GBDT and algebro-geometric approaches to explicit solutions and wave functions for nonlocal NLS

J. Michor and A.L. Sakhnovich

Abstract

We apply the GBDT version of the Bäcklund-Darboux transformation to the nonlocal NLS (focusing and defocusing cases). The matrix case is included and solutions in the form of rectangular $m_1 \times m_2$ matrix functions are dealt with. In the case of the trivial initial solution, wide classes of the wave functions and multipole solutions are constructed explicitly. Families of explicit examples are considered in detail. Some initial results and representations for the more complicated algebro-geometric solutions are obtained as well.

MSC(2010): 35B06, 35Q55, 14H70, 37K40

Keywords: Nonlocal nonlinear Schrödinger equation, explicit solution, wave function, Bäcklund-Darboux transformation, algebro-geometric solution.

1 Introduction

Nonlocal nonlinear integrable equations and, in particular, the nonlocal nonlinear Schrödinger equation (nonlocal NLS) have been actively studied during the last years (see the important papers [3, 15, 19, 20, 38] and references therein), starting from the article [2] by M.J. Ablowitz and Z.H. Musslimani. Interesting results and figures one can find also in the very recent work [27]. In our paper, we apply the GBDT and algebro-geometric approach to the most studied (in the scalar case) nonlocal NLS. However, the explicit expression for the wave function (as well as algebro-geometric considerations) are new even in the scalar case. Similar considerations may be successfully applied to many other important nonlocal integrable equations.

The nonlocal nonlinear Schrödinger equation (nonlocal NLS) is a special case of the coupled NLS:

$$\xi_t(x,t) + \mathrm{i}j\xi_{xx}(x,t) + 2\mathrm{i}j\xi(x,t)^3 = 0, \tag{1.1}$$

$$\xi := \begin{bmatrix} 0 & v_1 \\ v_2 & 0 \end{bmatrix}, \quad \xi_t := \frac{\partial}{\partial t}\xi(x,t), \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}; \quad \text{i.e.,} \tag{1.2}$$

$$v_{1t} + \mathrm{i}v_{1xx} + 2\mathrm{i}v_1v_2v_1 = 0, \quad v_{2t} - \mathrm{i}v_{2xx} - 2\mathrm{i}v_2v_1v_2 = 0, \tag{1.3}$$

where I_{m_k} is the $m_k \times m_k$ identity matrix. Indeed, setting in (1.3)

$$v_1(x,t) = u(x,t), \quad v_2(x,t) = -\sigma u(-x,t)^*, \quad \sigma = \mp 1, \tag{1.4}$$

we transform the first equation in (1.3) into the nonlocal matrix NLS:

$$\mathrm{i}u_t(x,t) - u_{xx} + 2\sigma u(x,t)u(-x,t)^*u(x,t) = 0. \tag{1.5}$$

Here u is an $m_1 \times m_2$ matrix function. Although the scalar case of the nonlocal NLS where $m_1 = m_2 = 1$ was considered before in the literature, the matrix case is of interest as well (see, e.g., [5, 20]), and we deal in the present paper with this more general situation. We note that under the assumptions (1.4) the first and second equations in (1.3) are equivalent, and so (1.5) is equivalent to (1.1). Although the cases $\sigma = -1$ and $\sigma = 1$ differ in some important aspects, we often formulate the results for the nonlocal NLS with $\sigma = -1$ and with $\sigma = 1$ simultaneously and the differences in the corresponding formulas are restricted to the values of σ and of $\varkappa = (1-\sigma)/2$.

The nonlocal NLS is closely related to the well-known PT-symmetric theory (see, e.g., [8, 9, 14, 35, 52] and references therein). In this paper, we use some ideas from [41], where the generalized Bäcklund-Darboux transformation (GBDT) was applied to the linear PT-symmetric Schrödinger equation, in order to apply GBDT to the nonlocal NLS (1.5). Bäcklund-Darboux transformations and commutation methods (see, e.g., [11, 12, 24, 26, 33, 34, 46, 50]) are well-known tools for explicitly solving integrable equations and spectral and scattering problems. In particular, GBDT (where *generalized eigenvalues* are $n \times n$ matrices with an arbitrary Jordan structure) allows to construct wide classes of explicit solutions and explicitly recover potentials from the rational Weyl functions and reflection coefficients (see [16, 17, 25, 39, 40, 42, 46] and references cited there).

In this article, we apply GBDT to construct a large class of explicit solutions of the nonlocal matrix NLS and corresponding explicit expressions for the wave functions. The construction of the wave functions is of interest in itself and for possible further applications to spectral and scattering results.

The more complicated class of algebro-geometric solutions is both interesting and important (see, e.g. [18, 21, 22, 28, 30, 36] for algebro-geometric solutions in the context of several different nonlinear evolution equations). Its elements can still be regarded as explicit solutions, even though their complexity increases due to the underlying analysis on hyperelliptic Riemann surfaces. Some initial results and representations of such solutions for the nonlocal NLS in the stationary case are given here.

Section 2 contains some necessary preliminary results on the GBDT approach. In Section 3, we apply GBDT to the nonlocal NLS. Section 4 is dedicated to the construction of explicit solutions of the nonlocal NLS, examples are considered in Section 5. A nonlocal analog of the important algebro-geometric Theorem 3.11 from [21] is presented in Section 6. The necessary results on algebro-geometric solutions are given in Appendix A.

As usually $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of natural, integer, real and complex numbers, respectively. The notation $\mathrm{diag}\{a_1, \ldots, a_n\}$ stands for the $n \times n$ diagonal matrix with the entries $a_1, \ldots, a_n$ on the main diagonal.

2 Preliminaries

The zero curvature representation

$$G_t(x,t,z) - F_x(x,t,z) + G(x,t,z)F(x,t,z) - F(x,t,z)G(x,t,z) = 0 \quad (2.1)$$

is an important modification of the famous Lax pairs (see [1, 49] and more historical remarks in [13]). System (2.1) is the compatibility condition for the auxiliary linear systems

$$w_x(x,t,z) = G(x,t,z)w(x,t,z), \quad w_t(x,t,z) = F(x,t,z)w(x,t,z). \qquad (2.2)$$

This fact is easily proved in one direction and in a more complicated way (see [47, Ch. 12] and [44]) in the opposite direction. The coupled NLS (1.1) admits representation (2.1) where G and F are matrix polynomials of the first and second orders (with respect to $z \in \mathbb{C}$):

$$G = -(zq_1 + q_0), \quad F = -(z^2Q_2 + zQ_1 + Q_0); \quad 2q_1 \equiv -Q_2 \equiv 2\mathrm{i}j, \qquad (2.3)$$

$$2q_0(x,t) = -Q_1(x,t) = 2j\xi(x,t), \quad Q_0(x,t) = \mathrm{i}(j\xi(x,t)^2 - \xi_x(x,t)). \qquad (2.4)$$

Here j and ξ have the form (1.2). *From here on in the text we consider* G, F, $\{q_k\}$ *and* $\{Q_k\}$ *as given by* (2.3) *and* (2.4). The auxiliary system $w_x = GW$ is the well-known Dirac system, also called Zakharov-Shabat or AKNS system.

Remark 2.1. *Note that the so called spectral* (*or scattering*) *parameter* z *in* (2.1)–(2.3) *is essential in the study of the scattering and Weyl-Titchmarsh functions of the auxiliary Dirac system. The evolution of the scattering functions is crucial in the Inverse Scattering Method for solving Cauchy problems for integrable nonlinear equations* (*see e.g. [1, 5, 13, 51] and various references therein*)*. The evolution of the Weyl-Titchmarsh functions is used in the study of the initial-boundary value problems* (*see [46] and references therein*)*.*

The results on the GBDT for the coupled NLS are derived in [43, Sec. 3]. Let us formulate some of them below. Each GBDT for system (1.1) is determined by the initial system itself and *five parameter matrices with complex-valued entries*: $n \times n$ ($n \in \mathbb{N}$) matrices A_1, A_2, and $S(0,0)$, and $n \times m$ matrices $\Pi_1(0,0)$, $\Pi_2(0,0)$ such that

$$A_1S(0,0) - S(0,0)A_2 = \Pi_1(0,0)\Pi_2(0,0)^*, \quad \det S(0,0) \neq 0, \quad m := m_1 + m_2. \qquad (2.5)$$

If (2.1) holds, then the following linear systems are compatible and (jointly with the initial values $S(0,0)$, $\Pi_1(0,0)$, and $\Pi_2(0,0)$) determine matrix functions $S(x,t)$, $\Pi_1(x,t)$, and $\Pi_2(x,t)$, respectively:

$$\Pi_{1,x} = \sum_{p=0}^{1} A_1^p \Pi_1 q_p, \quad \Pi_{1,t} = \sum_{p=0}^{2} A_1^p \Pi_1 Q_p \quad \left(\Pi_{1,x} := \frac{\partial}{\partial x}\Pi_1\right); \tag{2.6}$$

$$\Pi_{2,x} = -\sum_{p=0}^{1} (A_2^*)^p \Pi_2 q_p^*, \quad \Pi_{2,t} = -\sum_{p=0}^{2} (A_2^*)^p \Pi_2 Q_p^*; \tag{2.7}$$

$$S_x = \Pi_1 q_1 \Pi_2^*, \quad S_t = \sum_{p=1}^{2}\sum_{k=1}^{p} A_1^{p-k}\Pi_1 Q_p \Pi_2^* A_2^{k-1}. \tag{2.8}$$

Although the point $x=0$, $t=0$ is chosen above as the initial point, it is easy to see that any other point may be chosen for this purpose as well.

Consider $S(x,t)$, $\Pi_1(x,t)$, and $\Pi_2(x,t)$ in some domain D, for instance,

$$D = \{(x,t) : \, -\infty \le a_1 < x < a_2 \le \infty, \; -\infty \le b_1 < t < b_2 \le \infty\},$$

such that ξ is well-defined in D and satisfies (1.1) and such that $(0,0) \in D$. Introduce (in the points of invertibility of $S(x,t)$ in D) matrix functions

$$\widetilde{\xi} = \begin{bmatrix} 0 & \widetilde{v}_1 \\ \widetilde{v}_2 & 0 \end{bmatrix} := \xi + \mathrm{i}(jX_0 j - X_0), \quad X_0 := \Pi_2^* S^{-1}\Pi_1. \tag{2.9}$$

Proposition 2.2. *Let ξ satisfy the coupled NLS* (1.1)*. Then, in the points of invertibility of S, the matrix function $\widetilde{\xi}$ given by* (2.9) *satisfies the coupled NLS as well.*

Remark 2.3. *Proposition 2.2 was proved as [43, Proposition 3.1] earlier and checked recently using a program [48] developed by D.R. Popovych and based on the NCAlgebra package.*

Remark 2.4. *Relations* (2.5)–(2.8) *imply that the matrix identity*

$$A_1 S(x,t) - S(x,t) A_2 = \Pi_1(x,t)\Pi_2(x,t)^* \tag{2.10}$$

holds everywhere on D.

