi)-\mathcal{H}_{P(v)}(\psi_v,\pi_v)\\&=\frac{1}{2}\int d^3x\,(|\pi(x)|^2+|\nabla\psi(x)|^2)+(1+p_v^2)^{-1/2}\int d^3x\,\pi(x)\,p_v\cdot\nabla\psi(x)\\&-(1+p_v^2)^{-1/2}p_v\cdot m+(1+(p_v+m)^2)^{1/2}-(1+p_v^2)^{1/2}\,.\end{aligned}$$

It is easy to check that the expression in the third line is nonnegative. Then the lower bound (2.1.17) follows by using $|(1+p_v^2)^{-1/2}p_v|=|v|$.

ii) If $(\Psi,\Pi)\in\mathcal{F}$ is a critical point for $\mathcal{H}_{P(v)}$, then it satisfies

$$0=\Pi(x)+(1+\tilde p^2)^{-1/2}\tilde p\cdot\nabla\Psi(x)\,,\quad 0=-\Delta\Psi(x)+\rho(x)-(1+\tilde p^2)^{-1/2}\tilde p\cdot\nabla\Pi(x)\,,$$

where $\tilde p=P(v)+\displaystyle\int d^3x\,\Pi(x)\,\nabla\Phi(x)$. This system is equivalent to equation (2.1.5) for solitons in the case of the velocity $\tilde v=(1+\tilde p^2)^{-1/2}\tilde p$. Hence, $\Psi=\psi_{\tilde v}$, $\Pi=\pi_{\tilde v}$ and $P(\tilde v)=P(v)$.

It remains to check that $\tilde v=v$. Indeed, for the total momentum $P(v)$ of the soliton solution (2.1.3), the Parseval identity and (2.1.6) imply

$$P(v)=p_v+\int d^3x\;v\cdot\nabla\psi_v(x)\,\nabla\psi_v(x)=\frac{v}{\sqrt{1-v^2}}+(2\pi)^{-3}\int d^3k\frac{(v\cdot k)\hat\rho(k)\overline{k\hat\rho(k)}}{(k^2-(v\cdot k)^2)^2}\,.$$

Hence, $P(v)=\varkappa(|v|)v$ with $\varkappa(|v|)\geq 0$, and for $v\neq 0$ one has

$$|P(v)|=\frac{|v|}{\sqrt{1-v^2}}+\frac{1}{(2\pi)^3|v|}\int d^3k\frac{|(v\cdot k)\hat\rho(k)|^2}{(k^2-(v\cdot k)^2)^2}\,.$$

Since $|P(v)|=\varkappa(|v|)|v|$ is a monotone increasing function of $|v|\in[0,1)$, we conclude that $v=\tilde v$. □

Remark 2.1.7. Proposition 2.1.4 is not really needed for the proof of Theorem 2.1.1. However, the Proposition together with (2.1.14) and (2.1.16) show that (ψ_v,π_v) is a critical point and suggest an investigation of the stability through a lower bound as in (2.1.17). In Section 2.1.4 we sketch the derivation of Proposition 2.1.4 for sufficiently smooth solutions based only on the invariance of symplectic structure. We expect that a similar proposition holds for other translation invariant systems similar to (2.1.1).

2.1.2 Orbital stability of solitons

We follow [102] deducing orbital stability from the conservation of the Hamiltonian $\mathcal{H}_P$ together with its lower bound (2.1.17). For $|v| < 1$ denote

$$\delta = \delta(v) = \|\psi^0(x) - \psi_v(x - q^0)\| + \|\pi^0(x) - \pi_v(x - q^0)\| + |p^0 - p_v| \,. \tag{2.1.19}$$

Lemma 2.1.8. *Let $Y(t) = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ be a solution to (2.1.1) with an initial state $Y(0) = Y^0 = (\psi^0, \pi^0, q^0, p^0) \in \mathcal{E}$. Then for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that*

$$\|\psi(q(t) + x, t) - \psi_v(x)\| + \|\pi(q(t) + x, t) - \pi_v(x)\| + |p(t) - p_v| \le \varepsilon, \qquad t \in \mathbb{R} \tag{2.1.20}$$

provided $\delta \le \delta_\varepsilon$.

Proof. Denote by P^0 the total momentum of the considered solution $Y(t)$. There exists a soliton solution (2.1.3) corresponding to some velocity $\tilde v$ with the same total momentum $P(\tilde v) = P^0$. Then (2.1.19) implies that $|P^0 - P(v)| = |P(\tilde v) - P(v)| = \mathcal{O}(\delta)$. Hence also $|\tilde v - v| = \mathcal{O}(\delta)$ and

$$\|\psi^0(x) - \psi_{\tilde v}(x - q^0)\| + \|\pi^0(x) - \pi_{\tilde v}(x - q^0)\| + |p^0 - p_{\tilde v}| = \mathcal{O}(\delta) \,.$$

Therefore, denoting $(\Psi^0, Q^0, \Pi^0, P^0) = TY^0$, we have

$$\mathcal{H}_{P(\tilde v)}(\Psi^0\,, \Pi^0) - \mathcal{H}_{P(\tilde v)}(\psi_{\tilde v}\,,\, p_{\tilde v}) = \mathcal{O}(\delta^2) \,. \tag{2.1.21}$$

Total momentum and energy conservation imply that for $(\Psi(t), Q(t), \Pi(t), P^0) = TY(t)$

$$\mathcal{H}_{P(\tilde v)}(\Psi(t),\, \Pi(t)) = \mathcal{H}(TY(t)) = \mathcal{H}_{P(\tilde v)}(\Psi^0\,, \Pi^0) \ \text{ for } t \in \mathbb{R} \,.$$

Hence (2.1.21) and (2.1.17) with $\tilde v$ instead of v imply

$$\|\Psi(t) - \psi_{\tilde v}\| + \|\Pi(t) - \pi_{\tilde v}\| = \mathcal{O}(\delta) \tag{2.1.22}$$

uniformly in $t \in \mathbb{R}$. On the other hand, total momentum conservation implies

$$p(t) = P(\tilde v) + \langle \Pi(t), \nabla \Psi(t) \rangle \ \text{ for } t \in \mathbb{R} \,.$$

Therefore (2.1.22) leads to

$$|p(t) - p_{\tilde v}| = \mathcal{O}(\delta) \tag{2.1.23}$$

uniformly in $t \in \mathbb{R}$. Finally (2.1.22), (2.1.23) together imply (2.1.20) because $|\tilde v - v| = \mathcal{O}(\delta)$. □

2.1.3 Strong Huygens principle and soliton asymptotics

We combine the relaxation of the acceleration and orbital stability with the Strong Huygens principle to prove Theorem 2.1.1.

Proposition 2.1.9. *Let the assumptions of Theorem 2.1.1 be fulfilled. Then for every $\delta > 0$ there exist a $t_* = t_*(\delta)$ and a solution $Y_*(t) = (\psi_*(x,t), \pi_*(x,t), q_*(t), p_*(t)) \in C([t_*, \infty), \mathcal{E})$ to the system (2.1.1) such that*

i) $Y_*(t)$ coincides with $Y(t)$ in the future cone,

$$q_*(t) = \quad q(t) \quad \textit{for } \ t \ge t_*\,, \tag{2.1.24}$$

$$\psi_*(x,t) = \quad \psi(x,t) \quad \textit{for } \ |x - q(t_*)| < t - t_*\,. \tag{2.1.25}$$

ii) $Y_*(t_*)$ is close to a soliton $Y_{v,a}$ with some v and a,

$$\|Y_*(t_*) - Y_{v,a}\|_{\mathcal{E}} \le \delta \,. \tag{2.1.26}$$

Proof. The Kirchhoff formula gives

$$\psi(x,t) = \psi_r(x,t) + \psi_0(x,t), \qquad x \in \mathbb{R}^3,\ t > 0,$$

where

$$\psi_r(x,t) = -\int \frac{d^3y}{4\pi|x-y|}\rho(y - q(t-|x-y|)), \tag{2.1.27}$$

$$\psi_0(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} d^2y\, \pi(y,0) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)} d^2y\, \psi(y,0)\right). \tag{2.1.28}$$

Here $S_t(x)$ denotes the sphere $|y-x| = t$. Let us assume for simplicity that initial fields vanish. General case can be easily reduced to this situation using the strong Huygens principle. We will comment on this reduction at the end of the proof.

In the case of zero initial data the solution reduces to the retarded potential:

$$\psi(x,t) = \psi_r(x,t), \quad x \in \mathbb{R}^3, \quad t > 0.$$

We construct the solution $Y_*(t)$ as a modification of $Y(t)$. First, we modify the trajectory $q(t)$. The relaxation of acceleration (2.1.8) means that for any $\varepsilon > 0$ there exist $t_\varepsilon > 0$ such that

$$|\ddot{q}(t)| \le \varepsilon, \qquad t \ge t_\varepsilon.$$

Hence, the trajectory for large times locally tends to a straight line, i.e., for any fixed $T > 0$

$$q(t) = q(t_\varepsilon) + (t - t_\varepsilon)\dot{q}(t_\varepsilon) + r(t_\varepsilon, t), \quad \text{where} \quad \max_{t\in[t_\varepsilon, t_\varepsilon+T]} |r(t_\varepsilon,t)| \to 0, \quad t_\varepsilon \to \infty.$$

Denote $\lambda_\varepsilon(t) := q(t_\varepsilon) + \dot{q}(t_\varepsilon)(t - t_\varepsilon)$ and define modified trajectory as

$$q_*(t) = \left\{ \begin{array}{ll} \lambda_\varepsilon(t), & t \le t_\varepsilon \\ q(t), & t \ge t_\varepsilon \end{array} \right| , \tag{2.1.29}$$

Then

$$\ddot{q}_*(t) = \left\{ \begin{array}{ll} 0, & t < t_\varepsilon \\ \ddot{q}(t), & t > t_\varepsilon \end{array} \right| .$$

The next step we define the modified field as retarded potential of type (2.1.27)

$$\psi_*(x,t) = -\int \frac{d^3y}{4\pi|x-y|}\rho(y - q_*(t-|x-y|)), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \tag{2.1.30}$$

Lemma 2.1.10. *The right hand side of (2.1.30) depends on the trajectory $q_*(\tau)$ only from a bounded interval of time $\tau \in [t - T(x,t), t]$, where*

$$T(x,t) := \frac{R_\rho + |x - q(t)|}{1 - \overline{v}}. \tag{2.1.31}$$

Here $\overline{v} = \sup_{t\in\mathbb{R}} |\dot{q}(t)| < 1$ by (1.5.11).

Proof. This lemma is obvious geometrically, and its formal proof also is easy. The inegrand of (2.1.30) vanishes for $|y - q_*(t - |x - y|)| \geq R_\rho$ by (1.5.14). Therefore, the integral is spreaded over the region $|y - q_*(t - |x - y|)| \leq R_\rho$, which implies $|y - q_*(t) + q_*(t) - q_*(t - |x - y|)| \leq R_\rho$. Hence,

$$|y - q_*(t)| \leq R_\rho + \overline{v}|x - y|.$$

On the other hand, $|x - y| \leq |x - q_*(t)| + |y - q_*(t)|$, and hence,

$$|y - q_*(t)| \geq -|x - q_*(t)| + |x - y|.$$

Therefore,

$$-|x - q_*(t)| + |x - y| \leq R_\rho + \overline{v}|x - y|,$$

which implies

$$|x - y| \leq \frac{R_\rho + |x - q_*(t)|}{1 - \overline{v}}.$$

Now the lemma is proved. □

The potential (2.1.30) satisfies the wave equation

$$\ddot{\psi}_*(x, t) = \Delta\psi_*(x, t) - \rho(x - q_*(t)), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}.$$

We should still prove equations for the trajectory $q_*(t)$:

$$\dot{q}_*(t) = \frac{p_*(t)}{\sqrt{1 + p_*^2(t)}}, \qquad \dot{p}_*(t) = -\int \nabla\psi_*(x, t)\rho(x - q_*(t))\, dx, \qquad t > t_* \tag{2.1.32}$$

with sufficiently large $t_* \geq t_\varepsilon$. Let us note that the integral here is spreaded over the ball $|x - q_*(t)| \leq R_\rho$. Now Lemma 2.1.10 implies that $\psi_*(x, t)$ depends on the trajectory $q_*(\tau)$ only from a bounded interval $\tau \in [t - \overline{T}, t]$, where

$$\overline{T} := \frac{2R_\rho}{1 - \overline{v}}.$$

Let us define $t_* := t_\varepsilon + \overline{T}$. Then by Lemma 2.1.10

$$\psi_*(x, t) = \psi(x, t), \quad t > t_*, \quad |x - q_*(t)| \leq R_\rho$$

since $q_*(t) \equiv q(t)$ for $t > t_* - \overline{T} = t_\varepsilon$ by (2.1.29). Hence, equations (2.1.32) hold for $q_*(t)$ as well as for $q(t)$.

It remains to prove (2.1.26). The key observation is that outside the cone $K_\varepsilon := \{(x, t) \in \mathbb{R}^4 : |x - q(t_\varepsilon)| < t - t_\varepsilon\}$ the retarded potential (2.1.30) coincides with the soliton $\psi_{v,a}(x, t)$, where $v = \dot{q}(t_\varepsilon)$ and $a = q(t_\varepsilon)$ by our definition (2.1.29). In particular,

$$\psi(x, t_*) = \psi_{v,a}(x - a - vt_*), \qquad |x - q(t_\varepsilon)| > t_* - t_\varepsilon = \overline{T}.$$

In the ball $|x - q(t_*)| < \overline{T}$ the coincidence generally does not hold, but the difference of the left hand side with the right hand side converges to zero as $\varepsilon \to 0$ uniformly for $|x - q(t_*)| < \overline{T}$, and such uniform convergence holds for the gradient of the difference. This follows from the integral representation (2.1.30) by Lemma 2.1.10 since

$$\max_{t \in (t_* - T(x, t_*), t_*)} [|q_*(t) - \lambda_\varepsilon(t)| + |\dot{q}_*(t) - \dot{\lambda}_\varepsilon(t)|] \to 0, \qquad \varepsilon \to 0$$

by the relaxation of acceleration (2.1.8). It is important that $T(x,t_*)$ is bounded for $|x-q(t_*)|<\overline{T}$ by (2.1.31). This proves Proposition 2.1.9 in the case of zero initial data.

The next step is the proof for initial data with bounded support:

$$\psi(x,0)=\pi(x,0)=0, \qquad |x|>R_0.$$

Now we apply the strong Huygens principle: in this case the potential (2.1.28) vanishes in a future cone,

$$\psi_0(x,t)=0, \qquad |x|<t-R_0.$$

However, the estimate $|\dot q(t)|\le\overline{v}<1$ implies that the trajectory $(q(t),t)$ lies in this cone for all $t>t_0$. Hence, the solution for $t>t_0$ again reduces to the retarded potential and the needed conclusion follows.

Finally, arbitrary finite energy initial data admits a splitting in two summands: the first vanishing for $|x|>R_0$ and the second vanishing for $|x|<R_0-1$. The energy of the second summand is arbitrarily small for large R_0, and the energy of the corresponding potential (2.1.28) is conserved in time since it is a solution to free wave equation. Hence, its role is negligible for sufficiently large R_0. □

Now we can prove our main result.

Proof of Theorem 2.1.1 For every $\varepsilon>0$ there exists $\delta>0$ such that (2.1.26) implies by Lemma 2.1.8,

$$\|\psi_*(q_*(t)+x,t)-\psi_v(x)\|+\|\pi_*(q_*(t)+x,t)-\pi_v(x)\|+|\dot q_*(t)-v|\le\varepsilon \text{ for } t>t_*\,.$$

Therefore, (2.1.24) and (2.1.25) imply that for every $R>0$ and $t>t_*+\dfrac{R}{1-\overline{v}}$

$$\begin{aligned}&\|\psi(q(t)+x,\,t)-\psi_v(x)\|_R\,+\,\|\pi(q(t)+x,\,t)-\pi_v(x)\|_R\,+\,|\dot q(t)-v|\\ &=\|\psi_*(q_*(t)+x,t)-\psi_v(x)\|_R+\,\|\pi_*(q_*(t)+x,t)-\pi_v(x)\|_R+\,|\dot q_*(t)-v|\le\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we conclude (2.1.10). Theorem 2.1.1 is proved.

2.1.4 Invariance of symplectic structure

The canonical equivalence of the Hamiltoniansystems (2.1.1) and (2.1.15) can be seen from the Lagrangian viewpoint. We remain at the formal level. For a complete mathematical justification we would have to develop some theory of infinite dimensional Hamiltoniansystems which is beyond the scope of this book.

By definition we have $\mathcal{H}^T(\Psi,\Pi,Q,P)=\mathcal{H}(\psi,\pi,q,p)$ with the arguments related through the transformation T. To each Hamiltonian we associate a Lagrangian through the Legendre transformation

$$\begin{aligned}L(\psi,\dot\psi,q,\dot q)&=\ \langle\pi,\dot\psi\rangle+p\cdot\dot q-\mathcal{H}(\psi,\pi,q,p)\,, &&\dot\psi=D_\pi\mathcal{H}\,,\quad \dot q=D_p\mathcal{H}\,,\\ L^T(\Psi,\dot\Phi,Q,\dot Q)&=\ \langle\Pi,\dot\Psi\rangle+P\cdot\dot Q-\mathcal{H}^T(\Psi,\Pi,Q,P)\,, &&\dot\Psi=D_\Pi\mathcal{H}^T\,,\quad \dot Q=D_P\mathcal{H}^T\,.\end{aligned}$$

These Legendre transforms are well defined because the Hamiltonian functionals are convex in the momenta.

Lemma 2.1.11. *The following indentity holds,*

$$L^T(\Psi,\dot\Psi,Q,\dot Q)=L(\psi,\dot\psi,q,\dot q).$$

Proof. Clearly we have to check the invariance of the canonical form,

$$\langle \Pi, \dot{\Psi} \rangle + P \cdot \dot{Q} = \langle \pi, \dot{\psi} \rangle + p \cdot \dot{q} \,. \tag{2.1.33}$$

For this purpose we substitute

$$\left\{ \begin{array}{rclrcl} \Pi(x) & = & \pi(q+x), & \dot{\Psi}(x) & = & \dot{\psi}(q+x) + \dot{q} \cdot \nabla\psi(q+x) \\ P & = & p - \displaystyle\int \dot{\psi} \cdot \nabla\psi \, dx, & \dot{Q} & = & \dot{q} \end{array} \right|$$

Then the left hand side of (2.1.33) becomes

$$\langle \pi(q+x), \dot{\psi}(q+x) + \dot{q} \cdot \nabla\psi(q+x) \rangle + (p - \langle \pi(x), \nabla\psi(x) \rangle) \cdot \dot{q} = \langle \pi, \dot{\psi} \rangle + p \cdot \dot{q} \,.$$

The lemma is proved. □

This lemma implies that he corresponding action functionals are identical when transformed by T. Hence, finally, the two Hamiltonian systems (2.1.1) and (2.1.15) are equivalent since dynamical trajectories are stationary points of the respective action functionals.

2.1.5 Translation-invariant Maxwell-Lorentz system

In [58] asymptotics of type (2.1.8)–(2.1.10) were extended to the Maxwell-Lorentz translation-invariant system (1.6.1) without external fields. In this case, the Hamiltonian coincides with (1.6.3) where $V(x) \equiv 0$. The extension of methods [57] to this case required a new detailed analysis of the corresponding Hamiltonian structure which is necessary for the canonical transformation. Now the key role in applying strong Huygens principle is played by new estimates of long-time decay for oscillations of energy and total momentum for solutions of perturbed Maxwell-Lorentz system (estimates (4.24)–(4.25) in [58]).

2.2 The case of weak coupling

In [60] the soliton asymptotic of type (2.1.8)–(2.1.10) for the system (1.5.1)–(1.5.2) was proved in a stronger form for the case of a weak coupling

$$\|\rho\|_{L^2(\mathbb{R}^3)} \ll 1. \tag{2.2.1}$$

Namely, in [60] initial fields are considered with decay $|x|^{-5/2-\varepsilon}$, where $\varepsilon > 0$ (condition (2.2) in [60]) provided that $\nabla V(q) = 0$ for $|q| >$ const. Under these assumptions, more strong decay holds,

$$|\ddot{q}(t)| \le C(1 + |t|)^{-1-\varepsilon}, \qquad t \in \mathbb{R} \tag{2.2.2}$$

for 'outgoing solutions' that satisfy the condition

$$|q(t)| \to \infty, \qquad t \to \pm\infty. \tag{2.2.3}$$

With these assumptions asymptotics (2.1.8)–(2.1.10) can be significantly strengthen: now

$$\dot{q}(t) \to v_\pm, \qquad (\psi(x,t), \pi(x,t)) = (\psi_{v_\pm}(x - q(t)), \pi_{v_\pm}(x - q(t))) + W(t)\Phi_\pm + (r_\pm(x,t), s_\pm(x,t)),$$

where 'dispersion waves' $W(t)\Phi_\pm$ are solutions of a free wave equation shown on Fig. 2.1.

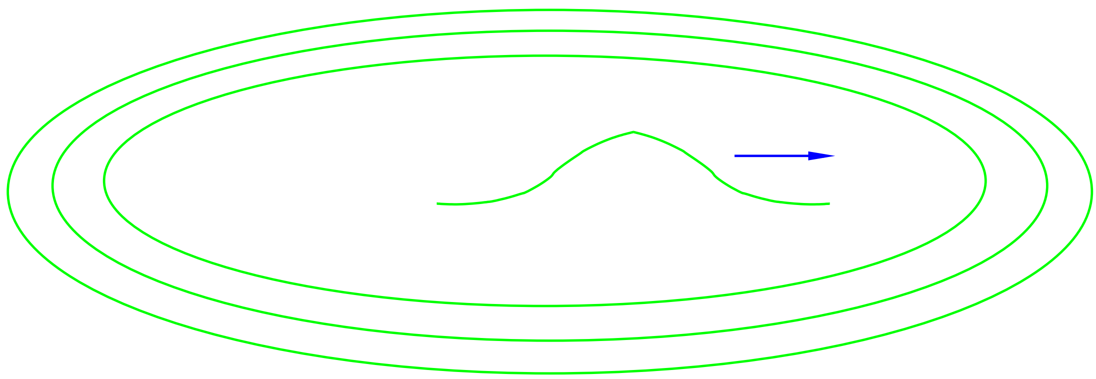

Figure 2.1: Soliton and dispersion waves

Now the remainder converges to zero in *global energy norm*:

$$\|\nabla r_\pm(q(t),t)\| + \|r_\pm(q(t),t)\| + \|s_\pm(q(t),t)\| \to 0, \qquad t \to \pm\infty.$$

This progress compared with local decay (2.1.10) is due to the fact that we identified a dispersion wave $W(t)\Phi_\pm$ under the condition of smallness (2.2.1). This identification is possible due to the rapid decay (2.2.2), in difference with (1.5.17).

All solitons propagate with velocities $v < 1$, and therefore they are spatially separated for large time from the dispersion waves $W(t)\Phi_\pm$, which propagate with unit velocity (Fig. 2.1).

The proofs rely on integral Duhamel representation and on rapid dispersion decay of solutions to free wave equation. Similar results were obtained in [126] for a system of type (1.5.1)–(1.5.2) with the Klein–Gordon equation, and in [59] for the Maxwell–Lorentz equations (1.6.1) with the same smallness condition (2.2.3) under assumption that $E^{\text{ext}}(x) = B^{\text{ext}}(x) = 0$ for $|x| >$ const. In [91], this result was extended to the Maxwell–Lorentz equations of type (1.6.1) with a rotating charge.

Remark 2.2.1. The results of [60, 91] imply A. Soffer's 'Grand Conjecture' [161, p. 460] in a moving frame for translation -invariant systems under the condition of smallness (2.2.1).

Open problem. Global attraction to solitons for the *relativistically-invariant* nonlinear wave equations

$$\ddot{\psi}(x,t) = -\Delta\psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}^n \tag{2.2.4}$$

is still an open problem. Numerical simulations [61] for the case $n = 1$ confirm the asymptotics (0.0.13) for a broad class of the nonlinearities, see Chapter 6.

Chapter 3

Global Attraction to Stationary Orbits

In this chapter we present with details the first results on global attraction to stationary orbits (0.0.15) obtained in [65]–[67]. The results concern the global attraction for 1D Klein–Gordon equation coupled to a nonlinear oscillator.

Besides the formal proof, we give in Section 3.9 an informal explanation of the *nonlinear radiation mechanism*.

In conclusion, we specify the general conjecture (0.0.6) which summarises all results on global attractors of Chapters 1, 2 and 3 (see Section 3.10).

3.1 Nonlinear Klein–Gordon equation

The first results on global attraction to stationary orbits (0.0.15) were established in [65]–[67] for the Klein–Gordon equation coupled to nonlinear oscillator

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{3.1.1}$$

Asymptotics (0.0.6) for this equation means the *single-frequency asymptotics* (0.0.15),

$$\psi(x,t) \sim \psi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty. \tag{3.1.2}$$

We consider complex solutions, identifying complex values $\psi \in \mathbb{C}$ with the real vectors $(\psi_1, \psi_2) \in \mathbb{R}^2$, where $\psi_1 = \operatorname{Re}\psi$ and $\psi_2 = \operatorname{Im}\psi$. Suppose that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and

$$F(\psi) = -\nabla U(\psi), \qquad \psi \in \mathbb{C}, \tag{3.1.3}$$

where U is a real function and $\nabla := (\partial_1, \partial_2)$. In this case the equation (3.1.1) is formally equivalent to the Hamiltonian system (1.1.2) in the Hilbert phase space $\mathcal{E} := H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. The Hamiltonian functional is

$$\mathcal{H}(\psi,\pi) = \frac{1}{2}\int \Big[|\pi(x)|^2 + |\psi'(x)|^2 + m^2|\psi(x)|^2\Big]\,dx + U(\psi(0)), \qquad (\psi,\pi) \in \mathcal{E}. \tag{3.1.4}$$

Let us write (3.1.1) in the vector form as

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \qquad t \in \mathbb{R}, \tag{3.1.5}$$

where $Y(t) = (\psi(t), \dot{\psi}(t))$. We assume that

$$\inf_{\psi\in\mathbb{C}} U(\psi) > -\infty. \tag{3.1.6}$$

In this case, a finite energy solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ exists and is unique for any initial state $Y(0) \in \mathcal{E}$. The a priori bound

$$\sup_{t\in\mathbb{R}}[\|\dot{\psi}(t)\|_{L^2(\mathbb{R})} + \|\psi(t)\|_{H^1(\mathbb{R})}] < \infty \tag{3.1.7}$$

holds due to conservation of the energy (3.1.4). Note that the confining condition of type (1.5.7) is no longer necessary, since conservation of energy (3.1.4) with $m > 0$ ensures the boundedness of solutions.

Further, we assume the $U(1)$-invariance of the potential:

$$U(\psi) = u(|\psi|), \qquad \psi \in \mathbb{C}. \tag{3.1.8}$$

Then the differentiation in (3.1.3) gives us that

$$F(\psi) = a(|\psi|)\psi, \qquad \psi \in \mathbb{C}, \tag{3.1.9}$$

and therefore

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \qquad \theta \in \mathbb{R}. \tag{3.1.10}$$

By ‘stationary orbits’ we mean solutions of the form

$$\psi(x,t) = \psi_\omega(x)e^{-i\omega t} \tag{3.1.11}$$

with $\omega \in \mathbb{R}$ and $\psi_\omega \in H^1(\mathbb{R})$. Each stationary orbit corresponds to some solution of the equation

$$-\omega^2\psi_\omega(x) = \psi''_\omega(x) - m^2\psi_\omega(x) + \delta(x)F(\psi_\omega(0)), \qquad x \in \mathbb{R}, \tag{3.1.12}$$

which is the *nonlinear eigenvalue problem* . Solutions $\psi_\omega \in H^1(\mathbb{R})$ of this equation have the form

$$\psi_\omega(x) = Ce^{-\varkappa|x|}, \qquad \varkappa := \sqrt{m^2-\omega^2} > 0,$$

and the constant C satisfies the nonlinear algebraic equation $2\varkappa C = F(C)$. Hence, the solutions ψ_ω exist for ω in some subset $\Omega \subset \mathbb{R}$ lying in the *spectral gap* $[-m, m]$. We denote the corresponding *solitary manifold* by $\mathcal{S}$:

$$\mathcal{S} = \{(e^{i\theta}\psi_\omega, -i\omega e^{i\theta}\psi_\omega) \in \mathcal{E} : \omega \in \Omega,\ \theta \in [0, 2\pi]\}. \tag{3.1.13}$$

Finally, suppose that the equation (3.1.1) is *strictly nonlinear:*

$$U(\psi) = u(|\psi|^2) = \sum_0^N u_j|\psi|^{2j}, \quad u_N > 0, \quad N \geq 2. \tag{3.1.14}$$

For example, the well-known *Ginzburg–Landau potential* $U(\psi) = a|\psi| - b|\psi|$ satisfies all the conditions (3.1.6), (3.1.8), and (3.1.14) for all $a, b > 0$.

Definition 3.1.1. i) $\mathcal{E}_F \subset H^1_{loc}(\mathbb{R}^3) \oplus L^2_{loc}(\mathbb{R}^3)$ is the space $\mathcal{E}$ endowed with the seminorms

$$\|Y\|_{\mathcal{E},R} := \|Y\|_{H^1(-R,R)} + \|Y\|_{L^2(-R,R)}, \qquad R = 1, 2, \dots \tag{3.1.15}$$

ii) Convergence in $\mathcal{E}_F$ is equivalent to convergence in every seminorm (3.1.15).

It is important that convergence in $\mathcal{E}_F$ is equivalent to convergence in the metric of type (1.2.9),

$$\mathrm{dist}[Y_1, Y_2] = \sum_{R=1}^\infty 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathcal{E},R}}{1 + \|Y_1 - Y_2\|_{\mathcal{E},R}}, \qquad Y_1, Y_2 \in \mathcal{E}. \tag{3.1.16}$$

Theorem 3.1.2. *Let the conditions* (3.1.3)*,* (3.1.6)*,* (3.1.8) *and* (3.1.14) *hold. Then any finite energy solution* $Y(t) = (\psi(t), \dot\psi(t)) \in C(\mathbb{R}, \mathcal{E})$ *of* (3.1.5) *is attracted to the solitary manifold* (see Fig. 2)*:*

$$Y(t) \xrightarrow{\mathcal{E}_F} \mathcal{S}, \qquad t \to \pm\infty, \tag{3.1.17}$$

where the attraction is in the sense of (1.2.18).

Generalizations and open questions

Generalizations: The global attraction to stationary orbits (3.1.17) was extended in [68] to the 1D Klein–Gordon equation coupled to N nonlinear oscillators

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \sum_{k=1}^N \delta(x - x_k)F_k(\psi(x_k, t)),\ x \in \mathbb{R}, \tag{3.1.18}$$

and in [64, 69, 70] it was extended to the nonlinear Klein–Gordon and Dirac equations with a non-local interaction in $\mathbb{R}^n$ with $n \geq 3$

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi + \sum_{k=1}^{N} \rho(x - x_k) F_k(\langle\psi(\cdot,t), \rho(\cdot - x_k)\rangle), \qquad (3.1.19)$$

$$i\dot{\psi}(x,t) = \big(- i\alpha \cdot \nabla + \beta m\big)\psi + \rho(x) F(\langle\psi(\cdot,t), \rho\rangle), \qquad (3.1.20)$$

under the Wiener condition (1.5.13). Here $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = \alpha_0$ are Dirac matrices.

Recently, the global attraction to stationary orbits (3.1.17) was extended in [72, 53, 73] to the 3D wave and Klein–Gordon equations with concentrated nonlinearities, and in [74] it was extended to the 1D Dirac equation coupled to a nonlinear oscillator.

In addition, the global attraction to stationary orbits (3.1.17) was extended in [63] to nonlinear space-time discrete Hamiltonian equations that are discrete approximations of equations of type (3.1.19), that is, they are the corresponding difference schemes. The proof relies on a new version in [71] of the Titchmarsh convolution theorem for distributions on a circle.

Open questions:

I. Global attraction (3.1.2) to stationary orbits with fixed frequencies $\omega_\pm$ has not yet been proved.

II. Global attraction to stationary orbits for nonlinear Schrödinger equations has also not been proved. In particular, such global attraction is not proved for the 1D Schrödinger equation coupled to a nonlinear oscillator

$$i\dot{\psi}(x,t) = -\psi''(x,t) + \delta(x) F(\psi(0,t)), \qquad x \in \mathbb{R}. \qquad (3.1.21)$$

The main difficulty is the infinite 'spectral gap' $(-\infty, 0)$ (see Remark 3.8.2).

III. Global attraction to stationary orbits is still an open problem for the *relativistically-invariant* nonlinear Klein–Gordon equations (2.2.4) in the case when $f(\psi) = -\nabla U(\psi)$ with $U(1)$-invariant potential $U(\psi) \equiv u(|\psi|)$.

3.2 Omega-limit trajectories

The proof of Theorem 3.1.2 is based on the general strategy of *omega-limit trajectories* first introduced in [65], and developed further in [66]–[74] and [63, 64].

Definition 3.2.1. An omega-limit trajectory for a given function $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is any limit function $Z(t)$ such that

$$Y(t+s_j) \xrightarrow{\mathcal{E}_F} Z(t), \qquad t \in \mathbb{R}, \tag{3.2.1}$$

as $s_j \to \infty$.

Definition 3.2.2. A function $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is *omega-compact* if for any sequence $s_j \to \infty$ there exists a subsequence $s_{j'} \to \infty$ such that (3.2.1) holds.

These concepts are useful in view of the following lemma, which lies at the basis of our approach.

Lemma 3.2.3. *Suppose that any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of (3.1.5) is omega-compact, and any omega-limit trajectory is a stationary orbit:*

$$Z(x,t) = (\psi_\omega(x)e^{-i\omega t}, -i\omega\psi_\omega(x)e^{-i\omega t}), \tag{3.2.2}$$

where $\omega \in \mathbb{R}$. Then the attraction (3.1.17) holds for each solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of (3.1.5).

Proof. We need to show that

$$\lim_{t\to\infty} \operatorname{dist}(Y(t), \mathcal{S}) = 0.$$

Assume by contradiction that there exists a sequence $s_j \to \infty$ such that

$$\operatorname{dist}(Y(s_j), \mathcal{S}) \geq \delta > 0 \quad \forall j \in \mathbb{N}. \tag{3.2.3}$$

According to the omega-compactness of the solution Y, the convergence (3.2.1) holds for some subsequence $s_{j'} \to \infty$ and some stationary orbit (3.2.2):

$$Y(t+s_j) \xrightarrow{\mathcal{E}_F} Z(t), \qquad t \in \mathbb{R}. \tag{3.2.4}$$

But this convergence with $t = 0$ contradicts (3.2.3), since $Z(0) \in \mathcal{S}$ by definition (3.1.13). □

For the proof of Theorem 3.1.2 it now suffices to check the conditions of Lemma 3.2.3:

I. Each solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ of (3.1.5) is omega-compact.

II. Any omega-limit trajectory is a stationary orbit (3.2.2).

We check these conditions by analyzing the Fourier transform of solutions with respect to time. The main steps of the proof are as follows:

(1) Spectral representation for solutions of the nonlinear equation (3.1.1):

$$\psi(t) = \frac{1}{2\pi}\int e^{-i\omega t}\tilde{\psi}(\omega)d\omega.$$

By the *spectrum* of a solution $\psi(t) := \psi(\cdot, t)$ we mean the support of its spectral density $\tilde{\psi}(\cdot)$, which is a tempered distribution of $\omega \in \mathbb{R}$ with values in H^1.

(2) The *absolute continuity* of the spectral density $\tilde{\psi}(\omega)$ *on the continuous spectrum* $(-\infty, -m) \cup (m, \infty)$ of the free Klein–Gordon equation. This is a nonlinear analogue of the Kato theorem on the absence of embedded eigenvalues.

(3) The *omega-compactness* of each solution.

(4) The reduction of the spectrum *of each omega-limit trajectory* to a subset of the *spectral gap* $[-m, m]$.

(5) Reduction of this spectrum to a *single point* using the *Titchmarsh convolution theorem.*

Below we follow this program, referring at some points to the papers [65] and [67] for technically important properties of quasimeasures.

3.3 The limiting absorption principle

It suffices to prove the global attraction to stationary orbits (3.1.17) only for positive times. For simplicity we consider only the solution $\psi(x,t)$ to equation (3.1.1) corresponding to zero initial data:

$$\psi(x,0)=0, \qquad \dot\psi(x,0)=0. \tag{3.3.1}$$

The general case of non-zero initial data can be reduced to this case by a trivial subtraction of the dispersion wave which is solution of the free Klein–Gordon equation with given initial data [65, 67]. We extend $\psi(x,t)$ and $f(t):=F(\psi(0,t))$ by zero for $t<0$

$$\psi_+(x,t):=\begin{cases}\psi(x,t), & t>0,\\ 0, & t<0,\end{cases} \qquad f_+(t):=\begin{cases} f(t), & t>0,\\ 0, & t<0.\end{cases} \tag{3.3.2}$$

From (3.1.1) and (3.3.1) it follows that these functions satisfy the equation

$$\ddot\psi_+(x,t)=\psi_+''(x,t)-m^2\psi_+(x,t)+\delta(x)f_+(t), \qquad (x,t)\in\mathbb{R}^2 \tag{3.3.3}$$

in the distribution sense.