The so called Darboux matrix corresponding to the transformation $\xi \to \widetilde{\xi}$ has (at each point (x,t) of invertibility of $S(x,t)$) the form of the Lev Sakhnovich's transfer matrix function (see [46,47] and references therein):

$$w_A(x,t,z) = I_m - \Pi_2(x,t)^* S(x,t)^{-1}(A_1 - zI_n)^{-1}\Pi_1(x,t). \tag{2.11}$$

In other words, we have the following statement (see [43, Sectons 2, 3]).

Proposition 2.5. *Let w satisfy the auxiliary systems* (2.2)*. Then, the function*

$$\widetilde{w}(x,t,z) = w_A(x,t,z)w(x,t,z) \tag{2.12}$$

satisfies the transformed system

$$\widetilde{w}_x(x,t,z) = \widetilde{G}(x,t,z)\widetilde{w}(x,t,z), \quad \widetilde{w}_t(x,t,z) = \widetilde{F}(x,t,z)\widetilde{w}(x,t,z), \tag{2.13}$$

where

$$\widetilde{G} = -(z\widetilde{q}_1 + \widetilde{q}_0), \quad F = -(z^2\widetilde{Q}_2 + z\widetilde{Q}_1 + \widetilde{Q}_0); \quad \widetilde{2}q_1 \equiv -\widetilde{Q}_2 \equiv 2\mathrm{i}j, \tag{2.14}$$

$$2\widetilde{q}_0(x,t) = -\widetilde{Q}_1(x,t) = 2j\widetilde{\xi}(x,t), \quad \widetilde{Q}_0(x,t) = \mathrm{i}(j\widetilde{\xi}(x,t)^2 - \widetilde{\xi}_x(x,t)), \tag{2.15}$$

and $\widetilde{\xi}$ is given by (2.9)*.*

3 GBDT for nonlocal NLS

In this section, we consider the case when the condition (1.4) is valid, and so the coupled NLS is reduced to the nonlocal NLS (1.5). In view of the first equality in (1.2), relations (1.4) are equivalent to

$$\xi(-x) = -\sigma\xi(x)^* \quad (\sigma = \mp 1). \tag{3.1}$$

1. Consider first the case $\sigma = -1$. Then, taking into account (2.3), (2.4), and (3.1) we have

$$q_1^* = -jq_1j, \quad q_0(x,t)^* = jq_0(-x,t)j; \tag{3.2}$$

$$Q_2^* = -jQ_2j, \quad Q_1(x,t)^* = jQ_1(-x,t)j, \tag{3.3}$$

$$\xi_x(-x,t) = -(\xi_x(x,t))^*, \quad Q_0(x,t)^* = -jQ_0(-x,t)j. \tag{3.4}$$

Relations (2.6) and (3.2) imply that

$$\big(\Pi_1(-x,t)j\big)_x = -\sum_{p=0}^{1} A_1^p\big(\Pi_1(-x,t)j\big)jq_p(-x,t)j$$

$$= -\sum_{p=0}^{1}(-A_1)^p\big(\Pi_1(-x,t)j\big)q_p(x,t)^*. \tag{3.5}$$

In the same way, formulas (2.6), (3.3), and (3.4) yield the equation

$$\big(\Pi_1(-x,t)j\big)_t = -\sum_{p=0}^{2}(-A_1)^p\big(\Pi_1(-x,t)j\big)Q_p(x,t)^*. \tag{3.6}$$

Comparing (2.7) with (3.5), (3.6) we see that in the case

$$\xi(-x) = \xi(x)^*, \quad A_2^* = -A_1 \tag{3.7}$$

we may set

$$\Pi(x,t) := \Pi_1(x,t), \quad \Pi_2(x,t) = \Pi(-x,t)j. \tag{3.8}$$

In view of (3.8), relations (2.8) take the form

$$S_x(x,t) = \mathrm{i}\Pi(x,t)\Pi(-x,t)^*, \tag{3.9}$$

$$S_t(x,t) = \sum_{p=1}^{2}\sum_{k=1}^{p}(-1)^{k-1}A^{p-k}\Pi(x,t)Q_p(x,t)j\Pi(-x,t)^*(A^*)^{k-1} \quad (A := A_1).$$

Thus, under condition

$$S(0,0) = S(0,0)^* \tag{3.10}$$

we have $S(0,t) = S(0,t)^*$, and so

$$S(-x,t) = S(x,t)^*. \tag{3.11}$$

According to (2.9) and (3.8) we have

$$\widetilde{\xi}(x,t) = \xi(x,t) + \mathrm{i}\big(\Pi(-x,t)^*S(x,t)^{-1}\Pi(x,t)j - j\Pi(-x,t)^*S(x,t)^{-1}\Pi(x,t)\big). \tag{3.12}$$

From (3.7), (3.11), and (3.12), it is immediate that

$$\widetilde{\xi}(-x,t)^* = \widetilde{\xi}(x,t). \tag{3.13}$$

Recall that by virtue of Proposition 2.2 the matrix function $\widetilde{\xi}$ satisfies the coupled NLS. The additional property (3.13) means that the block $\widetilde{u} := \widetilde{v}_1$ of $\widetilde{\xi}$ satisfies the nonlocal matrix NLS. In other words, we constructed a GBDT-transformed solution of the nonlocal matrix NLS (with $\sigma = -1$).

2. Let us formulate our result on GBDT for the nonlocal NLS for both cases $\sigma = \mp 1$.

Theorem 3.1. *Let an $m_1 \times m_2$ matrix function $u(x,t)$ satisfy the nonlocal NLS* (1.5)*, and assume that a triple of matrices $\{A,\, S(0,0), \Pi(0,0)\}$ with complex-valued entries, such that*

$$AS(0,0) + S(0,0)A^* = \Pi(0,0)j^{\varkappa}\Pi(0,0)^* \quad (\varkappa := (1-\sigma)/2), \tag{3.14}$$

is given, where A and $S(0,0) = S(0,0)^*$ are $n \times n$ matrices, $\det S(0,0) \neq 0$*, and $\Pi(0,0)$ is an $n \times m$ matrix.*

Introduce the matrix function $\xi(x,t)$ by the first equality in (1.2) *and by the relations* (1.4)*, and determine $\Pi(x,t)$ and $S(x,t)$ by their values $\Pi(0,0)$ and $S(0,0)$, respectively, at $(x,t) = (0,0)$ and by the equations*

$$\Pi_x(x,t) = \sum_{p=0}^{1} A^p \Pi(x,t) q_p(x,t), \quad \Pi_t(x,t) = \sum_{p=0}^{2} A^p \Pi(x,t) Q_p(x,t); \tag{3.15}$$

$$S_x(x,t) = \mathrm{i}\Pi(x,t)j^{\varkappa+1}\Pi(-x,t)^*, \tag{3.16}$$

$$S_t(x,t) = \sum_{p=1}^{2}\sum_{k=1}^{p}(-1)^{k-1}A^{p-k}\Pi(x,t)Q_p(x,t)j^{\varkappa}\Pi(-x,t)^*(A^*)^{k-1}, \tag{3.17}$$

where the coefficients $\{q_p\}$ and $\{Q_p\}$ are defined (*via ξ*) *in* (2.3) *and* (2.4)*.*

Then, the matrix function

$$\widetilde{u}(x,t) = u(x,t) - 2\mathrm{i}\begin{bmatrix} I_{m_1} & 0 \end{bmatrix}\Pi(-x,t)^* S(x,t)^{-1}\Pi(x,t)\begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \tag{3.18}$$

also satisfies (*in the points of invertibility of $S(x,t)$*) *the nonlocal NLS. That is, the equality*

$$\mathrm{i}\widetilde{u}_t(x,t) - \widetilde{u}_{xx} + 2\sigma\widetilde{u}(x,t)\widetilde{u}(-x,t)^*\widetilde{u}(x,t) = 0 \tag{3.19}$$

holds.

Proof. It is immediate that the right-hand side of (3.18) coincides with $\widetilde{v}_1$, and so for $\sigma = -1$ the statement of the theorem is already proved in paragraph 1 above.

Now, we assume that $\sigma = 1$ and prove our theorem in a similar way as for the case $\sigma = -1$. Namely, in view of the equality $\xi(x)^* = -\xi(-x)$ we have

$$q_1^* = -q_1, \quad q_0(x,t)^* = q_0(-x,t); \tag{3.20}$$

$$Q_2^* = -Q_2, \quad Q_1(x,t)^* = Q_1(-x,t), \quad Q_0(x,t)^* = -Q_0(-x,t) \tag{3.21}$$

(instead of the equalities (3.2)–(3.4) in the case $\sigma = -1$). Hence, relations (3.5) and (3.6) are substituted with

$$\big(\Pi_1(-x,t)\big)_x = -\sum_{p=0}^{1}(-A_1)^p\big(\Pi_1(-x,t)\big)q_p(x,t)^*, \tag{3.22}$$

$$\big(\Pi_1(-x,t)\big)_t = -\sum_{p=0}^{2}(-A_1)^p\big(\Pi_1(-x,t)\big)Q_p(x,t)^*. \tag{3.23}$$

Thus, we may set

$$A := A_1, \quad A_2 = -A^*, \quad \Pi(x,t) := \Pi_1(x,t), \quad \Pi_2(x,t) = \Pi(-x,t), \tag{3.24}$$

and formulas (2.8) take the form

$$S_x(x,t) = \mathrm{i}\Pi(x,t)j\Pi(-x,t)^*, \tag{3.25}$$

$$S_t(x,t) = \sum_{p=1}^{2}\sum_{k=1}^{p}(-1)^{k-1}A^{p-k}\Pi(x,t)Q_p(x,t)\Pi(-x,t)^*(A^*)^{k-1}. \tag{3.26}$$

In particular, under assumption (3.10) relations (3.25) and (3.26) yield the equality $S(0,t) = S(0,t)^*$ and (3.11). Finally, taking into account (2.9), (3.11), and (3.24) we derive

$$\widetilde{\xi}(-x,t)^* = -\widetilde{\xi}(x,t), \tag{3.27}$$

and so the block $\widetilde{u} := \widetilde{v}_1$ of the solution $\widetilde{\xi}$ of (1.1) satisfies the nonlocal matrix NLS (1.5) with $\sigma = 1$. □

The following corollary is immediate from the theorem's proof.

Corollary 3.2. *Under the conditions of Theorem 3.1, the identity* (2.10) *takes the form*

$$AS(x,t) + S(x,t)A^* = \Pi(x,t)j^{\varkappa}\Pi(-x,t)^*, \tag{3.28}$$

and the equality (3.11) *always holds.*

Remark 3.3. *According to* (2.11), (3.8)*, and* (3.24)*, the Darboux matrix for the nonlocal NLS has the form*

$$w_A(x,t,z) = I_m - j^{\varkappa}\Pi(-x,t)^*S(x,t)^{-1}(A - zI_n)^{-1}\Pi(x,t). \tag{3.29}$$

Moreover, in case of the nonlocal NLS, the inverse matrix function $w_A(z)^{-1}$ admits (*see, e.g., general formulas* (*1.75*) *and* (*1.76*) *in [46]*) *the reduction*

$$\begin{aligned} w_B(x,t,z) := w_A(x,t,z)^{-1} &= I_m - j^{\varkappa}\Pi(-x,t)^*(A^* + zI_n)^{-1}S(x,t)^{-1}\Pi(x,t) \\ &= j^{\varkappa}w_A(-x,t,-\overline{z})^*j^{\varkappa}. \end{aligned} \tag{3.30}$$

Taking into account Proposition 2.5, we see that the wave function (*i.e., the fundamental solution*) $\widetilde{w}$ *of the transformed system* (2.13)*, where $\widetilde{G}$ and $\widetilde{F}$ are given by* (2.14) *and* (2.15) *with*

$$\widetilde{\xi}(x,t) = \begin{bmatrix} 0 & \widetilde{u}(x,t) \\ -\sigma\widetilde{u}(-x,t)^* & \end{bmatrix}, \tag{3.31}$$

has the form

$$\widetilde{w}(x,t,z) = w_A(x,t,z)w(x,t,z).$$

Here w_A is given in (3.29) *and w is the fundamental solution of the initial system* (2.2).

4 Explicit solutions

For some special choices of the initial solution u of the nonlocal NLS, Theorem 3.1 allows us to construct wide families of other explicit solutions of the nonlocal NLS. Clearly, the trivial initial solution $u \equiv 0$ is the most popular choice in the construction of explicit solutions via Bäcklund-Darboux transformations. In particular, in the case $u \equiv 0$ the fundamental solution (wave function) $\widetilde{w}$ of the transformed systems (2.13) considered in Remark 3.3 takes the form

$$\widetilde{w}(x,t,z) = w_A(x,t,z)\mathrm{e}^{-\mathrm{i}(zx-2z^2t)j}. \tag{4.1}$$

Choosing $u \equiv 0$, one may set $D = \mathbb{R}^2$, that is, investigate $\widetilde{u}(x,t)$ at all real values of x and t. In Sections 4 and 5, we assume $u \equiv 0$ and study this case in greater detail.

Remark 4.1. *Below we find explicit expressions for* $\Pi(x,t)$ *and* $S(x,t)$ (*see formulas* (4.4), (4.8) *and* (4.15)). *In view of* (4.1) *and* (3.29), *it means that we have explicit expressions for the wave function* $\widetilde{w}(x,t,z)$. *The wave function is of independent interest and it is also important in the study of the scattering and Weyl-Titchmarsh functions of the auxiliary system* (*see Remark 2.1*). *Clearly, the examples of* A, $\Pi(x,t)$ *and* $S(x,t)$ *in Section 5 provide examples of the wave functions as well* (*see Remark 5.6*).

1. First, partition Π into $n \times m_1$ and $n \times m_2$ blocks and set:

$$u(x,t) \equiv 0, \quad \Pi(x,t) = \begin{bmatrix} \Lambda_1(x,t) & \Lambda_2(x,t) \end{bmatrix}, \quad \Pi(0,0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}. \tag{4.2}$$

In view of (1.4), (2.4), and the first equality in (4.2), we have

$$q_0 = Q_1 = Q_0 = 0,$$

and equations (3.15) take a simple form

$$\Pi_x = \mathrm{i}A\Pi j, \quad \Pi_t = -2\mathrm{i}A^2\Pi j. \tag{4.3}$$

Using partition (4.2) and relations (4.3), write down Π in an explicit form

$$\Pi(x,t) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}(xA-2tA^2)}\vartheta_1 & \mathrm{e}^{-\mathrm{i}(xA-2tA^2)}\vartheta_2 \end{bmatrix}. \tag{4.4}$$

Hence, relations (3.16) and (3.17) take the form

$$S_x(x,t) = \mathrm{i}\Big(\mathrm{e}^{\mathrm{i}(xA-2tA^2)}\vartheta_1\vartheta_1^*\,\mathrm{e}^{\mathrm{i}(xA^*+2t(A^*)^2)} \\ + (-1)^{\varkappa+1}\mathrm{e}^{-\mathrm{i}(xA-2tA^2)}\vartheta_2\vartheta_2^*\,\mathrm{e}^{-\mathrm{i}(xA^*+2t(A^*)^2)}\Big), \tag{4.5}$$

$$S_t(x,t) = 2\mathrm{i}\big(\Pi(x,t)j^{\varkappa+1}\Pi(-x,t)^*A^* - A\Pi(x,t)j^{\varkappa+1}\Pi(-x,t)^*\big) \\ = 2\mathrm{i}\big(\mathrm{e}^{\mathrm{i}(xA-2tA^2)}\big(\vartheta_1\vartheta_1^*A^* - A\vartheta_1\vartheta_1^*\big)\mathrm{e}^{\mathrm{i}(xA^*+2t(A^*)^2)} \\ + (-1)^{\varkappa+1}\mathrm{e}^{-\mathrm{i}(xA-2tA^2)}\big(\vartheta_2\vartheta_2^*A^* - A\vartheta_2\vartheta_2^*\big)\mathrm{e}^{-\mathrm{i}(xA^*+2t(A^*)^2)}\big). \tag{4.6}$$

Now, we see that the following corollary of Theorem 3.1 is valid.

Corollary 4.2. *To each triple of matrices* $\{A,\, S(0,0), \Pi(0,0)\}$*, such that* A *and* $S(0,0) = S(0,0)^*$ *are* $n \times n$ *matrices,* $\det S(0,0) \neq 0$*,* $\Pi(0,0)$ *is an* $n{\times}m$ *matrix and* (3.14) *holds, corresponds an explicit solution of the nonlocal matrix NLS* (1.5) *(with* $\sigma = \mp 1$*).*

This solution has the form

$$\widetilde{u}(x,t) = -2\mathrm{i}\vartheta_1^*\,\mathrm{e}^{\mathrm{i}(xA^*+2t(A^*)^2)}S(x,t)^{-1}\mathrm{e}^{-\mathrm{i}(xA-2tA^2)}\vartheta_2, \tag{4.7}$$

where ϑ_1 *and* ϑ_2 *are the blocks of* $\Pi(0,0)$ *(see* (4.2)*) and the derivatives of* $S(x,t)$ *are given explicitly by* (4.5) *and* (4.6)*. Thus, the matrix function* $S(x,t)$ *is recovered, for instance, by*

$$S(x,t) = S(0,0) + \int_0^t S_t(0,r)dr + \int_0^x S_x(r,t)dr. \tag{4.8}$$

In terms of the blocks ϑ_1 and ϑ_2 of $\Pi(0,0)$, the identity (3.14) may be rewritten in the form

$$AS(0,0) + S(0,0)A^* = \vartheta_1\vartheta_1^* + (-1)^{\varkappa}\vartheta_2\vartheta_2^* \quad (\varkappa := (1-\sigma)/2). \tag{4.9}$$

Remark 4.3. *Doubly periodic (i.e., both time and spatially periodic) solutions are of interest. Such solutions for the local NLS have been studied in the important papers [6, 31, 32]. Doubly periodic solutions appear also in the case of the nonlocal NLS. Indeed, let the conditions of Corollary 4.2 hold. Assume additionally that* A *is a diagonal matrix with the real commensurable entries on the main diagonal. More precisely, assume that*

$$A = \mathrm{diag}\{a_1, \dots, a_n\}, \quad a_i + a_k \neq 0 \quad (1 \leq i, k \leq n), \tag{4.10}$$

and for some c we have

$$c\,a_i \in \mathbb{Z} \quad (c > 0, \quad 1 \le i \le n). \tag{4.11}$$

Then $\widetilde{u}(x,t)$ given by (4.7) *is periodic with respect to x* (*with a period* $2\pi c$) *if only $S(x,t)$ is periodic with this period. Moreover, in view of* (4.10) *$S(x,t)$ is uniquely recovered from* (3.28) *using formula* (4.4) *for* $\Pi(x,t)$. *Thus, $S(x,t)$ is periodic with the period $2\pi c$ because $\Pi(x,t)$ given by* (4.4) *as well as $\Pi(-x,t)$ are periodic with the period $2\pi c$.*

In the same way one can show that under assumptions (4.10) *and* (4.11) *$\widetilde{u}(x,t)$ given by* (4.7) *is periodic with respect to t* (*with a period* πc^2). *The simplest doubly periodic solutions have the form* (5.4).

2. Let us introduce several block matrices: $\mathcal{P} = \begin{bmatrix} I_n & I_n \end{bmatrix}$,

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} A^2 & 0 \\ 0 & -A^2 \end{bmatrix}, \quad \vartheta = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}. \tag{4.12}$$

Using (4.12), formula (4.4) may be rewritten in the form

$$\Pi(x,t) = \mathcal{P}\mathrm{e}^{-2\mathrm{i}t\mathcal{B}}\mathrm{e}^{\mathrm{i}x\mathcal{A}}\vartheta. \tag{4.13}$$

Hence, taking into account (3.16), one can further simplify the procedure of constructing $S(x,t)$.