The Fourier–Laplace transform with respect to time. For tempered distributions $g(t)$, we let $\tilde g(\omega)$ denote their Fourier transform, which is defined for $g\in C_0^\infty(\mathbb{R})$ by

$$\tilde g(\omega)=\int_{\mathbb{R}} e^{i\omega t}g(t)\,dt, \qquad \omega\in\mathbb{R}.$$

The a priori estimates (3.1.7) imply that $\psi_+(x,t)$ and $f_+(t)$ are bounded functions of $t\in\mathbb{R}$ with values in the Sobolev space $H^1(\mathbb{R})$ and in $\mathbb{C}$, respectively. Therefore, their Fourier transforms are (by definition) *quasimeasures* with values in $H^1(\mathbb{R})$ and in $\mathbb{C}$, respectively [4]. Moreover, these Fourier transforms can be extended from the real axis to analytic functions in the upper complex half-plane $\mathbb{C}^+:=\{\omega\in\mathbb{C}:\ \operatorname{Im}\omega>0\}$ with values in $H^1(\mathbb{R})$ and in $\mathbb{C}$ respectively:

$$\tilde\psi_+(x,\omega)=\int_0^\infty e^{i\omega t}\psi(x,t)\,dt, \qquad \tilde f_+(\omega)=\int_0^\infty e^{i\omega t}f(t)\,dt, \qquad \omega\in\mathbb{C}^+.$$

Further, we have the following convergence of tempered distributions with values in H^1 and $\mathbb{C}$, respectively:

$$e^{-\varepsilon t}\psi_+(x,t)\to\psi_+(x,t), \qquad e^{-\varepsilon t}f_+(t)\to f_+(t), \qquad \varepsilon\to 0+\,.$$

Hence, their Fourier transforms also converge in the same sense:

$$\tilde\psi_+(x,\omega+i\varepsilon)\to\tilde\psi_+(x,\omega), \qquad \tilde f_+(\omega+i\varepsilon)\to\tilde f_+(\omega), \qquad \varepsilon\to 0+\,. \tag{3.3.4}$$

The analytic functions $\tilde\psi_+(x,\omega)$ and $\tilde f(\omega)$ grow (in norm) no faster than $|\operatorname{Im}\omega|^{-1}$ as $\operatorname{Im}\omega\to 0+$ in view of (3.1.7). Hence, their boundary values at $\omega\in\mathbb{R}$ are tempered distributions of small singularity: they are the second-order derivatives of continuous functions, as in the case of $\tilde f_+(\omega)=i/(\omega-\omega_0)$ with $\omega_0\in\mathbb{R}$, which corresponds to $f_+(t)=\theta(t)e^{-i\omega_0 t}$.

The limiting absorption principle. By (3.3.1), in terms of the Fourier transform the equation (3.3.3) becomes the stationary Helmholtz equation

$$-\,\omega^2\tilde\psi_+(x,\omega)=\tilde\psi_+''(x,\omega)-m^2\tilde\psi_+(x,\omega)+\delta(x)\tilde f_+(\omega), \qquad x\in\mathbb{R}. \tag{3.3.5}$$

This equation has two linearly independent solutions, but only one of them admits an analytic continuation to the upper complex half-plane $\operatorname{Im}\omega > 0$ with values in $H^1(\mathbb{R})$:

$$\tilde{\psi}_+(x,\omega) = -\tilde{f}_+(\omega)\frac{e^{ik(\omega)|x|}}{2ik(\omega)}, \qquad \operatorname{Im}\omega > 0. \tag{3.3.6}$$

Here $k(\omega) := \sqrt{\omega^2 - m^2}$, where the branch has a positive imaginary part for $\operatorname{Im}\omega > 0$. For the other branch this function *grows exponentially* as $|x| \to \infty$. Such an argument in the selection of solutions of stationary Helmholtz equations is known as the *limiting absorption principle* in diffraction theory [189, 10].

Spectral representation. We rewrite (3.3.6) in the form

$$\tilde{\psi}_+(x,\omega) = \tilde{\alpha}(\omega)e^{ik(\omega)|x|}, \qquad \operatorname{Im}\omega > 0, \qquad \text{where} \quad \alpha(t) := \psi_+(0,t). \tag{3.3.7}$$

It is a non-trivial fact that the identity (3.3.7) between analytic functions keeps its structure for their restrictions to the real axis, which are tempered distributions:

$$\tilde{\psi}_+(x,\omega + i0) = \tilde{\alpha}(\omega + i0)e^{ik(\omega+i0)|x|}, \qquad \omega \in \mathbb{R}, \tag{3.3.8}$$

where $\tilde{\psi}_+(\cdot,\omega + i0)$ and $\tilde{\alpha}(\omega + i0)$ are the corresponding quasimeasures with values in $H^1(\mathbb{R})$ and $\mathbb{C}$, respectively. The problem is that the factor $M_x(\omega) := e^{ik(\omega+i0)|x|}$ is not smooth with respect to ω at the points $\omega = \pm m$. Correspondingly, the identity (3.3.8) must be justified, based on quasimeasure theory [67].

Finally, the inversion of the Fourier transform can be written as

$$\psi_+(x,t) = \frac{1}{2\pi}\langle\tilde{\psi}_+(x,\omega + i0), e^{-i\omega t}\rangle = \frac{1}{2\pi}\langle\tilde{\alpha}(\omega + i0)e^{ik(\omega+i0)|x|}, e^{-i\omega t}\rangle, \quad x,t \in \mathbb{R}, \tag{3.3.9}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear duality between distributions and smooth bounded functions. The right-hand side exists by Theorem 3.4.1, see below.

3.4 A non-linear analogue of Kato's theorem

It turns out that the properties of the quasimeasures $\tilde{\alpha}(\omega + i0)$ with $|\omega| < m$ and that with $|\omega| > m$ differ significantly. This is because the set $\{i\omega : |\omega| \geq m\}$ is the continuous spectrum of the generator

$$A = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - m^2 & 0 \end{pmatrix},$$

which is the generator of the free Klein–Gordon equation. The following theorem plays a key role in the proof of our main Theorem 3.1.2. It is a nonlinear analogue of Kato's theorem on the absence of *embedded eigenvalues* in the continuous spectrum (see Remark 3.4.4 below). Let $\Sigma := \{\omega \in \mathbb{R} : |\omega| > m\}$. Below we will also write $\tilde{\alpha}(\omega)$ and $k(\omega)$ instead of $\tilde{\alpha}(\omega + i0)$ and $k(\omega + i0)$ for $\omega \in \mathbb{R}$.

Theorem 3.4.1. (see [67, Proposition 3.2]). *Let the conditions* (3.1.3), (3.1.6), *and* (3.1.8) *hold, and let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be any finite-energy solution of* (3.1.5). *Then the corresponding tempered distribution $\tilde{\alpha}(\omega)$ is absolutely continuous on Σ. Moreover, $\alpha \in L^1(\Sigma)$ and*

$$\int_{\Sigma} |\tilde{\alpha}(\omega)|^2 \, |\omega \, k(\omega)| \, d\omega < \infty. \tag{3.4.1}$$

Proof. We first explain the main idea of the proof. By (3.3.9), the function $\psi_+(x,t)$ is formally a 'linear combination' of the functions $e^{ik|x|}$ with the *amplitudes* $\hat{z}(\omega)$:

$$\psi_+(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{z}(\omega) e^{ik(\omega)|x|} e^{-i\omega t} \, d\omega, \qquad x \in \mathbb{R}.$$

For $\omega \in \Sigma$ the functions $e^{ik(\omega)|x|}$ have an infinite $L^2(\mathbb{R})$-norm, while $\psi_+(\cdot, t)$ has a finite $L^2(\mathbb{R})$-norm. This is possible only if the amplitude is absolutely continuous on Σ. This idea is suggested by the Fourier integral $f(x) = \int_{\mathbb{R}} e^{-ikx} g(k) dk$, which belongs to $L^2(\mathbb{R})$ if and only if $g \in L^2(\mathbb{R})$. For example, if one took $\hat{z}(\omega) = \delta(\omega - \omega_0)$ with $\omega_0 \in \Sigma$, then $\psi_+(\cdot, t)$ would have infinite L^2-norm.

The rigorous proof relies on estimates of Paley–Wiener type. Namely, the Parseval identity and (3.1.7) imply that

$$\int_{\mathbb{R}} \|\tilde{\psi}_+(\cdot, \omega + i\varepsilon)\|^2_{H^1(\mathbb{R})} \, d\omega = 2\pi \int_0^\infty e^{-2\varepsilon t} \|\psi_+(\cdot, t)\|^2_{H^1(\mathbb{R})} \, dt \leq \frac{\text{const}}{\varepsilon}, \qquad \varepsilon > 0. \tag{3.4.2}$$

On the other hand, we can estimate exactly the integral on the left-hand side of (3.4.2). Indeed, according to (3.3.9),

$$\tilde{\psi}_+(\cdot, \omega + i\varepsilon) = \tilde{\alpha}(\omega + i\varepsilon) e^{ik(\omega + i\varepsilon)|x|}.$$

Consequently, (3.4.2) gives us that

$$\varepsilon \int_{\mathbb{R}} |\tilde{\alpha}(\omega + i\varepsilon)|^2 \|e^{ik(\omega + i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} \, d\omega \leq \text{const}, \qquad \varepsilon > 0. \tag{3.4.3}$$

Here is a crucial observation about the asymptotics of the norm of $e^{ik(\omega + i\varepsilon)|x|}$ as $\varepsilon \to 0+$.

Lemma 3.4.2. i) *For* $\omega \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0+} \varepsilon \|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} = n(\omega) := \begin{cases} \omega k(\omega), & |\omega| > m \\ 0, & |\omega| < m \end{cases}, \tag{3.4.4}$$

where the norm in $H^1(\mathbb{R})$ *is chosen to be* $\|\psi_{H^1(\mathbb{R})}\| = \Big(\|\psi'_{L^2}\|^2 + m^2\|\psi\|^2_{L^2(\mathbb{R})}\Big)^{1/2}$.
ii) *For any* $\delta > 0$ *there exists an* $\varepsilon_\delta > 0$ *such that for* $|\omega| > m + \delta$ *and* $\varepsilon \in (0, \varepsilon_\delta)$,

$$\varepsilon \|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} \geq \frac{n(\omega)}{2}. \tag{3.4.5}$$

Proof. Let us compute the $H^1(\mathbb{R})$-norm using the Fourier representation. Namely, setting $k_\varepsilon = k(\omega + i\varepsilon)$ so that $\operatorname{Im} k_\varepsilon > 0$, we get $F_{x\to k}\left[e^{ik_\varepsilon|x|}\right] = 2ik_\varepsilon/(k_\varepsilon^2 - k^2)$ for $k \in \mathbb{R}$. Hence, by the Parseval identity and the Cauchy theorem on residues

$$\|e^{ik_\varepsilon|x|}\|^2_{H^1(\mathbb{R})} = \frac{2|k_\varepsilon|^2}{\pi}\int_{\mathbb{R}} \frac{(k^2+m^2)dk}{|k_\varepsilon^2 - k^2|^2} = -4\operatorname{Im}\left[\frac{(k_\varepsilon^2+m^2)\bar{k}_\varepsilon}{k_\varepsilon^2 - \bar{k}_\varepsilon^{\,2}}\right].$$

Substituting here $k_\varepsilon^2 = (\omega + i\varepsilon)^2 - m^2$, we get that

$$\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} = \frac{1}{\varepsilon}\operatorname{Re}\left[\frac{(\omega+i\varepsilon)^2\overline{k(\omega+i\varepsilon)}}{\omega}\right], \quad \varepsilon > 0, \quad \omega \in \mathbb{R}, \quad \omega \neq 0.$$

The limits (3.4.4) now follow, since the function $k(\omega)$ is real for $|\omega| > m$ but is purely imaginary for $|\omega| < m$. Therefore, the second assertion of the lemma also follows, since $n(\omega) > 0$ for $|\omega| > m$, and $n(\omega) \sim |\omega|^2$ for $|\omega| \to \infty$. □

Remark 3.4.3. Clearly, $n(\omega) \equiv 0$ for $|\omega| < m$ without any calculations, since in that case the function $e^{ik(\omega)|x|}$ decays exponentially in x, and hence, the $H^1(\mathbb{R})$-norm of $e^{ik(\omega+i\varepsilon)|x|}$ remains finite as $\varepsilon \to 0+$.

Substituting (3.4.5) into (3.4.3), we get that

$$\int_{\Sigma_\delta} |\tilde{\alpha}(\omega + i\varepsilon)|^2 \omega k(\omega)\, d\omega \leq 2C, \qquad 0 < \varepsilon < \varepsilon_\delta, \tag{3.4.6}$$

with the same C as in (3.4.3), and with the region $\Sigma_\delta := \{\omega \in \mathbb{R} : |\omega| > m + \delta\}$. We conclude that for each $\delta > 0$ the set of functions

$$g_\varepsilon(\omega) = \tilde{\alpha}(\omega + i\varepsilon)|\omega k(\omega)|^{1/2}, \qquad \varepsilon \in (0, \varepsilon_\delta),$$

is bounded in the Hilbert space $L^2(\Sigma_\delta)$, so by the Banach Theorem it is weakly compact. Hence, convergence of the distributions (3.3.4) implies weak convergence in $L^2(\Sigma_\delta)$:

$$g_\varepsilon \rightharpoonup g, \qquad \varepsilon \to 0+,$$

where the limit function $g(\omega)$ coincides with the distribution $\hat{z}(\omega)|\omega k(\omega)|^{1/2}$ restricted to Σ_δ. It remains to note that the norms of g in $L^2(\Sigma_\delta)$ with all $\delta > 0$ are bounded in view of (3.4.6), and this implies (3.4.1). Finally, $\tilde{\alpha}(\omega) \in L^1(\bar{\Sigma})$ by (3.4.1) and the Cauchy-Schwarz inequality. □

Remark 3.4.4. Theorem 3.4.1 is a nonlinear analogue of the Kato theorem on the absence of embedded eigenvalues in the continuous spectrum. Indeed, solutions of type $\psi_*(x)e^{-i\omega_* t}$ become $\psi_*(x)[\pi i\delta(\omega - \omega_*) + v.p.\frac{1}{i(\omega-\omega_*)}]$ in the Fourier–Laplace transform, and this is forbidden for $|\omega_*| > m$ by Theorem 3.4.1.

3.5 Splitting into dispersion and bound components

Theorem 3.4.1 presupposes a splitting of the solutions (3.3.9) into a 'dispersion component' and a 'bound component':

$$\begin{aligned}\psi_+(x,t) &= \frac{1}{2\pi}\int_\Sigma (1-\zeta(\omega))\tilde\alpha(\omega)e^{ik(\omega)|x|}e^{-i\omega t}d\omega + \frac{1}{2\pi}\langle\zeta(\omega)\tilde\alpha(\omega)e^{ik(\omega)|x|}, e^{-i\omega t}\rangle \\ &= \psi_d(x,t)+\psi_b(x,t), \qquad t>0, \qquad x\in\mathbb{R},\end{aligned} \tag{3.5.1}$$

where

$$\zeta(\omega)\in C_0^\infty(\mathbb{R}), \quad \text{and} \quad \zeta(\omega)=1 \quad \text{for} \quad \omega\in[-m-1,m+1].$$

Note that $\psi_d(x,t)$ is a dispersion wave, because

$$\psi_d(x,t) := \frac{1}{2\pi}\int_\Sigma (1-\zeta(\omega))e^{-i\omega t}\tilde\alpha(\omega)e^{ik(\omega)|x|}d\omega \to 0, \qquad t\to\infty$$

according to the Riemann–Lebesgue theorem, since $\alpha\in L^1(\Sigma)$ by Theorem 3.4.1. Moreover, it is easy to prove that

$$(\psi_d(\cdot,t),\dot\psi_d(\cdot,t))\to 0, \qquad t\to\infty \tag{3.5.2}$$

in the metric (3.1.16). Therefore, it remains to prove the attraction (3.1.17) for $Y_b(t) := (\psi_b(\cdot,t),\dot\psi_b(\cdot,t))$ instead of $Y(t)$:

$$Y_b(t)\xrightarrow{\mathcal{E}_F}\mathcal{S}, \qquad t\to\infty. \tag{3.5.3}$$

3.6 Omega-compactness

Here we establish the omega-compactness of the trajectory $Y_b(t)$, which is necessary for the application of Lemma 3.2.3. First, we note that the bound component $\psi_b(x,t)$ is a smooth function for $x \neq 0$, and

$$\partial_x^j \partial_t^l \psi_b(x,t) = \frac{1}{2\pi}\langle \zeta(\omega)(ik(\omega)\,\mathrm{sgn}\, x)^j \tilde{\alpha}(\omega) e^{ik(\omega)|x|}, (-i\omega)^l e^{-i\omega t}\rangle, \;\; t > 0,\, x \neq 0 \qquad (3.6.1)$$

for any $j, l = 0, 1, \ldots$. These formulas must be justified, since the function $k(\omega)$ is not smooth at the points $\omega = \pm m$. The needed justification is done in [65, 67] by a suitable development of the theory of quasimeasures. These formulae imply the boundedness of each derivative.

Lemma 3.6.1. (see [67, Proposition 4.1]). *For all $j, l = 0, 1, 2, \ldots$*

$$\sup_{x\neq 0}\; \sup_{t\in\mathbb{R}} |\partial_x^j \partial_t^l \psi_b(x,t)| < \infty. \qquad (3.6.2)$$

Proof. Note that in general the distribution $\tilde{\alpha}(\omega)$ is not a finite measure, since we only know that $\alpha(t) := \psi_+(0,t)$ is a bounded function by (3.3.7) and (3.1.7). To prove the lemma, it suffices to check that

$$\zeta(\omega)(ik(\omega)\,\mathrm{sgn}\, x)^j e^{ik(\omega)|x|}(-i\omega)^l = \tilde{g}_x(\omega),$$

where the function $g_x(\cdot)$ belongs to a bounded subset of $L^1(\mathbb{R})$ for $x \neq 0$ and $t \in \mathbb{R}$. This implies the lemma, since by the Parseval identity the right-hand side of (3.6.1) is the convolution

$$\langle \alpha(t-s), g_x(s)\rangle,$$

where $\alpha(t)$ is a bounded function. □

Remark 3.6.2. All the properties of quasimeasures used are justified in [65, 67] by similar arguments relying on the Parseval identity.

By the Ascoli–Arzelà theorem, for any sequence $s_j \to \infty$ there is a subsequence $s_{j'} \to \infty$ such that

$$\partial_x^j \partial_t^l \psi_b(x, s_{j'} + t) \to \partial_x^j \partial_t^l \beta(x,t), \qquad x \neq 0,\; t \in \mathbb{R} \qquad (3.6.3)$$

for any $j, l = 0, \ldots$, and this convergence is uniform on $|x| + |t| \le R$ with any $R > 0$. The estimates (3.6.2) imply that

$$\sup_{(x,t)\in\mathbb{R}^2} |\partial_x^j \partial_t^l \beta(x,t)| < \infty. \qquad (3.6.4)$$

Corollary 3.6.3. *Each solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to (3.1.5) is omega-compact. This follows from (3.5.1), (3.5.2), and (3.6.3).*

3.7 Reduction of spectrum to the spectral gap

The convergence of the functions (3.6.3) implies the convergence of their Fourier transforms:

$$\tilde{\psi}_b(x,\omega)e^{-i\omega s_{j'}} \to \tilde{\beta}(x,\omega), \qquad \forall x \in \mathbb{R} \tag{3.7.1}$$

in the sense of tempered distributions of $\omega \in \mathbb{R}$.

Lemma 3.7.1. *For any $x \in \mathbb{R}$*

$$\tilde{\beta}(x,\omega) = 0, \qquad |\omega| > m. \tag{3.7.2}$$

Proof. The convergence (3.7.1) and the representation (3.6.1) with $j = l = 0$ imply that

$$\zeta(\omega)\tilde{\alpha}(\omega)e^{ik(\omega)|x|}e^{-i\omega s_{j'}} \to \tilde{\beta}(x,\omega), \qquad \forall x \in \mathbb{R} \tag{3.7.3}$$

in the sense of tempered distributions of $\omega \in \mathbb{R}$. Moreover, this convergence takes place in the stronger *Ascoli–Arzelà topology* in the space of quasimeasures [67]. In addition, the function $e^{-ik(\omega)|x|}$ is a multiplier in the space of quasimeasures with this topology by Lemma B.3 of [67]). Therefore, (3.7.3) implies that

$$\zeta(\omega)\tilde{\alpha}(\omega)e^{-i\omega s_{j'}} \to \tilde{\gamma}(\omega) := \tilde{\beta}(x,\omega)e^{-ik(\omega)|x|}, \qquad \forall x \in \mathbb{R} \tag{3.7.4}$$

in the same topology of quasimeasures. Applying the same lemma again, we obtain

$$\beta(x,t) = \frac{1}{2\pi}\langle\tilde{\gamma}(\omega)e^{ik(\omega)|x|}, e^{-i\omega t}\rangle, \qquad (x,t) \in \mathbb{R}^2. \tag{3.7.5}$$

Note that

$$\beta(0,t) = \gamma(t). \tag{3.7.6}$$

Finally, the key observation is that (3.7.4) and Theorem 3.4.1 imply that

$$\operatorname{supp}\tilde{\gamma} \subset [-m,m] \tag{3.7.7}$$

by the Riemann–Lebesgue theorem. □

3.8 Reduction of spectrum to a single point

The question arises of the available means for verifying the representation (3.2.2) for omega-limit trajectories. We have no formulae for solutions of equation (3.1.1), and so the only hope is to use the nonlinear equation itself.

Equation for omega-limit trajectories and spectral inclusion

The key observation, albeit simple, is that $\beta(x,t)$ is a solution of the original nonlinear equation (3.1.1) *for all* $t \in \mathbb{R}$, despite the fact that $\psi_+(x,t)$ is a solution of the equation (3.1.1) only for $t > 0$, due to (3.3.2).

Lemma 3.8.1. *The function* $\beta(x,t)$ *satisfies the equation* (3.1.1)*:*

$$\ddot{\beta}(x,t) = \beta'(x,t) - m^2\beta(x,t) + \delta(x)F(\beta(0,t)), \qquad (x,t) \in \mathbb{R}^2. \tag{3.8.1}$$

Proof. This lemma follows by (3.5.2) and (3.6.3) in the limit as $s_{j'} \to \infty$ in the equation (3.1.1) for $\psi_+(x, s_{j'} + t) = \psi_d(x, s_{j'} + t) + \psi_b(x, s_{j'} + t)$ with $s_{j'} + t > 0$. □

Applying the Fourier transform to the equation (3.8.1), we now get the corresponding stationary *nonlinear Helmholtz equation*

$$-\omega^2\tilde{\beta}(x,\omega) = \tilde{\beta}''(x,\omega) - m^2\tilde{\beta}(x,\omega) + \delta(x)\tilde{f}(\omega), \qquad (x,\omega) \in \mathbb{R}^2, \tag{3.8.2}$$

where we define $f(t) := F(\beta(0,t)) = F(\gamma(t))$ in accordance with (3.7.6). From (3.1.9), we get that

$$f(t) = a(|\gamma(t)|)\gamma(t) = A(t)\gamma(t), \qquad A(t) := a(|\gamma(t)|), \qquad t \in \mathbb{R}.$$

Finally, in the Fourier transform we get the convolution $\tilde{f} = \tilde{A} * \tilde{\gamma}$, which exists by (3.7.7). Respectively, (3.8.2) is now

$$-\omega^2\tilde{\beta}(x,\omega) = \tilde{\beta}''(x,\omega) - m^2\tilde{\beta}(x,\omega) + \delta(x)[\tilde{A} * \tilde{\gamma}](\omega), \qquad (x,\omega) \in \mathbb{R}^2.$$

This identity implies the key **spectral inclusion**

$$\operatorname{supp} \tilde{A} * \tilde{\gamma} \subset \operatorname{supp} \tilde{\gamma}, \tag{3.8.3}$$

because $\operatorname{supp} \tilde{\beta}(x,\cdot) \subset \operatorname{supp} \tilde{\gamma}$ and $\operatorname{supp} \tilde{\beta}'(x,\cdot) \subset \operatorname{supp} \tilde{\gamma}$ in view of the representation (3.7.5).

We will derive below (3.2.2) from this inclusion, using a fundamental result of Harmonic Analysis – the Titchmarsh convolution theorem .

Titchmarsh convolution theorem

In 1926 E. C. Titchmarsh proved a theorem on the distribution of zeros of entire functions (see [82] and [76, p. 119]), which implies, in particular, the following corollary (see [6, Theorem 4.3.3]):

Theorem. *Let* $f(\omega)$ *and* $g(\omega)$ *be distributions of* $\omega \in \mathbb{R}$ *with bounded supports. Then*

$$[\operatorname{supp} f * g] = [\operatorname{supp} f] + [\operatorname{supp} g],$$

where $[X]$ *denotes the* **convex hull** *of a set* $X \subset \mathbb{R}$.

Note that in our situation, supp $\tilde{\gamma}$ is bounded by (3.7.7). Consequently, supp $\tilde{A}$ is also bounded, since $A(t) := a(|\gamma(t)|)$ is a polynomial in $|\gamma(t)|^2$ according to (3.1.14). Now the spectral inclusion (3.8.3) and Titchmarsh theorem imply that

$$[\text{supp } \tilde{A}] + [\text{supp } \tilde{\gamma}] \subset [\text{supp } \tilde{\gamma}],$$

whence it immediately follows in turn that $[\text{supp } \tilde{A}] = \{0\}$. Besides, $A(t) := a(|\gamma(t)|)$ is a bounded function due to (3.6.4), because $\gamma(t) = \beta(0,t)$. Therefore, $\tilde{A}(\omega) = C\delta(\omega)$, and hence

$$a(|\gamma(t)|) = C_1, \qquad t \in \mathbb{R}.$$

Now, the strict nonlinearity condition (3.1.14) implies that

$$|\gamma(t)| = C_2, \qquad t \in \mathbb{R}.$$

This immediately gives us that supp $\tilde{\gamma} = \{\omega_+\}$ by the same Titchmarsh theorem for the convolution $\tilde{\gamma} * \overline{\tilde{\gamma}} = C_3\delta(\omega)$. Therefore, $\tilde{\gamma}(\omega) = C_4\,\delta(\omega - \omega_+)$, and now (3.2.2) follows from (3.7.5).

Remark 3.8.2. In the case of the nonlinear Schrödinger equation (3.1.21), the Titchmarsh theorem does not work. The fact is that the continuous spectrum of the operator $-d^2/dx^2$ is the half-line $[0,\infty)$, so now the role of the 'spectral gap' is played by the unbounded interval $(-\infty, 0)$. Respectively, in this case the spectral inclusion (3.9.1) gives only that supp $\tilde{\beta}(x,\cdot) \subset (-\infty, 0)$, while the Titchmarsh theorem applies only to distributions with bounded supports.

3.9 On nonlinear radiation mechanism

Let us explain the informal arguments for global attraction to stationary orbits behind the formal proof of our main Theorem 3.1.2. The main part of the proof involves the study of the spectrum of omega-limit trajectories

$$\beta(x,t) = \lim_{s_{j'}\to\infty} \psi(x, s_{j'} + t).$$

Theorem 3.4.1 implies the spectral inclusion (3.7.7), which leads to

$$\operatorname{supp} \tilde\beta(x,\cdot) \subset [-m, m], \qquad x \in \mathbb{R}. \tag{3.9.1}$$

The Titchmarsh theorem then let us conclude that

$$\operatorname{supp} \tilde\beta(x,\cdot) = \{\omega_+\}. \tag{3.9.2}$$

These two inclusions are suggested by the following two informal arguments.

A. *Dispersion radiation in the continuous spectrum.*

B. *Nonlinear inflation of the spectrum and energy transfer from lower to higher harmonics.*

A. Dispersion radiation in the continuous spectrum. The inclusion (3.9.1) is due to the dispersion mechanism, which can be illustrated by the example of energy radiation in a wave field of a harmonic source with a frequency lying in the continuous spectrum. Namely, let us consider a one-dimensional linear Klein–Gordon equation with a *harmonic source*

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t) + b(x)e^{-i\omega_0 t}, \qquad x \in \mathbb{R}, \tag{3.9.3}$$

where the amplitude $b \in L^2(\mathbb{R})$ and the real frequency ω_0 is different from $\pm m$. In this case the *limiting amplitude principle* holds [189, 75, 78]:

$$\psi(x,t) \sim a(x)e^{-i\omega_0 t}, \qquad t \to \infty. \tag{3.9.4}$$

For the equation (3.9.3), this follows directly by the Fourier–Laplace transform in time

$$\tilde\psi(\omega,t) = \int_0^\infty e^{i\omega t}\psi(x,t)dt, \qquad x \in \mathbb{R}, \quad \operatorname{Im}\omega > 0. \tag{3.9.5}$$

Namely, applying this transform to equation (3.9.3), we get that

$$-\omega^2\tilde\psi(x,\omega) = \tilde\psi''(x,\omega) - m^2\tilde\psi(x,\omega) + \frac{b(x)}{i(\omega-\omega_0)}, \qquad x \in \mathbb{R}, \quad \operatorname{Im}\omega > 0,$$

where for the simplicity we assume zero initial data. Hence,

$$\tilde\psi(\cdot,\omega) = \frac{R(\omega)b}{i(\omega-\omega_0)} = \frac{R(\omega_0+i0)b}{i(\omega-\omega_0)} + \frac{R(\omega)b - R(\omega_0+i0)b}{i(\omega-\omega_0)}, \qquad \operatorname{Im}\omega > 0, \tag{3.9.6}$$

where

$$R(\omega) := (H-\omega^2)^{-1}$$

is the resolvent of the Schrödinger operator $H := -d^2/dx^2 + m^2$. This resolvent is a convolution operator with fundamental solution

$$-\frac{e^{ik(\omega)|x|}}{2ik(\omega)}, \qquad k(\omega) := \sqrt{\omega^2 - m^2} \in \overline{\mathbb{C}^+} \quad \text{for} \quad \omega \in \overline{\mathbb{C}^+},$$

as in (3.3.6). The last quotient of (3.9.6) is regular at $\omega = \omega_0$, and therefore its contribution is a dispersion wave, which decays like (3.5.2) in local energy seminorms. Consequently, the long-time asymptotics of $\psi(x,t)$ is determined by the middle quotient in (3.9.6). Therefore, (3.9.4) holds with the limiting amplitude $a(x) = R(\omega_0 + i0)b$. The Fourier transform of this limiting amplitude is equal to

$$\hat{a}(k) = -\frac{\hat{b}(k)}{k^2 + m^2 - (\omega_0 + i0)^2}, \qquad k \in \mathbb{R}.$$

This formula shows that the properties of the limiting amplitude differ significantly in the cases $|\omega_0| < m$ and $|\omega_0| \ge m$: $a(x) \in H^2(\mathbb{R})$ for $|\omega_0| < m$, however,

$$a(x) \notin L^2(\mathbb{R}) \quad \text{for} \quad |\omega_0| \ge m, \tag{3.9.7}$$

if $|\hat{b}(k)| \ge \varepsilon > 0$ in a neighborhood of the 'sphere' $|k|^2 + m^2 = \omega_0^2$ (which consists of two points in the 1D case). *This means the following.*

I. In the case $|\omega_0| \ge m$ the energy of the solution $\psi(x,t)$ tends to infinity for large times according to (3.9.4) and (3.9.7). This means that energy is transmitted from the harmonic source to the wave field!

II. Contrary, for $|\omega_0| < m$ the energy of the solution remains bounded, so there is no radiation.

It is this radiation in the case of $|\omega_0| \ge m$ that prohibits the presence of harmonics with such frequencies in omega-limit trajectories. Indeed, any omega-limit trajectory cannot radiate at all, since total energy is finite and bounded from below, and hence the radiation cannot last forever. These physical arguments make the inclusion (3.9.1) plausible, although a rigorous proof of it, as was seen above, requires special arguments.

Recall that the set $i\Sigma := \{i\omega_0 \in \mathbb{R}, |\omega_0| \ge m\}$ coincides with the continuous spectrum of the generator of the free Klein–Gordon equation. Radiation in the continuous spectrum is well known in the theory of waveguides. Namely, a waveguide can transmit only signals with a frequency $|\omega_0| > \mu$, where μ is a *threshold frequency*, which is an edge point of the continuous spectrum [77]. In our case, the waveguide occupies the 'entire space' $x \in \mathbb{R}$ and is described by the nonlinear Klein–Gordon equation (3.1.1) with the threshold frequency m.

B. Non-linear inflation of spectrum and energy transfer from lower to higher harmonics. Let us show that the single-frequency spectrum (3.9.2) is due to inflation of the spectrum by nonlinear functions. For example, consider the potential $U(\psi) = |\psi|^4$. Correspondingly, $F(\psi) = -\nabla_{\overline{\psi}} U(\psi) = -4|\psi|^2\psi$. We consider the sum$\psi(t) = e^{i\omega_1 t} + e^{i\omega_2 t}$ of two harmonics, whose spectrum is shown in Fig. 3.1:

We substitute this sum into the nonlinearity:

$$F(\psi(t)) \sim \psi(t)\overline{\psi(t)}\psi(t) = e^{i\omega_2 t}e^{-i\omega_1 t}e^{i\omega_2 t} + \ldots = e^{i(\omega_2+\Delta)t} + \ldots, \qquad \Delta := \omega_2 - \omega_1.$$

The spectrum of this expression contains harmonics with the new frequencies $\omega_1 - \Delta$ and $\omega_2 + \Delta$. As a result, all the frequencies $\omega_1 - \Delta$, $\omega_1 - 2\Delta, \ldots$ and $\omega_2 + \Delta$, $\omega_2 + 2\Delta, \ldots$ also

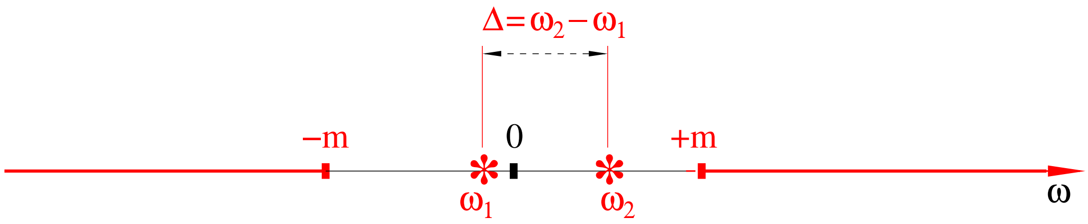

Figure 3.1: Two-point spectrum.

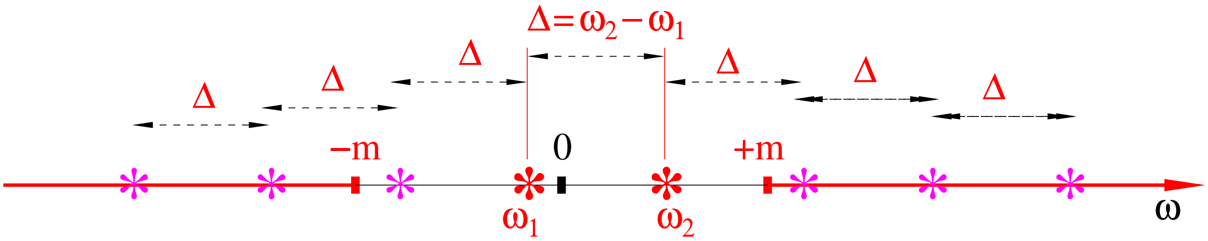

Figure 3.2: Non-linear inflation of spectrum.

will appear in the nonlinear dynamics described by (3.1.1) (see Fig. 3.2). Consequently, these frequencies will appear also in the nonlinear δ-function term which plays the role of a source.

As we already know, these frequencies lying in the continuous spectrum $|\omega| > m$ will surely cause energy radiation. This dispersion radiation will continue until the spectrum of the solution contains at least two different frequencies. It is this fact that prohibits the presence of two different frequencies in omega-limit trajectories, because total energy is finite, and thus the radiation cannot continue forever.

However, we underscore that

i) the precise meaning of the arguments "... until the spectrum of the solution contains at least two different frequencies." is established by our method of omega-limit trajectories;

ii) the inflation of the spectrum by a nonlinearity is justified by the Titchmarsh convolution theorem.

Nonlinear radiation mechanism. The above arguments physically mean the following binary *nonlinear radiation mechanism*:

I. The nonlinearity inflates the spectrum, which means energy transfer from lower to higher harmonics.

II. The dispersion radiation carries energy to infinity.

We have for the first time rigorously justified such a nonlinear radiation mechanism for the nonlinear $U(1)$-invariant Klein–Gordon and Dirac equations (3.1.1) and (3.1.18)–(3.1.20). Our numerical experiments demonstrate an analogous radiation mechanism for *relativistically-invariant* nonlinear wave equations (see Remark 6.1.1), however a rigorous proof is still missing.

3.10 Conjecture on attractors of G-invariant PDEs

Let us specify the conjecture (0.0.6) for *generic* Hamiltonian *G-invariant* PDEs in $\mathbb{R}^n$ of type (0.0.5) with a Lie symmetry group G acting on suitable Hilbert or Banach phase space $\mathcal{E}$ via a representation T. The Hamiltonian structure means that

$$F(\Psi) = JD\mathcal{H}(\Psi), \qquad J^* = -J, \tag{3.10.1}$$

where $\mathcal{H}$ denotes the corresponding Hamiltonian functional. The G-invariance means that

$$F(T(g)\Psi) = T(g)F(\Psi), \qquad \Psi \in \mathcal{E} \tag{3.10.2}$$

for all $g \in G$. In this case, for any solution $\Psi(t)$ to equations (0.0.5) the trajectory $T(g)\Psi(t)$ is also a solution.

Let us note that the theory of elementary particles deals systematically with the symmetry groups $SU(2)$, $SU(3)$, $SU(5)$, $SO(10)$ and others, and with the group $SU(4) \times SU(2) \times SU(2)$ which is the symmetry group of 'Grand Unification', see [222].

The conjecture (0.0.6) means that all solutions of type $e^{\hat{\lambda}t}\Psi$ with $\lambda \in \mathfrak{g}$ and $\Psi \in \mathcal{E}$ form a global attractor for **generic** G-invariant Hamiltonian nonlinear PDEs of type (0.0.5).

We still should specify the term **generic G-invariant equation** in Conjecture (0.0.6) (and in all results of Chapters 1, 2 and 3). Namely, this conjecture means that the asymptotics (0.0.6) hold for all solutions to an *open dense set* of G-invariant equations.

In particular, all asymptotics (0.0.7), (0.0.12), (0.0.15) and (0.0.16) hold under appropriate conditions, which define some 'open dense subset' of G-invariant equations with three types of the symmetry group G. The asymptotics can break down if these conditions fail — this corresponds to some 'exceptional equations': for example, global attraction (3.1.2) breaks down for the linear Schrödinger equations with at least two different eigenvalues.