Proposition 4.4. *Assume that $\mathcal{S}$ satisfies the identity*

$$\mathcal{A}\mathcal{S} + \mathcal{S}\mathcal{A}^* = \vartheta j^{\varkappa+1}\vartheta^*. \tag{4.14}$$

Then, we have

$$S(x,t) = C(t) + \mathcal{P}\mathrm{e}^{-2\mathrm{i}t\mathcal{B}}\mathrm{e}^{\mathrm{i}x\mathcal{A}}\mathcal{S}\mathrm{e}^{\mathrm{i}x\mathcal{A}^*}\mathrm{e}^{2\mathrm{i}t\mathcal{B}^*}\mathcal{P}^*, \tag{4.15}$$

where $C(t) = S(0,t) - \mathcal{P}\mathrm{e}^{-2\mathrm{i}t\mathcal{B}}\mathcal{S}\mathrm{e}^{2\mathrm{i}t\mathcal{B}^*}\mathcal{P}^*$.

Proof. Clearly, the right-hand side of (4.15) equals $S(0,t)$ at $x = 0$. Moreover, in view of (3.16), (4.13), and (4.14) the derivative of the right-hand side of (4.15) (with respect to x) equals $S_x(x,t)$. Thus, the proposition's statement is immediate. □

5 Examples

1. When $\sigma(A) \cap \sigma(-A^*) = \emptyset$ (where $\sigma(A)$ stands for the spectrum of A) the matrix function $S(x,t)$ is uniquely recovered from the identity (3.28). It is a convenient way to calculate some examples.

Example 5.1. *Assume that $m_1 = m_2 = 1$ and that $n = 1$, that is, A, ϑ_1, and ϑ_2 are scalars, and $S(x,t)$ and $\widetilde{u}(x,t)$ are scalar matrix functions. We set $A = a$ and fix a, ϑ_1, and ϑ_2 such that*

$$a + \overline{a} \neq 0, \quad \vartheta_1 \neq 0, \quad \vartheta_2 \neq 0. \tag{5.1}$$

Then, (3.28) *and* (4.4) *yield*

$$S(x,t) = \frac{1}{a+\overline{a}}\Big(\exp\big\{\mathrm{i}\big((a+\overline{a})x - 2(a^2-\overline{a}^2)t\big)\big\}|\vartheta_1|^2 + (-1)^{\varkappa}\exp\big\{-\mathrm{i}\big((a+\overline{a})x - 2(a^2-\overline{a}^2)t\big)\big\}|\vartheta_2|^2\Big). \tag{5.2}$$

Thus, formula (4.7) *for the solutions $\widetilde{u}$ of the nonlocal NLS* (3.19) *takes in*

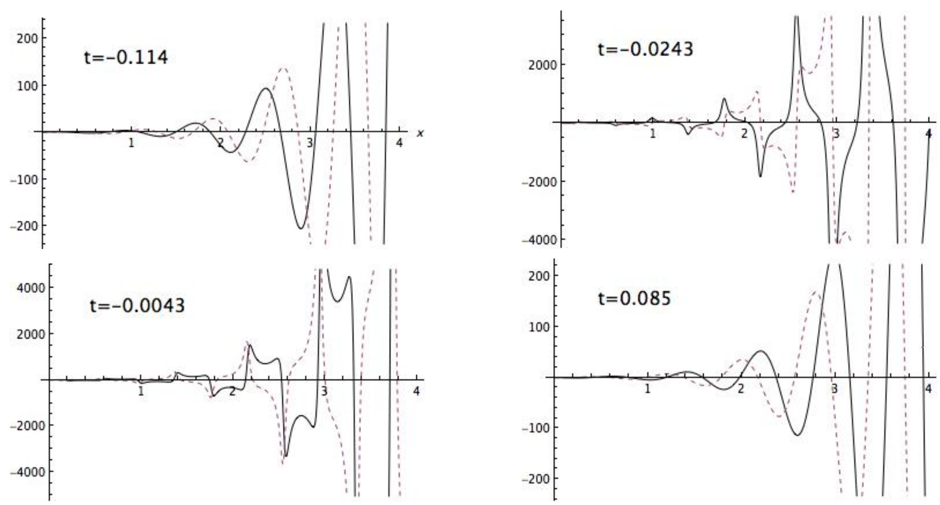


Figure 1: Sequence of time shots of real (black) and imaginary (dashed) part of solution (5.3) for $(a, \vartheta_1, \vartheta_2) = (4+\mathrm{i}, 2-\mathrm{i}, 1+\mathrm{i})$ in the defocusing case ($\varkappa = 0$).

our case the form

$$\begin{aligned}\widetilde{u}(x,t) =& \frac{-2\mathrm{i}}{S(x,t)} \exp\big\{\mathrm{i}\big((\overline{a}-a)x+2(a^2+\overline{a}^2)t\big)\big\}\overline{\vartheta_1}\vartheta_2 \\ =& -2\mathrm{i}(a+\overline{a})\exp\big\{-2\mathrm{i}a(x-2at)\big\}\overline{\vartheta_1}\vartheta_2 \\ & \times\big(|\vartheta_1|^2+(-1)^{\varkappa}\exp\big\{-2\mathrm{i}\big((a+\overline{a})x-2(a^2-\overline{a}^2)t\big)\big\}|\vartheta_2|^2\big)^{-1},\end{aligned} \tag{5.3}$$

where $S(x,t)$ *is given in* (5.2). *Solutions given in* (5.3) *are determined by four real valued parameters:* $\mathrm{Re}(a)$, $\mathrm{Im}(a)$, $|\vartheta_1/\vartheta_2|$ *and* $\arg(\vartheta_1/\vartheta_2)$.

The functions $\widetilde{u}$ above look similar to the interesting one-soliton solutions of nonlocal NLS studied in [2, 4]. However, there is also an essential difference because the one-soliton solutions in [2, 4] are periodic with respect to t. Instead of this property, we have the periodicity of $S(x,t)$ and of the denominator in (5.3) with respect to x. Solutions of the form (5.3) appear, for instance, in [10] (see also some further references therein).

Remark 5.2. *When* $a \neq \overline{a}$, *the singularities* (*or blow ups*) *of* $\widetilde{u}(x,t)$ (*i.e., zeros of* $S(x,t)$) *appear at one and only one value of* t. *Namely, they appear when* $|\vartheta_1|^2 = \mathrm{e}^{4\mathrm{i}(a^2-\overline{a}^2)t}|\vartheta_2|^2$. *For this* t, *the singularities appear with the periodicity* $T = \pi/(a+\overline{a})$ *with respect to* x, *compare Fig. 2.*

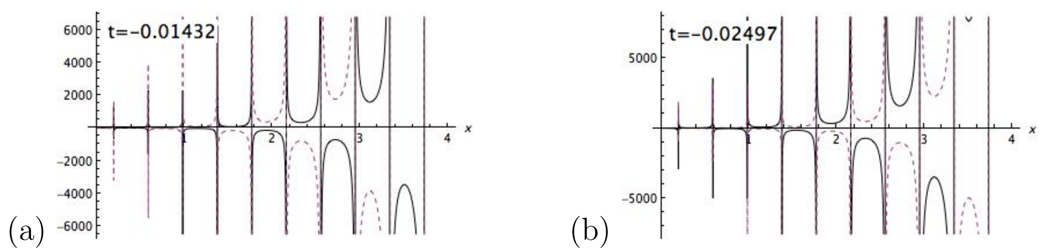


Figure 2: Time of blow up with $a = 4+\mathrm{i}$ and periodicity $T = \pi/8$ for the choice $(\vartheta_1, \vartheta_2) = (2-\mathrm{i}, 1+\mathrm{i})$ in (a) and $(\vartheta_1, \vartheta_2) = (8+\mathrm{i}/2, 3-2\mathrm{i})$ in (b), defocusing case.

When $a = \overline{a}$, *the conditions* (4.10) *and* (4.11) *hold, and we obtain the simplest doubly periodic solution, compare Fig. 3*

$$\widetilde{u}(x,t) = \frac{4a\overline{\vartheta_1}\vartheta_2\exp\big\{4\mathrm{i}a^2t\big\}}{\mathrm{i}\big(|\vartheta_1|^2\exp\{2\mathrm{i}ax\}+(-1)^{\varkappa}|\vartheta_2|^2\exp\{-2\mathrm{i}ax\}\big)}. \tag{5.4}$$

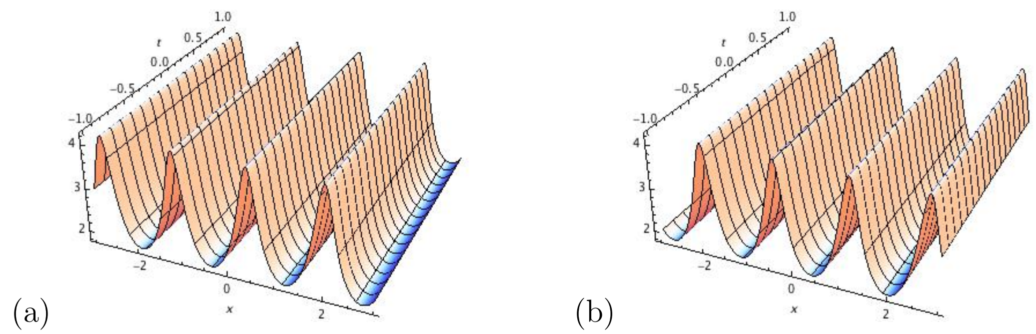


Figure 3: The periodic solution $|\tilde{u}|$ of (5.4) for $(a, \vartheta_1, \vartheta_2) = (1, 2 - \mathrm{i}, 1 + \mathrm{i})$ in the (a) defocusing and (b) focusing case ($\varkappa = 1$).

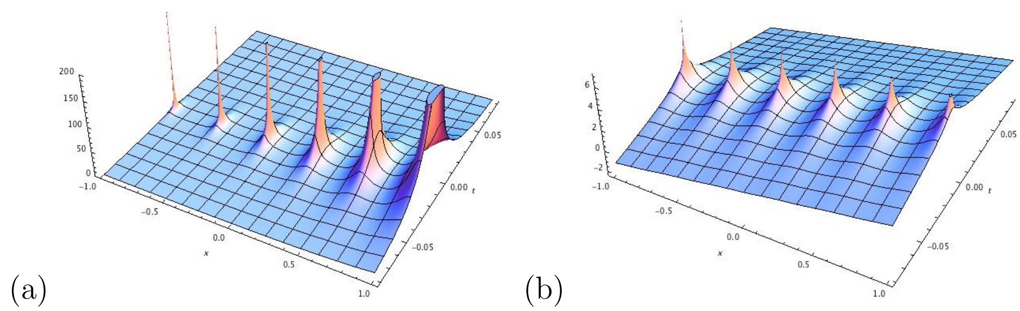


Figure 4: Solution (5.3) plotted as $|\tilde{u}|$ in (a) and $\log|\tilde{u}|$ in (b) for $(a, \vartheta_1, \vartheta_2) = (4 + \mathrm{i}, 2 - \mathrm{i}, 1 + \mathrm{i})$, defocusing case.

2. When we take non-diagonalisable matrices A, factors polynomial in x and t appear (in addition to the exponents) in the expressions for the constructed solutions [45]. Rational solutions are also constructed in this way [25, 42]. The so called "multipole" solutions are constructed using matrices A with Jordan cells of order more than one as well (see, e.g., [25]). For nonlocal NLS, we consider a simple particular case

$$A = aI_2 + A_0, \quad A_0 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad a + \overline{a} \neq 0; \quad \vartheta_1 = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} 0 \\ c \end{bmatrix}. \tag{5.5}$$

Example 5.3. *Assume that $m_1 = m_2 = 1$, $n = 2$, and that A, ϑ_1, and ϑ_2 are given by* (5.5)*. Here the solution $\widetilde{u}$ is again a scalar function but $S(x,t)$*

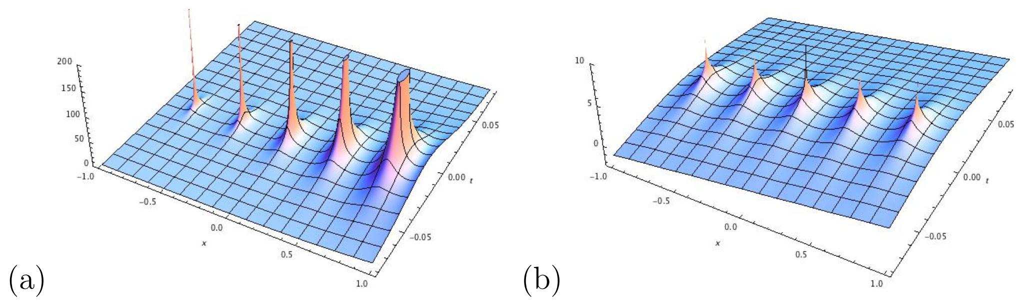


Figure 5: Same as in Fig. 4, focusing case.

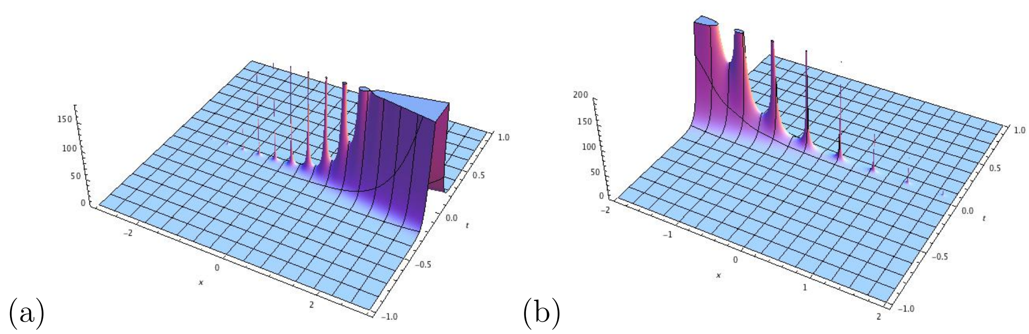


Figure 6: Solution $|\tilde{u}|$ of (5.3) with $(a, \vartheta_1, \vartheta_2) = (4 + \mathrm{i}, 3 + 5\mathrm{i}, 2 - \mathrm{i})$ in (a) and $(a, \vartheta_1, \vartheta_2) = (3 - \mathrm{i}, 8 - 5\mathrm{i}, 0.1 + 2\mathrm{i})$ in (b), defocusing case.

is a 2×2 *matrix function. The following relations for* $S = \{s_{ik}\}_{i,k=1}^2$ *are immediate from the identity* (3.28) *and the representation of* A *in* (5.5):

$$s_{22} = \omega_{22}/(a + \overline{a}), \quad s_{21} = (\omega_{21} - s_{22})/(a + \overline{a}),$$

$$s_{12} = (\omega_{12} - s_{22})/(a + \overline{a}), \quad s_{11} = (\omega_{11} - s_{12} - s_{21})/(a + \overline{a}); \tag{5.6}$$

$$\omega(x, t) = \{\omega_{ik}(x, t)\}_{i,k=1}^2 := \Pi(x, t) j^{\varkappa} \Pi(-x, t)^*. \tag{5.7}$$

After some simple calculations, using repeatedly (5.6) *we derive*

$$\det S = (a + \overline{a})^{-2} \big(\omega_{11}\omega_{22} - \omega_{12}\omega_{21} + (a + \overline{a})^{-2} \omega_{22}^2 \big). \tag{5.8}$$

Next, in view of (4.4) *and* (5.5), *we see that*

$$\Pi(x,t) = \begin{bmatrix} \mathrm{i}b(x-4at) & -\mathrm{i}c(x-4at) \\ b & c \end{bmatrix} \mathrm{e}^{\mathrm{i}(ax-2a^2t)j}. \tag{5.9}$$

Here we used the equalities $A_0^2 = 0$ *and* $A^2 = a^2 I_2 + 2aA_0$. *Relations* (5.7) *and* (5.9) *imply that*

$$\det \omega(x,t) = (-1)^{\varkappa+1} 4|bc|^2 (x-4at)(x+4\overline{a}t), \tag{5.10}$$

$$\omega_{22}(x,t) = |b|^2 \exp\{\mathrm{i}\big((a+\overline{a})x + 2(\overline{a}^2 - a^2)t\big)\} + (-1)^{\varkappa}|c|^2 \exp\{-\mathrm{i}\big((a+\overline{a})x + 2(\overline{a}^2 - a^2)t\big)\}. \tag{5.11}$$

Finally, (5.8), (5.10), *and* (5.11) *yield*

$$\det S(x,t) = (a+\overline{a})^{-4}\Big(|b|^4 \exp\{2\mathrm{i}\big((a+\overline{a})x + 2(\overline{a}^2 - a^2)t\big)\} + |c|^4 \exp\{-2\mathrm{i}\big((a+\overline{a})x + 2(\overline{a}^2 - a^2)t\big)\} + 2(-1)^{\varkappa}|bc|^2 + (-1)^{\varkappa+1} 4|bc|^2 (a+\overline{a})^2 (x-4at)(x+4\overline{a}t)\Big). \tag{5.12}$$

Note that the polynomial terms in the expression for $\det S(x,t)$ *make the study of zeros of* $\det S(x,t)$ (*that is, singularities of* $\widetilde{u}$) *much more complicated than in Example 5.1.*

Similar to the derivation of (5.9), *we rewrite* (4.7) (*in our case*) *as*

$$\widetilde{u}(x,t) = \frac{-2\mathrm{i}\,\overline{b}\,c}{\det S(x,t)} \exp\big\{\mathrm{i}\big((\overline{a}-a)x + 2(a^2+\overline{a}^2)t)\big)\big\} \begin{bmatrix} \mathrm{i}(x+4\overline{a}t) & 1 \end{bmatrix} \times \begin{bmatrix} s_{22}(x,t) & -s_{12}(x,t) \\ -s_{21}(x,t) & s_{11}(x,t) \end{bmatrix} \begin{bmatrix} -\mathrm{i}(x-4at) \\ 1 \end{bmatrix}, \tag{5.13}$$

where $\det S(x,t)$ *is given in* (5.12). *The expressions for* $\det S(x,t)$ *and for other terms on the right hand side of* (5.13) *will look more compact if we introduce the polynomial*

$$P(x,t) = \mathrm{i}\big((a+\overline{a})x + 2(\overline{a}^2 - a^2)t\big). \tag{5.14}$$

Then, relations (5.12) *and* (5.11) *may be rewritten as*

$$\det S(x,t) = (a+\overline{a})^{-4}\Big(|b|^4 \mathrm{e}^{2P(x,t)} + |c|^4 \mathrm{e}^{-2P(x,t)} + 2(-1)^{\varkappa}|bc|^2 + (-1)^{\varkappa+1} 4|bc|^2 (a+\overline{a})^2 (x-4at)(x+4\overline{a}t)\Big). \tag{5.15}$$

$$\omega_{22}(x,t) = |b|^2 \mathrm{e}^{P(x,t)} + (-1)^{\varkappa}|c|^2 \mathrm{e}^{-P(x,t)}. \tag{5.16}$$

In a similar way, taking into account (5.7) *and* (5.9)*, we construct other entries of* ω :

$$\omega_{11}(x,t) = -(x-4at)(x+4\overline{a}t)\big(|b|^2 e^{P(x,t)} + (-1)^{\varkappa}|c|^2 e^{-P(x,t)}\big), \tag{5.17}$$
$$\omega_{12}(x,t) = \mathrm{i}(x-4at)\big(|b|^2 e^{P(x,t)} + (-1)^{\varkappa+1}|c|^2 e^{-P(x,t)}\big), \tag{5.18}$$
$$\omega_{21}(x,t) = \mathrm{i}(x+4\overline{a}t)\big(|b|^2 e^{P(x,t)} + (-1)^{\varkappa+1}|c|^2 e^{-P(x,t)}\big). \tag{5.19}$$

Furthermore, relations (5.6) *imply that*

$$\begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix} = \frac{1}{a+\overline{a}} \begin{bmatrix} \omega_{22} & -\omega_{12} \\ -\omega_{21} & \omega_{11} \end{bmatrix} + \frac{1}{(a+\overline{a})^2} \begin{bmatrix} 0 & \omega_{22} \\ \omega_{22} & -\omega_{12}-\omega_{21} \end{bmatrix} + \frac{2}{(a+\overline{a})^3} \begin{bmatrix} 0 & 0 \\ 0 & \omega_{22} \end{bmatrix}. \tag{5.20}$$

In view of (5.16)–(5.20)*, after some simple calculations we rewrite* (5.13) *as*

$$\begin{aligned} \widetilde{u}(x,t) = &- 2\mathrm{i}\,\overline{b}\,c(a+\overline{a}) \exp\big\{\mathrm{i}\big((\overline{a}-a)x + 2(a^2+\overline{a}^2)t)\big)\big\} \\ &\times \Big(|b|^2 e^{P(x,t)}\big(8\mathrm{i}a(a+\overline{a})t - 2\mathrm{i}(a+\overline{a})x + 2\big) \\ &\quad + (-1)^{\varkappa}|c|^2 e^{-P(x,t)}\big(8\mathrm{i}\overline{a}(a+\overline{a})t + 2\mathrm{i}(a+\overline{a})x + 2\big)\Big) \\ &\times \Big(|b|^4 e^{2P(x,t)} + |c|^4 e^{-2P(x,t)} + 2(-1)^{\varkappa}|bc|^2 \\ &\quad + (-1)^{\varkappa+1}4|bc|^2(a+\overline{a})^2(x-4at)(x+4\overline{a}t)\Big)^{-1}. \end{aligned} \tag{5.21}$$

A related family of solutions depending on one complex parameter is also constructed in [10].