The general situation is as follows. Let a Lie group G_1 be a (proper) subgroup of some larger Lie group G_2. So, the G_2-invariant equations form an 'exceptional subset' among all G_1-invariant equations, and the corresponding asymptotics (0.0.6) may be completely different. For example, the trivial group $\{e\}$ is a subgroup in $U(1)$ and in $\mathbb{R}^n$, while the asymptotics (0.0.12) and (0.0.15) may differ significantly from (0.0.7).

Conjecture (0.0.6) is confirmed by all rigorous results [43]–[74] presented in previous sections of this book. The results concern a list of model equations of type (0.0.5) with the following four basic symmetry groups: the trivial group $\{e\}$, the group of translations $\mathbb{R}^n$, the unitary group $U(1)$, and the orthogonal group $SO(3)$. In these cases, the asymptotics (0.0.6) read as (0.0.7), (0.0.12), (0.0.15), and (0.0.16), respectively.

Conjecture (0.0.6) suggests to define *stationary G-orbits* for equations (0.0.5) as solutions of type

$$\Psi(t) = e^{\hat{\lambda}t}\Psi, \qquad t \in \mathbb{R}, \tag{3.10.3}$$

where $\lambda \in \mathfrak{g}$. This definition leads to the corresponding *nonlinear eigenvalue problem*

$$F(\Psi) = \hat{\lambda}\Psi. \tag{3.10.4}$$

In particular, for the case of unitary symmetry group $U(1)$ the Lie algebra is $\mathfrak{g} = \mathbb{R}$, and λ is a real number. On the other hand, for the symmetry group $G = SU(3)$, the generator λ is a skew-Hermitian 3×3-matrix.

Empirical evidence. The conjecture (0.0.6) agrees with the Gell-Mann–Ne'eman theory of baryons [220, 221]. Namely, in 1961 Gell-Mann and Ne'eman suggested the symmetry group $SU(3)$ and other ones for the strong interaction of baryons relying on the discovered parallelism between empirical data for the baryons, and the 'Dynkin scheme' of the Lie algebra $\mathfrak{g} = su(3)$ with 8 generators (the famous 'eightfold way').

This theory resulted in the scheme of quarks and in the development of Quantum Chromodynamics [222], and in the prediction of a new baryon with prescribed values of its mass and decay products. This particle (the Ω^--hyperon) was promptly discovered experimentally [223].

The elementary particles seem to describe long-time asymptotics of quantum fields. Hence, this empirical correspondence between the baryons and generators of the Lie algebra of the symmetry group presumably gives an evidence in favour of our general conjecture on attractors (0.0.6).

Chapter 4

Asymptotic Stability of Solitons

More precisely we should phrase 'asymptotic stability of solitary manifolds' which means a local attraction, i.e. for states sufficiently close to such manifold.

In Sections 4.1 and 4.2 we describe general strategies introduced by A. Soffer and M. Weinstein, and by V.S. Buslaev and G. Perelman for proving such local attraction,

In Sections 4.3 and 4.4 we give a brief survey of related results.

In final Section 4.5 we give a concise and streamlined proof of the result [113] illustrating general strategy of V.S. Buslaev and G. Perelman in the case of 1D Schrödinger equation coupled to a nonlinear oscillator.

4.1 Orthogonal projection

This strategy arose in 1985–1992 in the pioneering work of A. Soffer and M. Weinstein [162, 163, 171], see the review [161]. The results concern nonlinear $U(1)$-invariant Schrödinger equations with real potential $V(x)$

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + \lambda|\psi(x,t)|^p\psi(x,t), \qquad x \in \mathbb{R}^n, \tag{4.1.1}$$

where $\lambda \in \mathbb{R}$, $p = 3$ or 4, $n = 2$ or $n = 3$, and $\psi(x,t) \in \mathbb{C}$. The corresponding Hamiltonian functional reads

$$\mathcal{H} = \int \big[\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V(x)|\psi(x)|^2 + \frac{\lambda}{p}|\psi(x)|^p\big]\,dx.$$

For $\lambda = 0$, the equation (4.1.1) is linear. It is assumed that the discrete spectrum of the short range Schrödinger operator $H := -\Delta + V(x)$ is a single point $\omega_* < 0$, and the point zero is neither an eigenvalue nor a resonance for H. Let $\phi_*(x)$ denote the corresponding ground state:

$$H\phi_*(x) = \omega_*\phi_*(x). \tag{4.1.2}$$

Then $C\phi_*(x)e^{-i\omega_* t}$ are periodic solutions for all complex constants C. Corresponding phase curves are circles, filling the complex plane.

For nonlinear equations (4.1.1) with a small real $\lambda \neq 0$, it turns out that a wonderful *bifurcation* occurs: small neighborhood of the zero of the complex plane turns into an analytic invariant solitary manifold $\mathcal{S}$ which is still filled with invariant circles which are trajectories of *stationary orbits* of type (3.1.11),

$$\psi(x,t) = \psi_\omega(x)e^{-i\omega t} \tag{4.1.3}$$

whose frequencies ω are close to ω_*.

Remark 4.1.1. *Now all these solutions $\psi_\omega(x)e^{-i\omega t}$ are called as ground states.*

The main result of [162, 163] (see also [155]) is long-time attraction to one of these ground states for any solution of equation (4.1.1) with sufficiently small $\lambda > 0$ in the case of small initial data:

$$\psi(x,t) = \psi_\pm(x)e^{-i\omega_\pm t} + r_\pm(x,t), \tag{4.1.4}$$

where the remainder decay in weighted norms: for $\sigma > 2$

$$\|\langle x\rangle^{-\sigma} r_\pm(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0, \qquad t \to \pm\infty,$$

where $\langle x\rangle := (1+|x|)^{1/2}$. The proof relies on linearisation of the dynamics and decomposition of solutions into two components

$$\psi(t) = e^{-i\Theta(t)}(\psi_{\omega(t)} + \phi(t)),$$

with the orthogonality condition [162, (3.2) and (3.4)]:

$$\langle \psi_{\omega(t)}, \phi(t)\rangle = 0. \tag{4.1.5}$$

This orthogonality and dynamics (4.1.1) imply the *modulation equations* for $\omega(t)$ and $\gamma(t)$, where $\gamma(t) := \Theta(t) - \int_0^t \omega(s)ds$ (see (3.2) and (3.9a)–(3.9b) from [162]). The orthogonality (4.1.5) implies that the component $\phi(t)$ lies in the continuous spectral space of the

Schrödinger operator $H(\omega_0) := -\Delta + V + \lambda|\psi_{\omega_0}|^p$, which leads to time-decay of $\phi(t)$ (see [162, (4.2a) and (4.2b)]). Finally, this decay implies the convergence $\omega(t) \to \omega_\pm$ and the asymptotics (4.1.4).

These results and methods were further developed in numerous works for nonlinear Schrödinger, wave and Klein–Gordon equations with potentials under various spectral assumptions on linearised dynamics, see [108, 113, 141, 155, 164, 165, 171].

4.2 Symplectic projection

Genuine breakthrough in the theory of asymptotic stability was achieved in 1990-2003 by V.S. Buslaev, G. Perelman and C. Sulem [110, 111, 112], who first extended asymptotics of type (4.1.4) to 1D translation-invariant Schrödinger equation

$$i\dot\psi(x,t) = -\psi''(x,t) - F(\psi(x,t)), \qquad x \in \mathbb{R} \tag{4.2.1}$$

without smallness conditions on the nonlinearity and initial data.

The equation is assumed to be $U(1)$-invariant. The latter means that the nonlinear function $F(\psi) = -\nabla_{\overline{\psi}}U(\psi)$ satisfies identities (3.1.8)–(3.1.10). Also the following condition is assumed

$$U(\psi) = \mathcal{O}(|\psi|^{10}), \qquad \psi \to 0, \tag{4.2.2}$$

which is required probably by a failure of suitable technique. Under some simple additional conditions on the potential U (see below), there exist *stationary orbits* which are finite energy solutions of the form

$$\psi(x,t) = \psi_0(x)e^{i\omega_0 t}, \tag{4.2.3}$$

with $\omega_0 > 0$. The amplitude $\psi_0(x)$ satisfies the corresponding stationary equation

$$-\omega_0\psi_0(x) = -\psi_0''(x) - F(\psi_0(x)), \qquad x \in \mathbb{R}, \tag{4.2.4}$$

which implies the 'energy conservation'

$$\frac{|\psi_0'(x)|^2}{2} + U_e(\psi_0(x)) = E, \tag{4.2.5}$$

where the 'effective potential' $U_e(\psi) = U(\psi) + \omega_0\frac{|\psi|^2}{2} \sim \omega_0\frac{|\psi|^2}{2}$ as $\psi \to 0$ by (4.2.2). For the existence of finite energy solution (4.2.3), the graph of the effective potential $U_e(\psi)$ should be similar to Fig. 4.1. The finite energy solution is defined by (4.2.5) with the constant $E = U_e(0)$ since for other E the solutions to (4.2.5) do not converge to zero as $|x| \to \infty$. This equation with $E = U_e(0)$ implies that

$$\frac{|\psi_0'(x)|^2}{2} = U_e(0) - U_e(\psi_0(x)) \sim \frac{\omega_0}{2}\psi_0^2(x). \tag{4.2.6}$$

Hence, for finite energy solutions

$$\psi_0(x) \sim e^{-\sqrt{\omega_0}|x|}, \qquad |x| \to \infty. \tag{4.2.7}$$

It is easy to verify that the following functions are also solutions, (*moving solitons*)

$$\psi_{\omega,v,a,\theta}(x,t) = \psi_\omega(x - vt - a)e^{i(\omega t + kx + \theta)}, \qquad \omega = \omega_0 - v^2/4, \qquad k = v/2. \tag{4.2.8}$$

The set of all such solitons with parameters ω, v, a, θ forms a 4-dimensional smooth submanifold $\mathcal{S}$ in the Hilbert phase space $\mathcal{X} := L^2(\mathbb{R})$. Moving solitons (4.2.8) are obtained from standing soliton (4.2.3) by the Galilean transformation

$$G(a,v,\theta) : \psi(x,t) \mapsto \varphi(x,t) = \psi(x - vt - a, t)e^{i(-\frac{v^2}{4}t + \frac{v}{2}x + \theta)}. \tag{4.2.9}$$

It is easy to verify that the Schrödinger equation (4.2.1) is invariant with respect to this symmetry group.

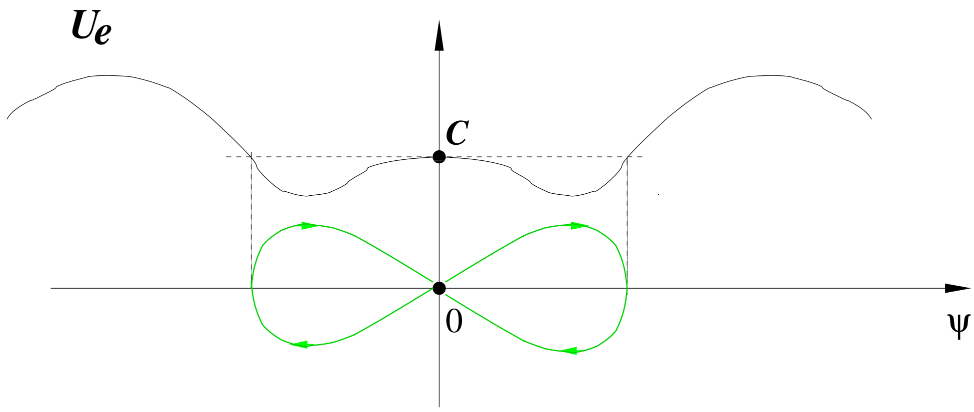

Figure 4.1: Reduced potential and soliton.

Linearisation of the Schrödinger equation (4.2.1) on the stationary orbit (4.2.3) is obtained by substitution $\psi(x,t) = (\psi_0(x) + \chi(x))e^{-i\omega_0 t}$ and retaining terms of the first order in χ. This linearised equation contains χ and $\overline{\chi}$, and hence, it is not linear over the field of complex numbers. This follows from the fact that the nonlinearity of $F(\psi)$ is not complex-analytic due to the $U(1)$-invariance (3.1.8). Complexification of this linearised equation reads

$$\dot{\Psi}(x,t) = C_0\Psi(x,t), \qquad C_0 = -jH_0, \tag{4.2.10}$$

where j is a real 2×2 matrix, representing the multiplier i, $\Psi(x,t)\in\mathbb{C}^2$, and $H_0 = -d^2/dx^2 + \omega_0 + V(x)$, where $V(x)$ is a real matrix potential, which decreases exponentially as $|x|\to\infty$ due to (4.2.7).

Note that the operator $C_0 = C_{\omega_0,0,0,0}$ corresponds to the linearisation on the soliton (4.2.8) which is one of the solitons (4.2.8) corresponding to parameters $\omega=\omega_0$, and $a=v=\theta=0$. Similar operators $C_{\omega,a,v,\theta}$, corresponding to linearisation at solitons (4.2.8) with various parameters ω, a, v, θ, are connected with C_0 via the differential of the Galilean transformation (4.2.9). Therefore, their spectral properties completely coincide. In particular, their continuous spectrum coincides with $(-i\infty, -i\omega_0]\cup[i\omega_0, i\infty)$.

Main results of [110, 111, 112] are asymptotics of type (4.1.4) for solutions with initial data close to the solitary manifold $\mathcal{S}$:

$$\psi(x,t) = \psi_\pm(x - v_\pm t)e^{-i(\omega_\pm t + k_\pm x)} + W(t)\Phi_\pm + r_\pm(x,t), \qquad \pm t > 0, \tag{4.2.11}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation, $\Phi_\pm$ are some scattering states of finite energy, and $r_\pm$ are remainder terms which decay to zero in a global norm:

$$\|r_\pm(\cdot,t)\|_{L^2(\mathbb{R})} \to 0, \qquad t\to\pm\infty. \tag{4.2.12}$$

These asymptotics were obtained under following assumptions on the spectrum of the generator B_0:

U1. The discrete spectrum of the operator C_0 consists of exactly three eigenvalues 0 and $\pm i\lambda$, and

$$\lambda < \omega_0 < 2\lambda. \tag{4.2.13}$$

This condition means that the discrete mode can interact with the continuous spectrum already in the first order of perturbation theory.

U2. The edge points $\pm i\omega_0$ of the continuous spectrum are neither eigenvalues, nor resonances of C_0.

U3. Furthermore, it is assumed the condition [112, (1.0.12)], which means a strong coupling of discrete and continuous spectral components, providing energy radiation, similarly to the Wiener condition (1.5.13). The condition [112, (1.0.12)] ensures that the interaction of discrete component with continuous spectrum does not vanish in the first order of perturbation theory. This condition is a nonlinear version of the Fermi Golden Rule [158], which was introduced by I.M. Sigal in the context of nonlinear PDEs [79].

Examples of potentials satisfying all these conditions are constructed in [139].

In 2001, Cuccagna extended results of [110, 111, 112] to nD translation-invariant Schrödinger equations in the dimensions $n \geq 2$, [114].

Method of symplectic projection in the Hilbert phase space. Novel approach [110, 111, 112] relies on *symplectic projection* of solutions onto the solitary manifold. This means that

$$Z := \psi - S \quad \text{is symplectic-orthogonal to the tangent space} \quad \mathcal{T} := T_S\mathcal{S}$$

for the projection $S := P\psi$. This projection is correctly defined in a small neighborhood of $\mathcal{S}$ because $\mathcal{S}$ is a *symplectic manifold*, i.e. the corresponding symplectic form is non-degenerate on the tangent spaces $T_S\mathcal{S}$.

Thus a solution $\psi(t)$ for each $t > 0$ decomposes as $\psi(t) = S(t) + Z(t)$, where $S(t) := P\psi(t)$, and the dynamics is linearised on the soliton $S(t)$. Similarly, for each $t \in \mathbb{R}$ the total Hilbert phase space $\mathcal{X} := L^2(\mathbb{R})$ is splitted as $\mathcal{X} = \mathcal{T}(t) \oplus \mathcal{Z}(t)$, where $\mathcal{Z}(t)$ is **symplectic-orthogonal** complement to the tangent space $\mathcal{T}(t) := T_{S(t)}\mathcal{S}$. The corresponding equation for the *transversal component* $Z(t)$ reads

$$\dot{Z}(t) = A(t)Z(t) + N(t),$$

where $A(t)Z(t)$ is the linear part, and $N(t) = \mathcal{O}(\|Z(t)\|^2)$ is the corresponding nonlinear part.

The main difficulties in studying this equation are as follows: i) it is *non-autonomous*, and ii) the generators $A(t)$ *are not self-adjoint* (see Appendix in [137]). It is important that $A(t)$ are *Hamiltonian operators*, for which the existence of spectral decomposition is provided by the Krein–Langer theory of J-selfadjoint operators [145, 148]. In [137, 138] we have developed a special version of this theory providing the corresponding eigenfunction expansion which is necessary for the justification of the approach [110, 111, 112]. The main steps of this strategy are as follows.

• **modulation equations.** The parameters of the soliton $S(t)$ satisfy **modulation equations**: for example, for the speed $v(t)$ we have

$$\dot{v}(t) = M(\psi(t)),$$

where $M(\psi) = \mathcal{O}(\|Z\|^2)$ for small norms $\|Z\|$. This means that the parameters change 'superslowly' near the solitary manifold, like adiabatic invariants.

• **Tangent and transversal components.** The *transversal component* $Z(t)$ in the splitting $\psi(t) = S(t) + Z(t)$ belongs to the *transversal subspace* $\mathcal{Z}(t)$. The tangent space $\mathcal{T}(t)$ is the root space of the generator $A(t)$ and corresponds to the 'unstable spectral point' $\lambda = 0$. The key observation is that

i) the transversal subspace $\mathcal{Z}(t)$ is **invariant** with respect to the generator $A(t)$, since the subspace $\mathcal{T}(t)$ is invariant, and $A(t)$ is the Hamiltonian operator;

ii) moreover, the transversal subspace $\mathcal{Z}(t)$ does not contain the tangent vectors corresponding to the unstable eigenvalue $\lambda = 0$.

• **Continuous and discrete components.** The transversal component allows further splitting $Z(t) = z(t) + f(t)$, where $z(t)$ and $f(t)$ belong, respectively, to discrete and continuous spectral subspaces $\mathcal{Z}_d(t)$ and $\mathcal{Z}_c(t)$ of $A(t)$ in the space $\mathcal{Z}(t) = \mathcal{Z}_d(t) + \mathcal{Z}_c(t)$.

• **Poincare normal forms and Fermi Golden Rule.** The component $z(t)$ satisfies a nonlinear equation, which is reduced to Poincare normal form up to higher order terms [112, Equations (4.3.20)]. The normal form allowed to obtain some 'conditional decay' for $z(t)$ using the Fermi Golden Rule [112, (1.0.12)]. For the relativistically-invariant Ginzburg–Landau equation, a similar reduction was done in [136, Equations (5.18)].

• **Method of majorants.** A skillfull combination of the conditional decay for $z(t)$ with the superslow evolution of the soliton parameters allows one to prove the decay for $f(t)$ and $z(t)$ by the method of majorants. Finally, this decay implies the asymptotics (4.2.11)–(4.2.12).

Remark 4.2.1. i) The role of the symplectic projection in the theory of V.S. Buslaev, G. Perelman and C. Sulem [110, 111, 112] probably was suggested by the theory of orbital stability of M. Grillakis, J. Shatah and W. Strauss [103, 104] which extends to Hamiltonian PDEs the stability theory of finite-dimensional Hamiltonian systems with symmetry groups, see [13, 14]. The last theory, in its own turn, is dating back to H. Poincaré who established the theory of stability of the fixed points of the reduced dynamics, which he called *relative equilibria*, [15].

ii) The difference of the theory [110, 111, 112] with [103, 104] is as follows.

i) The linearised dynamics in [103, 104] is stable in the transversal directions because the positive spectrum is away from zero and hence, the conserved Hamiltonian serves as the Lyapunov function in these directions.

ii) On the other hand, in [110, 111, 112] the positive spectrum of this transversal dynamics is not away from zero. However, the asymptotic stability holds since the positive spectrum is absolute continuous.

4.3 Generalisations and applications

N-soliton solutions. The methods and results of [112] were developed in [149, 150, 151, 152, 153, 156, 157, 159, 160] for N-soliton solutions for translation-invariant nonlinear Schrödinger equations.

Multiphoton radiation. In [116] Cuccagna and Mizumachi extended methods and results of [112] to the case when the inequality (4.2.13) is changed to

$$N\lambda < \omega_0 < (N+1)\lambda,$$

with some natural $N > 1$, and the corresponding analogue of condition U3 holds. It means, that the interaction of discrete modes with a continuous spectrum occurs only in the N-th order of perturbation theory. The decay rate of the remainder term (4.2.12) worsens with growing N.

Linear equations coupled to nonlinear oscillators and particles. The methods and results of [112] were extended i) in [113, 141] to the Schrödinger equation coupled to a nonlinear $U(1)$-invariant oscillator, ii) in [125, 127] to systems (2.1.1) and (1.6.1) with zero external fields, and iii) in [126, 134, 140] to similar translation-invariant systems of the Klein–Gordon, Schrödinger and Dirac equations coupled to a particle. The survey of these results can be found in [124].

For example, article [127] concerns solutions to the system (2.1.1) with initial data close to a solitary manifold (2.1.3) in weighted norm

$$\|\psi\|_\sigma^2 = \int \langle x\rangle^{2\sigma}|\psi(x)|^2 dx$$

with sufficiently large $\sigma > 0$. Namely, the initial state is close to soliton (2.1.3) with some parameters v_0, a_0:

$$\|\nabla\psi(x,0) - \nabla\psi_{v_0}(x-a_0)\|_\sigma + \|\psi(x,0) - \psi_{v_0}(x-a_0)\|_\sigma + \|\pi(x,0) - \pi_{v_0}(x-a_0)\|_\sigma$$
$$+|q(0)-a_0| + |\dot q(0) - v_0| \le \varepsilon,$$

where $\sigma > 5$, and $\varepsilon > 0$ is sufficiently small. Moreover, the Wiener condition (1.5.13) is assumed for $k \ne 0$. Additionally, let

$$\partial^\alpha \hat\rho(0) = 0, \quad |\alpha| \le 5,$$

that is equivalent to equalities

$$\int x^\alpha \rho(x)\, dx = 0, \quad |\alpha| \le 5.$$

Under these conditions, the main results of [127] are the asymptotics

$$\ddot q(t) \to 0, \quad \dot q(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \qquad t \to \pm\infty$$

(cf. (2.1.8) and (2.1.11)) and the attraction to solitons (2.1.9), where the remainder now decays in *global weighted norms* in the comoving frame (cf. (2.1.10)):

$$\|\nabla r_\pm(q(t)+x,t)\|_{-\sigma} + \|r_\pm(q(t)+x,t)\|_{-\sigma} + \|s_\pm(q(t)+x,t)\|_{-\sigma} \to 0, \qquad t \to \pm\infty.$$

Relativistically-invariant equations. In [107, 109, 144, 135, 136] the asymptotic stability of solitary manifolds was established for the first time for *relativistically-invariant* nonlinear equations. Namely, in [107] and [144, 135, 136] asymptotics of the type (4.2.11) were obtained for 1D relativistically-invariant nonlinear wave equations (2.2.4) with potentials of the Ginzburg–Landau type, and in [109] for relativistically-invariant nonlinear Dirac equations. In [139] we have constructed examples of potentials providing all spectral properties of the linearised dynamics imposed in [144, 135, 136].

In [137, 138] we have justified the eigenfunction expansions for nonselfadjoint Hamiltonian operators which were used in [144, 135, 136]. For the justification we have developed a special version of the Krein–Langer theory of J-selfadjoint operators [145, 148].

Cherenkov radiation. The article [122] concerns a system of type (2.1.1) with the Schrödinger equation instead of the wave equation (system (1.9)–(1.10) in [122]). This system is considered as a model of the Cherenkov radiation. The main result of [122] is long-time convergence to a soliton with the sonic speed for initial solitons with a supersonic speed in the case of a weak interaction (the 'Bogolyubov limit') and small initial field. The asymptotic stability of solitary manifolds for very close system with the Schrödinger equation was established in [134].

4.4 Further generalisations

The results on asymptotic stability of solitary manifolds were developed in different directions.

Systems with several bound states. Articles [106, 115, 168, 169, 170] concern asymptotic stability of stationary orbits (4.1.3) for the nonlinear Schrödinger, Klein–Gordon and wave equations in the case of several simple eigenvalues of the linearisation. The typical assumptions are as follows:

i) the endpoint of continuous spectrum is neither an eigenvalue nor a resonance for linearised equation;

ii) the eigenvalues of the linearised equation satisfy several non-resonance conditions;

iii) a new version of the Fermi Golden Rule.

One typical difficulty is possible long stay of solutions near metastable tori which correspond to approximate resonances. Great efforts are being made to show that the role of metastable tori decreases like $t^{-1/2}$ as $t \to \infty$. The typical result is the long-time asymptotics 'ground state + dispersion wave' in the norm $H^1(\mathbb{R}^3)$ for solutions close to the ground state.

General Theory of Relativity. The article [123] concerns so-called 'kink instability' of self-similar and spherically symmetric solutions of the equations of the General Theory of Relativity with a scalar field, as well as with a 'hard fluid' as sources. The authors constructed examples of self-similar solutions that are unstable to the kink perturbations.

The article [117] examines linear stability of slowly rotating Kerr solutions for the Einstein equations in vacuum. In [167] a pointwise damping of solutions to the wave equation is investigated for the case of stationary asymptotically flat space-time in the three-dimensional case.

In [105] the Maxwell equations are considered outside slowly rotating Kerr black hole. The main results are: i) boundedness of a positive definite energy on each hypersurface $t = \text{const}$ and ii) convergence of each solution to a stationary Coulomb field.

In [118] the pointwise decay was proved for linear waves against the Schwarzschild black hole.

Method of concentration compactness. In [130] the concentration compactness method was used for the first time to prove global well-posedness, scattering and blow-up of solutions to critical focusing nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) - |\psi(x,t)|^{\frac{4}{n-2}}\psi(x,t), \qquad x \in \mathbb{R}^n$$

in the radial case. Later on, these methods were extended in [119, 121, 131, 146] to general non-radial solutions and to nonlinear wave equations of the type

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + |\psi(x,t)|^{\frac{4}{n-2}}\psi(x,t), \qquad x \in \mathbb{R}^n.$$

One of the main results is splitting of the set of initial states, close to the critical energy level, into three subsets with certain long-term asymptotics: either a blow-up in a finite time, or an asymptotically free wave, or the sum of the ground state and an asymptotically free wave. All three alternatives are possible; all nine combinations with $t \to \pm\infty$ are also possible. Lectures [154] give excellent introduction to this area. The articles [120, 132] concern super-critical nonlinear wave equations.

Recently, these methods and results were extended to critical wave mappings [129, 128, 146, 147]. The 'decay onto solitons' is proved: every 1-equivariant finite-energy wave mapping of exterior of a ball with Dirichlet boundary conditions into three-dimensional sphere exists globally in time and dissipates into a single stationary solution of its own topological class.

4.5 The Schrödinger equation coupled to an oscillator

In this section we illustrate the strategy of V.S. Buslaev and G. Perelman [110, 111] (which was also used in [112] and in many works cited in Sections 4.2, 4.3 and 4.4) by application to 1D Schrödinger equation coupled to a nonlinear oscillator, see [113]. The coupled system is invariant with respect to the phase rotation group $U(1)$. For initial states close to a stationary orbit, the solution converges to a sum of another stationary orbit and dispersive wave which is a solution to the free Schrödinger equation. The proofs are complete and rely on the strategy of [110, 111]: the linearisation of the dynamics on the solitary manifold, the symplectic orthogonal projection and method of majorants.

4.5.1 Introduction

Our main goal is the study of asymptotic stability of 'quantum stationary states' for a model $U(1)$-invariant nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \quad x \in \mathbb{R}. \tag{4.5.1}$$

Here $\psi(x,t)$ is a continuous complex-valued wave function and F is a continuous function, the dots stand for the derivatives in t and the primes in x. All derivatives and the equation are understood in the distribution sense. Physically, equation (4.5.1) describes the system of the free Schrödinger equation coupled to a nonlinear oscillator located at the point $x = 0$; F is a nonlinear 'oscillator force'.

We assume that $F(\psi) = -\nabla U(\psi)$ where $U(\psi) = u(|\psi|)$. Then (4.5.1) defines a $U(1)$-invariant Hamiltoniansystem and admits finite energy solutions of type $\psi_\omega(x)e^{i\omega t}$ called *stationary orbits* which are *nonlinear eigenfunctions*. The stationary orbits constitute a two-dimensional *solitary manifold* in the Hilbert phase space of finite energy states of the equation. We prove the asymptotics of type

$$\psi(\cdot,t) \sim \psi_{\omega_\pm}e^{i\omega_\pm t} + W(t)\Phi_\pm, \quad t \to \pm\infty, \tag{4.5.2}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation, $\Phi_\pm \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ are the corresponding asymptotic scattering states, and the remainder converges to zero as $\mathcal{O}(|t|^{-1/2})$ in the global norm of $C_b(\mathbb{R}) \cap L^2(\mathbb{R})$. Here $C_b(\mathbb{R})$ is the space of bounded continuous functions $\mathbb{R} \to \mathbb{C}$. The asymptotics hold for the solutions with initial states close to the *stable part* of the solitary manifold, extending the methods and results of [110, 111, 112] to the equation (4.5.1).

Let us note that we impose conditions which are more general than the standard ones in the following respects:

i) We do not hypothesize any spectral properties of the linearised equation, and do not require any smallness condition on the initial state (only closeness to the solitary manifold).

ii) The stable part of the solitary manifold is characterised by a condition on the nonlinearity (4.5.17). The relation of this to the standard criterion for orbital stability $\partial_\omega \int |\psi_\omega(x)|^2 dx > 0$ (see [103, 104] and references therein) will be discussed below.

This progress is possible on account of the simplicity of our model which allows an exact analysis of all spectral properties of the linearisation.

Let us note the following two main novelties in our approach to the uniform decay of the dynamics in transversal directions to the solitary manifold. First, we calculate exactly all needed spectral properties of corresponding generator. Second, we do not use a spectral representation of the generator. Instead, we develop the Jensen–Kato approach applying directly the Zygmund type Lemma 6.1 (cf. [184, Lemma 10.2]) to the Laplace integral of the resolvent. We expect that the development would be promising for more general problems.

This section is organized as follows. In Section 4.5.2 some notation and definitions are given. In Section 4.5.3 we describe all nonzero stationary orbits and formulate the main theorem. The linearisation on a stationary orbit is carried out in Section 4.5.4. In Sections 4.5.5 and 4.5.6, we construct the spectral representation for the linearised equation. In Section 4.5.7 we establish the time decay for the linearised equation in the continuous spectrum. In Section 4.5.9 the modulation equations for the parameters of the soliton are displayed. The decay of the transversal component is proved in Sections 4.5.10 and 4.5.11. In Section 4.5.12 we obtain the soliton asymptotics (4.5.2). In Appendix we study the resolvent of linearised equation.

In conclusion, we expect that the asymptotics (4.5.2) holds for *any* finite energy solution of the equation (4.5.1), however this is still open problem.

4.5.2 Notation and definitions

We identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $(\psi_1, \psi_2) \in \mathbb{R}^2$ and assume that the vector version $\mathbf{F}$ of the oscillator force F admits a real-valued potential,

$$\mathbf{F}(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{R}^2, \quad U \in C^2(\mathbb{R}^2). \tag{4.5.3}$$

Then (4.5.1) is formally a Hamiltonian system with Hamiltonian

$$\mathcal{H}(\psi) = \frac{1}{2}\int |\psi'|^2 dx + U(\psi(0)). \tag{4.5.4}$$

which is conserved for sufficiently regular finite energy solutions. We assume that the potential $U(\psi)$ satisfies the inequality

$$U(z) \ge A - B|z|^2 \quad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \tag{4.5.5}$$

Our key assumption concerns the $U(1)$-invariance of the oscillator, where $U(1)$ stands for the rotation group $e^{i\theta}$, $\theta \in [0, 2\pi]$ acting by phase rotation $\psi \mapsto e^{i\theta}\psi$. Namely, we assume that

$$U(\psi) = u(|\psi|^2), \qquad u \in C^2(\mathbb{R}) \tag{4.5.6}$$

(cf. [27, 28]). In this case

$$F(\psi) = a(|\psi|^2)\psi, \qquad \psi \in \mathbb{C}, \qquad a \in C^1(\mathbb{R}). \tag{4.5.7}$$

Therefore,

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \quad \theta \in [0, 2\pi], \tag{4.5.8}$$

and $F(0) = 0$. This rotation symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to (4.5.1) if $\psi(x,t)$ is. The equation is $U(1)$-invariant in the sense of [103, 104], and the Nöther theorem implies the *charge conservation*:

$$\mathcal{Q}(\psi) = \int |\psi|^2 dx = \text{const}. \tag{4.5.9}$$

The main subject of this section is an analysis of asymptotic stability of 'quantum stationary orbits', or *solitary waves* in the sense of [103, 104], which are finite energy solutions of the form

$$\psi(x,t) = \psi_\omega(x)e^{i\omega t}, \quad \omega \in \mathbb{R}. \tag{4.5.10}$$

The frequency ω and the amplitude $\psi_\omega(x)$ solve the following *nonlinear eigenvalue problem*:

$$-\omega\psi_\omega(x) = -\psi_\omega''(x) - \delta(x)F(\psi_\omega(0)), \quad x \in \mathbb{R}. \tag{4.5.11}$$

which follows directly from (4.5.1) and (4.5.7) since $\omega \in \mathbb{R}$.

Definition 4.5.1. *$\mathcal{S}$ denotes the set of all nonzero solutions $\psi_\omega(x) \in H^1(\mathbb{R})$ to (4.5.11) with all possible $\omega \in \mathbb{R}$.*

Here $H^1(\mathbb{R}) = H^1$ denotes the Sobolev space of complex valued measurable functions with $\int (|\psi'|^2 + |\psi|^2)dx < \infty$. We give below in section 4.5.3 a complete analysis of the set $\mathcal{S}$ of all nonzero stationary orbits $\psi_\omega(x)$ by an explicit calculation: it consists of functions $C(\omega)e^{-\sqrt{\omega}|x|+i\theta}$ with $C > 0$, $\omega = \omega(C) > 0$ and any $\theta \in [0, 2\pi]$, and C restricted to lie in a set which, in the case of polynomial F, is a finite union of one-dimensional intervals. Notice that $C = 0$ corresponds to the zero function $\psi(x) = 0$ which is always a solitary wave as $F(0) = 0$, and for $\omega \le 0$ only the zero stationary orbit exists.

Our main results describe the large time bechavior of the global solutions whose existence is guaranteed by the following theorem, which is proved in [133].

Theorem 4.5.2. *i) Let conditions (4.5.3) and (4.5.5) hold. Then for any initial state $\psi(0) \in H^1$ there exist a unique solution $\psi(\cdot) \in C_b(\mathbb{R}, H^1)$ to the equation (4.5.1).*
ii) The following a priori bound holds:

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1} < \infty. \tag{4.5.12}$$

The functional spaces we are going to consider are the weighted Banach spaces L^p_β, $p \in [1, \infty)$, $\beta \in \mathbb{R}$ of complex valued measurable functions with the norm

$$\|u\|_{L^p_\beta} = \|(1 + |x|)^\beta u(x)\|_{L^p}. \tag{4.5.13}$$

4.5.3 Solitary waves and the main theorem

Lemma 4.5.3. *The set of all nonzero stationary orbits is given by*

$$\mathcal{S} = \Big\{\psi_\omega e^{i\theta} = Ce^{i\theta-\sqrt{\omega}|x|} : \omega > 0, \quad C > 0, \quad \sqrt{\omega} = a(C^2)/2 > 0, \quad \theta \in [0, 2\pi]\Big\}.$$

Proof. Let us calculate all stationary orbits (4.5.10). The equation (4.5.11) implies $\psi''(x) = \omega\psi(x)$, $x \neq 0$, hence the formula $\psi(x) = C_\pm e^{\sqrt{\omega}x}$ gives two linearly independent solutions in each of the two regions $\pm x > 0$ depending on which branch of $\sqrt{\omega}$ is chosen. Since $\psi(x) \in L^2$ it is necessary that $\omega > 0$ and the branch is chosen with $\pm\sqrt{\omega} > 0$ for $\pm x < 0$. Furthermore, since $\psi'(x) \in L^2$, the function $\psi(x)$ is continuous, hence $C_- = C_+ = C$ and the solutions are of the form

$$\psi(x) = Ce^{-\varkappa|x|}, \quad \varkappa = \sqrt{\omega} > 0, \quad \omega > 0. \tag{4.5.14}$$

Finally we get an algebraic equation for the constant C equating the coefficients of $\delta(x)$ in both sides of (4.5.11):

$$0 = \psi'(0+) - \psi'(0-) + F(\psi(0)). \tag{4.5.15}$$

This implies $0 = -2\varkappa C + F(C)$, or equivalently,

$$\varkappa = \frac{F(C)}{2C} = \frac{a(C^2)}{2}. \tag{4.5.16}$$

□

Corollary 4.5.4. *The set $\mathcal{S}$ is a smooth manifold with co-ordinates $\theta \in \mathbb{R} \mod 2\pi$ and $C > 0$ such that $a(C^2) > 0$.*

Remark 4.5.5. *We will analyse only the stationary orbits with $a'(C) \neq 0$. On the manifold $\mathcal{S}$ we have $\omega = \varkappa^2$ with $\varkappa = a(C^2)/2$ according to (4.5.16). Hence, the parameters θ, ω locally also are smooth coordinates on $\mathcal{S}$ at the points with $a' = a'(C) \neq 0$ since $\omega' = 2\varkappa\varkappa' = aa'C \neq 0$ then, see Fig. 2.*

The stationary orbits is a trajectory $\psi_{\omega(t)}(x)e^{i\theta(t)} = Ce^{-\sqrt{\omega(t)}|x|}e^{i\theta(t)}$, where the parameters satisfy the equation $\dot{\theta} = \omega$, $\dot{\omega} = 0$. The stationary orbit $t \mapsto e^{i\omega t}\psi_\omega(x)$ maps out in time an orbit $\theta \mapsto e^{i\theta}\psi_\omega(x)$ of the $U(1)$ symmetry group. This group acts on the Hilbert phase space $H^1(\mathbf{R})$ preserving the Hamiltonian (4.5.4) and the symplectic form (4.5.40); in other words the stationary orbits (4.5.10) are relative equilibria of the corresponding Hamiltonian system.