We note that if we choose $\vartheta_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}$ instead of ϑ_1 given in (5.5), the solutions (5.3) appear again in the case $n=2$ (i.e., one should avoid the simplest choice of ϑ_1, ϑ_2 in order to construct a new class of solutions).

Remark 5.4. *Various cases of Examples 5.1 and 5.3 are treated in Figures 1–8. Similar to the local NLS equations, the focusing and defocusing cases for the nonlocal NLS* (1.5) *are considered in the literature as well. The focusing case corresponds to the value* $\sigma = -1$ *in* (1.5) *and the defocusing*

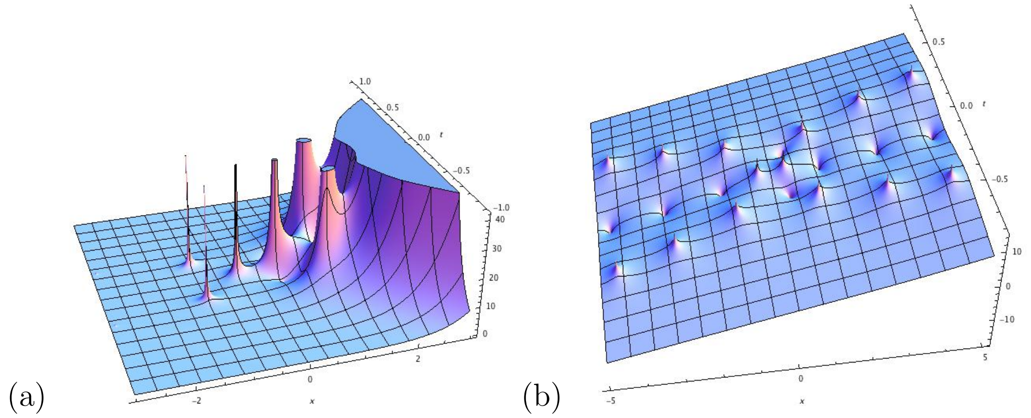


Figure 7: The multipole solution (5.21) plotted as $|\tilde{u}|$ in (a) and $\log|\tilde{u}|$ in (b) for $P(x,t) = 8t + 2ix$ with $a = b = c = 1 + \mathrm{i}$, defocusing case.

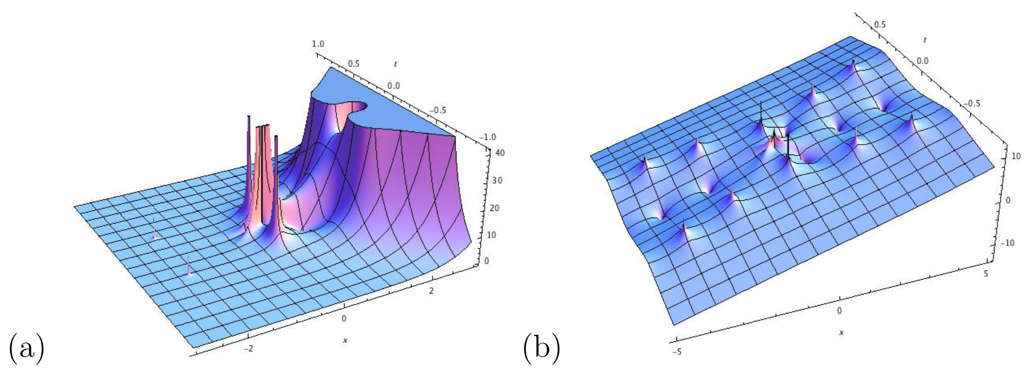
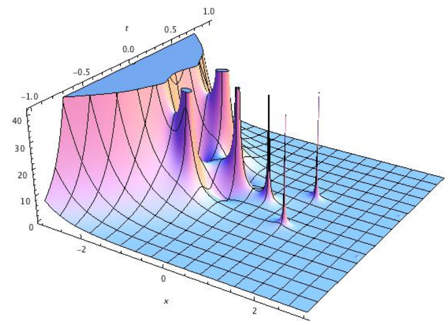
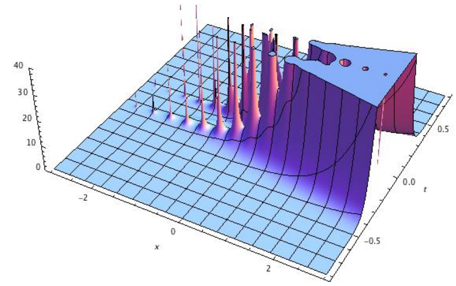


Figure 8: Same as in Fig. 7, focusing case.

case corresponds to $\sigma = 1$. Equivalently, the focusing case corresponds to $\varkappa = 1$ ($\varkappa = (1-\sigma)/2$) and the defocusing case corresponds to $\varkappa = 0$ in the examples. However, we see that there is no great difference between the focusing and defocusing cases for the nonlocal NLS: compare Fig. 4 with Fig. 5 or Fig. 7 with Fig. 8. The influence of other parameters is much more essential (see Fig. 6 and Fig. 9). Both in the focusing and defocusing cases there are doubly periodic solutions (see Fig. 3). One can also notice that the behaviour of the singularities in the multipole (twopole) case is much more complicated than for the one-soliton solution (compare Fig. 2–6 with Fig. 7–9).



(a)



(b)

Figure 9: Solution $|\tilde{u}|$ of (5.21) with $a = b = c = 1 - \mathrm{i}$ in (a) and $(a, b, c) = (4 + \mathrm{i}, 6 - 4\mathrm{i}, 2 - \mathrm{i})$ in (b), defocusing case.

Finally, in the example below we construct the simplest family of multi-component solutions of the nonlocal NLS.

Example 5.5. *Assume that* $m_1 = 2$*,* $m_2 = 1$*, and* $n = 1$*. Then, the solutions* $\widetilde{u}$ *of* (3.19) *are* 2×1 *vector functions and* $S(x,t)$ *are scalar functions. Introduce the parameters* A *and* $\Pi(0,0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}$ *by the equalities*

$$A = a \quad (a + \overline{a} \neq 0), \quad \vartheta_1 = \begin{bmatrix} b_1 & b_2 \end{bmatrix}, \quad \vartheta_2 = c. \tag{5.22}$$

Then, (3.28) *and* (4.4) *yield*

$$S(x,t) = \frac{1}{a + \overline{a}} \Big(\exp\big\{\mathrm{i}\big((a+\overline{a})x - 2(a^2 - \overline{a}^2)t\big)\big\}(|b_1|^2 + |b_2|^2) + (-1)^{\varkappa} \exp\big\{ - \mathrm{i}\big((a+\overline{a})x - 2(a^2 - \overline{a}^2)t\big)\big\}|c|^2\Big). \tag{5.23}$$

Hence, using again formula (4.7) (*and slightly modifying the results of Example 5.1*) *we derive*

$$\widetilde{u}(x,t) = \frac{-2\mathrm{i}\, c(a+\overline{a}) \exp\big\{ - 2\mathrm{i}a(x - 2at)\big\}}{|b_1|^2 + |b_2|^2 + (-1)^{\varkappa} \exp\big\{ - 2\mathrm{i}\big((a+\overline{a})x - 2(a^2 - \overline{a}^2)t\big)\big\}|c|^2} \begin{bmatrix} \overline{b_1} \\ \overline{b_2} \end{bmatrix}.$$

Compare (5.2) *and* (5.23) *to see that the behaviour of the singularities of* $\widetilde{u}$ *from Example 5.1, which we discussed in Remark 5.2, coincides with the behaviour of the singularities of* $\widetilde{u}$ *in the present example.*

Remark 5.6. *We note that formulas* (5.2), (5.14)–(5.20)*, and* (5.23) *provide precise expressions for* $S(x,t)^{-1}$ *in the Examples 5.1, 5.3, and 5.5, respectively. Thus, in view of the important formulas* (4.1) *and* (3.29) *as well as relations* (4.4) *and* (5.9)*, the wave functions* $\widetilde{w}$ *are constructed for these examples as well.*

6 Algebro-geometric solutions

In this section we discuss algebro-geometric solutions for the nonlocal NLS in the scalar case $m_1 = m_2 = 1$. The coupled NLS (1.3) in the scalar case is given by

$$\begin{cases} v_{1t} + \mathrm{i}v_{1xx} + 2\mathrm{i}v_1^2 v_2 = 0 \\ v_{2t} - \mathrm{i}v_{2xx} - 2\mathrm{i}v_2^2 v_1 = 0 \end{cases} \quad \Leftrightarrow \quad G_t - F_x + [G, F] = 0, \tag{6.1}$$

where the matrix polynomials G, F in (6.1) defined by (2.2)–(2.4) now read

$$G(z) = \begin{bmatrix} -\mathrm{i}z & -v_1 \\ v_2 & \mathrm{i}z \end{bmatrix}, \quad F(z) = \mathrm{i} \begin{bmatrix} 2z^2 - v_1 v_2 & -2\mathrm{i}z v_1 + v_{1x} \\ 2\mathrm{i}z v_2 + v_{2x} & -2z^2 + v_1 v_2 \end{bmatrix}. \tag{6.2}$$

System (6.1) is also known as the AKNS system, which was introduced by Ablowitz, Kaup, Newell, and Segur in 1974. Algebro-geometric solutions are well known for the AKNS hierarchy, see for example Gesztesy and Holden [21] and references therein. By definition, algebro-geometric AKNS solutions (or potentials) are the set of solutions of the stationary AKNS system

$$\text{s-AKNS}_1(v_1, v_2) = \begin{pmatrix} \frac{\mathrm{i}}{2} v_{1xx} - \mathrm{i}v_1^2 v_2 + c_1(-v_{1x}) + c_2(-2\mathrm{i}v_1) \\ -\frac{\mathrm{i}}{2} v_{2xx} + \mathrm{i}v_1 v_2^2 + c_1(-v_{2x}) + c_2(2\mathrm{i}v_2) \end{pmatrix} = 0, \tag{6.3}$$

with c_ℓ ranging in $\mathbb{C}$. More details can be found in Appendix A.