Let us denote $N(C) = \mathcal{Q}(\psi_\omega(x))$ with $\omega = \varkappa^2$, and $\varkappa = a(C^2)/2$ according to (4.5.16). It is easy to compute that $N(C) = C^2/\varkappa$. We now differentiate:

$$N'(C) = \frac{2C}{\varkappa} - \frac{C^2\varkappa'}{\varkappa^2}.$$

Differentiating the identity (4.5.16), we obtain $\varkappa' = a'C$. Thus, again by (4.5.16),

$$N'(C) = \frac{2C}{\varkappa}(1 - \frac{a'C^2}{a}) \neq 0$$

if $C > 0$, $a > 0$ and $a' \neq a/C^2$. Therefore noticing that $N'(C) = \omega'(C)\partial_\omega \mathcal{Q}(\psi_\omega)$ with $\omega'(C) = 2\varkappa\varkappa' = aa'C$, we obtain the following result

Lemma 4.5.6. *For $C > 0$, $a > 0$ we have*

$$\partial_\omega \mathcal{Q}(\psi_\omega) < 0 \quad \textit{if } a' \in (-\infty, 0) \cup (a/C^2, +\infty),$$

and

$$\partial_\omega \mathcal{Q}(\psi_\omega) > 0 \quad \textit{if } 0 < a' < a/C^2.$$

Remark 4.5.7. (i) Orbital stability of stationary orbits is a much studied subject (see [103, 104] for very general theorems in this area, and [166] for an approach more similar to that taken in this section). The standard condition for orbital stability ([103, 104]) for the present problem would read $\partial_\omega \mathcal{Q}(\psi_\omega) > 0$; this is expected to be a necessary and sufficient condition for orbital stability when the Hessian of the augmented Hamiltonian ([166]) has a single negative eigenvalue. In the present problem it can be easily calculated that this Hessian is non-negative when $a' < 0$ and thus the standard condition is not necessarily relevant if $a' < 0$. Indeed Theorem 4.5.9 asserts stability in the case $a' < 0$. Restricting to $a' > 0$, in which case the Hessian does have a single negative eigenvalue, the calculation above shows that orbital stability is expected to hold when $a' < a/C^2$. In this section we will work under the spectral condition (4.5.18) which, for $a' > 0$, is slightly stricter: it is imposed to ensure that the linearisation has no discrete spectrum except zero (which is always present on account of the circular symmetry of the problem). If $a/\sqrt{2}C^2 < a' < a/C^2$ there are two purely imaginary eigenvalues of the linearised operator. It is intended to treat this case in a later publication thus extending our proof of asymptotic stability to the entire range

$$-\infty < a' < a/C^2. \tag{4.5.17}$$

For $a' > a/C^2$ the linearised operator has a positive eigenvalue and the stationary orbit is linearly unstable.

(ii) It is explained at the end of Section 4.5.4 that (4.5.6) can be interpreted as saying the restriction of the symplectic form (4.5.40) to the tangent space to $\mathcal{S}$ is non-degenerate (i.e. $\mathcal{S}$ satisfies the condition to be a symplectic submanifold).

Definition 4.5.8. *We say the stationary orbit $\psi_\omega(x)e^{i\theta} = Ce^{-\sqrt{\omega}|x|+i\theta}$, $C > 0$ satisfies the spectral condition if $\omega > 0$ and (cf. Remark 4.5.5)*

$$a'(C^2) \in (-\infty, 0) \cup (0, a(C^2)/(\sqrt{2}C^2)). \tag{4.5.18}$$

Let us denote by $W(t)$ the dynamical group of the free Schrödinger equation: $W(t)f$ is defined by the Fourier representation for all tempered distributions f. Our main theorem is the following:

Theorem 4.5.9. *Let conditions (4.5.3), (4.5.5) and (4.5.6) hold, $\beta \geq 2$ and $\psi(x,t) \in C(\mathbb{R}, H^1)$ be the solution to the equation (4.5.1) with initial state $\psi(0) \in H^1 \cap L^1_\beta$ which is close to a stationary orbit $\psi_{\omega_0}e^{i\theta_0} = C_0e^{-\sqrt{\omega_0}|x|+i\theta_0}$ with $C_0 > 0$ and $\omega_0 > 0$:*

$$d := \|\psi(0) - \psi_{\omega_0}e^{i\theta_0}\|_{H^1\cap L^1_\beta} \ll 1. \tag{4.5.19}$$

Assume further that the spectral condition (4.5.18) holds for the stationary orbit with $C = C_0$. Then for sufficiently small $d > 0$ the solution admits the following asymptotics:

$$\psi(\cdot, t) = \psi_{\omega_\pm}e^{i\omega_\pm t} + W(t)\Phi_\pm + r_\pm(t), \quad t \to \pm\infty, \tag{4.5.20}$$

where $\Phi_\pm \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ are the corresponding asymptotic scattering states, and

$$\|r_\pm(t)\|_{C_b(\mathbb{R})\cap L^2(\mathbb{R})} = \mathcal{O}(|t|^{-1/2}), \quad t \to \pm\infty. \tag{4.5.21}$$

Remark 4.5.10. It is possible to derive further information about the structure of $\Phi_\pm$ and $r_\pm(t)$ as discussed towards the end of section 10.

4.5.4 Linearisation on the stationary orbit

As the first step in the proof of main theorem, let us linearise the nonlinear Schrödinger equation (4.5.1) on a stationary orbit $e^{i(\omega t+\theta)}\psi_\omega(x)$, with $\psi_\omega(x) = Ce^{-\varkappa|x|}$ where $\varkappa = \sqrt{\omega} > 0$ and $C > 0$. Substituting

$$\psi(x,t) = e^{i(\omega t+\theta)}(\psi_\omega(x) + \chi(x,t)) \tag{4.5.22}$$

to (4.5.1), we obtain,

$$-\omega\chi(x,t)+i\dot\chi(x,t)=-\chi''(x,t)-\delta(x)[F(C+\chi(0,t))-F(C)] \tag{4.5.23}$$

Use the representation (4.5.7) to write

$$\begin{aligned}F(C+\chi)-F(C) &= a(|C+\chi|^2)(C+\chi)-a(|C|^2)C\\ &= a((C+\chi)(\overline{C}+\overline{\chi}))(C+\chi)-a(|C|^2)C\\ &= a(|C|^2)\chi+a'(|C|^2)C(C\overline{\chi}+\overline{C}\chi)+\mathcal{O}(|\chi|^2)\\ &= a(C^2)\chi+a'(C^2)C^2(\overline{\chi}+\chi)+\mathcal{O}(|\chi|^2)\end{aligned} \tag{4.5.24}$$

since $C\geq 0$. Hence, the first order part of (4.5.23) is given by

$$\begin{aligned}i\dot\chi(x,t) &= -\chi''(x,t)+\omega\chi(x,t)\\ &\quad -\delta(x)[a(C^2)\chi(0,t)+a'(C^2)C^2 2\mathrm{Re}\,\chi(0,t)].\end{aligned} \tag{4.5.25}$$

Now it is evident that the first order part is not linear over the complex field. On the other hand, it is linear over the real field. Hence, it would be useful to rewrite (4.5.25) in the real form. Namely, identify $\chi=\chi_1+i\chi_2\in\mathbb{C}$ with the real vector $(\chi_1,\chi_2)\in\mathbb{R}^2$ and denote it again by χ. Then (4.5.25) becomes the system

$$\begin{aligned}j\dot\chi(x,t) &= -\chi''(x,t)+\omega\chi(x,t)\\ &\quad -\delta(x)[a(C^2)E+2a'(C^2)C^2P_1]\chi(0,t),\end{aligned} \tag{4.5.26}$$

where E is the unit 2×2-matrix, P_1 is the projector in $\mathbb{R}^2$ acting as $\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}\mapsto\begin{pmatrix}\chi_1\\ 0\end{pmatrix}$ and j is the 2×2 matrix

$$j=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{4.5.27}$$

Respectively, we also rewrite (4.5.1) in the real form

$$j\dot\psi(x,t)=-\psi''(x,t)-\delta(x)\mathbf{F}(\psi(0,t)), \tag{4.5.28}$$

as an equation for $\mathbb{R}^2$-valued function $\psi(x,t)$ with $\mathbf{F}(\psi)\in\mathbb{R}^2$ which is the real vector version of $F(\psi)\in\mathbb{C}$. Then the linearisation (4.5.26) reads as the system

$$j\dot\chi(x,t)=-\chi''(x,t)+\omega\chi(x,t)-\delta(x)\mathbf{F}'((C,0))\chi(0,t), \tag{4.5.29}$$

where $\mathbf{F}'$ is the differential of the map $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$,

$$\mathbf{F}'((C,0))=aE+bP_1,\qquad a:=a(C^2),\quad b:=2a'(C^2)C^2. \tag{4.5.30}$$

In order to apply the Laplace transform the next step is to complexify the system (4.5.29) i.e. to consider it as a system of equations for the complex functions

$\chi_1(x,t), \chi_2(x,t)$, so $\chi(x,t) \in \mathbb{C}^2$ for any fixed (x,t). This gives a system which is linear over the complex field allowing application of the Laplace transform. To write this system more concisely let us denote the linear operator

$$\mathbf{B} := -\frac{d^2}{dx^2} + \omega - \delta(x)\mathbf{F}'((C,0)) = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{D}_1 &= -\frac{d^2}{dx^2} + \omega - \delta(x)[a+b], \\ \mathbf{D}_2 &= -\frac{d^2}{dx^2} + \omega - \delta(x)a. \end{aligned} \tag{4.5.31}$$

The system (4.5.29) then reads as

$$\dot{\chi}(x,t) = \mathbf{C}\chi(x,t), \qquad \mathbf{C} := j^{-1}\mathbf{B} = \begin{pmatrix} 0 & \mathbf{D}_2 \\ -\mathbf{D}_1 & 0 \end{pmatrix}. \tag{4.5.32}$$

Theorem 4.5.2 generalises to the equation (4.5.32): the equation admits unique solution $\chi(x,t) \in C_b(\mathbb{R}, H^1)$ for every initial function $\chi(x,0) = \chi_0 \in H^1$. Denote by $e^{\mathbf{C}t}$ the dynamical group of equation (4.5.32) acting in the space H^1.

4.5.5 Laplace transform

Equation (4.5.32) can be solved by the Laplace transform $\tilde{\chi}(x,\omega) := \int_0^\infty e^{-\lambda t}\chi(x,t)dt$. The Laplace transform is analytic function in the complex halfplane $\operatorname{Re}\lambda > 0$ with the values in H^1 since the solution is bounded in H^1. This implies that the resolvent $\mathbf{R}(\lambda) := (\mathbf{C}-\lambda)^{-1}$ is also analytic for $\operatorname{Re}\lambda > 0$, with values in the space of bounded operators on H^1. From the inversion of the Laplace transform we obtain

$$e^{\mathbf{C}t} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\mathbf{R}(\lambda+\varepsilon)\, d\lambda, \qquad t > 0, \tag{4.5.33}$$

for any $\varepsilon > 0$, where the integral converges in the sense of distributions of $t \in \mathbb{R}$.

We assume that the spectral condition (4.5.18) holds from now on. Then the resolvent admits analytic continuation from $\operatorname{Re}\lambda > 0$ to the complex plain with the cuts $\mathcal{C}_+ = [i\omega, i\infty)$, $\mathcal{C}_- = (-i\infty, -i\omega]$, and with the pole of order two at $\lambda = 0$ as detailed in Section 4.5.13. Furthermore, for $\lambda \in \mathcal{C}_+ \cup \mathcal{C}_-$, the resolvent $\mathbf{R}(\lambda \pm \varepsilon)$ has right and left limits $\mathbf{R}(\lambda \pm 0)$ as $\varepsilon \to 0$. Then (4.5.33) implies that for any $r \in (0,\omega)$

$$e^{\mathbf{C}t} = -\frac{1}{2\pi i}\int_{|\lambda|=r} e^{\lambda t}\mathbf{R}(\lambda)\, d\lambda - \frac{1}{2\pi i}\int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}\big(\mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0)\big)\, d\lambda \tag{4.5.34}$$

by the Cauchy theorem. Setting $t = 0$, we obtain that

$$1 = -\frac{1}{2\pi i}\int_{|\lambda|=r} \mathbf{R}(\lambda)\, d\lambda - \frac{1}{2\pi i}\int_{\mathcal{C}_+\cup\mathcal{C}_-} \big(\mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0)\big)\, d\lambda = \mathbf{P}^0 + \mathbf{P}^c, \tag{4.5.35}$$

where $\mathbf{P}^0$ and $\mathbf{P}^c$ stands for the corresponding Riesz projections (see [17]) onto, respectively, the generalised null space of $\mathbf{C}$, and onto the continuous spectral subspace. We will show in the next section that $\mathbf{P}^0$ is the symplectic projection, and therefore, $\mathbf{P}^c$ is also the symplectic projection. The projectors $\mathbf{P}^0$, $\mathbf{P}^c$ commute with $\mathbf{C}$ and with the group $e^{\mathbf{C}t}$. Let us note that

$$\left\{ \begin{array}{lcl} \mathbf{P}^0 e^{\mathbf{C}t} & = & -\frac{1}{2\pi i} \displaystyle\int\limits_{|\lambda|=r} e^{\lambda t}\mathbf{R}(\lambda)\, d\lambda, \\ \mathbf{P}^c e^{\mathbf{C}t} & = & -\frac{1}{2\pi i} \displaystyle\int\limits_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}\big(\mathbf{R}(\lambda+0)-\mathbf{R}(\lambda-0)\big)\, d\lambda \end{array} \right| . \tag{4.5.36}$$

The first equation holds since both sides are one-parameter groups of operators , and their derivatives at $t = 0$ coincide. The second equation follows from (4.5.34) and the fact that $1 = \mathbf{P}^0 + \mathbf{P}^c$ by (4.5.35). Therefore, (4.5.34) becomes

$$e^{\mathbf{C}t} = \mathbf{P}^0 e^{\mathbf{C}t} + \mathbf{P}^c e^{\mathbf{C}t}. \tag{4.5.37}$$

4.5.6 Invariant subspace of discrete spectrum

Here we prove that $\mathbf{P}^0$ is the symplectic projection onto the tangent space of the solitary manifold $\mathcal{S}$ at the stationary orbit $e^{j\theta}\psi_\omega$. The real form of the stationary orbit is $e^{j\theta}\Phi_\omega$ where $\Phi_\omega = (\psi_\omega(x), 0)$. The tangent space to $\mathcal{S}$ at the point $e^{j\theta}\Phi_\omega$ with parameters ω, θ is the linear span of the derivatives with respect to θ and ω cf. Remark 4.5.5) i.e.

$$T_{\omega,\theta}\mathcal{S} \equiv \text{linear span}\Big\{ je^{j\theta}\Phi_\omega(x), e^{j\theta}\partial_\omega\Phi_\omega(x)\Big\}.$$

Notice that the operator $\mathbf{C}$ corresponds to $\theta = 0$ since we have extracted the phase factors $e^{i\theta}$ from the solution in the process of linearisation (4.5.22). The tangent space to $\mathcal{S}$ at the point Φ_ω with parameters $(\omega, 0)$ is spanned by the vectors

$$T_0(\omega) := j\Phi_\omega, \quad T_1(\omega) := \partial_\omega\Phi_\omega. \tag{4.5.38}$$

Observe that (4.5.11) and its derivative in ω give the following identities:

$$\mathbf{D}_2\psi_\omega = 0 \qquad \mathbf{D}_1(\partial_\omega\psi_\omega) = -\psi_\omega. \tag{4.5.39}$$

These formulae imply that the vectors T_0 and T_1 lie in the generalised null space of the non-self-adjoint operator $\mathbf{C}$ defined in (4.5.32) and in fact Theorem 4.5.38 *ii*) implies:

Lemma 4.5.11. *Let the spectral condition (4.5.18) hold. Then the generalised null space of* $\mathbf{C}$ *is two dimensional, is spanned by* T_0, T_1*, and*

$$\mathbf{C}T_0 = 0 \qquad \mathbf{C}T_1 = T_0.$$

We also introduce the symplectic form Ω for the real vectors ψ and η by the integral

$$\Omega(\psi, \eta) = \int \langle j\psi, \eta\rangle dx = \int \big(\psi_1\eta_2 - \psi_2\eta_1\big) dx, \tag{4.5.40}$$

where $\langle \cdot, \cdot, \rangle$ stands for the scalar product in $\mathbb{R}^2$. Since $a' \neq a/C^2$ then by Lemma 4.5.6

$$\boldsymbol{\mu_\omega} = -\Omega(T_0, T_1) = \frac{1}{2}\partial_\omega \int |\psi_\omega|^2 dx \neq 0. \tag{4.5.41}$$

Hence, the symplectic form Ω is nondegenerate on the tangent space $T_{\omega,0}\mathcal{S}$, i.e. $T_{\omega,0}\mathcal{S}$ is a symplectic subspace. Therefore, there exists a symplectic projection operator from $L^2(\mathbb{R})$ onto $T_{\omega,0}\mathcal{S}$.

Lemma 4.5.12. *The operator $\mathbf{P}^0$, defined in (4.5.35), is precisely the symplectic projection from $L^2(\mathbb{R})$ onto $T_{\omega,0}\mathcal{S}$, and, furthermore, it may be represented by the formula*

$$\mathbf{P}^0\psi = b_0 T_0 + b_1 T_1 \quad \textit{with} \quad -\boldsymbol{\mu_\omega} b_0 = \Omega(\psi, T_1), \quad \boldsymbol{\mu_\omega} b_1 = \Omega(\psi, T_0). \tag{4.5.42}$$

Proof. The coincidence of both definition (4.5.35) and (4.5.42) of operator $\mathbf{P}^0$ follows by the Cauchy residue theorem from the formulas (4.5.45)-(4.5.47) for the resolvent. □

Corollary 4.5.13. $\mathbf{P}^c = 1 - \mathbf{P}^0$ *is also symplectic projection.*

Remark 4.5.14. *Since $T_0(\omega), T_1(\omega)$ lie in $H^1(\mathbb{R})$ the operator $\mathbf{P}^0$ extends uniquely to define a continuous linear map $H^{-1}(\mathbb{R}) \to T_{\omega,0}\mathcal{S}$, which is still designated $\mathbf{P}^0$. In particular this operator can be applied to the Dirac measure $\delta(x)$.*

Using the Taylor expansion for the $e^{\lambda t}$ at $\lambda = 0$ and the identity $\lambda\mathbf{R}(\lambda) = \mathbf{C}\mathbf{R}(\lambda) - 1$, we obtain by (4.5.36)

$$\mathbf{P}^0 e^{\mathbf{C}t} = (1 + \mathbf{C}t)\mathbf{P}^0 \tag{4.5.43}$$

Remark 4.5.15. *On the generalised null space itself $\mathbf{C}^2 = 0$ by Lemma 4.5.11 and so the group $e^{t\mathbf{C}}$ reduces to $1 + \mathbf{C}t$ as usual for the exponential of the nilpotent part of an operator.*

4.5.7 Time decay in continuous spectrum

From formulas (4.5.37, (4.5.43) we see that the solutions $\chi(t) = e^{\mathbf{C}t}\chi_0$ of the linearised equation (4.5.32) do not decay as $t \to \infty$ if $\mathbf{P}^0\chi_0 \neq 0$. On the other hand, we do expect time decay of $\mathbf{P}^c\chi(t)$, as a consequence of the Laplace representation (4.5.36) for $\mathbf{P}^c e^{\mathbf{C}t}$:

$$\mathbf{P}^c e^{\mathbf{C}t} = -\frac{1}{2\pi i} \int\limits_{\mathcal{C}_+ \cup \mathcal{C}_-} e^{\lambda t}\big(\mathbf{R}(\lambda + 0) - \mathbf{R}(\lambda - 0)\big)\, d\lambda. \tag{4.5.44}$$

The decay for the oscillatory integral is obtained from the analytic properties of $\mathbf{R}(\lambda)$ for $\lambda \in \mathcal{C}_+ \cup \mathcal{C}_-$. The resolvent $\mathbf{R}(\lambda)$ is an integral operator with matrix-valued integral kernel

$$\mathbf{R}(\lambda, x, y) = \Gamma(\lambda, x, y) + P(\lambda, x, y), \tag{4.5.45}$$

where the columns of matrices Γ and P are given in (4.5.134), (4.5.135), (4.5.137), (4.5.138):

$$\Gamma(\lambda, x, y) = \begin{pmatrix} \dfrac{1}{4k_+} & -\dfrac{1}{4k_-} \\ \dfrac{i}{4k_+} & \dfrac{i}{4k_-} \end{pmatrix} \begin{pmatrix} e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)} & -i\big(e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)}\big) \\ e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)} & i\big(e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)}\big) \end{pmatrix}, \tag{4.5.46}$$

$$P(\lambda,x,y)=\frac{1}{2D}\begin{pmatrix} e^{ik_+|x|} & e^{ik_-|x|} \\ ie^{ik_+|x|} & -ie^{ik_-|x|}\end{pmatrix}\begin{pmatrix} i\alpha-2k_- & i\beta \\ -i\beta & -i\alpha+2k_+\end{pmatrix}\begin{pmatrix} e^{ik_+|y|} & -ie^{ik_+|y|} \\ e^{ik_-|y|} & ie^{ik_-|y|}\end{pmatrix}. \quad (4.5.47)$$

Here $k_\pm(\lambda) = \sqrt{-\omega \mp i\lambda}$ is the square root defined with cuts in the complex λ plane so that $k_\pm(\lambda)$ is an analytic function on $\mathbb{C}\setminus\mathcal{C}_\pm$ and $\operatorname{Im} k_\pm(\lambda) > 0$ for $\lambda \in \mathbb{C}\setminus\mathcal{C}_\pm$. The constants α, β and $D = D(\lambda)$ are given by the formulas

$$\alpha = a + b/2,\ \beta = b/2,\ D = 2i\alpha(k_+ + k_-) - 4k_+k_- + \alpha^2 - \beta^2.$$

Recall from Section 4.5.13 that $D(\lambda) \neq 0$ for $\lambda \in \mathcal{C}_+ \cup \mathcal{C}_-$. Clearly in order to understand the decay of $\mathbf{P}^c e^{t\mathbf{C}}$, it is crucial to study the behaviour of $\mathbf{R}(\lambda,x,y)$ near the branch points $\lambda = \pm i\omega$ (where $k_\pm$ vanish).

We deduce time decay for the group $\mathbf{P}^c e^{t\mathbf{C}}$ by means of the following version of Lemma 10.2 from [184], which is itself based on Zygmund's lemma [192, p.45].

Let $\mathcal{F} : [0,\infty) \to \mathbf{B}$ be a C^2 function with values in a Banach space $\mathbf{B}$. Let us define the $\mathbf{B}$-valued function

$$I(t) = \int_0^\infty e^{-it\zeta}\mathcal{F}(\zeta)\, d\zeta.$$

Lemma 4.5.16. *Suppose that $\mathcal{F}(0) = 0$, and for some $\delta > 0$*

$$\mathcal{F}'' \in L^1(\delta,\infty;\mathbf{B}), \quad (4.5.48)$$

and

$$\mathcal{F}''(\zeta) = \mathcal{O}(\zeta^{p-2}), \quad \zeta \downarrow 0 \quad (4.5.49)$$

in the norm of $\mathbf{B}$ *for some $p \in (0,1)$. Then $I(t) \in C_b(\varepsilon,\infty;\mathbf{B})$ for any $\varepsilon > 0$, and*

$$I(t) = \mathcal{O}(t^{-1-p}) \quad \text{as} \quad t \to \infty$$

in the norm of $\mathbf{B}$.

For $\beta \geq 2$ let us introduce a Banach space $\mathcal{M}_\beta$, which is the subset of distributions which are linear combinations of L^1_β functions and multiples of the Dirac distribution at the origin with the norm:

$$\|\psi + C\delta(x)\|_{\mathcal{M}_\beta} := \|\psi\|_{L^1_\beta} + |C|. \quad (4.5.50)$$

We will apply Lemma 4.5.16 to the function $\mathcal{F}(\lambda) = \mathbf{R}(\lambda+0) - \mathbf{R}(\lambda-0)$ with values in the Banach space $\mathcal{B} = B(\mathcal{M}_\beta, L^\infty_{-\beta})$, the space of continuous linear maps $\mathcal{M}_\beta \to L^\infty_{-\beta}$ for any $\beta \geq 2$.

Theorem 4.5.17. *Assume that the spectral condition (4.5.18) holds so that $\lambda = 0$ is the only point in the discrete spectrum of the operator $\mathbf{C} = \mathbf{C}(\omega)$. Then for $\beta \geq 2$*

$$\|\mathbf{P}^c e^{\mathbf{C}t}\|_{\mathcal{B}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \quad (4.5.51)$$

First we use the formulas (4.5.44) and (4.5.45) to obtain

$$-2\pi i\mathbf{P}^c e^{\mathbf{C}t} = \int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(\Gamma(\lambda+0)-\Gamma(\lambda-0))\,d\lambda + \int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(P(\lambda+0)-P(\lambda-0))\,d\lambda. \quad (4.5.52)$$

Next we apply Lemma 4.5.16 to each summand in the RHS of (4.5.52) separately. Then Theorem 4.5.17 immediately follows from the two lemmas below.

Lemma 4.5.18. *If the assumption of Theorem 4.5.17 hold then*

$$\int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(\Gamma(\lambda+0)-\Gamma(\lambda-0))\,d\lambda = \mathcal{O}(t^{-3/2}),\quad t\to\infty \tag{4.5.53}$$

in the norm $\mathcal{B}$.

Proof. We consider only the integral over $\mathcal{C}_+$ since the integral over $\mathcal{C}_-$ can be handled in the same way. The point $\lambda = i\omega$ is the branch point for k_+, therefore, if $\lambda\in\mathcal{C}_+$ then since k_- is continuous across $\mathcal{C}_+$

$$\Gamma(\lambda+0)-\Gamma(\lambda-0)=\Gamma^+(\lambda+0)-\Gamma^+(\lambda-0),$$

where Γ^+ is the sum of those terms in Γ which involve k_+. Let us consider, for example, Γ^+_{11}. The expression (4.5.46) implies for $y>0$ that

$$\Gamma^+_{11}(\lambda,x,y)=\begin{cases}0, & x\le 0,\\[2mm] \dfrac{e^{ik_+y}(e^{-ik_+x}-e^{ik_+x})}{4k_+}, & 0\le x\le y,\\[2mm] \dfrac{e^{ik_+x}(e^{-ik_+y}-e^{ik_+y})}{4k_+}, & x\ge y.\end{cases}$$

For $\lambda\in\mathcal{C}_+$, the root $k_+=\sqrt{-\omega-i\lambda}$ is real, and $k_+(\lambda+0)=-k_+(\lambda-0)$. Then, for $y>0$,

$$\Gamma^+_{11}(\lambda+0,x,y)-\Gamma^+_{11}(\lambda-0,x,y) \;=\; -\Theta(x)\frac{\sin(\sqrt{\zeta}|x|)\sin(\sqrt{\zeta}|y|)}{\sqrt{\zeta}}, \tag{4.5.54}$$

where $\zeta=-\omega-i\lambda$, and $\Theta(x)=1$ for $x>0$ and zero otherwise. The second derivative of the function $f(\zeta)=\dfrac{\sin(\sqrt{\zeta}|x|)\sin(\sqrt{\zeta}|y|)}{\sqrt{\zeta}}$ satisfies

$$|f''(\zeta)| = |-\frac{\sin(\sqrt{\zeta}|x|)\sin(\sqrt{\zeta}|y|)(|x|^2+|y|^2)}{4\zeta\sqrt{\zeta}}+\frac{2\cos(\sqrt{\zeta}|x|)\cos(\sqrt{\zeta}|y|)|x||y|}{4\zeta\sqrt{\zeta}}$$
$$-\frac{\sin(\sqrt{\zeta}|x|)\cos(\sqrt{\zeta}|y|)|y|+\cos(\sqrt{\zeta}|x|)\sin(\sqrt{\zeta}|y|)|x|}{2\zeta^2}| \le \frac{C(1+|x|^2)(1+|y|^2)}{\zeta\sqrt{\zeta}}.$$

For $y<0$ an identical calculation leads to the same bound. Therefore the operator valued function $\mathcal{F}(\zeta)=\Gamma^+_{11}(\lambda+0)-\Gamma^+_{11}(\lambda-0)$ satisfies the conditions (4.5.48) and (4.5.49) of Lemma 4.5.16 with $\zeta=-\omega-i\lambda$, $p=1/2$ and $\mathbf{B}=\mathcal{B}$. □

Next we consider the second summand on the RHS of (4.5.52).

Lemma 4.5.19. *In the situation of Theorem 4.5.17*

$$\int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(P(\lambda+0)-P(\lambda-0))\,d\lambda = \mathcal{O}(t^{-3/2}), \tag{4.5.55}$$

in the norm $\mathcal{B}$.

Proof. We consider only the integral over $\mathcal{C}_+$ and one component of the matrix P, for example, P_{11}:

$$P_{11}(\lambda,x,y)=\frac{(i\alpha-2k_-)e^{ik_+(|x|+|y|)}+(-i\alpha+2k_+)e^{ik_-(|x|+|y|)}+i\beta(e^{ik_-|y|+ik_+|x|}-e^{ik_+|y|+ik_-|x|})}{2i\alpha(k_++k_-)-4k_+k_-+\alpha^2-\beta^2}.$$

Denote $\zeta=-\omega-i\lambda$, then $k_+=\sqrt{\zeta}$, $k_-=\sqrt{-2\omega-\zeta}$. A Taylor expansion in $\sqrt{\zeta}$ as $\zeta\to 0$, $\quad\mathrm{Im}\,\zeta\geq 0$ implies

$$P_{11}(\lambda,x,y)=P_0+P_1(x,y)\zeta^{1/2}+P_2(x,y)\mathcal{O}(\zeta),$$

where $|P_j(x,y)|\leq C_j(1+|x|^j)(1+|y|^j)$, $j=1,2$. Therefore, if $\lambda\in\mathcal{C}_+$ then

$$\mathcal{F}(\zeta)=P_{11}(\lambda+0)-P_{11}(\lambda-0)=\mathcal{O}(\zeta^{1/2}),\quad \zeta\to 0$$

in the norm of $\mathcal{B}$. Similarly, differentiating two times the function $P_{11}(\lambda,x,y)$ in λ, we obtain that

$$\mathcal{F}''(\zeta)=-P''_{11}(\lambda+0)+P''_{11}(\lambda-0)=\mathcal{O}(\zeta^{-3/2}),\quad i,j=1,2,\quad \zeta\to 0$$

in the norm of $\mathcal{B}$. Moreover, $\mathcal{F}''(\zeta)\sim\zeta^{-3/2}$ as $\zeta\to\infty$. Therefore, the function $\mathcal{F}(\zeta)$ satisfies the conditions (4.5.48) and (4.5.49) of Lemma 4.5.16 with $p=1/2$ and $\mathbf{B}=\mathcal{B}$. □

4.5.8 Bounds for small times

As a starting point for the method of majorants in Section 4.5.10 we will need also some estimates on the dynamical group $e^{\mathbf{C}t}$ for small t. First note that the function $e^{\mathbf{C}t}\chi_0$ belongs to $C_b(\mathbb{R},H^1)$. This follows from a theorem analogous to Theorem 4.5.2 for solutions $e^{\mathbf{C}t}\chi_0$ of the linearised equation (4.5.32), with initial condition $\chi_0\in H^1$. Moreover, energy and charge conservation imply that

$$\|e^{\mathbf{C}t}\chi_0\|_{H^1}\leq c\|\chi_0\|_{H^1},\quad t\in\mathbb{R}. \tag{4.5.56}$$

For a further application in section 4.5.11 we need a bound for the action of $e^{\mathbf{C}t}$ on the Dirac distribution $\delta=\delta(x)$.

Thus let $\chi_\delta(x,t)$ be the solution to the linearised equation (4.5.25) with $\chi_\delta(x,0)=\delta(x)$ and $e^{\mathbf{C}t}\delta$ its real vector version. Note that, by Theorem 4.5.17, we have $e^{\mathbf{C}t}\delta\in C_b(\varepsilon,\infty;L^\infty_{-\beta})$, for every $\varepsilon>0$, and $\beta\geq 2$. The next lemma gives the small t behaviour:

Lemma 4.5.20. *The following bound holds*

$$\|e^{\mathbf{C}t}\delta\|_{L^\infty}=\mathcal{O}(t^{-1/2}),\quad t\to 0. \tag{4.5.57}$$

Proof. By the Duhamel representation for the solution to (4.5.25), we obtain

$$\chi_\delta(x,t)=W_\omega(t)\delta-\int_0^t ds\Big(a\chi_\delta(0,s)+b\mathrm{Re}\,(\chi_\delta(0,s)\Big)W_\omega(t-s)\delta \tag{4.5.58}$$

where a and b are defined by (4.5.30), and $W_\omega(t)$ is the dynamical group of the modified Schrödinger equation

$$i\dot{\chi}(x,t)=-\chi''(x,t)+\omega\chi(x,t). \tag{4.5.59}$$

Note that

$$W_\omega(t)\delta = \frac{1}{\sqrt{4\pi i t}}\ e^{i\frac{x^2}{4t}-i\omega t} \tag{4.5.60}$$

Therefore (4.5.58) becomes

$$\chi_\delta(x,t) = \frac{1}{\sqrt{4\pi i t}}\ e^{i\frac{x^2}{4t}-i\omega t} - \int\limits_0^t \frac{1}{\sqrt{4\pi i(t-s)}}\ e^{i\frac{x^2}{4(t-s)}-i\omega(t-s)}\Big(a\chi_\delta(0,s)+b\mathrm{Re}\,(\chi_\delta(0,s)\Big)ds. \tag{4.5.61}$$

Denote $\varsigma(x,t) = \sqrt{t}\ \chi_\delta(x,t)$. Then

$$\varsigma(x,t) = \frac{1}{\sqrt{4\pi i}}\ e^{i\frac{x^2}{4t}-i\omega t} - \sqrt{t}\int\limits_0^t \frac{1}{\sqrt{4\pi i(t-s)s}}\ e^{i\frac{x^2}{4(t-s)}-i\omega(t-s)}\Big(a\varsigma(0,s)+b\mathrm{Re}\,(\varsigma(0,s)\Big)ds. \tag{4.5.62}$$

Therefore,

$$\|\varsigma(t)\|_{L^\infty} \le \frac{1}{2\sqrt{\pi}} + \frac{1}{2}\sqrt{\pi t}(|a|+|b|)\int\limits_0^t \frac{1}{\pi\sqrt{(t-s)s}}\ \|\varsigma(s)\|_{L^\infty}ds, \qquad t>0. \tag{4.5.63}$$

Since $\displaystyle\int_0^t \frac{ds}{\pi\sqrt{(t-s)s}} = 1$, we obtain the bound

$$\|\varsigma(t)\|_{L^\infty} \le \frac{1}{2\sqrt{\pi}}\frac{1}{1-\frac{1}{2}\sqrt{\pi t}(|a|+|b|)}$$

if t is sufficiently small. □

4.5.9 Modulation equations

In this section we present the modulation equations which allow a construction of solutions $\psi(x,t)$ of the equation (4.5.1) close at each time t to a soliton i.e. to one of the functions

$$Ce^{i\theta-\sqrt{\omega}|x|}, \qquad C=C(\omega)>0$$

in the set $\mathcal{S}$ described in section 4.5.3 with time varying ('modulating') parameters $(\omega,\theta) = (\omega(t),\theta(t))$. It will be assumed that $\psi(x,t)$ is a given weak solution of (4.5.1) as provided by Theorem 4.5.2, so that the map $t\to\psi(\,\cdot\,,t)$ is continuous into $H^1(\mathbb{R})$. The modulation equations follow from the ansatz for the solution which is explained next. Recall that we defined

$$\Phi_\omega(x) \equiv \big(Ce^{-\sqrt{\omega}|x|},0\big) = \big(\psi_\omega,0\big) \tag{4.5.64}$$

so that $\psi(x,t) = e^{j\theta(t)}\Phi_{\omega(t)}(x)$ is a solution of (4.5.28) if and only if $\dot\theta=\omega$ and $\dot\omega=0$. Here it is to be understood that $C=C(\omega(t))$ is determined from $\omega(t)$ via (4.5.16). We look for a solution to (4.5.28) in the form

$$\psi(x,t) = e^{j\theta(t)}\big(\Phi_{\omega(t)}(x)+\chi(x,t)\big) = e^{j\theta(t)}\Psi(x,t), \quad \Psi(x,t) = \Phi_{\omega(t)}(x)+\chi(x,t). \tag{4.5.65}$$

Since this is a solution of (4.5.28) as long as $\chi \equiv 0$ and $\dot{\theta} = \omega$ and $\dot{\omega} = 0$ it is natural to look for solutions in which χ is small and

$$\theta(t) = \int_0^t \omega(s)ds + \gamma(t)$$

with γ treated perturbatively. Observe that so far this representation is underdetermined since for any $\big(\omega(t), \theta(t)\big)$ it just amounts to a definition of χ; it is made unique by restricting $\chi(t)$ to lie in the image of the projection operator onto the continuous spectrum $\mathbf{P}^c_t = \mathbf{P}^c(\omega(t))$ or equivalently that

$$\mathbf{P}^0_t \chi(t) = 0, \quad \mathbf{P}^0_t = \mathbf{P}^0(\omega(t)) = I - \mathbf{P}^c(\omega(t)) \tag{4.5.66}$$

(The projection operators are as defined in (4.5.35) and (4.5.42)). An equivalent formulation of (4.5.66) is to say that $e^{j\theta}\chi$ is required to lie in the symplectic normal space $N_{\omega(t),\theta(t)}\mathcal{S}$. This is equivalent to imposition of the following orthogonality conditions (at each time t):

$$\Omega(\chi(t), T_0(\omega(t)) = 0 = \Omega(\chi(t), T_1(\omega(t)), \tag{4.5.67}$$

where Ω is the symplectic form introduced previously. Writing $\chi(t) = (\chi_1(t), \chi_2(t))$ the orthogonality conditions reduce to

$$\int \chi_1(x,t) C e^{-\sqrt{\omega}|x|}\, dx = 0, \quad \text{and} \quad \int \chi_2(x,t)\partial_\omega (C e^{-\sqrt{\omega}|x|})\, dx = 0. \tag{4.5.68}$$

Now we give a system of *modulation equations* for $\omega(t)$, $\gamma(t)$ which ensure the conditions (4.5.68) are preserved by the time evolution.

Lemma 4.5.21. *(i) Assume given a solution of (4.5.28) with regularity as described in theorem 4.5.2, which can be written in the form (4.5.65)–(4.5.66) with continuously differentiable $\omega(t)$, $\theta(t)$. Then*

$$\dot{\chi} = \mathbf{C}\chi - \dot{\omega}\partial_\omega \Phi_\omega + \dot{\gamma} j^{-1}(\Phi_\omega + \chi) + \mathbf{Q} \tag{4.5.69}$$

where $\mathbf{Q}(\chi, \omega) = -\delta(x) j^{-1}\big(\mathbf{F}(\Phi_\omega + \chi) - \mathbf{F}(\Phi_\omega) - \mathbf{F}'(\Phi_\omega)\chi\big)$, and

$$\dot{\omega} = \frac{\langle \mathbf{P}^0 \mathbf{Q}, \Psi\rangle}{\langle \partial_\omega \Phi_\omega - \partial_\omega \mathbf{P}^0 \chi, \Psi\rangle} \tag{4.5.70}$$

$$\dot{\gamma} = \frac{\langle j\mathbf{P}^0(\partial_\omega \Phi_\omega - \partial_\omega \mathbf{P}^0\chi), \mathbf{P}^0\mathbf{Q}\rangle}{\langle \partial_\omega \Phi_\omega - \partial_\omega \mathbf{P}^0 \chi, \Psi\rangle}, . \tag{4.5.71}$$

where $\mathbf{P}^0 = \mathbf{P}^0(\omega(t))$ is the projection operator defined in (4.5.42) and $\partial_\omega \mathbf{P}^0 = \partial_\omega \mathbf{P}^0(\omega)$ evaluated at $\omega = \omega(t)$.

(ii) Conversely given ψ a solution of (4.5.28) as in theorem 4.5.2 and continuously differentiable functions $\omega(t)$, $\theta(t)$ which satisfy (4.5.70)–(4.5.71) then χ defined by (4.5.65) satisfies (4.5.69) and the condition (4.5.66) holds at all times if it holds initially.

Proof. This can be proved as in [112, Prop.2.2]. □

It remains to show, for appropriate initial data close to a soliton, that there exist solutions to (4.5.70)–(4.5.71), at least locally. To achieve this observe that if the spectral condition (4.5.18) holds then by Lemma 4.5.6 the denominator appearing on the right hand side of (4.5.70) and (4.5.71) does not vanish for small $\|\chi\|_{L^1_\beta}$. This is because

$$\langle \partial_\omega \psi_s, \psi_s \rangle = \frac{1}{2}\partial_\omega \int |\psi_\omega|^2 dx \neq 0 \tag{4.5.72}$$

as discussed in section 4.5.3. This has the consequence that the orthogonality conditions really can be satisfied for small χ because they are equivalent to a locally well posed set of ordinary differential equations for $t \to (\theta(t), \omega(t))$. This implies the following corollary:

Corollary 4.5.22. *(i) In the situation of (i) in the previous lemma assume that (4.5.72) holds. If $\|\chi\|_{L^p_\beta}$ is sufficiently small for some p, β the right hand sides of (4.5.70) and (4.5.71) are smooth in θ, ω and there exists continuous $\mathcal{R} = \mathcal{R}(\omega, \chi)$ such that*

$$|\dot\gamma(t)| \le \mathcal{R}|\chi(0,t)|^2, \qquad |\dot\omega(t)| \le \mathcal{R}|\chi(0,t)|^2.$$

(ii) Assume given ψ, a solution of (4.5.28) as in Theorem 4.5.2. If ω_0 satisfies (4.5.72) and $\chi(x,0) = e^{-j\theta_0}\psi(x,0) - \Phi_{\omega_0}(x)$ is small in some L^p_β norm and satisfies (4.5.66) there is a time interval on which there exist C^1 functions $t \mapsto \big(\omega(t), \gamma(t)\big)$ which satisfy (4.5.70)–(4.5.71).

4.5.10 Time decay for the transversal dynamics

In this section we state our Theorem 4.5.24 on the time decay of the transversal component $\chi(t)$ in the nonlinear setting, leaving the proof to the next section. Theorem 4.5.24 will be used to prove the main theorem in Section 4.5.12. First we represent the initial data ψ_0 in a convenient form for application of the modulation equations: the next Lemma will allow us to assume that (4.5.66) holds initially without loss of generality.

Lemma 4.5.23. *In the situation of Theorem 4.5.9 there exists a stationary orbit $\psi_{\tilde\omega_0} = \tilde C_0 e^{-\sqrt{\tilde\omega_0}|x|}$ satisfying the spectral condition (4.5.18) such that in vector form*

$$\psi_0(x) = e^{j\tilde\theta_0}(\Phi_{\tilde\omega_0}(x) + \chi_0(x)), \quad \Phi_{\tilde\omega_0} = (\psi_{\tilde\omega_0}, 0),$$

and for $\chi_0(x)$ we have

$$\mathbf{P}^0(\tilde\omega_0)(\chi_0) = 0, \tag{4.5.73}$$

and

$$\|\chi_0\|_{L^1_\beta \cap H^1} = \tilde d = O(d) \quad \textit{as } d \to 0.$$

Proof. By (4.5.67), the condition (4.5.73) is equivalent to the pair of equations

$$\Omega(e^{-j\tilde\theta_0}\psi_0 - \Phi_{\tilde\omega_0}, T_0(\tilde\omega_0)) = 0, \quad \Omega(e^{-j\tilde\theta_0}\psi_0 - \Phi_{\tilde\omega_0}, T_1(\tilde\omega_0)) = 0,$$

where $T_0(\omega) = j\Phi_\omega$, $T_1(\omega) = \partial_\omega\Phi_\omega$. For ψ_0 sufficiently close (in L^1_β) to $e^{j\theta_0}\Phi_{\omega_0}$ the existence of $\tilde\theta_0, \tilde\omega_0$ follows by a standard application of the implicit function theorem if we show that the Jacobian matrix

$$\begin{pmatrix} \partial_\omega\Omega(e^{-j\theta}\psi_0 - \Phi_\omega, j\Phi_\omega) & \partial_\omega\Omega(e^{-j\theta}\psi_0 - \Phi_\omega, \partial_\omega\Phi_\omega) \\ \partial_\theta\Omega(e^{-j\theta}\psi_0 - \Phi_\omega, j\Phi_\omega) & \partial_\theta\Omega(e^{-j\theta}\psi_0 - \Phi_\omega, \partial_\omega\Phi_\omega) \end{pmatrix}, \tag{4.5.74}$$

with $\psi_0 = e^{j\theta_0}\Phi_{\omega_0}$, is non-degenerate at $\omega = \omega_0$ and $\theta = \theta_0$. But this is equivalent to the non-degeneracy of the matrix

$$\begin{pmatrix} \Omega(\partial_\omega \Phi_{\omega_0}, j\Phi_{\omega_0}) & 0 \\ 0 & \Omega(j\Phi_{\omega_0}, \partial_\omega \Phi_{\omega_0}) \end{pmatrix} \tag{4.5.75}$$

which holds by (4.5.72). □

In Section 4.5.12 we will show that our main Theorem 4.5.9 can be derived from the following time decay of the transversal component $\chi(t)$:

Theorem 4.5.24. *Let all the assumptions of Theorem 4.5.9 hold. For d sufficiently small there exist C^1 functions $t \mapsto \big(\omega(t), \gamma(t)\big)$ defined for $t \geq 0$ such that the solution $\psi(x,t)$ of (4.5.28) can be written as in (4.5.65-4.5.66) with (4.5.70-4.5.71) satisfied, and there exists a number $\overline{M} > 0$, depending only on the initial data, such that*

$$M(T) = \sup_{0\leq t\leq T} [(1+t)^{3/2}\|\chi(t)\|_{L^\infty_{-\beta}} + (1+t)^3\big(|\dot\gamma| + |\dot\omega|\big)] \leq \overline{M}, \tag{4.5.76}$$

uniformly in $T > 0$, and $\overline{M} = O(d)$ as $d \to 0$.

Remarks 4.5.25. *(0) This theorem will be deduced from Proposition 4.5.26 in the next section.*

(i) Theorem 4.5.2 implies that the norms in the definition of M are continuous functions of time (and so M is also).

(ii) The result holds also for negative time with appropriate changes since $\psi(x,t)$ solves (4.5.1) if and only if $\overline{\psi}(x,-t)$ does.

(iii) The result implies in particular that $t^3|\dot\theta - \omega| + t^3|\dot\omega| \leq C$, hence $\omega(t)$ and $\theta(t) - t\omega_+$ should converge as $t \to \infty$ while $\psi(x,t) - e^{j\theta(t)}\Phi_{\omega(t)}(x)$ have limit zero in $L^\infty_{-\beta}(\mathbb{R})$.

(iv) The notation $\chi(t)$ indicates the function $x \mapsto \chi(x,t)$ as usual.

4.5.11 Decay in transversal directions

In this section we prove Theorem 4.5.24. Let us write the initial data in the form

$$\psi_0(x) = e^{j\theta_0}(\Phi_{\omega_0}(x) + \chi_0(x)). \tag{4.5.77}$$

with $d = \|\chi_0\|_{L^1_\beta \cap H^1}$ sufficiently small. By Lemma 4.5.23 we can assume that $\mathbf{P}^0(\omega_0)(\chi_0) = 0$ without loss of generality. Then the local existence asserted in Corollary 4.5.22 implies the existence of an interval $[0, t_1]$ on which are defined C^1 functions $t \mapsto \big(\omega(t), \gamma(t)\big)$ satisfying (4.5.70)-(4.5.71) and such that $M(t_1) = \rho$ for some $t_1 > 0$ and $\rho > 0$. By continuity we can make ρ as small as we like by making d and t_1 small.

Proposition 4.5.26. *In the situation of Theorem 4.5.24 let $M(t_1) \leq \rho$ for some $t_1 > 0$ and $\rho > 0$. Then there exist numbers d_1 and ρ_1, independent of t_1, such that*

$$M(t_1) \leq \rho/2 \tag{4.5.78}$$

if $d = \|\chi_0\|_{L^1_\beta \cap H^1} < d_1$ and $\rho < \rho_1$.

Proof of Theorem 4.5.24. Assuming the truth of Proposition 4.5.26 for now Theorem 4.5.24 will follow from the next argument:
Consider the set $\mathcal{T}$ of $t_1 \geq 0$ such that $\big(\omega(t), \gamma(t)\big)$ are defined on $[0, t_1]$ and $M(t_1) \leq \rho$. This set is relatively closed by continuity. On the other hand, (4.5.78) and Corollary 4.5.22 with sufficiently small ρ and d imply that this set is also relatively open, and hence $\sup \mathcal{T} = +\infty$, completing the proof of Theorem 4.5.24.

In the remaining part of the section we prove Proposition 4.5.26.

Frozen linearised equation

A crucial part of the proof of Proposition 4.5.26 is the estimation of the first term in M, for which purpose it is necessary to make use of the dispersive properties obtained in sections 4.5.6 and 4.5.7. Rather than study directly (4.5.69), whose linear part is non-autonomous, it is convenient (following [110, 111]) to introduce a second ansatz, a small modification of (4.5.65), which leads to an autonomous linearised equation. This new ansatz for the solution is

$$\psi(x,t) = e^{j\theta}(\Phi_\omega(x)+e^{-j(\theta-\tilde{\theta})}\eta), \quad \text{where } \tilde{\theta}(t) = \omega_1 t+\theta_0,\ \theta_0 = \theta_0 \text{ and } \omega_1 = \omega(t_1) \quad (4.5.79)$$

so that, $\eta = e^{j(\theta-\tilde{\theta})}\chi$ and $\chi = e^{-j(\theta-\tilde{\theta})}\eta$. Since

$$\dot{\chi} = e^{-j(\theta-\tilde{\theta})}\Big(\dot{\eta} - j(\omega+\dot{\gamma}-\omega_1)\eta\Big)$$

equation (4.5.69) implies

$$\dot{\eta} = j^{-1}(\omega_1-\omega)\eta+e^{j(\theta-\tilde{\theta})}\mathbf{C}\Big(e^{-j(\theta-\tilde{\theta})}\eta\Big)+e^{j(\theta-\tilde{\theta})}\Big(j^{-1}\dot{\gamma}\Phi_\omega-\dot{\omega}\partial_\omega\Phi_\omega+\mathbf{Q}[e^{-j(\theta-\tilde{\theta})}\eta]\Big). \quad (4.5.80)$$

The matrices $\mathbf{C}$ and $e^{j\phi}$, where $\phi = \theta - \tilde{\theta}$, do not commute hence we need the following lemma:

Lemma 4.5.27.

$$\mathbf{C}e^{j\phi} - e^{j\phi}\mathbf{C} = \delta(x) b \sin\phi\ \sigma,\ \text{where } \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ b = 2a'C^2. \quad (4.5.81)$$

Proof.

$$\mathbf{C}e^{j\phi} - e^{j\phi}\mathbf{C} = \begin{pmatrix} 0 & \mathbf{D}_2 \\ -\mathbf{D}_1 & 0 \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 0 & \mathbf{D}_2 \\ -\mathbf{D}_1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (\mathbf{D}_2-\mathbf{D}_1)\sin\phi & 0 \\ 0 & (\mathbf{D}_1-\mathbf{D}_2)\sin\phi \end{pmatrix} = \begin{pmatrix} \delta(x)b\sin\phi & 0 \\ 0 & -\delta(x)b\sin\phi \end{pmatrix}.$$

□

Using Lemma 4.5.27 we rewrite equation (4.5.80) as

$$\dot{\eta} = j^{-1}(\omega_1-\omega)\eta + \mathbf{C}\eta + e^{j(\theta-\tilde{\theta})}\Big(-\delta(x)b\sin(\theta-\tilde{\theta})\sigma\eta + j^{-1}\dot{\gamma}\Phi_\omega - \dot{\omega}\partial_\omega\Phi_\omega + \mathbf{Q}[e^{-j(\theta-\tilde{\theta})}\eta]\Big).$$

To obtain a perturbed *autonomous* equation we rewrite the first two terms on the RHS by freezing the coefficients at $t = t_1$. Note that

$$j^{-1}(\omega_1-\omega) + \mathbf{C} = \mathbf{C_1} - j^{-1}\delta(x)(V-V_1),$$

where $V = a + bP_1$, $V_1 = V(t_1)$, and $\mathbf{C_1} = \mathbf{C}(t_1)$. The equation for η now reads

$$\begin{aligned} \dot{\eta} &= \mathbf{C_1}\eta - j^{-1}\delta(x)(V-V_1)\eta \\ &\quad + e^{j(\theta-\tilde{\theta})}\Big(-\delta(x)b\sin(\theta-\tilde{\theta})\sigma\eta + j^{-1}\dot{\gamma}\Phi - \dot{\omega}\partial_\omega\Phi_\omega + \mathbf{Q}[e^{-j(\theta-\tilde{\theta})}\eta]\Big) \end{aligned} \quad (4.5.82)$$

The first term is now independent of t; the idea is that if there is sufficiently rapid convergence of $\omega(t)$ as $t \to \infty$ the other remaining terms are small *uniformly with respect to* t_1. Finally the equation (4.5.82) can be written in the following *frozen form*

$$\dot{\eta} = \mathbf{C_1}\eta + \mathbf{f_1} \tag{4.5.83}$$

where

$$\begin{aligned} \mathbf{f_1} &= -j^{-1}\delta(x)(V - V_1)\eta \\ &\quad + e^{j(\theta-\tilde{\theta})}\Big(-\delta(x)b\sin(\theta-\tilde{\theta})\sigma\eta + j^{-1}\dot{\gamma}\Phi - \dot{\omega}\partial_\omega\Phi_\omega + \mathbf{Q}[e^{-j(\theta-\tilde{\theta})}\eta]\Big). \end{aligned} \tag{4.5.84}$$

Remark 4.5.28. The advantage of (4.5.83) over (4.5.69) is that it can be treated as a perturbed autonomous linear equation, so that the estimates from section 4.5.6 can be used directly. The additional terms in $\mathbf{f_1}$ can be estimated as small uniformly in t_1: see lemma 4.5.29 below. This is the reason for introduction of the second ansatz (4.5.79).

Lemma 4.5.29. *In the situation of Proposition 4.5.26 there exists $c > 0$, independent of t_1, such that for $0 \le t \le t_1$*

$$|a(t) - a_1| + |b(t) - b_1| + |\theta(t) - \tilde{\theta}(t)| \le c\rho.$$

Proof. By (4.5.76)

$$\sup_{0\le t\le t_1}(1+t^3)(|\dot{\gamma}(t)| + |\dot{\omega}(t)|) \le M(t_1) = \rho. \tag{4.5.85}$$

Therefore

$$|a(t) - a(t_1)| = |\int_t^{t_1} \dot{a}(\tau)d\tau| \le c\Big(\sup_{0\le\tau\le t_1}(1+\tau^2)|\dot{\omega}(\tau)|\Big)\int_t^{t_1}\frac{d\tau}{1+\tau^2} \le c\rho,$$

since $|\dot{a}(\tau)| \le c|\dot{\omega}(\tau)|$. The difference $|b(t) - b(t_1)|$ can be estimated similarly. Next

$$\begin{aligned} \theta(t) - \tilde{\theta}(t) &= \int_0^t \omega(\tau)d\tau + \gamma(t) - \omega(t_1)t - \gamma(0) = \int_0^t (\omega(\tau) - \omega(t_1))d\tau + \int_0^t \dot{\gamma}(\tau)d\tau \\ &= -\int_0^t\int_\tau^{t_1}\dot{\omega}(s)ds d\tau + \int_0^t \dot{\gamma}(\tau)d\tau. \end{aligned} \tag{4.5.86}$$

By (4.5.85) the first summand on RHS of (4.5.86) can be estimated as

$$\begin{aligned} \int_0^t\int_{t_1}^\tau |\dot{\omega}(s)|ds\,d\tau &\le \int_0^t\int_\tau^{t_1}(1+s)^{2+\varepsilon}|\dot{\omega}(s)|\frac{1}{(1+s)^{2+\varepsilon}}ds\,d\tau \\ &\le c\sup_{0\le s\le t_1}(1+s)^{2+\varepsilon}|\dot{\omega}(s)|\int_0^t\int_\tau^{t_1}\frac{1}{(1+s)^{2+\varepsilon}}ds\,d\tau \le c\rho \end{aligned}$$

since the last integral is bounded for $t \in [0, t_1]$. Finally, for the second summand on the RHS of (4.5.86) inequality (4.5.85) implies

$$|\int_0^t \dot{\gamma}(\tau)d\tau| \le c\sup_{0\le\tau\le t_1}(1+\tau^2)|\dot{\gamma}(\tau)|\int_t^{t_1}\frac{d\tau}{1+\tau^2} \le c\rho$$

□

Projection onto discrete and continuous spectral spaces

From sections 4.5.6 and 4.5.7 we have information concerning $U(t) = e^{\mathbf{C_1}t}$, in particular decay on the subspace orthogonal to the (two dimensional) generalized null space. It is therefore necessary to introduce a further decomposition to take advantage of this. Recall, by comparing (4.5.65) and (4.5.79) that

$$\eta = e^{j(\theta-\tilde{\theta})}\chi \quad \text{and} \quad \mathbf{P}^0_t\chi(t) = 0 \tag{4.5.87}$$

Introduce the symplectic projections $\mathbf{P}^0_1 = \mathbf{P}^0_{t_1}$ and $\mathbf{P}^c_1 = \mathbf{P}^c_{t_1}$ onto the discrete and continuous spectral subspaces defined by the operator $\mathbf{C_1}$ and write, at each time $t \in [0, t_1]$:

$$\eta(t) = g(t) + h(t) \tag{4.5.88}$$

with $g(t) = \mathbf{P}^0_1\eta(t)$ and $h(t) = \mathbf{P}^c_1\eta(t)$. The following lemma shows that it is only necessary to estimate $h(t)$.

Lemma 4.5.30. *In the situation of Proposition 4.5.26, assume*

$$\sup_{0\le t\le t_1}\big(|\omega(t)-\omega_1| + |\theta(t)-\theta_1(t)|\big) = \Delta$$

is sufficiently small. Then for $0 \le t \le t_1$ there exists a linear operator $\Xi(t)$, bounded on $L^\infty_{-\beta}\cap H^1$, and $c(\Delta,\omega_1)$ such that $\eta(t) = \Xi(t)h(t)$, and

$$c(\Delta,\omega_1)^{-1}|||h||_{L^\infty_{-\beta}\cap H^1} \le \|\eta\|_{L^\infty_{-\beta}\cap H^1} \le c(\Delta,\omega_1)|||h||_{L^\infty_{-\beta}\cap H^1}. \tag{4.5.89}$$

Proof. Explicitly we write

$$\eta(t) = h(t) + g(t), \qquad g(t) = b_0(t)T_0(\omega_1) + b_1(t)T_1(\omega_1) \tag{4.5.90}$$

where b_0, b_1 are chosen at each time t to ensure that $\Omega(h(t), T_0(\omega_1)) = \Omega(h(t), T_1(\omega_1)) = 0$. Using the fact that (since $\mathbf{P}^0_t\chi(t) = 0$)

$$\Omega\big(e^{-j(\theta-\tilde{\theta})}\eta, T_0(\omega(t)\big) = 0 = \Omega\big(e^{-j(\theta-\tilde{\theta})}\eta, T_1(\omega(t)\big)$$

this means that b_0, b_1 are determined by

$$-\boldsymbol{\mu}_{\boldsymbol{\omega_1}}b_0(t) = \Omega\big(\eta(t), T_1(\omega_1)\big) = \Omega\big(\eta(t), T_1(\omega_1) - e^{j(\theta-\tilde{\theta})}T_1(\omega(t))\big) \tag{4.5.91}$$

$$\boldsymbol{\mu}_{\boldsymbol{\omega_1}}b_1(t) = \Omega\big(\eta(t), T_0(\omega_1)\big) = \Omega\big(\eta(t), T_0(\omega_1) - e^{j(\theta-\tilde{\theta})}T_0(\omega(t))\big). \tag{4.5.92}$$

From these it follows that there exists $c > 0$ such that $\|g(t)\|_{L^\infty_{-\beta}\cap H^1} \le c\Delta\|\eta(t)\|_{L^\infty_{-\beta}\cap H^1}$ and hence (4.5.89) follows as claimed. □

Proof of Proposition 4.5.26

To prove Proposition 4.5.26 we explain how to estimate both terms in M, (4.5.76), to be $\le \rho/4$, uniformly in t_1.
Estimation of the second term in M. As in Corollary 4.5.22 we have

$$|\dot{\gamma}(t)| + |\dot{\omega}(t)| \le c_0|\chi(0,t)|^2 \le c_0\frac{M(t)^2}{(1+|t|)^3}, \quad t \le t_1,$$

since $|\chi(0,t)| \le \|\chi(t)\|_{L^\infty_{-\beta}}$. Finally let $\rho_1 < 1/(4c_0)$ to complete the estimate for the second term in M as $\le \rho/4$.
Estimation of the first term in M. By Lemma 4.5.30 it is enough to estimate h. Let us apply the projection $\mathbf{P}_1^c$ to both sides of (4.5.83). Then the equation for h reads

$$\dot{h} = \mathbf{C_1} h + \mathbf{P_1^c f_1} \tag{4.5.93}$$

Now to estimate h we use the Duhamel representation:

$$h(t) = U(t)h(0) + \int_0^t U(t-s)\mathbf{P}_1^c \mathbf{f_1}(s)ds, \quad t \le t_1. \tag{4.5.94}$$

with $U(t) = e^{\mathbf{C_1}t}$ the one parameter group just introduced. Recall that $\mathbf{P}_1^0 h(t) = 0$ for $t \in [0, t_1]$. Therefore

$$\|U(t)h(0)\|_{L^\infty_{-\beta}} \le c(1+t)^{-3/2}\|h(0)\|_{L^1_\beta \cap H^1} \le c(1+t)^{-3/2}\|\eta(0)\|_{L^1_\beta \cap H^1}. \tag{4.5.95}$$

by Theorem 4.5.17 and inequalities (4.5.56) and (4.5.89). Let us estimate the integrand on the right-hand side of (4.5.94).

Lemma 4.5.31. *The integrand in (4.5.94) satisfies the following bound: for $0 < s < t$*

$$\|U(t-s)\mathbf{P}_1^c \mathbf{f_1}(\mathbf{s})\|_{L^\infty_{-\beta}} \le c\frac{1}{(t-s)^{1/2}(1+t-s)}\Big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\Big). \tag{4.5.96}$$

Proof. We consider two different cases : $t-s > \nu$, and $0 < t-s < \nu$, where $\nu = \nu(a,b)$ is defined in Lemma 4.5.20.
i) $t - s > \nu$ **:** We use the representation (4.5.84) for $\mathbf{f_1}$ and apply Theorem 4.5.17, Corollary 4.5.22 and Lemma 4.5.29 to obtain that for $t \le t_1$

$$\begin{aligned}
\|U(t-s)\mathbf{P}_1^c\mathbf{f_1}\|_{L^\infty_{-\beta}} &\le c(\nu)(1+t-s)^{-3/2}\|\mathbf{P}_1^c(\mathbf{f_1}(t))\|_{\mathcal{M}_\beta} \\
&\le c(\nu)(1+t-s)^{-3/2}\Big(|\eta(0,t)|^2 + \rho|\eta(0,t)|\Big) \\
&\le c(\nu)(1+t-s)^{-3/2}\Big(\|\eta(t)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(t)\|_{L^\infty_{-\beta}}\Big).
\end{aligned} \tag{4.5.97}$$

ii) $0 < t-s < \nu$ **:** We denote $\mathbf{Q} = \delta(x)\tilde{\mathbf{Q}}$, and represent $\mathbf{f_1}(x,s)$ as

$$\mathbf{f_1}(x,s) = p(s)\delta(x) + q(s)\Phi_\omega + r(s)\partial_\omega\Phi_\omega. \tag{4.5.98}$$

where

$$p(s) = -j^{-1}(V - V_1)\eta(0,s) + e^{j(\theta-\tilde{\theta})}\Big(b\sin(\theta-\tilde{\theta})\sigma\eta(0,s) + \tilde{\mathbf{Q}}[e^{-j(\theta-\tilde{\theta})}\eta(0,s)]\Big),$$

is an $\mathbb{R}^2$ valued function of time, and

$$q(s) = -e^{-j(\theta-\tilde{\theta})}j\dot{\gamma}, \quad r(s) = -e^{-j(\theta-\tilde{\theta})}\dot{\omega}$$

are (2×2) matrix valued functions of time. Writing $\|\cdot\|$ for both the standard Euclidean and operator norms on these, we have, by Lemma 4.5.29,

$$\|p(s)\| \le c\Big(|\eta(0,s)|^2 + \rho|\eta(0,s)|\Big) \le c\Big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\Big)$$

and by Corollary 4.5.22

$$\|q(s)\|, \|r(s)\| \le c|\eta(0,s)|^2 \le c\|\eta(s)\|^2_{L^\infty_{-\beta}}.$$

Applying projector $\mathbf{P}_1^c$ to $\mathbf{f_1}$ we obtain

$$\mathbf{P}_1^c\mathbf{f_1}(x,s) = p(s)\delta(x) + q(s)\Phi_\omega + r(s)\partial_\omega\Phi_\omega - \mathbf{P}_1^0\mathbf{f_1}(x,s). \tag{4.5.99}$$

By Lemma 4.5.20 for sufficiently small ν we obtain

$$\|U(t-s)p(s)\delta(x)\|_{L^\infty_{-\beta}} \le \|U(t-s)p(s)\delta(x)\|_{L^\infty} \le c(\nu)\|p(s)\|(t-s)^{-1/2}$$

$$\le \quad c(\nu)(t-s)^{-1/2}\Big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\Big), \quad 0 < t-s < \nu. \tag{4.5.100}$$

By inequality (4.5.56) we have

$$\|U(t-s)\Big(q(s)\Phi_\omega + r(s)\partial_\omega\Phi_\omega\Big)\|_{L^\infty_{-\beta}} \le c\|U(t-s)\Big(q(s)\Phi_\omega + r(s)\partial_\omega\Phi_\omega\Big)\|_{H^1}$$

$$\le \quad c\Big(\|q(s)\|\|\Phi_\omega\|_{H^1} + (\|r(s)\|\|\partial_\omega\Phi_\omega\|_{H^1}\Big) \le c\|\eta(s)\|^2_{L^\infty_{-\beta}}, \quad 0 \le t-s < \nu. \tag{4.5.101}$$

The definition (4.5.42) of the projector $\mathbf{P}_1^0$ implies immediately that

$$\|\mathbf{P}_1^0\mathbf{f_1}\|_{H^1} \le c\Big(\|p(s)\| + \|q(s)\| + \|r(s)\|\Big)$$

Then, similarly to (4.5.101), we obtain

$$\|U(t-s)\mathbf{P}_1^0\mathbf{f_1}\|_{L^\infty_{-\beta}} \le c\big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\big), \quad 0 \le t-s < \nu \tag{4.5.102}$$

Finally, (4.5.99)-(4.5.102) imply

$$\|U(t-s)\mathbf{P}_1^c\mathbf{f_1}\|_{L^\infty_{-\beta}} \le c(t-s)^{-1/2}\big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\big), \quad 0 < t-s < \nu. \tag{4.5.103}$$

From (4.5.97) and (4.5.103) inequality (4.5.96) follows. □

Now (4.5.89), (4.5.94), (4.5.95) and (4.5.96) imply

$$\|\eta(t)\|_{L^\infty_{-\beta}} \le c(1+t)^{-3/2}\|\eta(0)\|_{L^1_\beta\cap H^1} + c_1\int\limits_0^t \frac{ds}{(t-s)^{1/2}(1+t-s)}\Big(\|\eta(s)\|^2_{L^\infty_{-\beta}} + \rho\|\eta(s)\|_{L^\infty_{-\beta}}\Big)$$

Multiply by $(1+t)^{3/2}$ to deduce

$$(1+t)^{3/2}\|\eta(t)\|_{L^\infty_{-\beta}} \le cd + c_1\int\limits_0^t \frac{(1+t)^{3/2}(1+s)^{-3}}{(t-s)^{1/2}(1+t-s)}(1+s)^3\|\eta(s)\|^2_{L^\infty_{-\beta}}ds \tag{4.5.104}$$

$$+ c_1\rho\int\limits_0^t \frac{(1+t)^{3/2}(1+s)^{-3/2}}{(t-s)^{1/2}(1+t-s)}(1+s)^{3/2}\|\eta(s)\|_{L^\infty_{-\beta}}ds$$

since $\|\eta(0)\|_{L^1_\beta\cap H^1}\le d$. Introduce the majorant

$$m(t):=\sup_{[0,t]}(1+s)^{3/2}\|\eta(s)\|_{L^\infty_{-\beta}},\quad t\le t_1$$

and hence

$$\begin{aligned} m(t) &\le cd+c_1m^2(t)\int_0^t \frac{(1+t)^{3/2}(1+s)^{-3}}{(t-s)^{1/2}(1+t-s)}\,ds\\ &\quad +\rho c_1 m(t)\int_0^t \frac{(1+t)^{3/2}(1+s)^{-3/2}}{(t-s)^{1/2}(1+t-s)}\,ds. \end{aligned}\tag{4.5.105}$$

It easy to see (by splitting up the integrals into $s<t/2$ and $s\ge t/2$) that both these integrals are bounded independent of t. Thus (4.5.105) implies that there exist c,c_2,c_3, independent of t_1, such that

$$m(t)\le cd+\rho c_2 m(t)+c_3m^2(t),\quad t\le t_1.$$

Recall that $m(t_1)\le\rho\le\rho_1$ by assumption. Therefore this inequality implies that $m(t)$ is bounded for $t\le t_1$, and moreover,

$$m(t)\le c_4 d,\quad t\le t_1$$

if d and ρ are sufficiently small. The constant c_4 does not depend on t_1. We choose d in (4.5.19) small enough that $d<\rho/(4c_4)$. Therefore,

$$\sup_{[0,t_1]}(1+t)^{3/2}\|\eta(t)\|_{L^\infty_{-\beta}}<\rho/4$$

if d and ρ are sufficiently small. This bounds the first term as $<\rho/4$ by (4.5.87) and hence $M(t_1)<\rho/2$, completing the proof of Proposition 4.5.26.

4.5.12 Soliton asymptotics

Here we prove our main Theorem 4.5.9 using the bounds (4.5.76). For the solution $\psi(x,t)$ to (4.5.1) let us define the accompanying soliton as $s(x,t)=\psi_{\omega(t)}(x)e^{i\theta(t)}$, where $\dot\theta(t)=\omega(t)+\dot\gamma(t)$. Then for the difference $z(x,t)=\psi(x,t)-s(x,t)$ we obtain easily from equations (4.5.1) and (4.5.11)

$$i\dot z(x,t)=-z''(x,t)+\dot\gamma s(x,t)-i\dot\omega\partial_\omega s(x,t)-\delta(x)\Big(F(\psi(x,t))-F(s(x,t))\Big).\tag{4.5.106}$$

Then

$$\begin{aligned} z(t)=&\,W(t)z(0)\\ &+\int_0^t W(t-\tau)\Big[\dot\gamma s(\tau)-i\dot\omega\partial_\omega s(\tau)-\delta(x)\Big(F(\psi(0,\tau))-F(s(0,\tau))\Big)\Big]d\tau, \end{aligned}\tag{4.5.107}$$

where $z(t) = z(\cdot,t)$, $s(t) = s(\cdot,t)$, and $W(t)$ is the dynamical group of the free Schrödinger equation. Since $\gamma(t)-\gamma_+$, $\omega(t)-\omega_+ = \mathcal{O}(t^{-2})$, and therefore $\theta(t)-\omega_+t-\gamma_+ = \mathcal{O}(t^{-1})$ for $t\to\infty$, to establish the asymptotic behaviour (4.5.20) it suffices to prove that

$$z(t) = W(t)\Phi_+ + r_+(t) \tag{4.5.108}$$

with some $\Phi_+ \in C_b(\mathbb{R})\cap L^2(\mathbb{R})$ and $\|r_+(t)\|_{C_b(\mathbb{R})\cap L^2(\mathbb{R})} = \mathcal{O}(t^{-1/2})$. Denote $g(t) = \dot\gamma s(t) - i\dot\omega\partial_\omega s(t)$, $h(t) = F(\psi(0,t)) - F(s(0,t))$ and rewrite equation (4.5.107) as

$$z(t) = W(t)z(0) + W(t)\int_0^t W(-\tau)g(\tau)d\tau - W(t)\int_0^t W(-\tau)\delta(x)h(\tau)d\tau. \tag{4.5.109}$$

Let us recall that $\dot\omega(t)$, $\dot\gamma(t)\sim t^{-3}$ as $t\to\infty$. Hence, for the second summand in RHS of (4.5.109) we have

$$\begin{aligned} W(t)\int_0^t W(-\tau)g(\tau)d\tau &= W(t)\int_0^\infty W(-\tau)g(\tau)d\tau - W(t)\int_t^\infty W(-\tau)g(\tau)d\tau \\ &= W(t)\phi_1 + r_1(t), \end{aligned} \tag{4.5.110}$$

where, from the unitarity in H^1 of the dynamical group $W(t)$ and the t^{-3} decay of $\dot\omega$ and $\dot\gamma$, we infer that $\phi_1 = \int_0^\infty W(-\tau)g(\tau)d\tau \in H^1$, and $\|r_1(t)\|_{H^1} = \mathcal{O}(t^{-2})$, $t\to\infty$.

Consider now the last summand onn the RHS of (4.5.109). Note that $W(t)\delta(x) = \dfrac{e^{ix^2/(4t)}}{\sqrt{4\pi i t}}$, and $|h(t)|\le c|\chi(0,t)|\le c(1+t)^{-3/2}$ by (4.5.76). Therefore

$$\begin{aligned} W(t)\int_0^t W(-\tau)\delta(x)h(\tau)d\tau &= W(t)\int_0^\infty \frac{e^{-ix^2/(4\tau)}}{\sqrt{-4\pi i\tau}}\,h(\tau)d\tau - \int_t^\infty \frac{e^{ix^2/(4(t-\tau))}}{\sqrt{4\pi i(t-\tau)}}\,h(\tau)d\tau \\ &= W(t)\phi_2 + r_2(t). \end{aligned} \tag{4.5.111}$$

Evidently, $\phi_2 = \int_0^\infty \dfrac{e^{-ix^2/(4\tau)}}{\sqrt{-4\pi i\tau}}\,h(\tau)d\tau \in C_b$, and $\|r_2(t)\|_{C_b} = \mathcal{O}(t^{-1})$, $t\to\infty$.