We call solutions of the stationary nonlocal NLS equation

$$\text{s-nNLS}_\pm(u) = \frac{\mathrm{i}}{2} u_{xx}(x) \mp \mathrm{i}u(x)^2 \overline{u(-x)} + \tilde{c}_1(-u_x(x)) + \tilde{c}_2(-2\mathrm{i}u(x)) = 0 \tag{6.4}$$

with $\tilde{c}_\ell$ ranging in $\mathbb{R}$, algebro-geometric nonlocal NLS solutions. Note that the plus sign in (6.4), denoted by s-nNLS$_+$, corresponds to the defocusing case, while the minus sign in (6.4), denoted by s-nNLS$_-$, corresponds to the focusing case. Such solutions can be recast as a particular case of algebro-geometric AKNS solutions by applying the following modified symmetry reduction to (6.3).

Lemma 6.1. *Let $e_0 \in \mathbb{R}$. If $u(x,t)$ satisfies* s-nNLS$_\pm(u) = 0$ *with $\tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$, then $v_1(x,t)$ and $v_2(x,t)$ defined by*

$$v_1(x,t) = u(x,t)e^{\mathrm{i}e_0 x}, \qquad v_2(x,t) = \pm\overline{u(-x,t)}e^{-\mathrm{i}e_0 x} \tag{6.5}$$

satisfy s-AKNS$_1(v_1, v_2) = 0$ *with constants c_1, c_2 given by*

$$c_1 = \tilde{c}_1 - e_0, \qquad c_2 = \tilde{c}_2 - \frac{1}{4}e_0^2 - \frac{1}{2}e_0 c_1.$$

The converse statement is also true.

In a natural manner one can associate a hyperelliptic Riemann surface with (6.3), as described in (A.18). The modified symmetry reduction (6.5) now implies certain constraints on the branch points of this surface, namely, the set of zeros $\{E_j\}$ of $R_4(z) = \prod_{j=0}^{3}(z - E_j)$ can either be real valued or consists of complex conjugate pairs. This follows from inserting (6.5) into (A.13)–(A.15) which yields

$$R_4(z) = \overline{R_4(\overline{z})}. \tag{6.6}$$

This relation implies (either one of) the following constraints on the set of zeros of $R_4(z)$ after possible relabeling:

$$\begin{aligned}
&\text{(i)} \quad E_0 < E_1 < E_2 < E_3, \quad E_j \in \mathbb{R},\\
&\text{(ii)} \quad E_0, \overline{E_0}, E_1, \overline{E_1} \in \mathbb{C} \setminus \mathbb{R},\\
&\text{(iii)} \quad E_0 < E_1, \quad E_0, E_1 \in \mathbb{R}, \quad E_2, \overline{E_2} \in \mathbb{C} \setminus \mathbb{R}.
\end{aligned}$$

Theorem 6.2. *Assume either* (i), (ii), *or* (iii) *and choose the homology basis $\{a_1, b_1\}$ according to Theorem A.2. Moreover, assume that Δ in* (A.30) *satisfies*

$$\mathrm{Re}(\Delta) = \frac{1}{2}\chi \pmod{\mathbb{Z}}, \quad \chi \in \{0, 1\}. \tag{6.7}$$

Then $u(x)$ represents a stationary nonlocal NLS solution if and only if A in (A.29) *satisfies the constraint*

$$\mathrm{Im}(A) = \frac{1}{2}\chi \mathrm{Im}(\tau) \pmod{\mathbb{Z}}, \quad \chi \in \{0,1\}. \tag{6.8}$$

Proof. First assume (i). Given E_j, $j = 0, \dots, 3$, and $\{a_1, b_1\}$, the constants c_ℓ and e_0 are uniquely determined by (A.19) and (A.24). Define the antiholomorphic involution $\rho_+ : (z, y) \mapsto (\overline{z}, \overline{y})$ as in [21, Example A.35 (i)]. One infers that the symmetric Riemann surface $(\mathcal{K}_1, \rho_+)$ is of dividing type (compare [21, Def. A.33]) and hence

$$\begin{aligned} &r = 2, \quad \rho_+(a_1) = a_1, \quad \rho_+(b_1) = -b_1, \\ &\overline{\tau} = -\tau, \quad U_0^{(2)} \in \mathbb{R}, \quad \overline{\theta(z)} = \theta(\overline{z}). \end{aligned}$$

Thus B defined in (A.30) is purely imaginary, $\overline{B} = -B$. So if $u(x)$ satisfies s-nNLS$_\pm(u) = 0$, then by Lemma 6.1 and Theorem A.1, the functions $v_1(x)$ and $v_2(x)$ admit representations (A.27) and (A.28). Applying (6.5) yields

$$\pm \frac{C_2}{\overline{C_1}} = \frac{\theta(A+Bx)\overline{\theta(A-Bx-\Delta)}}{\theta(A+Bx+\Delta)\overline{\theta(A-Bx)}} = \frac{\theta(A+Bx)\theta(\overline{A}+Bx-\overline{\Delta})}{\theta(A+Bx+\Delta)\theta(\overline{A}+Bx)}. \tag{6.9}$$

Equation (6.9) is equivalent to

$$A = \overline{A} + m_1 + n_1\tau$$

for some $n_1 \in \mathbb{Z}$ and arbitrary $m_1 \in \mathbb{Z}$, and hence

$$\mathrm{Im}(A) = \frac{1}{2} n_1 \mathrm{Im}(\tau), \quad n_1 \in \mathbb{Z},$$

and $m_1 = 0$. Similarly, one obtains

$$A + \Delta = \overline{A} - \overline{\Delta} + m_2 + n_2\tau$$

for some $n_2 \in \mathbb{Z}$ and arbitrary $m_2 \in \mathbb{Z}$, and hence

$$\mathrm{Re}(\Delta) = \frac{1}{2} m_2, \quad m_2 \in \mathbb{Z}. \tag{6.10}$$

Replacing A by $A+m+n\tau$ with $m, n \in \mathbb{Z}$ then yields (6.8) and (6.7). In case (ii), $(\mathcal{K}_1, \rho_+)$ is again of dividing type and, in particular, $\overline{B} = -B$. For (iii), $(\mathcal{K}_1, \rho_+)$ is of nondividing type and $U_0^{(2)} \in \mathbb{R}$ follows from [21, (C.37), (C.39), (C.33)]. Hence the same arguments as before yield (6.7) and (6.8). □

Remark 6.3. *Given E_j as in* (i)–(iii)*, we do not know if we get a solution of* s-nNLS$_+(u) = 0$ *or* s-nNLS$_-(u) = 0$ *by the constraints on A. This has to be determined a priori, that is, there should be a correspondence between the location of the E_j's and the defocusing/focusing nNLS equation. For comparison, the stationary nonlinear Schrödinger potentials correspond to condition* (i) *in the defocusing case and to* (ii) *in the focusing case (see [21, Lemmas 3.15, 3.18]).*

To see that condition (6.7) for Δ can indeed be satisfied, consider case (i), that is, $E_0 < E_1 < E_2 < E_3$. Then Δ can be rewritten in terms of elliptic integrals (see [22, Ex. 1.27]; L is the period lattice defined in the appendix):

$$\Delta = \int_{P_{\infty_+}}^{P_{\infty_-}} \omega_1 = -2 \int_{P_0}^{P_{\infty_+}} \omega_1 = -\frac{F(\ell, k)}{F(1,k)} \pmod{L},$$

where $F(z,k)$ denotes the Jacobi integral of the first kind

$$F(z,k) = \int_0^z \frac{dx}{((1-x^2)(1-k^2x^2))^{1/2}}$$

with values

$$k = \left(\frac{(E_2 - E_1)(E_3 - E_0)}{(E_3 - E_1)(E_2 - E_0)} \right)^{1/2} \in (0,1), \quad \ell = \left(\frac{E_3 - E_1}{E_3 - E_0} \right)^{1/2}.$$

The choice $E_0 = 1, E_1 = 2, E_2 = 3$, and $E_4 = 4$ yields $\Delta = -1/2$ as desired.

Acknowledgments. The authors are grateful to F. Gesztesy for helpful discussions and to the referees for helpful remarks. This research was supported by the Austrian Science Fund (FWF) under Grants No. P29177 and V120.

A Algebro-geometric AKNS solutions

Following [21], we give a brief introduction to algebro-geometric AKNS solutions and their underlying Riemann surface and describe the theta function representation of such solutions which we need for Theorem 6.2. The analog of these formulas for the nonlinear Schrödinger equation was first published by Its and Kotlyarov [29]. Since then, many authors presented varying approaches to algebro-geometric solutions of the nonlinear Schrödinger and AKNS equations, see for instance E. Belokolos and V. Enol'skii [7], Gesztesy and Ratnaseelan [23] or Previato [37] and references therein.