Moreover, $\phi_2\in L^2$, and $\|r_2(t)\|_{L^2} = \mathcal{O}(t^{-1/2})$, $t\to\infty$. To see that this is indeed true change variable to $\tau = 1/u$ in the definition to get:

$$\phi_2(x) = \frac{1}{\sqrt{-4\pi i}}\int_0^\infty e^{-iux^2/4}\,\eta(u)\,du, \qquad \eta(u) = h(1/u)/u^{3/2}. \tag{4.5.112}$$

Now $h(t)$ is bounded and it follows from the decay of $h(t)$ that $\eta(u)$ is bounded as $u\to 0$. Therefore $\eta(u)$ is square integrable and so by the Parseval theorem ϕ_2 is square integrable as a function of $y = x^2$, and hence also as a function of x (since $dy = 2xdx$ and ϕ_2 is a bounded continuous function). Next we have $r_2(t) = -W(t)R(t)$ with

$$R(x,t) = \frac{1}{\sqrt{-4\pi i}}\int_0^{1/t} e^{-iux^2/4}\,\eta(u)\,du = \frac{1}{\sqrt{-4\pi i}}F_{u\to x^2/4}\zeta_t(u)\eta(u),$$

where $\zeta_t(u)$ is the characteristic function of the interval $(0,1/t)$. The function $\eta(u)$ is bounded, hence $\|\zeta_t\eta\|_{L^2} = ct^{-1/2}$ and therefore $\|r_2(t)\|_{L^2} = \mathcal{O}(t^{-1/2})$, $t \to \infty$. To conclude, using (4.5.109), (4.5.110), and (4.5.111) we obtain (4.5.108) with $\phi_+ = z(0)+\phi_1+\phi_2$ and $r_+(t) = r_1(t) + r_2(t)$. The $t \to -\infty$ case is handled in an identical way.

Now Theorem 4.5.9 is proved.

Remark 4.5.32. The expression (4.5.112) for ϕ_2 as a Fourier transform implies immediately that $|\phi_2|$, and hence $|\Phi_+|$ also, tend to 0 as $|x| \to \infty$ by the Riemann–Lebesgue theorem. This same expression could be used with Zygmund's lemma to obtain more detailed decay properties of ϕ_2 and hence of Φ_+. The decay rate would be determined essentially by the regularity of the function $h(t)$ in addition to the decay rate of the initial data.

4.5.13 The kernel and poles of the resolvent

In this section we calculate the resolvent and its poles.

The kernel of the resolvent

The derivation of the time decay of the solution to the linearised equation (4.5.25) in section 4.5.6 required an analysis of the smoothness and singularities of the resolvent $\mathbf{R}(\lambda)$ and its asymptotics as $\lambda \to \infty$. Here we will construct its matrix integral kernel explicitly

$$\mathbf{R}(\lambda, x, y) = \begin{pmatrix} R_{11}(\lambda, x, y) & R_{12}(\lambda, x, y) \\ R_{21}(\lambda, x, y) & R_{22}(\lambda, x, y) \end{pmatrix}. \tag{4.5.113}$$

It is the solution to the equation

$$(\mathbf{C} - \lambda)\mathbf{R}(\lambda, x, y) = \delta(x-y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.5.114}$$

Calculation of first column For the first column $R_I(\lambda, x, y) := \begin{pmatrix} R_{11}(\lambda, x, y) \\ R_{21}(\lambda, x, y) \end{pmatrix}$ of the matrix $\mathbf{R}(\lambda, x, y)$ we obtain

$$(\mathbf{C} - \lambda)R_I(\lambda, x, y) = \delta(x-y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.5.115}$$

If $x \neq 0$ and $x \neq y$, (4.5.115) takes the form (cf. (4.5.31), (4.5.32))

$$\begin{pmatrix} -\lambda & \mathbf{D}_2 \\ -\mathbf{D}_1 & -\lambda \end{pmatrix} R_I(\lambda, x, y) = \begin{pmatrix} -\lambda & -\dfrac{d^2}{dx^2} + \omega \\ \dfrac{d^2}{dx^2} - \omega & -\lambda \end{pmatrix} R_I(\lambda, x, y) = 0. \tag{4.5.116}$$

The general solution is a linear combination of exponential solutions of type $e^{ikx}v$. Substituting into (4.5.116), we get

$$\begin{pmatrix} -\lambda & k^2 + \omega \\ -k^2 - \omega & -\lambda \end{pmatrix} v = 0. \tag{4.5.117}$$

For nonzero vectors v, the determinant of the matrix vanishes,

$$\lambda^2 + (k^2 + \omega)^2 = 0. \tag{4.5.118}$$

Then $k_\pm^2 + \omega = \mp i\lambda$. Finally, we obtain four roots $\pm k_\pm(\lambda)$ with

$$k_\pm(\lambda) = \sqrt{-\omega \mp i\lambda}, \tag{4.5.119}$$

where the square root is defined as an analytic continuation from a neighborhood of the zero point $\lambda = 0$ taking the positive value of Im $\sqrt{-\omega}$ at $\lambda = 0$. We choose the cuts in the complex plane λ from the branching points to infinity: the cut $\mathcal{C}_+ := [i\omega, i\infty)$ for $k_+(\lambda)$ and the cut $\mathcal{C}_- := [-i\omega, -i\infty)$ for $k_-(\lambda)$. Then

$$\operatorname{Im} k_\pm(\lambda) > 0, \qquad \lambda \in \mathbb{C} \setminus \mathcal{C}_\pm. \tag{4.5.120}$$

It remains to derive the vector $v = (v_1, v_2)$ which is solution to (4.5.117):

$$v_2 = -\frac{k_\pm^2 + \omega}{\lambda} v_1 = \frac{\pm i\lambda}{\lambda} v_1 = \pm i v_1.$$

Therefore, we have two corresponding vectors $v_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ and we get four linearly independent exponential solutions

$$v_+ e^{\pm ik_+ x} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{\pm ik_+ x}, \qquad v_- e^{\pm ik_- x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\pm ik_- x}.$$

Now we can solve the equation (4.5.115). First we rewrite it using the representations (4.5.32) and (4.5.31) for the operator $\mathbf{C}$,

$$\begin{pmatrix} -\lambda & -\dfrac{d^2}{dx^2} + \omega \\ \dfrac{d^2}{dx^2} - \omega & -\lambda \end{pmatrix} \begin{pmatrix} R_{11}(\lambda, x, y) \\ R_{21}(\lambda, x, y) \end{pmatrix} = \delta(x-y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \delta(x) \begin{pmatrix} 0 & a \\ -a-b & 0 \end{pmatrix} \begin{pmatrix} R_{11}(\lambda, 0, y) \\ R_{21}(\lambda, 0, y) \end{pmatrix}$$

Let us consider $y > 0$ for the concreteness. Then the RHS vanishes in the open intervals $(-\infty, 0)$,$(0, y)$ and (y, ∞). Hence, for the parameter λ outside the cuts $C_\pm$, the solution admits the representation

$$R_I(\lambda, x, y) = \begin{cases} A_+ e^{-ik_+ x} v_+ + A_- e^{-ik_- x} v_-, & x < 0, \\ B_+^- e^{-ik_+ x} v_+ + B_-^- e^{-ik_- x} v_- + B_+^+ e^{ik_+ x} v_+ + B_-^+ e^{ik_- x} v_-, & 0 < x < y, \\ C_+ e^{ik_+ x} v_+ + C_- e^{ik_- x} v_-, & x > y \end{cases}$$

since by (4.5.120), the exponent $e^{-ik_\pm x}$ decays for $x \to -\infty$, and similarly, $e^{ik_\pm x}$ decays for $x \to \infty$. Next we need eight equations to calculate the eight constants $A_+, \ldots, C_-$. We have two continuity equations and two jump conditions for the derivatives at the points $x = 0$ and $x = y$. These four vector equations give just eight scalar equations for the calculation.

Continuity at $x = y$: $R_I(y - 0, y) = R_I(y + 0, y)$, i.e.

$$B_+^- v_+ / e_+ + B_-^- v_- / e_- + B_+^+ v_+ e_+ + B_-^+ v_- e_- = C_+ v_+ e_+ + C_- v_- e_-,$$

where $e_\pm := e^{ik_\pm y}$. It is equivalent to

$$\begin{cases} B_+^- / e_+ + B_+^+ e_+ = C_+ e_+, \\ B_-^- / e_- + B_-^+ e_- = C_- e_-. \end{cases} \tag{4.5.121}$$

Continuity at $x = 0$: $R_I(-0, y) = R_I(+0, y)$, i.e.

$$A_+v_+ + A_-v_- = B_+^-v_+ + B_-^-v_- + B_+^+v_+ + B_-^+v_-$$

that is equivalent to

$$\begin{cases} A_+ = B_+^- + B_+^+, \\ A_- = B_-^- + B_-^+. \end{cases} \tag{4.5.122}$$

Jump at $x = y$: $R_I'(y+0, y) = R_I'(y-0, y) + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, where prime denotes the derivative in x. Substituting (4.5.121), we get

$$ik_+C_+v_+e_+ + ik_-C_-v_-e_- =$$

$$-ik_+B_+^-v_+/e_+ - ik_-B_-^-v_-/e_- + ik_+B_+^+v_+e_+ + ik_-B_-^+v_-e_- + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4.5.123}$$

Noting that

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{v_+ - v_-}{2}i, \tag{4.5.124}$$

we get

$$\begin{cases} ik_+C_+e_+ = -ik_+B_+^-/e_+ + ik_+B_+^+e_+ + \dfrac{i}{2}, \\ ik_-C_-e_- = -ik_-B_-^-/e_- + ik_-B_-^+e_- - \dfrac{i}{2}. \end{cases} \tag{4.5.125}$$

After substituting of $C_\pm$ from (4.5.121), the constants $B_\pm^+$ cancel and we get

$$B_+^- = \frac{e_+}{4k_+}, \qquad B_-^- = -\frac{e_-}{4k_-}. \tag{4.5.126}$$

Jump at $x = 0$: $R_I'(+0, y) = R_I'(-0, y) - \begin{pmatrix} a+b & 0 \\ 0 & a \end{pmatrix} R_I(-0, y)$. Substituting (4.5.121), we get

$$\begin{aligned} & -ik_+B_+^-v_+ - ik_-B_-^-v_- + ik_+B_+^+v_+ + ik_-B_-^+v_- \\ = \; & -ik_+A_+v_+ - ik_-A_-v_- - M(A_+v_+ + A_-v_-), \end{aligned} \tag{4.5.127}$$

where M is the matrix $\begin{pmatrix} a+b & 0 \\ 0 & a \end{pmatrix}$. Note that

$$\begin{cases} Mv_+ = \alpha v_+ + \beta v_- \\ Mv_- = \alpha v_- + \beta v_+ \end{cases}, \qquad \text{where} \quad \alpha = a + \frac{b}{2}, \quad \beta = \frac{b}{2}. \tag{4.5.128}$$

Then (4.5.127) becomes

$$\begin{cases} -ik_+B_+^- + ik_+B_+^+ = -ik_+A_+ - A_+\alpha - A_-\beta, \\ -ik_-B_-^- + ik_-B_-^+ = -ik_-A_- - A_+\beta - A_-\alpha. \end{cases}$$

Substituting here (4.5.122), we get after cancellations,

$$\begin{cases} (2ik_+ + \alpha)B_+^+ + \beta B_-^+ = -\alpha B_+^- - \beta B_-^- \\ \beta B_+^+ + (2ik_- + \alpha)B_-^+ = -\beta B_+^- - \alpha B_-^- \end{cases}$$

Hence, the solution is given by

$$\begin{pmatrix} B_+^+ \\ B_-^+ \end{pmatrix} = -\frac{1}{D}\begin{pmatrix} 2ik_- + \alpha & -\beta \\ -\beta & 2ik_+ + \alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} B_+^- \\ B_-^- \end{pmatrix}, \qquad (4.5.129)$$

where D is the determinant

$$D := (2ik_+ + \alpha)(2ik_- + \alpha) - \beta^2, \qquad (4.5.130)$$

and B_+^-, B_-^- are given by (4.5.126). The formulas (4.5.126) and (4.5.129) imply

$$\left|\begin{cases} B_+^+ &= \frac{1}{2D}\Big(-\dfrac{2ik_-\alpha + \alpha^2 - \beta^2}{2k_+}e_+ + i\beta e_-\Big) \\ B_-^+ &= \frac{1}{2D}\Big(-i\beta e_+ + \dfrac{2ik_+\alpha + \alpha^2 - \beta^2}{2k_-}e_-\Big) \end{cases}\right| . \qquad (4.5.131)$$

Using the identities

$$2ik_-\alpha + \alpha^2 - \beta^2 = D - 2ik_+\alpha + 4k_+k_-, \quad 2ik_+\alpha + \alpha^2 - \beta^2 = D - 2ik_-\alpha + 4k_+k_-,$$

we rewrite (4.5.131) as

$$\left|\begin{cases} B_+^+ &= -\dfrac{e_+}{4k_+} + \dfrac{1}{2D}\Big((i\alpha - 2k_-)e_+ + i\beta e_-\Big) \\ B_-^+ &= \dfrac{e_-}{4k_-} - \dfrac{1}{2D}\Big(i\beta e_+ + (i\alpha - 2k_+)e_-\Big) \end{cases}\right| . \qquad (4.5.132)$$

Finally, the formulas (4.5.121)–(4.5.122), (4.5.126) and (4.5.132) give the first column $R_I(\lambda, x, y)$ of the resolvent for $y > 0$:

$$R_I(\lambda, x, y) = \Gamma_I(\lambda, x, y) + P_I(\lambda, x, y), \qquad (4.5.133)$$

where

$$\Gamma_I(\lambda, x, y) = \frac{1}{4k_+}(e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)})v_+ - \frac{1}{4k_-}(e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)})v_-, \qquad (4.5.134)$$

and

$$\begin{aligned} P_I(\lambda, x, y) &= \frac{1}{2D}\Big[\Big((i\alpha - 2k_-)e^{ik_+(|x|+|y|)} + i\beta e^{i(k_+|x|+k_-|y|)}\Big)v_+ \\ &- \Big(i\beta e^{i(k_-|x|+k_+|y|)} + (i\alpha - 2k_+)e^{ik_-(|x|+|y|)}\Big)v_-\Big] \end{aligned} \qquad (4.5.135)$$

Calculation of second column The second column is given by similar formulas with the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ instead of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in (4.5.121). Then $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ in (4.5.123) is changed by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Respectively, (4.5.124) is changed by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{v_- + v_+}{2}.$$

Hence, we have now change $i/2$ by $1/2$ in the first equation of (4.5.125) and $-i/2$ by $1/2$ in the second one. Respectively, (4.5.126) for the second column reads

$$B_+^- = -\frac{ie_+}{4k_+}, \qquad B_-^- = -\frac{ie_-}{4k_-}.$$

Then the second column $R_{II}(\lambda, x, y)$ of the resolvent reads:

$$R_{II}(\lambda, x, y) = \Gamma_{II}(\lambda, x, y) + P_{II}(\lambda, x, y), \tag{4.5.136}$$

where

$$\Gamma_{II}(\lambda, x, y) = -\frac{i}{4k_+}(e^{ik_+|x-y|} - e^{ik_+(|x|+|y|)})v_+ - \frac{i}{4k_-}(e^{ik_-|x-y|} - e^{ik_-(|x|+|y|)})v_-, \tag{4.5.137}$$

and

$$\begin{aligned} P_{II}(\lambda, x, y) &= \frac{i}{2D}\Big[\Big(-(i\alpha - 2k_-)e^{ik_+(|x|+|y|)} + i\beta e^{i(k_+|x|+k_-|y|)}\Big)v_+ \\ &+ \Big(i\beta e^{i(k_-|x|+k_+|y|)} - (i\alpha - 2k_+)e^{ik_-(|x|+|y|)}\Big)v_-\Big] \end{aligned} \tag{4.5.138}$$

Note, that if $y < 0$ we get the same formulas.

The poles of the resolvent

The poles of the resolvent correspond to the roots of the determinant (4.5.130),

$$D(\lambda) := \alpha^2 + 2i\alpha(k_+ + k_-) - 4k_+k_- - \beta^2 = 0. \tag{4.5.139}$$

with $k_\pm$ as in (4.5.119)-(4.5.120). Thus $D(\lambda)$ is an analytic function on $\mathbb{C} \setminus \mathcal{C}_- \cup \mathcal{C}_+$. Since there are two possible values for the square roots in $k_\pm$ there is a corresponding four-sheeted function $\tilde{D}(\lambda)$ analytic on a four sheeted cover of $\mathbb{C}$ which is branched over $\mathcal{C}_-$ and $\mathcal{C}_+$. We call the sheet defined by (4.5.120) the *physical sheet*.

We will reduce the equation (4.5.139) to the solution of two successive quadratic equations. These can be solved explicitly but the process involves squaring and thus actually produces zeros of the function $\tilde{D}(\lambda)$ rather than of $D(\lambda)$. Therefore we will then have to check whether or not the roots do actually lie on the physical sheet.

Step i)

Denote by $\sigma = k_+ + k_-$. Then

$$\sigma^2 = 2k_+k_- - 2\omega \tag{4.5.140}$$

by (4.5.119), hence (4.5.139) gives the *first quadratic equation*:

$$\alpha^2 + 2i\alpha\sigma - 2(\sigma^2 + 2\omega) - \beta^2 = 0.$$

Rewrite it as

$$\sigma^2 - i\alpha\sigma = \frac{\alpha^2 - \beta^2}{2} - 2\omega =: \delta \tag{4.5.141}$$

Finally,

$$\sigma = \frac{i\alpha}{2} \pm \sqrt{\delta - \frac{\alpha^2}{4}}, \tag{4.5.142}$$

where the root is choosen arbitrarily.

Further let us express the roots in ω. Since $a = 2\sqrt{\omega}$, $\alpha = a + b/2$, $\beta = b/2$ then substituting δ from (4.5.141), we obtain

$$\delta - \frac{\alpha^2}{4} = \frac{\alpha^2}{4} - \frac{\beta^2}{2} - 2\omega = \frac{(a+b/2)^2}{4} - \frac{b^2}{8} - \frac{a^2}{2} = -\frac{a^2}{4} - \frac{b^2}{16} + \frac{ab}{4} = -\frac{1}{16}(2a-b)^2 < 0.$$

Now (4.5.142) reads

$$\sigma = \frac{i\alpha}{2} \pm \frac{i}{4}(2a-b) = \frac{i}{4}\Big[(2a+b) \pm (2a-b)\Big] = i\gamma_j,\ j = 1, 2, \tag{4.5.143}$$

where $\gamma_j \in \mathbb{R}$, and

$$\gamma_1 = a = a(C^2), \qquad \gamma_2 = b/2 = a'(C^2)C^2. \tag{4.5.144}$$

Step ii)

It remains to calculate the corresponding spectral parameter λ. First, the quadratic equation (4.5.140) implies by (4.5.143) that

$$4(k_+k_-)^2 = (2\omega + \sigma^2)^2 = (2\omega - \gamma_j^2)^2,\ j = 1, 2. \tag{4.5.145}$$

On the other hand,

$$k_+k_- = \sqrt{-\omega + i\lambda}\sqrt{-\omega - i\lambda}, \tag{4.5.146}$$

hence (4.5.145) gives the *second quadratic equation*

$$4(\omega^2 + \lambda^2) = (2\omega - \gamma_j^2)^2.$$

Therefore,

$$\lambda^2 = \frac{(2\omega - \gamma_j^2)^2 - 4\omega^2}{4} = -\frac{\gamma_j^2(4\omega - \gamma_j^2)}{4}.$$

Finally, we obtain four roots

$$\lambda_j = i\frac{\gamma_j}{2}\sqrt{4\omega - \gamma_j^2}, \tag{4.5.147}$$

where $j \in \{1, 2\}$ and the square root can takes two opposite values.

Corollary 4.5.33. *The four-sheeted function $\tilde{D}(\lambda)$ has the following roots (zeros):*
i) $j = 1$ gives $\lambda_1 = 0$ since $4\omega = a^2 = \gamma_1$.
ii) If $|\gamma_2| < 2\sqrt{\omega}$, then both $j = 2$ roots $\pm i|\lambda_2|$ are pure imaginary.
iii) If $|\gamma_2| > 2\sqrt{\omega}$, then both $j = 2$ roots $\pm|\lambda_2|$ are real: one positive and one negative.

Remark 4.5.34. Note that a priori we can meet the wrong sign of $\operatorname{Im} k_\pm$ squaring (4.5.146) which is why the above calculation yields roots of $\tilde{D}(\lambda)$ rather than the physical branch $D(\lambda)$. Since the formulas (4.5.133)-(4.5.138) involve only $D(\lambda)$ it is important to know which of these are actually roots of $D(\lambda)$ and also to know the multiplicities. This is done in the next two sections.

Discrete spectrum $\lambda = 0$

In order to check that the roots of $\tilde{D}(\lambda)$ given in Corollary 4.5.33 are actually roots of $D(\lambda)$ it suffices to check explicitly that $D(\lambda)$ vanishes (with the assumption that we are on the physical branch defined by Im $k_\pm > 0$ for $\lambda \in \mathbb{C} \setminus \mathcal{C}_\pm$.

For $j = 1$ we have $\gamma = \gamma_1 = a = 2\sqrt{\omega}$ and then $\lambda_1 = 0$. For $j = 2$ we have $\gamma = \gamma_2 = a'C^2$. If $|\gamma_2| = 2\sqrt{\omega}$ (equivalently $|a'| = a/C^2$) or $\gamma_2 = 0$ (equivalently $a' = 0$), we have $\lambda_2 = 0$.

Let us check that $\lambda = 0$ is a root of $D(\lambda)$:

$$D(0) = \alpha^2 - \beta^2 + 2i\alpha 2i\sqrt{\omega} + 4\omega = (a + b/2)^2 - b^2/4 - 2(a + b/2)a + a^2 = 0$$

since $k_\pm = i\sqrt{\omega}$. Now let us compute $D'(\lambda)$:

$$D'(\lambda) = i\alpha(\frac{i}{\sqrt{-\omega + i\lambda}} + \frac{-i}{\sqrt{-\omega - i\lambda}}) - (\frac{2i}{\sqrt{-\omega + i\lambda}} \cdot \sqrt{-\omega - i\lambda} + \frac{-2i}{\sqrt{-\omega - i\lambda}} \cdot \sqrt{-\omega + i\lambda}).$$

Hence $D'(0) = 0$ and $\lambda = 0$ is the root of $D(\lambda)$ of multiplicity at least 2. Further calculation shows that the Taylor series for D near zero takes the form:

$$D(\lambda) = (\frac{1}{\omega} - \frac{b}{4\omega^{3/2}})\lambda^2 + O(\lambda^4). \tag{4.5.148}$$

Therefore $\lambda = 0$ is the root of $D(\lambda)$ of multiplicity 4 if and only if $b = 4\sqrt{\omega}$, i.e. $a' = a/C^2$, and we have proved the following lemma:

Lemma 4.5.35. *If $a' = a/C^2$ then $\lambda = 0$ is a root of the determinant $D(\lambda)$ with multiplicity 4, otherwise $\lambda = 0$ is a root of the determinant $D(\lambda)$ with multiplicity 2.*

Nonzero discrete spectrum

Now let us check whether the roots $\lambda = \lambda_2 \neq 0$ corresponding $\gamma = \gamma_2 \notin \{0, \pm 2\sqrt{\omega}\}$ lie on the physical branch. We analyze two different cases: $0 < |\gamma_2| < 2\sqrt{\omega}$ and $|\gamma_2| > 2\sqrt{\omega}$.
I.The case $\mathbf{0} < |\gamma_\mathbf{2}| < \mathbf{2}\sqrt{\omega}$ (equivalently $0 < |a'| < a/C^2$).
Since $4\omega - \gamma_2^2 > 0$, the corresponding roots λ_2 are pure imaginary by (4.5.147). Moreover, $|\lambda_2| \leq \omega$. Indeed, (4.5.147) implies

$$\omega^2 - |\lambda_2|^2 = \omega^2 + \gamma_2^4/4 - \gamma_2^2\omega = (\omega - \gamma_2^2/2)^2 \geq 0.$$

Hence $-\omega \mp i\lambda_2 \leq 0$ and $k_\pm$ are pure imaginary with nonnegative imaginary part, that is

$$k_+k_- \leq 0 \text{ and Im } (\mathrm{k}_+ + \mathrm{k}_-) > 0. \tag{4.5.149}$$

The equations (4.5.145) and (4.5.140) imply

$$|k_+k_-| = \frac{1}{4}|a^2 - 2(a')^2C^4|, \;\; (k_+ + k_-)^2 = -2\omega + 2k_+k_- = -\frac{a^2}{2} + 2k_+k_-. \tag{4.5.150}$$

In order to obtain k_+k_- and $k_+ + k_-$ from the last two equations we have to divide the set $0 < |a'| < a/C^2$ onto three subsets:

$$(-a/C^2, a/C^2) \setminus \{0\} = (-a/C^2, -a/\sqrt{2}C^2] \cup \Big((-a/\sqrt{2}C^2, a/\sqrt{2}C^2) \setminus \{0\}\Big) \cup [\frac{a}{\sqrt{2}C^2}, \frac{a}{C^2}).$$

1) First consider the case $a' \in [\frac{a}{\sqrt{2}C^2}, \frac{a}{C^2})$. Then (4.5.149) and (4.5.150) imply

$$k_+k_- = \frac{1}{4}(a^2 - 2(a')^2C^4).$$

$$(k_+ + k_-)^2 = -\frac{a^2}{2} + \frac{a^2}{2} - (a')^2C^4 = -(a')^2C^4,$$
$$k_+ + k_- = ia'C^2,$$

and using (4.5.139), we obtain

$$D(\lambda_2) = (a + a'C^2)^2 - (a'C^2)^2 + 2i(a + a'C^2)(k_+ + k_-) - 4k_+k_-$$
$$= a^2 + 2aa'C^2 - 2(a + a'C^2)a'C^2 - a^2 + 2(a')^2C^4 = 0.$$

Note that each γ_2 defines two values λ_2 up to factor ± 1. If we replace λ_2 by $-\lambda_2$, k_+ and k_- change places and our calculation remains valid. Therefore, both values of λ_2 are roots of $D(\lambda)$.

2) Further consider $a' \in (-\frac{a}{C^2}, -\frac{a}{\sqrt{2}C^2}]$. In this case

$$k_+k_- = \frac{1}{4}(a^2 - 2(a')^2C^4), \quad k_+ + k_- = -ia'C^2.$$

Then we have

$$D(\lambda_2) = a^2 + 2aa'C^2 + 2(a + a'C^2)a'C^2 - a^2 + 2(a')^2C^4 = 4a'C^2(a + a'C^2) \neq 0$$

since $a' \neq 0$ and $a' \neq -a/C^2$. Therefore in this case both values of λ_2 are not the roots of $D(\lambda)$.

3) Finally consider $0 < |a'| < \frac{a}{\sqrt{2}C^2}$. Then (4.5.149)-(4.5.150 imply that

$$k_+k_- = -\frac{1}{4}(a^2 - 2(a')^2C^4) < 0,$$

$$(k_+ + k_-)^2 = -a^2 + (a')^2C^4 < 0,$$
$$k_+ + k_- = i\sqrt{a^2 - (a')^2C^4}.$$

Then we have

$$D(\lambda_2) = a(a + 2a'C^2) - 2(a + a'C^2)\sqrt{a^2 - (a')^2C^4} + a^2 - 2(a')^2C^4. \tag{4.5.151}$$

To solve the equation $D(\lambda_2) = 0$ with respect to a', divide the RHS of (4.5.151) by $C^4 \neq 0$ and denote $p = a/C^2 > 0$. Then we get the equation

$$p^2 + pa' - (a')^2 = (p + a')\sqrt{p^2 - (a')^2},\ 0 < |a'| < p/\sqrt{2}. \tag{4.5.152}$$

Squaring both side of (4.5.152), we get

$$2(a')^4 - p^2(a')^2 = 0$$

The equation has no solutions for $0 < |a'| < p/\sqrt{2}$ and hence $D(\lambda_2)$ does not vanish.

Corollary 4.5.36. *i)* $D(\lambda_2) = 0$ *if* $a' \in [\frac{a}{\sqrt{2}C^2}, \frac{a}{C^2})$.
ii) $D(\lambda_2) \neq 0$ *if* $a' \in (-\frac{a}{C^2}, \frac{a}{\sqrt{2}C^2}) \setminus \{0\}$.

II. The case $|\gamma_2| > 2\sqrt{\omega}$ (equivalently $|a'| > a/C^2$).
Since $4\omega - \gamma_2^2 < 0$, the corresponding roots (4.5.147) are real: $\lambda_2^- < 0 < \lambda_2^+$, $\lambda_2^- = -\lambda_2^+$. It is easy to prove that $k_\pm$ take the form:

$$k_\pm = \pm\mu + i\nu, \quad \nu > 0.$$

Therefore

$$k_+k_- = -\mu^2 - \nu^2 < 0, \quad k_+ + k_- = 2i\nu \tag{4.5.153}$$

1) First consider the case $a' > a/C^2$.Then by (4.5.150) and (4.5.153)

$$k_+k_- = \frac{1}{4}(a^2 - 2(a')^2C^4), \quad (k_+ + k_-)^2 = -(a')^2C^4, \quad k_+ + k_- = ia'C^2.$$

Therefore

$$D(\lambda_2) = a(a + 2a'C^2) + 2i(a + a'C^2)(k_+ + k_-) - 4k_+k_- =$$
$$a(a + 2a'C^2) - 2(a + a'C^2)a'C^2 - a^2 + 2(a')^2C^4 = 0$$

and then λ_2 are real roots of $D(\lambda)$. Hence, the case $a' > a/C^2$ is **linearly unstable**.
2) Further consider the case $a' < -a/C^2$. Then

$$k_+k_- = \frac{1}{4}(a^2 - 2(a')^2C^4) < 0, \quad k_+ + k_- = -ia'C^2,$$

$$D(\lambda_2) = a^2 + 2aa'C^2 + 2(a + a'C^2)a'C^2 - a^2 + 2(a')^2C^4 = 4a'C^2(a + a'C^2) \neq 0$$

Therefore, in this case λ_2 are not roots of $D(\lambda)$.

Corollary 4.5.37. *i) In the unstable case* $a' > a/C^2$*: both* λ_2 *are roots of* $D(\lambda)$.
ii) If $a' < -a/C^2$ *then neither of the* λ_2 *are roots of* $D(\lambda)$.

Summarising, we have proved the following result

Theorem 4.5.38. *i) If* $a' \in (-\infty, a/(\sqrt{2}C^2))$ *the only root of* $D(\lambda)$ *is* $\lambda = 0$ *with multiplicity* 2.
ii) If $a' \in [a/\sqrt{2}C^2, a/C^2)$*, there are four roots of* $D(\lambda)$*: zero (multiplicity two) and* $\pm i|\lambda_2|$ *(pure imaginary) with* λ_2 *as in (4.5.147).*
iii) If $a' = +a/C^2$*, the only root of* $D(\lambda)$ *is* $\lambda = 0$ *multiplicity* 4.
iv) If $a' \in (a/C^2, +\infty)$*, there are four roots of* $D(\lambda)$*: zero (multiplicity two) and* $\pm|\lambda_2|$ *with* λ_2 *as in (4.5.147). In particular there exists a positive root (linear instability).*

Remark 4.5.39. Imagine reducing a' starting from a value greater than a/C^2. Initially there are two real roots, $\pm|\lambda_2|$, which approach zero as $a' \to a/C^2$ from above, giving rise to an increase of the multiplicity of the $\lambda = 0$ root to four when $a' = a/C^2$. As a' is reduced further below a/C^2 these two roots reappear as a pair of conjugate pure imaginary roots which move from zero to $\pm i\omega$ as a' goes from a/C^2 to $a/\sqrt{2}C^2$. When $a' = a/\sqrt{2}C^2$ these two roots touch the branch point (end of the continuous spectrum) and move onto an 'unphysical branch' (on which the conditions (4.5.120) do not hold). As a' is reduced further these roots do not return to the physical branch and thus even when their magnitude becomes zero they do not coalesce with the physical $\lambda = 0$ root to increase its multiplicity and most importantly the spectrum is pure continuous apart from zero for $a' < a/C^2$.

Chapter 5

Adiabatic effective dynamics of solitons

In this chapter we present without proofs the results of [87] on *adiabatic effective dynamics* for the wave-particle system (1.5.1)–(1.5.2) in the case of *slowly varying external potential*. We also discuss the related *mass-energy equivalence*.

5.1 Solitons in slowly varying external potentials

In this section we describe the first result [87] on adiabatic effective dynamics. The solitons (2.1.3) are solutions to the system (1.5.1)–(1.5.2) with zero external potential $V(x) \equiv 0$. The asymptotic stability of the corresponding solitary manifold, proved in [127], means the *soliton-like asymptotics*

$$\psi(x,t) \approx \psi_{v(t)}(x - q(t)), \qquad t \in \mathbb{R} \tag{5.1.1}$$

for any solution with initial state sufficiently close to this manifold. On the other hand, solutions of this form may exist even for the system (1.5.1)–(1.5.2) with nonzero external potential if this potential is slowly varying:

$$|\nabla V(q)| \le \varepsilon \ll 1. \tag{5.1.2}$$

In this case, the total momentum (2.1.2) is generally not conserved, but its slow evolution and the (fast) evolution of the parameter $q(t)$ in (5.1.1) can be described in terms of some finite-dimensional Hamiltonian dynamics.

Namely, denote by $P = P_v$ the total momentum of the soliton $S_{v,Q}$ in the notation (2.1.7). It is important that the map $\mathcal{P} : v \mapsto P_v$ is an isomorphism of the ball $|v| < 1$ on R^3. Therefore, we can consider Q, P as global coordinates on the solitary manifold $\mathcal{S}$. We define effective Hamiltonian functional

$$\mathcal{H}_{\text{eff}}(Q, P_v) \equiv \mathcal{H}(S_{v,Q}), \qquad Q, P_v \in \mathbb{R}^3, \tag{5.1.3}$$

where $\mathcal{H}$ is the total Hamiltonian (1.5.4). This functional allows the splitting $\mathcal{H}_{\text{eff}}(Q,\Pi) = E(\Pi)+V(Q)$ since the first integral in (1.5.4) does not depend on Q while the last integral vanishes on the solitons. Hence, the corresponding Hamiltonian equations read

$$\dot Q(t) = \nabla E(\Pi(t)), \qquad \dot\Pi(t) = -\nabla V(Q(t)). \tag{5.1.4}$$

The main result of [87] is the following theorem.

Theorem 5.1.1. *Let condition (5.1.2) hold, the initial state $S_0 = (\psi_0, \pi_0, q_0, p_0) \in \mathcal{S}$ is a soliton with total momentum P_0, and $\psi(x,t), \pi(x,t), q(t), p(t)$ of the system (1.5.1)–(1.5.2). Then the following 'adiabatic asymptotics' holds*

$$|q(t) - Q(t)| \le C_0, \quad |P(t) - \Pi(t)| \le C_1\varepsilon \quad \text{for} \quad |t| \le C\varepsilon^{-1}, \tag{5.1.5}$$

$$\sup_{t\in\mathbb{R}} \Big[\|\nabla[\psi(q(t)+x,t) - \psi_{v(t)}(x)]\|_R + \|\pi(q(t)+x,t) - \pi_{v(t)}(x)\|_R \Big] \le C\varepsilon, \tag{5.1.6}$$

where $P(t)$ denotes total momentum (2.1.2), $v(t) = \mathcal{P}^{-1}(\Pi(t))$, and $(Q(t),\Pi(t))$ is the solution to the effective Hamiltonian equations (5.1.4) with initial conditions

$$Q(0) = q(0), \qquad \Pi(0) = P(0).$$

Note that such relevance of effective dynamics (5.1.4) is due to the consistency of Hamiltonian structures:

1) The effective Hamiltonian (5.1.3) is a restriction of the Hamiltonian functional (1.5.4) onto the soliton manifold $\mathcal{S}$.

2) As shown in [87], the canonical form of the Hamiltonian system (5.1.4) is also a restriction onto $\mathcal{S}$ of canonical form of the system (1.5.1)–(1.5.2): formally

$$P\,dQ = \big[p\,dq + \int dx\,\pi(x)\,d\psi(x)\big]\Big|_{\mathcal{S}}.$$

Therefore, the total momentum P is canonically conjugate to the variable Q on the solitary manifold $\mathcal{S}$. This fact justifies definition (5.1.3) of the effective Hamiltonian as a function of the total momentum P_v, and not of the particle momentum p_v.

One of the important results of [87] is the following 'effective dispersion relation':

$$E(\Pi) \sim \frac{\Pi^2}{2(1+m_e)} + \text{const}, \qquad |\Pi| \ll 1. \tag{5.1.7}$$

It means that non-relativistic mass of a slow soliton increases due to an interaction with the field by the amount

$$m_e = -\frac{1}{3}\langle \rho, \Delta^{-1}\rho\rangle. \tag{5.1.8}$$

This increment is proportional to the field energy of a soliton in rest

$$\mathcal{H}(\Delta^{-1}\rho, 0, 0, 0) = -\frac{1}{2}\langle \rho, \Delta^{-1}\rho\rangle,$$

which agrees with the Einstein mass-energy equivalence principle (see below).

Remark 5.1.2. The relation (5.1.7) gives only a hint that m_e is an increment of the effective mass. The true *dynamical justification* for such an interpretation is given by the adiabatic asymptotics (5.1.5)–(5.1.6) which demonstrate the relevance of the effective dynamics (5.1.4).

Generalizations. In [88], the asymptotics (5.1.5), (5.1.6) were extended to solitons of the Maxwell–Lorentz equations (1.6.1) with small external fields.

After the papers [87, 88] suitable adiabatic effective dynamics was obtained in [85, 86] for nonlinear Hartree and Schrödinger equations with slowly varying external potentials. Similar effective dynamics in presence of small external fields later was constructed i) in [84, 89, 90] - for nonlinear systems of Einstein–Dirac, Chern–Simon–Schrödinger, Klein–Gordon–Maxwell systems, and ii) in [91] - for Maxwell–Lorentz equations with rotating charge. Similar adiabatic effective dynamics was established in [83] for electron in second-quantized Maxwell field in presence of a slowly varying external potential.

The results of numerical simulation [61] (see the next chapter) confirm the adiabatic effective dynamics of solitons (5.1.6) for relativistically-invariant 1D nonlinear wave equations.

5.2 Mass–Energy equivalence

In the case of the Maxwell–Lorentz equations [88] the increment of nonrelativistic mass also turns out to be proportional to the energy of the static soliton's own field.

Such equivalence of the self-energy of a particle with its mass was first discovered by M. Abraham in 1902: he obtained by direct calculation that electromagnetic self-energy $E_{\rm own}$ of an electron at rest adds

$$m_e = \frac{4}{3} E_{\rm own}/c^2$$

to its non-relativistic mass (see [203, 204], and also [213, pp 216–217]). It is easy to see that this self-energy is infinite for a point electron at the origin with a charge density $\delta(x)$, because in this case, the Coulomb electrostatic field $|E(x)| = C/|x|^2$ so the integral in (1.6.3) diverges around $x = 0$. This means that the field mass for a point electron is infinite, which contradicts experiment. That's why M. Abraham introduced the model of electrodynamics with 'extended electron' (1.6.1), whose self-energy is finite.

At the same time, M. Abraham conjectured that the *entire mass* of an electron is due to its own electromagnetic energy; that is, $m = m_e$: "... *matter disappeared, only energy remains* ... ", see [210, pp 63, 87, 88] (smile :)).

This conjecture was justified in 1905 by A. Einstein, who discovered the famous universal relation $E = m_0 c^2$, which follows from Special Theory of Relativity [206]. The doubtful factor $\frac{4}{3}$ in the M. Abraham formula is due to nonrelativistic character of the system (1.6.1). According to modern view, about 80% of the electron mass is of electromagnetic origin [207].

Chapter 6

Numerical Simulation of Solitons

In this chapter we describe the results of joint work with A.P. Vinnichenko (1945–2009) on numerical simulation of i) global attraction to solitons (0.0.12) and (0.0.13), and ii) adiabatic effective dynamics of solitons (5.1.6) for relativistically-invariant 1D nonlinear wave equations. Additional information can be found in [61].

6.1 Kinks of relativistically-invariant equations

First let us describe numerical simulations of solutions to relativistically-invariant 1D nonlinear wave equations with a polynomial nonlinearity

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \qquad \text{where} \quad F(\psi) := -\psi^3 + \psi. \tag{6.1.1}$$

Since $F(\psi) = 0$ for $\psi = 0, \pm 1$, there are three stationary states: $S(x) \equiv 0, +1, -1$. This equation is formally equivalent to a Hamiltonian system (1.1.2) with the Hamiltonian

$$H(\psi,\pi) = \int [\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x))]\,dx \tag{6.1.2}$$

where the potential is $U(\psi) = \frac{\psi^4}{4} - \frac{\psi^2}{2} + \frac{1}{4}$. This Hamiltonian is finite for functions $(\psi,\pi) \in \mathcal{E}$, where $\mathcal{E} = H^1_c \oplus L^2$ (see (1.1.4)), for which the convergence

$$\psi(x) \to \pm 1, \qquad |x| \to \infty$$

is sufficiently fast.

The potential $U(\psi)$ has minima at $\psi = \pm 1$ and a maximum at $\psi = 0$. Correspondingly, two finite energy solutions $\psi = \pm 1$ are stable, and the solution $\psi = 0$ with infinite energy is unstable. Such potentials with two wells are called potentials of Ginzburg–Landau type.

In addition to the constant stationary solutions $S(x) \equiv 0, +1, -1$, there is also a non-constant solution $S(x) = \tanh x/\sqrt{2}$, which is called a ‘kink’. Its shifts and reflections $\pm S(\pm x - a)$ are also stationary solutions, as well as their Lorentz transforms

$$\pm S(\gamma(\pm x - a - vt)), \qquad \gamma = 1/\sqrt{1 - v^2}, \quad |v| < 1.$$

These are uniformly moving ‘travelling waves’ (that is, solitons). The kink is strongly compressed when the velocity v is close to ± 1. This compression is known as the ‘Lorentz contraction’.

Numerical Simulation. Our numerical experiments show a decay of finite energy solutions to a finite set of kinks and dispersion waves outside the kinks, which corresponds to the asymptotics of type (0.0.13). The result of one of the experiments is shown in Fig. 6.1: a finite energy solution of the equation (6.1.1) decays to three kinks. The vertical line is the time axis, and the horizontal line is the space axis. The spatial scale redoubles at $t = 20$ and $t = 60$.

The red colour corresponds to the values $\psi > 1 + \varepsilon$, the blue colour to the values $\psi < -1 - \varepsilon$, and the yellow colour to the intermediate values $-1 + \varepsilon < \psi < 1 - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Thus, the yellow stripes represent the kinks, while the blue and red zones outside the yellow stripes are filled with dispersion waves.

For $t = 0$ the solution begins with a rather chaotic behaviour, when there are no visible kinks. After 20 seconds, three separate kinks appear, which subsequently move almost uniformly.

The Lorentz contraction. The left kink moves to the left at a low velocity $v_1 \approx 0.24$, the central kink is almost standing, because its velocity $v_2 \approx 0.02$ is very small, and the right kink moves very fast with velocity $v_3 \approx 0.88$. The Lorentz spatial contraction $\sqrt{1 - v_k^2}$ is clearly visible in this picture: the central kink is the widest, the left is a bit narrower, and the right one is quite narrow.

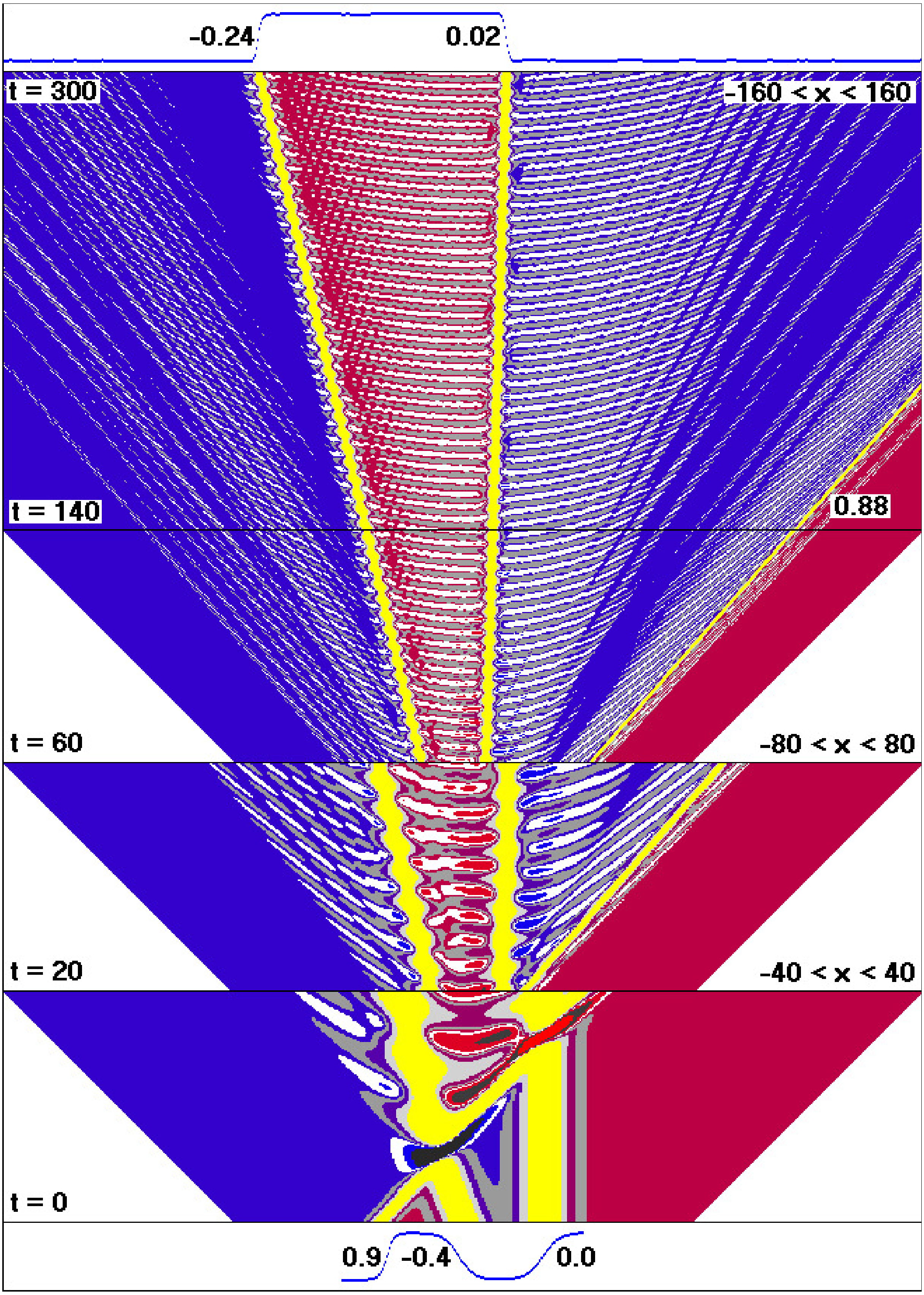

Figure 6.1: Decay to three kinks.

The Einstein time-delay. The Einstein time-delay is also very pronounced. Namely, all three kinks pulsate because of the presence of a non-zero eigenvalue in the equation linearised on the kink. Indeed, substituting $\psi(x,t) = S(x) + \varepsilon\varphi(x,t)$ in (6.1.1), we get in the first-order approximation the linearised equation

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - 2\varphi(x,t) - V(x)\varphi(x,t), \tag{6.1.3}$$

where the potential

$$V(x) = 3S^2(x) - 3 = -\frac{3}{\cosh^2 x/\sqrt{2}}$$

decays exponentially for large $|x|$. It is very fortunate that for this potential the spectrum of the corresponding *Schrödinger operator*

$$H := -\frac{d^2}{dx^2} + 2 + V(x)$$

is well known [62]. Namely, the operator H is non-negative, and its continuous spectrum is the interval $[2, \infty)$. It turns out that H also has a two-point discrete spectrum: the points $\lambda = 0$ and $\lambda = 3/2$. It is this non-zero eigenvalue that is responsible for the pulsations that we observe for the central slow kink, with frequency $\omega_2 \approx \sqrt{3/2}$ and period $T_2 \approx 2\pi/\sqrt{3/2} \approx 5$. On the other hand, for the fast kinks the ripples are much slower, that is, the corresponding period is longer. This time-delay agrees numerically with the Lorentz formulas, which confirms the relevance of these results of numerical simulation.

Dispersion waves. An analysis of dispersion waves provides additional confirmation. Namely, the space outside the kinks in Fig. 6.1 is filled with dispersion waves whose values are very close to ± 1, with an accuracy of 0.01. These waves satisfy with high accuracy the linear Klein–Gordon equation obtained by linearisation of the Ginzburg–Landau equation (6.1.1) on the stationary solutions$\psi_\pm \equiv \pm 1$:

$$\ddot{\varphi}(x,t) = \varphi''(x,t) + 2\varphi(x,t).$$

The corresponding dispersion relation $\omega^2 = k^2 + 2$ determines the group velocities of high-frequency wave packets:

$$\omega'(k) = \frac{k}{\sqrt{k^2+2}} = \pm\frac{\sqrt{\omega^2 - 2}}{\omega}. \tag{6.1.4}$$

These wave packets are clearly visible in Fig. 6.1 as straight lines whose propagation velocities converge to ± 1. This convergence is explained by the high-frequency limit $\omega'(k) \to \pm 1$ as $\omega \to \pm\infty$. For example, for dispersion waves emitted by the central kink the frequencies $\omega = \pm n\omega_2 \to \pm\infty$ are generated by the polynomial nonlinearity in (6.1.1) in accordance with Fig. 3.2.

Remark 6.1.1. These observations of dispersion waves agree with the radiation mechanism in Section 3.9.

The nonlinearity in (6.1.1) is chosen exactly because of the well-known spectrum of the linearised equation (6.1.3). In numerical experiments [61], more general nonlinearities of Ginzburg–Landau type have also been considered. The results were qualitatively the same: for 'any' initial data of finite energy, the solution decays for large times to a sum of kinks and dispersion waves. Numerically, this is clearly visible, but rigorous justification remains an open problem.

6.2 Numerical observation of soliton asymptotics

Besides the kinks the numerical experiments [61] also revealed soliton asymptotics of type (0.0.13) and adiabatic effective dynamics of the form (5.1.6) for complex solutions of the 1D relativistically-invariant nonlinear wave equations (2.2.4). polynomial potentials of the form

$$U(\psi) = a|\psi|^{2m} - b|\psi|^{2n}, \tag{6.2.1}$$

were considered with $a, b > 0$ and $m > n = 2, 3, \ldots$. Correspondingly,

$$F(\psi) = 2am|\psi|^{2m-2}\psi - 2bn|\psi|^{2n-2}\psi. \tag{6.2.2}$$

The parameters a, b, m, n were taken as follows,

N	a	m	b	n
1	1	3	0.61	2
2	10	4	2.1	2
3	10	6	8.75	5

Various 'smooth' initial functions $\psi(x, 0), \dot{\psi}(x, 0)$ with supports on the interval $[-20, 20]$ were considered. The second-order difference scheme with $\Delta x \sim 0.01$ and $\Delta t \sim 0.001$ was employed. In all cases, the asymptotics of type (0.0.13) were observed with the numbers 0, 1, 3 and 5 of solitons for $t > 100$.

6.3 Adiabatic effective dynamics of relativistic solitons

In the numerical experiments [61] the adiabatic effective dynamics of the form (5.1.6) was also observed for soliton-like solutions of type (5.1.1) of the 1D equations (2.2.4) with a slowly varying external potential (5.1.2):

$$\ddot{\psi}(x,t) = \psi''(x,t) - \psi(x,t) + F(\psi(x,t)) - V(x)\psi(x,t), \qquad x \in \mathbb{R}. \tag{6.3.1}$$

This equation is formally equivalent to the Hamiltonian system (1.1.2) with the Hamiltonian

$$H_V(\psi,\pi) = \int [\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x)) + \frac{1}{2}V(x)|\psi(x)|^2]\,dx. \tag{6.3.2}$$

The soliton-like solutions are of the form (cf. (5.1.1))

$$\psi(x,t) \approx e^{i\Theta(t)}\phi_{\omega(t)}(\gamma_{v(t)}(x-q(t))). \tag{6.3.3}$$

The numerical experiments [61] qualitatively confirm the adiabatic effective Hamiltonian dynamics for the parameters Θ, ω, q, and v, but it has not yet been rigorously justified.

Figure 6.2 represents a solution to equation (6.3.1) with the potential (6.2.1), where $a = 10$, $m = 6$ and $b = 8.75$, $n = 5$. The potential is $V(x) = -0.2\cos(0.31x)$ and the initial conditions are

$$\psi(x,0) = \phi_{\omega_0}(\gamma_{v_0}(x-q_0)), \qquad \dot{\psi}(x,0) = 0, \tag{6.3.4}$$

where $v_0 = 0$, $\omega_0 = 0.6$ and $q_0 = 5.0$. We note that the initial state does not belong to the solitary manifold. The effective width (half-amplitude) of the solitons is in the range $[4.4, 5.6]$. It is quite small when compared with the spatial period of the potential $2\pi/0.31 \sim 20$. The results of the numerical simulations are shown in Fig. 6.2.

• The blue and green colours represent a dispersion wave with values $|\psi(x,t)| < 0.01$, while the red colour represents the top of a soliton with values $|\psi(x,t)| \in [0.4, 0.8]$.

• The soliton trajectory ('red snake') corresponds to oscillations of a classical particlein the potential $V(x)$.

• For $0 < t < 140$ the solution is rather distant from the solitary manifold, and the radiation is rather intense.

• For $3020 < t < 3180$ the solution approaches the solitary manifold, and the radiation weakens. The oscillation amplitude of the soliton is almost unchanged over a long time, confirming the Hamiltonian type of the effective dynamics.

• However, for $5260 < t < 5420$ the amplitude of the soliton oscillation is halved. This suggests that on a large time scale the deviation from Hamiltonian effective dynamics becomes essential. Consequently, the effective dynamics gives a good approximation only on an adiabatic time scale of type $t \sim \varepsilon^{-1}$.

• The deviation of the effective dynamics from being Hamiltonian is due to radiation, which plays the role of dissipation.

• The radiation is realised as dispersion waves, which carry energy to infinity. The dispersion waves combine into uniformly moving wave packets with a discrete set of group velocities, as in Fig. 6.1. The magnitude of the solution is of order ~ 1 on the trajectory of the soliton, while the values of the dispersion waves is less than 0.01 for $t > 200$, so

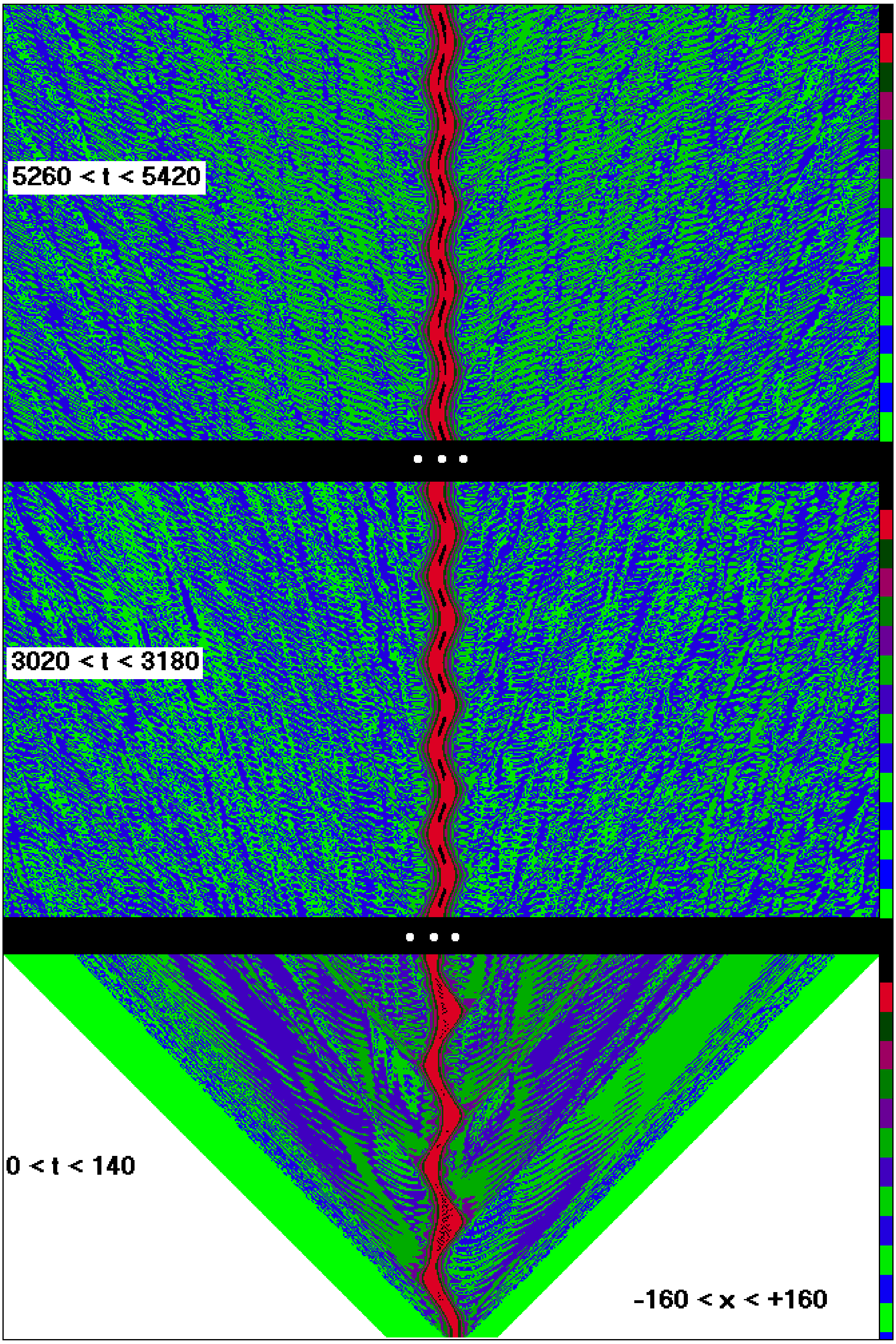

Figure 6.2: Adiabatic effective dynamics of relativistic solitons.

that their energy density does not exceed 0.0001. The amplitude of the dispersion waves decays at large times.

• In the limit as $t \to \pm\infty$ the soliton should converge to a static position corresponding to a local minimum of the potential $V(x)$. However, the numerical observation of this 'ultimate stage' is hopeless, since the rate of the convergence decays with the decrease of the radiation.

Chapter 7

Dispersion Decay

In this chapter we give i) a brief survey of basic results on the dispersion decay (Section 7.1), and ii) new short and simplified proof of the fundamental results on the $L^1 \to L^\infty$ dispersion decay established by J.-L. Journé, A. Soffer and C.D. Sogge in [185] for the Schrödinger equation (7.1.2) with $n \geq 3$ (Section 7.2).

7.1 The Schrödinger and Klein–Gordon equations

7.1.1 Dispersion decay in weighted Sobolev norms.

A powerful systematic approach to dispersion decay in weighted Sobolev norms for the Schrödinger equation with potential was proposed by S. Agmon, A. Jensen and T. Kato [172, 184]. This theory was extended by many authors to wave, Klein–Gordon and Dirac equations and to the corresponding discrete equations, see [10, 142, 143], [173]–[183] and [185]–[198], [199, 200] and references therein.

7.1.2 $L^1 - L^\infty$ decay

$$\|P_c\psi(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-n/2}\|\psi(0)\|_{L^1(\mathbb{R}^n)}, \qquad t > 0 \tag{7.1.1}$$

for solutions of linear Schrödinger equation

$$i\dot\psi(x,t) = H\psi(x,t) := (-\Delta + V(x))\psi(x,t), \qquad x \in \mathbb{R}^n \tag{7.1.2}$$

with $n \ge 3$ was established for the first time by J.-L. Journé, A. Soffer and C.D. Sogge [185] provided that $\lambda = 0$ is neither an eigenvalue nor resonance for H. The potential $V(x)$ is sufficiently smooth and rapidly decays as $|x| \to \infty$. Here P_c is an orthogonal projection onto continuous spectral space of the operator H. This result was generalised later by many authors, see below.

In [200] a decay of type (7.1.1) and Strichartz estimates were established for 3D Schrödinger equations (7.1.2) with "rough" and time-dependent potentials $V = V(x,t)$ (in stationary case $V(x)$ belongs to both the Rollnik class and the Kato class). Similar estimates were received in [176] for 3D Schrödinger and wave equations with (stationary) Kato class potentials.

In [180] the 4D Schrödinger equations (7.1.2) are considered for the case when there is a resonance or an eigenvalue at zero energy. In particular, in the case of an eigenvalue at zero energy, there is a time-dependent operator F_t of rank 1, such that $\|F_t\|_{L^1\to L^\infty} \le 1/\log t$ for $t > 2$, and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \le Ct^{-1}, \qquad t > 2.$$

Similar dispersion estimates were proved also for solutions to 4D wave equation with a potential.

In [182, 183] the Schrödinger equation (7.1.2) is considered in $\mathbb{R}^n$ with $n \ge 5$ when there is an eigenvalue at the zero point of the spectrum. It is shown, in particular, that there is a time-dependent rank one operator F_t such that $\|F_t\|_{L^1\to L^\infty} \le C|t|^{2-n/2}$ for $|t| > 1$, and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \le C|t|^{1-n/2}, \qquad |t| > 1.$$

With a stronger decay of the potential, the evolution admits an operator-valued expansion

$$e^{itH}P_c(H) = |t|^{2-n/2}A_{-2} + |t|^{1-n/2}A_{-1} + |t|^{-n/2}A_0,$$

where A_{-2} and A_{-1} are finite rank operators $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, while A_0 maps weighted L^1 spaces to weighted L^∞ spaces. Main members A_{-2} and A_{-1} equal to zero under certain conditions of the orthogonality of the potential V to eigenfunction with zero energy. Under the same orthogonality conditions, the remainder term $|t|^{-n/2}A_0$ also maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, and therefore, the group $e^{itH}P_c(H)$ has the same dispersion decay as free evolution, despite its eigenvalue at zero.

7.1.3 $L^p - L^q$ decay for the Klein–Gordon equation

Such decay was first established in [199] for solutions of the free Klein–Gordon equation $\ddot{\psi} = \Delta\psi - \psi$ with initial state $\psi(0) = 0$:

$$\|\psi(t)\|_{L^q} \le Ct^{-d}\|\dot{\psi}(0)\|_{L^p}, \qquad t > 1, \tag{7.1.3}$$

where $1 \le p \le 2$, $1/p + 1/q = 1$, and $d \ge 0$ is a piecewise-linear function of $(1/p, 1/q)$. The proofs use the Riesz interpolation theorem.

In [175], the estimates (7.1.3) were extended to solutions of perturbed Klein–Gordon equation

$$\ddot{\psi} = \Delta\psi - \psi + V(x)\psi$$

with $\psi(0) = 0$. The authors show that (7.1.3) holds for $0 \le 1/p - 1/2 \le 1/(n+1)$. The smallest value of p and the fastest decay rate d occurs when $1/p = 1/2 + 1/(n+1)$, $d = (n-1)/(n+1)$. The result is proved under the assumption that the potential V is smooth and small in a suitable sense. For example, the result true when $|V(x)| \le c(1+|x|^2)^{-\sigma}$, where $c > 0$ is sufficiently small. Here $\sigma > 2$ for $n = 3$, $\sigma > n/2$ for odd $n \ge 5$, and $\sigma > (2n^2 + 3n + 3)/4(n+1)$ for even $n \ge 4$. The results also apply to the case when $\psi(0) \ne 0$.

7.1.4 $L^p - L^q$ decay for the Schrödinger equation

The seminal article [185] concerns $L^p - L^q$ decay of solutions to the Schrödinger equation (7.1.2). It is assumed that $(1 + |x|^2)^\alpha V(x)$ is a multiplier in the Sobolev spaces H^η for some $\eta > 0$ and $\alpha > n + 4$, and the Fourier transform of V belongs to $L^1(\mathbb{R}^n)$. Under this conditions, the main result of [185] is the following theorem: if $\lambda = 0$ is neither an eigenvalue nor a resonance for H, then

$$\|P_c\psi(t)\|_{L^q} \le Ct^{-n(1/p-1/2)}\|\psi(0)\|_{L^p}, \qquad t > 1, \tag{7.1.4}$$

where $1 \le p \le 2$ and $1/p + 1/q = 1$. Proofs are based on $L^1 - L^\infty$ decay (7.1.1) and the Riesz interpolation theorem.

In [202] estimates (7.1.4) were proved for the Schrödinger equation (7.1.2) under suitable conditions on the decay of $V(x)$ i) with $1 \le p \le 2$ if $\lambda = 0$ is neither an eigenvalue nor a resonance for H, and ii) with all $3/2 < p \le 2$ otherwise.

7.1.5 The Strichartz estimates

The Strichartz estimates were extended i) in [173] to the Schrödinger magnetic equations in $\mathbb{R}^n$ with $n \ge 3$, ii) in [174] - to wave equations with a magnetic potential in $\mathbb{R}^n$ for $n \ge 3$, and iii) in [177] - to wave equation in $\mathbb{R}^3$ with potentials of the Kato class.

7.2 $L^1 - L^\infty$ decay for 3D Schrödinger equation

In this section we give new short and simplified proof of the $L^1 - L^\infty$ dispersion decay (7.1.1) for the Schrödinger equation (7.1.2), first established in [185] for $n \geq 3$. We restrict ourselves by the case $n = 3$,

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t), \qquad x \in \mathbb{R}^3. \tag{7.2.1}$$

Another approach to the proof of this decay in the cases $n = 1$ and $n = 3$ was suggested by M. Goldberg and W. Schlag [181].

Our approach considerably simplifies the arguments of [185] and of [181]. We suppose that the potential $V(x)$ is a continuous real function, and

$$|V(x)| \leq C\langle x\rangle^{-\beta}, \qquad \langle x\rangle = (1 + |x|^2)^{1/2}, \qquad x \in \mathbb{R}^3 \tag{7.2.2}$$

for some $\beta > 3$. As in [185] and [181] we consider the 'regular case' when the point 0 is neither eigenvalue nor resonance for the Schrödinger operator $H = -\Delta + V(x)$. Equivalently, the truncated resolvent of the operator $\mathcal{H}$ is bounded at the edge point of the continuous spectrum.

Theorem 7.2.1. *Let condition (7.2.2) with $\beta > 3$ holds. Then in the regular case*

$$\|e^{itH}P_c(H)\|_{L^1\to L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1, \tag{7.2.3}$$

where $P_c(H)$ is the Riesz projection onto the continuous spectrum of H.

This theorem immediately implies the decay in weighted norms

$$\|\psi\|_{L^p_\sigma} = \|\langle x\rangle^\sigma\psi\|_{L^p}, \qquad \sigma \in \mathbb{R}.$$

Corollary 7.2.2. *Let (7.2.2) hold and $\sigma > 3/2$. Then in the regular case,*

$$\|e^{itH}P_c(H)\|_{L^2_\sigma\to L^2_{-\sigma}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}. \tag{7.2.4}$$

Indeed, for any bounded operator $K : L^1 \to L^\infty$ and any $f \in L^2_\sigma$ with $\sigma > 3/2$, one has

$$\|Kf\|_{L^2_{-\sigma}} \leq C\|Kf\|_{L^\infty} \leq C\|K\|_{L^1\to L^\infty}\|f\|_{L^1} \leq C_1\|K\|_{L^1\to L^\infty}\|f\|_{L^2_\sigma}.$$

Remark 7.2.3. *For $\sigma > 5/2$ the dispersion decay of type (7.2.4) for the 3D Schrödinger equation was established first by A. Jensen and T. Kato [184].*

Our proofs follow general strategy of [172, 181, 184] which relies on the spectral Fourier representation

$$e^{iHt}P_c(H) = \frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}\Big[R(\omega + i0) - R(\omega - i0)\Big]d\omega, \tag{7.2.5}$$

where $R(\omega) = (H - \omega)^{-1}$ is the resolvent of the Schrödinger operator H.

We verify the decay (7.2.3) of the integral (7.2.5) developing a streamlined version of the approach [181]. First note that this integral generally does not converge in the operator norm $L^2_\sigma \to L^2_{-\sigma}$ due to the slow decay of the resolvent in this norm like $\sim \omega^{-1/2}$

by the results of S. Agmon, A. Jensen and T. Kato [172, 184]. On the other hand, this integral converges in the sense of distributions, that is the integrals over intervals $[0, L]$ are tempered distributions $W_L(t)$ which converge as $L \to \infty$. Thus, (7.2.3) will follow from uniform in $L \geq 1$ estimates

$$\|W_L(t)\|_{L^1 \to L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1. \tag{7.2.6}$$

Note that for the free Schrödinger operator the decay holds since its integral kernel is bounded for $|t| \geq 1$ and decays uniformly in space:

$$\|e^{-i\Delta t}\|_{L^1 \to L^\infty} = \sup_{x,y \in \mathbb{R}^3} \left| \frac{e^{i|x-y|^2/4t}}{(4\pi i t)^{3/2}} \right| \leq C|t|^{-3/2}, \quad |t| \geq 1. \tag{7.2.7}$$

7.2.1 Properties of the resolvent

Here we collect the properties of the resolvent $R(\omega) = (H - \omega)^{-1}$ obtained in [172, 184] (see also [189] where the full proofs of these properties can be found). We suppose that the condition (7.2.2) holds with some $\beta > 1$. Then

R1. $R(\omega) : L^2 \to L^2$ is a meromorphic function of $\omega \in \mathbb{C} \setminus [0, \infty)$; the poles of $R(\omega)$ are located at a finite set of eigenvalues $\omega_j < 0$.

R2. For $\omega > 0$ and $\sigma > 1/2$ there exist the limits $R(\omega \pm i0)$ such that

$$\|R(\omega \pm i\varepsilon) - R(\omega \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} \to 0, \quad \varepsilon \to 0+.$$

R3. Let $\beta > 3$. Then for $\sigma > 1/2 + k$

$$\|R^{(k)}(\omega)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(|\omega|^{-\frac{1+k}{2}}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty), \quad k = 0, 1, 2. \tag{7.2.8}$$

R4. Let $\beta > 2$. Then in the regular case, $R^\pm(\omega) := R(\omega \pm i0)$ are continuous operator functions of $\omega \geq 0$ with the values in $B(L^2_\sigma, L^2_{-\sigma})$ for any $\sigma_1, \sigma_2 > 1/2$ with $\sigma_1 + \sigma_2 > 2$.

R5. Let $\beta > 3$. Then in the regular case,

$$\|R^\pm(\omega)\|_{L^2_{\sigma_1} \to L^2_{-\sigma_2}} = \mathcal{O}(1), \quad \omega \to 0, \quad \sigma_1, \sigma_2 > 1/2, \quad \sigma_1 + \sigma_2 > 2, \tag{7.2.9}$$

$$\|\partial^k_\omega R^\pm(\omega)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(|\omega|^{\frac{1}{2} - k}), \quad \omega \to 0, \quad \sigma > 1/2 + k, \quad k = 1, 2. \tag{7.2.10}$$

In particular, all these properties hold for the free resolvent $R_0(\omega) = (-\Delta - \omega)^{-1}$.

The integral kernels of $R^\pm_0(\lambda^2)$ have an explicit representation

$$R^\pm_0(\lambda^2, x, y) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}. \tag{7.2.11}$$

The asymptotics (7.2.9)–(7.2.10) imply

Lemma 7.2.4. *Let (7.2.2) hold with $\beta > 3$. Then in the regular case*

$$\|\partial^k_\lambda R^\pm(\lambda^2)\|_{L^2_{\sigma_1} \to L^2_{-\sigma_2}} \leq C(1 + \lambda)^{-1}, \quad \lambda \geq 0, \tag{7.2.12}$$

where $\sigma_1, \sigma_2 > 1/2$, $\sigma_1 + \sigma_2 > 2$ for $k = 0$, and $\sigma_1, \sigma_2 > 1/2 + k$ for $k = 1, 2$.

Proof. First, note that

$$\|\partial_\lambda^k R^\pm(\lambda^2)\|_{L^2_{\sigma_1}\to L^2_{-\sigma_2}}=\mathcal{O}(1),\quad \lambda\to 0, \tag{7.2.13}$$

where $\sigma_1,\sigma_2>1/2$, $\sigma_1+\sigma_2>2$ for $k=0$, and $\sigma_1,\sigma_2>1/2+k$ for $k=1,2$. Indeed, asymptotics (7.2.13) with $k=0$ follows from (7.2.9). Next we apply the formulas (see for example [189, Formulas (17.9), (17.11)])

$$\begin{aligned} R' &= R_0'+RVR_0'+R_0'VR+RVR_0'VR,\\ R'' &= R_0''+RVR_0''+R_0''VR+RVR_0''VR+2R'VR_0'+2R'VR_0'VR \end{aligned} \tag{7.2.14}$$

These formulas and (7.2.9), (7.2.10) imply asymptotics (7.2.13) with $k=1,2$. Similarly, asymptotics (7.2.8) together with formulas (7.2.14) imply

$$\|\frac{\partial^k}{\partial\lambda^k}R^\pm(\lambda^2)\|_{L^2_{\sigma_1}\to L^2_{-\sigma_2}}=\mathcal{O}(\lambda^{-1}),\quad \lambda\to\infty,\quad \sigma_1,\sigma_2>1/2+k,\quad k=0,1,2. \tag{7.2.15}$$

□

7.2.2 The Born series

The identity $R(\lambda^2)-R_0(\lambda^2)=-R(\lambda^2)VR_0(\lambda^2)$ implies that

$$R(\lambda^2)=(1-R(\lambda^2)V)R_0(\lambda^2).$$

The iteration yields the finite Born series [181]

$$R^\pm=\sum_{k=0}^N(R_0^\pm V)^kR_0^\pm+(R_0^\pm V)^NR^\pm VR_0^\pm,\quad N\ge 0. \tag{7.2.16}$$

Substituting the expansion with N=2 into spectral representation (7.2.5), we obtain

$$e^{itH}P_c(H)=\sum_{j=0}^2 S_j(t)+Z(t),$$

where $S_0(t)=e^{-i\Delta t}$, and

$$S_j(t)=\frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}((R_0^+(\omega)V)^jR_0^+(\omega)-(R_0^-(\omega)V)^jR_0^-(\omega))d\omega,\quad j=1,2, \tag{7.2.17}$$

$$Z(t)=\frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}((R_0^+(\omega)V)^2R^+(\omega)VR_0^+(\omega)-(R_0^-(\omega)V)^2R^-(\omega)VR_0^-(\omega)). \tag{7.2.18}$$

By (7.2.7),

$$\|S_0(t)\|_{L^1\to L^\infty}\le C|t|^{-\frac{3}{2}},\quad |t|\ge 1.$$

It remains to prove similar decay for $S_j(t)$, $j=1,2$ and for $Z(t)$.

7.2.3 The decay of $S_1(t)$ and $S_2(t)$

Lemma 7.2.5. *Let condition (7.2.2) hold with $\beta > 3$. Then for $j = 1, 2$*

$$\|S_j(t)\|_{L^1\to L^\infty} \le C|t|^{-\frac{3}{2}}, \quad |t| \ge 1. \tag{7.2.19}$$

Proof. Similarly to (7.2.6) is suffices to prove the decay

$$\sup_{L\ge 1} \|S_j(t,L)\|_{L^1\to L^\infty} \le C|t|^{-\frac{3}{2}}, \quad |t| \ge 1, \quad j = 1, 2, \tag{7.2.20}$$

where

$$S_j(t,L) = \frac{1}{\pi i}\int_0^\infty e^{-i\lambda^2 t}\chi(\lambda/L)[(R_0^+(\lambda^2)V)^j R_0^+(\lambda^2) - (R_0^-(\lambda^2)V)^j R_0^-(\lambda^2)]\lambda d\lambda. \tag{7.2.21}$$

and $\chi(\lambda) \in C_0^\infty(\mathbb{R})$ with $\chi(\lambda) = 1$ for $|\lambda| \le 1$.

The representation (7.2.11) implies

$$\sup_{x,y\in\mathbb{R}^3} |R_0^+(\lambda^2,x,y) - R_0^-(\lambda^2,x,y)| \le C\lambda, \qquad \lambda \ge 0. \tag{7.2.22}$$

Hence,

$$\begin{aligned}
&\sup_{x,y\in R^3} |[R_0^+VR_0^+ - R_0^-VR_0^-](\lambda^2,x,y)| \\
&\le \sup_{x,y\in R^3} |[(R_0^+ - R_0^-)VR_0^+](\lambda^2,x,y)| + \sup_{x,y\in R^3} |[R_0^-V(R_0^+ - R_0^-)](\lambda^2,x,y)| \\
&\le C\lambda \sup_{x,y\in R^3} \int \Big(\frac{|V(z)|}{|x-z|} + \frac{|V(z)|}{|y-z|}\Big)dz \le C_1\lambda, \qquad \lambda \ge 0.
\end{aligned} \tag{7.2.23}$$

Similarly,

$$\begin{aligned}
&\sup_{x,y\in R^3} |[(R_0^+V)^2R_0^+) - (R_0^-V)^2R_0^-](\lambda^2,x,y)| \\
&\le C\lambda \sup_{x,y\in R^3} \iint \Big(\frac{|V(x_1)V(y_1)|}{|x_1-y_1||y_1-y|} + \frac{|V(x_1)V(y_1)|}{|x-x_1||x_1-y_1|} + \frac{|V(x_1)V(y_1)|}{|x-x_1||y-y_1|}\Big)dx_1dy_1 \le C\lambda.
\end{aligned}$$

Therefore, we can integrate by parts in (7.2.21):

$$\begin{aligned}
S_j(t,L) &= \frac{1}{2\pi t}\int_0^\infty e^{-i\lambda^2 t}\partial_\lambda\Big(\chi(\lambda/L)[(R_0^-(\lambda^2)V)^j R_0^-(\lambda^2) - (R_0^+(\lambda^2)V)^j R_0^+(\lambda^2)](\lambda^2)\Big)d\lambda \\
&= \frac{1}{2\pi t}(T_j^-(t,L) - T_j^+(t,L)).
\end{aligned} \tag{7.2.24}$$

It remains to prove that

$$\sup_{L\ge 1}\sup_{x,y} |T_j^\pm(t,L,x,y)| \le C|t|^{-1/2}, \quad |t| \ge 1, \quad j = 1, 2. \tag{7.2.25}$$

We have

$$\begin{aligned}
|T_1^\pm(t,L,x,y)| &\le \frac{1}{L}\Big|\int \frac{V(z)}{|z-y||z-x|}\Big(\int_0^\infty e^{-i\psi_1^\pm(\lambda)t}\chi'(\lambda/L)d\lambda\Big)dz\Big| \\
&+\Big|\int_{\mathbb{R}^3} \Big(\frac{V(z)}{|z-y|} + \frac{V(z)}{|z-x|}\Big)\Big(\int_0^\infty e^{-i\psi_1^\pm(\lambda)t}\chi(\lambda/L)d\lambda\Big)dz\Big|,
\end{aligned}$$

$$|T_2^{\pm}(t,L,x,y)| \le \frac{1}{L}|\iint \frac{|V(x_1)||V(y_1)|}{|x-x_1||x_1-y_1||y_1-y|}\Big(\int\limits_0^{\infty} e^{-i\psi_2^{\pm}(\lambda)t}\chi'(\lambda/L)d\lambda\Big)dz|$$

$$+|\iint\Big(\frac{V(x_1)V(y_1)}{|x_1-y_1||y_1-y|}+\frac{V(x_1)V(y_1)}{|x-x_1||x_1-y_1|}+\frac{V(x_1)V(y_1)}{|x-x_1||y-y_1|}\Big)\Big(\int\limits_0^{\infty} e^{-i\psi_2^{\pm}(\lambda)t}\chi(\lambda/L)d\lambda\Big)dx_1dy_1|$$

where

$$\psi_1^{\pm}(\lambda)=\lambda^2\mp\lambda(|x-z|+|z-y|)/t,\quad \psi_2^{\pm}(\lambda)=\lambda^2\mp\lambda(|x-x_1|+|x_1-y|+|y_1-y|)/t$$

with

$$\partial_\lambda^2\psi_j^{\pm}(\lambda)=2,\quad \lambda>0,\quad j=1,2$$

Then (7.2.25) follows by Van der Corput lemma (see [201, Chapter VIII, Proposition II and Corollary]). □

7.2.4 The decay of $Z(t)$

Now the proof of Theorem 7.2.1 is reduced to the proof of the following proposition

Proposition 7.2.6. *Let the conditions of Theorem 7.2.1 hold. Then*

$$\|Z(t)\|_{L^1\to L^\infty}\le C|t|^{-\frac{3}{2}},\quad |t|\ge 1. \tag{7.2.26}$$

Proof. Using the definition (7.2.28), we represent $Z(t)$ as

$$Z(t)=\frac{1}{\pi i}\int_0^{\infty} e^{-i\lambda^2 t}\big(\Lambda^+(\lambda)-\Lambda^-(\lambda)\big)\lambda d\lambda. \tag{7.2.27}$$

Here

$$\Lambda^{\pm}(\lambda)=(R_0^{\pm}(\lambda^2)V)^2R^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)=R_0^{\pm}(\lambda^2)V\Pi^{\pm}(\lambda)VR_0^{\pm}(\lambda^2), \tag{7.2.28}$$

where we denote $\Pi^{\pm}(\lambda)=R_0^{\pm}(\lambda^2)VR^{\pm}(\lambda^2)$.

First we prove some properties of $\Lambda^{\pm}(\lambda)$. We denote by $a+$ any number $a+\varepsilon$ with an arbitrary small, but fixed $\varepsilon>0$.

Lemma 7.2.7. *Let (7.2.2) holds with some $\beta>3$. Then in the regular case,*

$$\|\Lambda^+(\lambda)-\Lambda^-(\lambda)\|_{L^1\to L^\infty}\to 0,\quad \lambda\to 0. \tag{7.2.29}$$

Proof. For any $f,g\in L^1$, we obtain

$$|\langle f,(\Lambda^+(\lambda)-\Lambda^-(\lambda))g\rangle|\le|\langle V(R_0^-(\lambda^2)-R_0^+(\lambda^2))f,\Pi^+(\lambda)VR_0^+(\lambda^2)g\rangle|$$

$$+|\langle VR_0^+(\lambda^2)f,(\Pi^+(\lambda)-\Pi^-(\lambda))VR_0^+(\lambda^2)g\rangle|$$

$$+|\langle VR_0^+(\lambda^2)f,\Pi^-(\lambda)V(R_0^+(\lambda^2)-R_0^-(\lambda^2))g\rangle|$$

$$\le\|V(R_0^-(\lambda^2)-R_0^+(\lambda^2))f\|_{L^2_{1+}}\|\Pi^+(\lambda)\|_{L^2_{1+}\to L^2_{-1-}}\|VR_0^+(\lambda^2)g\|_{L^2_{1+}}$$

$$+\|VR_0^+(\lambda^2)f\|_{L^2_{1+}}\|(\Pi^+(\lambda)-\Pi^-(\lambda))\|_{L^2_{1+}\to L^2_{-1-}}\|VR_0^+(\lambda^2)g\|_{L^2_{1+}}$$

$$+\|VR_0^+(\lambda^2)f\|_{L^2_{1+}}\|\Pi^-(\lambda)\|_{L^2_{1+}\to L^2_{-1-}}\|V(R_0^+(\lambda^2)-R_0^-(\lambda^2))g\|_{L^2_{1+}}$$

since $(R_0^\pm)^* = R_0^\mp$. For any $0 \le \sigma \le \beta - 1/2$, we have

$$\|VR_0^\pm(\lambda^2)f\|^2_{L^2_\sigma} \le \int V^2(x)\langle x\rangle^{2\sigma}|\int R_0^\pm(\lambda^2,x,y)f(y)dy|^2dx$$
$$\le C \iint |f(y_1)||f(y_2)|\Big(\int \frac{\langle x\rangle^{2\sigma-2\beta}}{|x-y_1||x-y_2|}dx\Big)dy_1dy_2 \le C_1\|f\|^2_{L^1}. \quad (7.2.30)$$

Further, for any $0 \le \sigma \le \beta - 3/2$,

$$\|V(R_0^\pm(\lambda^2) - R_0^\mp(\lambda^2))f\|^2_{L^2_\sigma} \le C\lambda^2 \iiint \langle x\rangle^{2\sigma-2\beta}|f(y_1)||f(y_2)|dxdy_1dy_2 \le C_1\lambda^2\|f\|^2_{L^1} \quad (7.2.31)$$

by (7.2.22). Finally,

$$\|\Pi^+(\lambda) - \Pi^-(\lambda)\|_{L^2_{1+}\to L^2_{-1-}} \to 0, \quad \lambda \to 0. \quad (7.2.32)$$

Indeed, property **R4** implies

$$\|R_0^+(\lambda^2) - R_0^-(\lambda^2)\|_{L^2_{1+}\to L^2_{-1-}} \to 0, \quad \|R^+(\lambda^2) - R^-(\lambda^2)\|_{L^2_{1+}\to L^2_{-1-}} \to 0, \quad \lambda \to 0,$$

while $V : L_{-1-} \to L_{1+}$ is bounded for $\beta > 2$. □

Lemma 7.2.8. *Let (7.2.2) hold with some $\beta > 3$. Then in the regular case,*

$$\|\partial_\lambda^k \Lambda^\pm(\lambda)\|_{L^1\to L^\infty} \le C(1+\lambda)^{-2}, \quad \lambda \ge 0, \quad k = 0, 1. \quad (7.2.33)$$

Proof. We omit the signs $\pm$ not to overburden the exposition. For example, $R_0(\lambda^2)$ means $R_0^+(\lambda^2)$ or $R_0^-(\lambda^2)$, $\Pi(\lambda) = \Pi^\pm(\lambda)$, etc. First, we show that

$$\|\Pi(\lambda)\|_{L^2_{\sigma_1}\to L^2_{-\sigma_2}} + \|\partial_\lambda\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}} + \|\partial_\lambda^2\Pi_N^\pm(\lambda)\|_{L^2_{\frac52+}\to L^2_{-\frac52-}} \le C(1+\lambda)^{-2}, \ \lambda \ge 0, \quad (7.2.34)$$

where $\sigma_1, \sigma_2 > 1/2$, $\sigma_1 + \sigma_2 > 2$. Indeed, (7.2.12)–(7.2.15) imply

$$\begin{aligned}\|\Pi(\lambda)\|_{L^2_{\sigma_1}\to L^2_{-\sigma_2}} &\le \|R_0(\lambda^2)\|_{L^2_{3/2+}\to L^2_{-\sigma_2}}\|V\|_{L^2_{-3/2-}\to L^2_{3/2+}}\|R(\lambda^2)\|_{L^2_{\sigma_1}\to L^2_{-3/2-}}\\ &\le C(1+\lambda)^{-2}.\end{aligned}$$

Further, (7.2.12)–(7.2.15) imply

$$\|\partial_\lambda\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}} \le \|\partial_\lambda R_0(\lambda^2)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|V\|_{L^2_{-\frac32-}\to L^2_{\frac32+}}\|R(\lambda^2)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}$$
$$+\|R_0(\lambda^2)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|V\|_{L^2_{-\frac32-}\to L^2_{\frac32+}}\|\partial_\lambda R(\lambda^2)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}} \le C(1+\lambda)^{-2},$$

$$\|\partial_\lambda^2\Pi(\lambda)\|_{L^2_{\frac52+}\to L^2_{-\frac52-}} \le \|\partial_\lambda^2 R_0(\lambda^2)\|_{L^2_{\frac52+}\to L^2_{-\frac52-}}\|V\|_{L^2_{-\frac12-}\to L^2_{\frac52+}}\|R(\lambda^2)\|_{L^2_{\frac52+}\to L^2_{-\frac12-}}$$
$$\|\partial_\lambda R_0(\lambda^2)\|_{L^2_{\frac32+}\to L^2_{-\frac52-}}\|V\|_{L^2_{-\frac32-}\to L^2_{\frac32+}}\|\partial_\lambda R(\lambda^2)\|_{L^2_{\frac52+}\to L^2_{-\frac32-}}$$
$$+\|R_0(\lambda^2)\|_{L^2_{\frac12+}\to L^2_{-\frac52-}}\|V\|_{L^2_{-\frac52-}\to L^2_{\frac12+}}\|\partial_\lambda^2 R(\lambda^2)\|_{L^2_{\frac52+}\to L^2_{-\frac52-}} \le C(1+\lambda|^{-2}.$$

Now, (7.2.30) and (7.2.34) imply for any $f, g \in L^1$ and $k = 0, 1$

$$|\langle f, R_0(\lambda^2)V\partial_\lambda^k\Pi(\lambda)VR_0(\lambda^2)g\rangle| = |\langle VR_0^*(\lambda^2)f, \partial_\lambda^k\Pi(\lambda)VR_0(\lambda^2)g\rangle|$$

$$\le \|VR_0^*(\lambda^2)f\|_{L^2_{\frac32+}}\|\partial_\lambda^k\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|VR_0(\lambda^2)g\|_{L^2_{\frac32+}} \le C(1+\lambda)^{-2}\|f\|_{L^1}\|g\|_{L^1}. \quad (7.2.35)$$

Further, for any $0 \le \sigma \le \beta - 3/2$, we obtain

$$\|V\partial_\lambda R_0(\lambda^2)f\|^2_{L^2_\sigma} \le C\int \langle x\rangle^{2\sigma-2\beta}\Big(\int |f(y)|dy\Big)^2 dx \le C_1\|f\|^2_{L^1}. \tag{7.2.36}$$

Hence, (7.2.30) and (7.2.34) imply for any $f, g \in L^1$ and $\lambda \ge 0$

$$|\langle f, \partial_\lambda R_0(\lambda^2)V\Pi(\lambda)VR_0(\lambda^2)g\rangle| = |\langle V\partial_\lambda R_0^*(\lambda^2)f, \Pi(\lambda)VR_0(\lambda^2)g\rangle|$$

$$\le \|V\partial_\lambda R_0^*(\lambda^2)f\|_{L^2_{1+}}\|\Pi(\lambda)\|_{L^2_{1+}\to L^2_{-1-}}\|VR_0(\lambda^2)g\|_{L^2_{1+}} \le C\|f\|_{L^1}\|g\|_{L^1}(1+\lambda)^{-2} \tag{7.2.37}$$

Similarly,

$$|\langle f, R_0(\lambda^2)V\Pi(\lambda)V\partial_\lambda R_0(\lambda^2)g\rangle| \le C\|f\|_{L^1}\|g\|_{L^1}(1+\lambda)^{-2}, \quad \lambda \ge 0.$$

Then (7.2.33) follows by definition (7.2.28) of Λ. □

Due to Lemma 7.2.8, the integrand in(7.2.27) is a differentiable operator function of $\lambda \ge 0$ with values in the space of bounded operators mapping L^1 into L^∞. Moreover, due to Lemmas 7.2.7 and 7.2.8, we can integrate by parts,

$$Z(t) = \frac{1}{2\pi t}\int_0^\infty e^{-i\lambda^2 t}\partial_\lambda\big(\Lambda^-(\lambda) - \Lambda^+(\lambda)\big)d\lambda = \frac{1}{2\pi t}(Q^-(t) - Q^+(t)).$$

Here

$$Q^\pm(t,x,y) = \int\limits_0^\infty \Big(e^{-i\varphi_1 t}K_1^\pm(\lambda,x,y) + e^{-i\varphi_2^\pm t}K_2^\pm(\lambda,x,y)e^{-i\varphi_3^\pm t}K_3^\pm(\lambda,x,y))\Big)d\lambda,$$

where we denote

$$\varphi_1(\lambda) = -\lambda^2, \quad \varphi_2^\pm(\lambda) = -\lambda^2 \mp \lambda|x|/t, \quad \varphi_3^\pm(\lambda) = -\lambda^2 \mp \lambda|y|/t,$$

$$K_1^\pm(\lambda) = R_0^\pm(\lambda^2)V\partial_\lambda\Pi^\pm_{N-1}(\lambda)VR_0^\pm(\lambda^2),$$

$$K_2^\pm(\lambda) = (G^\mp(\lambda))^*V\Pi^\pm_N(\lambda)VR_0^\pm(\lambda^2), \quad K_3^\pm(\lambda) = R_0^\pm(\lambda^2)V\Pi^\pm_N(\lambda)VG^\pm(\lambda),$$

and $G^\pm(\lambda)$ is the operator with the kernel

$$G^\pm(\lambda,x,y) = \mp\frac{e^{\pm i\lambda(|x-y|-|y|)}}{4\pi i}, \quad \lambda \ge 0. \tag{7.2.38}$$

It remains to prove that

$$\sup_{x,y}|Q^\pm(t,x,y)| \le C|t|^{-\frac{1}{2}}, \quad |t| \ge 1. \tag{7.2.39}$$

To this end, we estimate the functions $K_j^\pm$, $j = 1,2,3$.

Lemma 7.2.9. *Let (7.2.2) hold with some $\beta > 3$. Then for $N \ge 1$, in the regular case,*

$$\|\partial_\lambda^k K_j(\lambda)\|_{L^1\to L^\infty} \le C(1+\lambda)^{-2}, \quad \lambda \ge 0, \quad k = 0,1, \quad j = 1,2,3. \tag{7.2.40}$$

Proof. We omit the signs $\pm$ again.
1) Note that (7.2.40) for K_1 is exactly (7.2.35) with $k=1$. Further, estimates (7.2.34), (7.2.30) and (7.2.36) imply

$$|\langle f,\partial_\lambda K_1(\lambda)g\rangle|\le\|V\partial_\lambda R_0^*(\lambda^2)f\|_{L^2_{\frac32+}}\|\partial_\lambda\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|VR_0(\lambda^2)g\|_{L^2_{\frac32+}}$$

$$+\|VR_0^*(\lambda^2)f\|_{L^2_{\frac32+}}\|\partial_\lambda\Pi_N(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|V\partial_\lambda R_0(\lambda^2)g\|_{L^2_{\frac32+}}$$

$$+\|VR_0^*(\lambda^2)f\|_{L^2_{\frac52+}}\|\partial_\lambda^2\Pi_N(\lambda)\|_{L^2_{\frac52+}\to L^2_{-\frac52-}}\|VR_0(\lambda^2)g\|_{L^2_{\frac52+}}\le C(1+\lambda)^{-2}\|f\|_{L^1}\|g\|_{L^1},$$

2) Now we estimate $\partial_\lambda^k K_2(\lambda)$. Note that

$$|\partial_\lambda^k G(\lambda,x,y)|\le|x|^k/(4\pi),\quad k=0,1,2,... \tag{7.2.41}$$

Then, similarly to (7.2.36), we obtain for $0\le\sigma\le\beta-k-3/2$

$$\|V\partial_\lambda^k G(\lambda)f\|_{L^2_\sigma}\lesssim\|f\|_{L^1},\quad k=0,1. \tag{7.2.42}$$

Hence, (7.2.34), (7.2.30) and (7.2.36) imply

$$\begin{aligned}|\langle f,K_2(\lambda)g\rangle| &\le \|VG(\lambda^2)f\|_{L^2_{1+}}\|\Pi(\lambda)\|_{L^2_{1+}\to L^2_{-1-}}\|VR_0(\lambda^2)g\|_{L^2_{1+}}\\ &\le C(1+\lambda)^{-2}\|f\|_{L^1}\|g\|_{L^1}.\end{aligned}$$

$$|\langle f,\partial_\lambda K_2(\lambda)g\rangle|\le\|V\partial_\lambda G(\lambda^2)f\|_{L^2_{\frac12+}}\|\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac12-}}\|VR_0(\lambda^2)g\|_{L^2_{\frac32+}}$$

$$+\|VG(\lambda^2)f\|_{L^2_{1+}}\|\Pi(\lambda)\|_{L^2_{1+}\to L^2_{-1-}}\|V\partial_\lambda R_0(\lambda^2)g\|_{L^2_{1+}}$$

$$+\|VG(\lambda^2)f\|_{L^2_{\frac32+}}\|\partial_\lambda\Pi(\lambda)\|_{L^2_{\frac32+}\to L^2_{-\frac32-}}\|VR_0(\lambda^2)g\|_{L^2_{\frac32+}}\le C(1+\lambda)^{-2}\|f\|_{L^1}\|g\|_{L^1}. \tag{7.2.43}$$

The estimates for K_3 can been obtained similarly. □

Corollary 7.2.10. *For $k=0,1$ the integral kernels $\partial_\lambda^k K_j(\lambda,x,y)$ belong to $L^\infty(\mathbb{R}^6)$, and*

$$\|\partial_\lambda^k K_j(\lambda,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^6)}\le C(1+\lambda)^{-2},\quad j=1,2,3,\quad \lambda\ge0.$$

Proof. The distributional kernel $A(x,y)$ of any bounded linear operator $A:L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)$ belongs to $L^\infty(\mathbb{R}^6)$, and

$$\|A(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^6)}=\|A\|_{L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)}.$$

This follows from the estimate $|\langle A,\phi\rangle|\le C\|\phi\|_{L^1(\mathbb{R}^6)}$ for $\phi\in L^1(\mathbb{R}^6)$ and from the duality $(L^1(\mathbb{R}^6))^*=L^\infty(\mathbb{R}^6)$. □

Applying Corollary 7.2.10, we obtain

$$\sup_{x,y}|Q^\pm(t,x,y)|\le C|t|^{-\frac12}\int_0^\infty(1+|\lambda|)^{-2}d\lambda\le C_1|t|^{-\frac12},\quad |t|\ge1.$$

by Van der Corput lemma. □

Chapter 8

Attractors and Quantum Mechanics

In this chapter we discuss possible relations of the foregoing results on global attractors of nonlinear Hamiltonian equations to some mathematical problems of Quantum Theory.

These results were suggested by fundamental postulates of quantum theory, primarily the Bohr postulate on transitions between *quantum stationary states*. As a result, we have introduced general summarising conjecture (0.0.6). Here we discuss possible relation of this conjecture to dynamical treatment of the Bohr postulates in the context of semiclassical nonlinear Maxwell–Schrödinger and Maxwell–Dirac equations.

8.1 Bohr's postulates

In 1913 N. Bohr suggested the following two postulates which give the 'Columbus solution' of the problem of stability and radiation of atoms and molecules [217]:

B1c. *Atoms and molecules are permanently on some stationary orbits* $|E_n\rangle$ *with energies* E_n*, and sometimes make transitions between the orbits,*

$$|E_n\rangle \mapsto |E_{n'}\rangle. \tag{8.1.1}$$

B2. *Such transition is followed by radiation of an electromagnetic wave of frequency* $\omega_{nn'} = \omega_{n'} - \omega_n$, *where* $\omega_k = E_k/\hbar$ (8.1.2)

Both these postulates should become theorems in discovered in 1925–1926 Quantum Theory of E. Schrödinger and W. Heisenberg. However, this did not happen till now, while both postulates are still actively used in quantum theory. This lack of theoretical clarity hinders the progress in the theory (e.g., in superconductivity and in nuclear reactions), and in numerical simulation of many engineering processes (e.g., of laser radiation and quantum amplifiers) since a computer can solve dynamical equations but cannot take postulates into account.

The juxtaposition of the quantum postulates (8.1.1) and (8.1.2) with the Schrödinger theory rises the following questions.

I. Why quantum stationary states (or *quantum stationary orbits*) in the Schrödinger theory are identified with wave functions of the form (3.1.11),

$$\psi(x,t) = \psi_\omega(x)e^{-i\omega t}\ ? \tag{8.1.3}$$

II. Whether Bohr's transitions (8.1.1) between these *quantum stationary states* allow a dynamical description?

Note that exactly the expression (8.1.3) implies that the amplitudes $\psi_\omega(x)$ are eigenfunctions of the Schrödinger operator.

The same questions arise in each other dynamical model: Quantum Field Theory, Chromodynamics, and so on. However, the answer is not found till now.

The theory of attractors of Hamiltonian nonlinear PDEs, presented in this book, suggests that

i) the form (8.1.3) of quantum stationary states is due to the $U(1)$-symmetry of the Schrödinger theory, that is of the coupled Maxwell–Schrödinger equations (8.2.1).

ii) the transitions can be interpreted as the global attraction (3.1.2) of all trajectories of a quantum system to an attractor formed by *stationary orbits* of type (8.1.3).

Moreover, the amplitudes of these stationary orbits are solutions of the *nonlinear eigenvalue problem* of type (3.1.12), which is *approximately linear* in a variety of cases due to the smallness of the interaction constant (the Sommerfeld constant) .

We expect that other fundamental postulates of Quantum Theory also allow suitable interpretation in the framework of the theory of attractors for nonlinear Hamiltonian PDEs: *wave-particle duality* (L. de Broglie, 1924), and *probabilistic interpretation* (M. Born, 1927). More details can be found in [214].

8.2 On dynamical interpretation of quantum jumps

The simplest dynamical interpretation of the postulate **B1** is the global attraction to stationary orbits (3.1.2) for all finite energy quantum trajectories $\psi(t)$. This means that stationary orbits form a global attractor of the corresponding quantum dynamics. However, this global attraction to stationary orbits contradicts the linear Schrödinger equation due to the superposition principle. Thus, Bohr's transitions **B1** in the linear theory do not exist.

It is natural to suggest that the global attraction to stationary orbits (3.1.2) holds for a nonlinear modification of the linear Schrödinger theory. On the other hand, it turns out that even the original Schrödinger theory is nonlinear, because it involves interaction with the Maxwell field. The corresponding *semiclassical* nonlinear Maxwell–Schrödinger system was introduced essentially in the first Schrödinger's articles [218] (see also Sections 4.2 and 12.4.2 of [213]):

$$\left\{\begin{array}{l} i\hbar\dot\psi(x,t) = \frac{1}{2\mathrm{m}}[-i\hbar\nabla - \frac{e}{c}(\mathbf{A}(x,t) + \mathbf{A}^{\rm ext}(x,t))]^2\psi + e[A_0(x,t) + A_0^{\rm ext}(x,t)]\psi \\ \Box A_\nu(x,t) = 4\pi J_\nu(x,t), \qquad \nu = 0,1,2,3 \end{array}\right| , \quad (8.2.1)$$

where $\Box$ is the d'Alembert wave operator $\frac{1}{c^2}\partial_t^2 - \Delta$. The Maxwell equations are written here in the 4-dimensional form and in *unrationalized Gaussian units* (cgs) (or Heaviside–Lorentz units). The physical constants in these units are approximately equal to

$$e = -4.8\times 10^{-10}\text{esu}, \quad \mathrm{m} = 9.1\times 10^{-28}\text{g}, \quad \hbar = 1.1\times 10^{-27}\text{erg·s}, \quad c = 3.0\times 10^{10}\text{cm/s}. \quad (8.2.2)$$

(see [212, p. 781] and [219, p. 221]). Further, $A = (A_0, \mathbf{A}) = (A_0, A_1, A_2, A_3)$ denotes 4-dimensional potential of the Maxwell field in the Lorentz gauge $\dot A_0/c + \nabla\cdot\mathbf{A} = 0$,

$A^{\rm ext} = (A_0^{\rm ext}, \mathbf{A}^{\rm ext})$ is an external 4-potential, and $J = (\rho, j/c)$ is the 4-dimensional current density. To make these equations a closed system, we must also express the density of charge and currents via the wave function:

$$J_0(x,t) = |\psi(x,t)|^2; J_k(x,t) = [(-i\nabla_k + A_k(x,t) + A_k^{\rm ext}(x,t))\psi(x,t)] \cdot \psi(x,t) \qquad (8.2.3)$$

for $k = 1, 2, 3$, and '$\cdot$' denotes the scalar product of two-dimensional real vectors corresponding to complex numbers. In particular, these expressions provide the continuity equation $\dot\rho + \operatorname{div} j = 0$ for any solution of the Schrödinger equation with arbitrary real potentialss [213, Section 3.4].

System (8.2.1) is nonlinear in (ψ, A) although the Schrödinger equation is formally linear in ψ. It can be written as (0.0.5) in the case of *static external potentials*

$$A^{\rm ext}(x,t) \equiv A^{\rm ext}(x). \qquad (8.2.4)$$

In this case the system (8.2.1) is G-invariant with the symmetry group $G = U(1)$ acting as

$$T(e^{i\theta})(\psi(x),\ A(x)) := (\psi(x)e^{i\theta},\ A(x)). \qquad (8.2.5)$$

The symmetry means that for any solution $(\psi(x,t), A(x,t))$ of (8.2.1) and any $\theta \in \mathbb{R}$ the functions

$$T(e^{i\theta})(\psi(x,t),\ A(x,t)) := (\psi(x,t)e^{i\theta},\ A(x,t)) \qquad (8.2.6)$$

are also solutions that can be easy verified. In particular, the 4-current (8.2.3) is invariant under this action. Now the 'stationary G-orbits' (3.10.3) for the nonlinear hyperbolic system (8.2.1) are solutions of type

$$(\psi(x)e^{-i\omega t},\ A(x)). \qquad (8.2.7)$$

The same remarks apply to the Maxwell—Dirac system introduced by Dirac in 1927:

$$\left\{\begin{array}{l} \displaystyle\sum_{\nu=0}^{3} \gamma^\nu [i\nabla_\nu - A_\nu(x,t) - A_\nu^{\rm ext}(x,t)]\psi(x,t) = m\psi(x,t) \\ \Box A_\nu(x,t) = J_\nu(x,t) := \overline{\psi(x,t)}\gamma^0\gamma_\nu\psi(x,t), \quad \nu = 0,\dots,3 \end{array}\right| \quad x \in \mathbb{R}^3, \qquad (8.2.8)$$

where $\nabla_0 := \partial_t$.

We suggest that the Bohr transitions **B1** for the systems (8.2.1) and (8.2.8) with a static external potentials (8.2.4) can be interpreted as the *single-frequency asymptotics*

$$(\psi(x,t),\ A(x,t)) \sim (\psi_\pm(x)e^{-i\omega_\pm t},\ A_\pm(x,t)), \qquad t \to \pm\infty \qquad (8.2.9)$$

for every finite energy solution, where the asymptotics hold in local energy norms. These asymptotics correspond to our general conjecture (0.0.6) with the symmetry group $G = U(1)$ and its representation (8.2.5).

Stationary G-orbits (8.2.7) are solutions to the *nonlinear eigenvalue problem*

$$\left\{\begin{array}{l} \hbar\omega\psi(x) = \dfrac{1}{2\mathrm{m}}[-i\hbar\nabla - \dfrac{e}{c}(\mathbf{A}(x) + \mathbf{A}^{\rm ext}(x))]^2\psi(x) + e[A_0(x) + A_0^{\rm ext}(x)]\psi(x) \\ -\Delta A_\nu(x) = 4\pi J_\nu(x), \qquad \nu = 0,1,2,3 \end{array}\right| . \qquad (8.2.10)$$

The existence of these stationary G-orbits for the Maxwell–Schrödinger equations was established by G.M. Coclite and V. Georgiev [29] for the case of Coulomb external potentials

$$A_0^{\rm ext} = -\frac{eZ}{|x|}, \qquad \mathbf{A}^{\rm ext}(x) \equiv 0. \tag{8.2.11}$$

For the Maxwell–Dirac system the existence of stationary G-orbits was established by M. Esteban, V. Georgiev and E. Séré in the case of *zero external potentials* [30].

Remark 8.2.1. The *nonlinear eigenvalue problem* (8.2.10) reduces to traditional linear eigenvalue problem for the Schrödinger operator if one neglect the 'own Maxwell potentials' $A_0(x)$ and $\mathbf{A}(x)$ in the first equation. The solution of this linear eigenvalue problem with the normalisation

$$\int |\psi(x)|^2 dx = 1 \tag{8.2.12}$$

can be considered as the first approximation. Further one can apply perturbation procedure solving the Poisson equations in (8.2.10) with currents (8.2.3) defined with the first approximation, and adding their solutions to the external potentials, and so on. The convergence of this procedure is not proved, though it gives satisfactory results in a variety of cases.

Furthermore, in the case of *zero external potentials* Maxwell–Schrödinger system is translation-invariant, while the Maxwell–Dirac system is relativistically-invariant. Respectively, for their solutions one should expect the soliton asymptotics of type (0.0.13) in global energy norms as $t \to \pm\infty$:

$$\psi(x,t) \sim \sum_k \psi_\pm^k(x - v_\pm^k t) e^{i\Phi_\pm^k(x,t)} + \varphi_\pm(x,t), \tag{8.2.13}$$

$$A(x,t) \sim \sum_k A_\pm^k(x - v_\pm^k t) + A_\pm(x,t). \tag{8.2.14}$$

Here $\Phi_\pm^k(x,t)$ are suitable phase functions, and each soliton $(\psi_\pm^k(x - v_\pm^k t)e^{i\Phi_\pm^k(x,t)}, A_\pm^k(x - v_\pm^k t))$ is a solution of the corresponding 'nonlinear eigenvalue problem', while $\varphi_\pm(x,t)$ and $A_\pm(x,t)$ represent some dispersion waves which are solutions to the free Schrödinger and Maxwell equations respectively.

The asymptotics (8.2.9) and (8.2.13) are not proved yet for the Maxwell–Schrödinger and Maxwell–Dirac systems (8.2.1) and (8.2.8). One could expect that these asymptotics should follow by suitable modification of the arguments from Chapter 3 which give a rigorous justification of similar arguments for $U(1)$-invariant equations (3.1.1) and (3.1.18)–(3.1.20). However, a rigorous justification for the systems (8.2.1) and (8.2.8) is still an open problem.

8.3 Bohr's postulates by perturbation theory

The remarkable success of the Schrödinger theory was the explanation of the Bohr postulates in the case of *static external potentials* by *perturbation theory* applied to the *coupled Maxwell–Scrödinger equations* (8.2.1). Namely, as a first approximation, the time-dependent fields $\mathbf{A}(x,t)$ and $A^0(x,t)$ in the Schrödinger equation of the system (8.2.1) can be neglected:

$$i\hbar\dot\psi(x,t) = H\psi(x,t) := \frac{1}{2m}[-i\hbar\nabla - \frac{e}{c}\mathbf{A}_{\rm ext}(x)]^2\psi(x,t) + eA^0_{\rm ext}(x)\psi(x,t), \tag{8.3.1}$$

For 'sufficiently good' external potentials and initial conditions any finite energy solution can be expanded in eigenfunctions

$$\psi(x,t) = \sum_n C_n \psi_n(x) e^{-i\omega_n t} + \psi_c(x,t), \qquad \psi_c(x,t) = \int C(\omega) e^{-i\omega t} d\omega, \tag{8.3.2}$$

where integration is performed over the continuous spectrum of the Schrödinger operator H, and for any $R > 0$

$$\int_{|x|<R} |\psi_c(x,t)|^2 dx \to 0, \quad t \to \pm\infty, \tag{8.3.3}$$

see, for example, [189, Theorem 21.1]. The substitution of this expansion into the expression for current density (8.2.3) gives the series

$$J(x,t) = \sum_{nn'} J_{nn'}(x) e^{-i\omega_{nn'} t} + c.c. + J_c(x,t), \tag{8.3.4}$$

where $J_c(x,t)$ has a continuous frequency spectrum. Therefore, the currents on the right hand side of the Maxwell equation from (8.2.1) contains, besides the continuous spectrum, only discrete frequencies $\omega_{nn'}$. Hence, the discrete spectrum of the corresponding Maxwell field also contains only these frequencies $\omega_{nn'}$. This proves the Bohr rule **B2** *in the first order of perturbation theory*, since this calculation ignores the inverse effect of radiation onto the atom.

Moreover, these arguments also suggest to treat the jumps (8.1.1) as the *single-frequency asymptotics* (8.2.9) for solutions to the Schrödinger equation *coupled to the Maxwell equations.*

Namely, the currents (8.3.4) on the right of the Maxwell equation from (8.2.1) produce the radiation when non-zero frequencies $\omega_{nn'}$ are present. This is due to the fact that $\mathbb{R} \setminus 0$ is a subset of absolutely continuous spectrum of the Maxwell equations.

However, this radiation cannot last forever, since it irrevocably carries the energy to infinity while the total energy is finite. Hence in the long-time limit only $\omega_{nn'} = 0$ should survive, which means exactly the single-frequency asymptotics (8.2.9) by (8.3.3).

Remark 8.3.1. Of course, these perturbation arguments cannot provide a rigorous justification of the long-time asymptotics (4.5.21) for the coupled Maxwell–Schrödinger equations. In [63]–[74], we have justified similar single-frequency asymptotics for a list of model $U(1)$-invariant nonlinear PDEs, see Chapter 3. Nevertheless, for the coupled Maxwell–Schrödinger equation such justification is still an open problem.

8.4 Conclusion

The discussion above suggests that N. Bohr's postulates cannot be interpreted in the framework of linear Schrödinger equation alone, but admit a hypothetical explanation in the framework of the coupled Maxwell–Schrödinger equations. In [214] we also suggest a mathematical treatment of other fundamental postulates of Quantum Theory relying on the coupled Maxwell–Schrödinger equations: of L. de Broglie's wave-particle duality and of M. Born's probabilistic interpretation.

It seems, the absence of suitable treatment of these postulates in the framework of linear theory was the cause of heated discussions by A. Einstein with N. Bohr and other physicists [205]. Note that W. Heisenberg began developing a nonlinear theory of elementary particles [208, 209].

According to many expert physicists, a mathematical analysis of problems of Quantum Mechanics is useless because its area of applicability is limited, as are its capabilities. However, the purpose of our discussions is not in improving the physical theory. Our goal is to prepare a mathematical ground for approach to some open questions of Quantum Theory which are not accessible with perturbation technique. For instance, to the questions of nuclei classification and of nuclear reactions. We suppose that the nuclei are 'quantum stationary states' of suitable nonlinear equations, i.e. points of the corresponding global attractor.

Note, the second-quantized MS system is the main subject of Quantum Electrodynamics [216]. Our specific attention to the *semiclassical* Maxwell–Schrödinger and Maxwell–Dirac systems is due to the fact that for these systems there is an extensive empirical material: on atomic spectra, electron diffraction, on crystals and their thermal and electric conductivity, etc. Therefore, one can try to find possible mathematical description of these phenomena in the framework of these systems. So these semiclassical systems serve as a testing ground for a development of the mathematical theory.

Similar questions also exist on a higher level in the Quantum Field Theory [216]. However, they obviously cannot be clarified until these questions are understood in a simpler context of semiclassical theory.

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