The stationary AKNS system (6.3) is equivalent to the stationary zero-curvature equation

$$\text{s-AKNS}_1(v_1, v_2) = 0 \quad \Leftrightarrow \quad -V_{2,x} + [U, V_2] = 0, \tag{A.11}$$

where

$$U(z) = \begin{bmatrix} -\mathrm{i}z & v_1 \\ v_2 & \mathrm{i}z \end{bmatrix}, \quad V_2(z) = \mathrm{i} \begin{bmatrix} -G_2(z) & F_1(z) \\ -H_1(z) & G_2(z) \end{bmatrix}. \tag{A.12}$$

The polynomials $G_2(z)$, $F_1(z)$, and $H_1(z)$ incorporate the constants c_ℓ,

$$G_2(z) = z^2 + \frac{1}{2} v_1 v_2 + c_1 z + c_2, \tag{A.13}$$

$$F_1(z) = -\mathrm{i} v_1 z + \frac{1}{2} v_{1,x} + c_1(-\mathrm{i} v_1), \tag{A.14}$$

$$H_1(z) = \mathrm{i} v_2 z + \frac{1}{2} v_{2,x} + c_1(\mathrm{i} v_2). \tag{A.15}$$

The stationary zero-curvature equation in (A.11) yields that

$$\big(G_2^2 - F_1 H_1\big)_x = 0, \tag{A.16}$$

and hence $G_2^2 - F_1 H_1$ is x-independent, implying $G_2^2 - F_1 H_1 = R_4$, where the integration constant R_4 is a monic polynomial of degree 4. If $E_0, \dots, E_3$ denote its zeros, then

$$R_4(z) = \prod_{m=0}^{3} (z - E_m), \quad \{E_m\}_{m=0}^{3} \subset \mathbb{C}. \tag{A.17}$$

In this manner we can associate a hyperelliptic curve $\mathcal{K}_1$ of genus 1 with (6.3) defined by

$$\mathcal{K}_1 : \mathcal{F}_1(z, y) = y^2 - R_4(z) = 0. \tag{A.18}$$

The curve $\mathcal{K}_1$ is compactified by joining two points at infinity, $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$; we denote the compactification again by $\mathcal{K}_1$. On $\mathcal{K}_1 \setminus \{P_{\infty_+}, P_{\infty_-}\}$, points P are represented as pairs $P=(z,y)$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_1$ satisfying $\mathcal{F}_1(z,y)=0$. The complex structure on $\mathcal{K}_1$ is then defined in the usual way (see for example [21, App. C]). Hence $\mathcal{K}_1$ becomes a two-sheeted hyperelliptic Riemann surface of genus 1. We emphasize that by fixing the curve $\mathcal{K}_1$ (i.e., by fixing $E_0,\dots,E_3$), the integration constants c_1, c_2 in (6.3) are uniquely determined,

$$c_1=-\frac{1}{2}(E_0+\cdots+E_3), \qquad c_2=-\frac{c_1^2}{8}+\frac{1}{2}\sum_{m,n=0;m<n}^{3} E_mE_n. \tag{A.19}$$

Let $\mu(x)$ and $\nu(x)$ denote the zeros of $F_1(z)$ and $H_1(z)$ in (A.14) and (A.15),

$$F_1(z)=-\mathrm{i}v_1(z-\mu), \qquad H_1(z)=\mathrm{i}v_2(z-\nu). \tag{A.20}$$

We lift $\mu(x)$ and $\nu(x)$ to $\mathcal{K}_1$ by defining

$$\hat{\mu}(x)=(\mu(x),G_2(\mu(x),x))\in\mathcal{K}_1, \qquad \hat{\nu}(x)=(\nu(x),-G_2(\nu(x),x))\in\mathcal{K}_1.$$

Choose a homology basis $\{a_1,b_1\}$ on $\mathcal{K}_1$ and denote by ω_1 the corresponding normalized holomorphic differential, that is,

$$\int_{a_1}\omega_1=1, \quad \int_{b_1}\omega_1=\tau\in\mathbb{C}. \tag{A.21}$$

Note that $\mathrm{Im}(\tau)>0$. Let $\Xi=\frac{\tau}{2}+\frac{1}{2}$ be the Riemann constant. The Riemann theta function associated with $\mathcal{K}_1$ is given by

$$\theta(z)=\sum_{m\in\mathbb{Z}}\exp(2\pi\mathrm{i}mz+\pi\mathrm{i}m^2\tau). \tag{A.22}$$

Without loss of generality we choose the branch point $P_0=(E_0,0)$ as a base point. Let $\omega^{(2)}_{P_{\infty_\pm},0}$ be a normalized differential of the second kind satisfying

$$\int_{a_1}\omega^{(2)}_{P_{\infty_\pm},0}=0, \qquad \omega^{(2)}_{P_{\infty_\pm},0}=(\zeta^{-2}+O(1))d\zeta \text{ as } P\to P_{\infty_\pm}, \tag{A.23}$$

where ζ denotes the local coordinate $\zeta=1/z$ for P near $P_{\infty_\pm}$. Then

$$\int_{P_0}^{P}\left(\omega^{(2)}_{P_{\infty_+},0}-\omega^{(2)}_{P_{\infty_-},0}\right)=\mp\left(\zeta^{-1}+\frac{e_0}{2}+e_1\zeta+O(\zeta^2)\right) \text{ as } P\to P_{\infty_\pm}. \tag{A.24}$$

In addition, we denote the b_1-period of this difference by

$$U_0^{(2)} = \frac{1}{2\pi\mathrm{i}} \int_{b_1} \left(\omega^{(2)}_{P_{\infty_+},0} - \omega^{(2)}_{P_{\infty_-},0} \right). \tag{A.25}$$

Finally, we turn to divisors, the Jacobi variety, and the Abel map for divisors in our setting. A divisor $\mathcal{D}$ on $\mathcal{K}_1$ is a map $\mathcal{D} : \mathcal{K}_1 \to \mathbb{Z}$, where $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_1$. We define the positive divisor $\mathcal{D}_Q$ by

$$\mathcal{D}_Q : \mathcal{K}_1 \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{if } P = Q, \\ 0 & \text{if } P \neq Q, \end{cases} \qquad Q \in \mathcal{K}_1,$$

and denote the set of all divisors on $\mathcal{K}_1$ by $\mathrm{Div}(\mathcal{K}_1)$. The Jacobi variety $J(\mathcal{K}_1)$ of $\mathcal{K}_1$ is defined by $J(\mathcal{K}_1) = \mathbb{C}/L$, where L is the period lattice $L = \{z \in \mathbb{C} \mid z = n + m + \tau,\ n, m \in \mathbb{Z}\}$. The Abel map for divisors is then defined by

$$\alpha_{P_0} : \mathrm{Div}(\mathcal{K}_1) \to J(\mathcal{K}_1), \quad \mathcal{D} \mapsto \alpha_{P_0}(\mathcal{D}) = \sum_{P \in K_1} \mathcal{D}(P) \int_{P_0}^{P} \omega_1. \tag{A.26}$$

With these quantities at hand, the algebro-geometric AKNS solutions admit the following representation in terms of Riemann theta functions, compare [21, Thm. 3.11].

Theorem A.1. *Suppose that $v_1, v_2 \in C^\infty(\Omega)$ are nonzero and satisfy the stationary AKNS system* (6.3) *on Ω. In addition, assume the affine part of $\mathcal{K}_1$ to be nonsingular and let $x, x_0 \in \Omega$, where $\Omega \subseteq \mathbb{R}$ is an open interval. Then*

$$v_1(x) = C_1 \frac{\theta(A + Bx - \Delta)}{\theta(A + Bx)} \exp(\mathrm{i}e_0 x), \tag{A.27}$$

$$v_2(x) = C_2 \frac{\theta(A + Bx + \Delta)}{\theta(A + Bx)} \exp(-\mathrm{i}e_0 x), \tag{A.28}$$

where

$$A = \Xi - \int_{P_0}^{P_{\infty_+}} \omega_1 + \mathrm{i}U_0^{(2)} x_0 + \alpha_{P_0}(\mathcal{D}_{\hat{\mu}(x_0)}), \tag{A.29}$$

$$B = -\mathrm{i}U_0^{(2)}, \qquad \Delta = \int_{P_{\infty_+}}^{P_{\infty_-}} \omega_1. \tag{A.30}$$

The constants $e_0 \in \mathbb{C}$ *and* Δ, B *are uniquely determined by* $\mathcal{K}_1$ *(and its homology basis), the constant* A *is in one-to-one correspondence with the Dirichlet datum* $\hat{\mu}(x_0)$ *at the point* x_0*. The constants* $C_1, C_2 \in \mathbb{C}$ *are given by*

$$C_1 = v_1(x_0) \frac{\theta(A + Bx_0)}{\theta(A + Bx_0 - \Delta)} \exp(-\mathrm{i}e_0 x_0), \tag{A.31}$$

$$C_2 = \frac{4}{v_1(x_0)\omega_0^2} \frac{\theta(A + Bx_0 - \Delta)}{\theta(A + Bx_0)} \exp(\mathrm{i}e_0 x_0), \tag{A.32}$$

and satisfy the constraint

$$C_1 C_2 = \frac{4}{\omega_0^2}. \tag{A.33}$$

Note that the free constant $v_1(x_0)$ in (A.31) cannot be determined since the AKNS equations are invariant with respect to scale transformations, $(v_1(x,t), v_2(x,t)) \mapsto (av_1(x,t), a^{-1}v_2(x,t))$ for $a \in \mathbb{C} \setminus \{0\}$.

We conclude this appendix with the following result used in the characterization of algebro-geometric nonlocal NLS solutions. The genus $g = 1$ case of [21, Theorem A.36 (i)] reads

Theorem A.2. *Let* $(\mathcal{K}_1, \rho)$ *be a symmetric Riemann surface, i.e., let* ρ *be an antiholomorphic involution on* $\mathcal{K}_1$*. There exists a canonical homology basis* $\{a_1, b_1\}$ *on* $\mathcal{K}_1$ *with intersection index* $a_1 \circ b_1 = 1$ *such that the* 2×2 *matrix* S *of complex conjugation of the action of* ρ *on* $H_1(\mathcal{K}_1, \mathbb{Z})$ *in this basis is given by*

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} \rho(a_1) & \rho(b_1) \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a_1 & -b_1 \end{bmatrix}.$$

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur 1974 The inverse scattering transform – Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53**, 249–315

[2] M.J. Ablowitz and Z.H. Musslimani 2013 Integrable nonlocal nonlinear Schrödinger equation, *Phys. Rev. Lett.* **110**, Paper 064105

[3] M.J. Ablowitz and Z.H. Musslimani 2017 Integrable nonlocal nonlinear equations, *Stud. Appl. Math.* **139**, 7–59

[4] M.J. Ablowitz, X-D. Luo, and Z.H. Musslimani 2018 Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions, *J. Math. Phys.* **59**, Paper 011501

[5] M.J. Ablowitz, B. Prinari, and A.D. Trubatch 2004 *Discrete and continuous nonlinear Schrödinger systems*, (Cambridge: Cambridge University Press)

[6] Akhmediev N.N. and Korneev V.I. 1986 Modulation instability and periodic solutions of the nonlinear Schrödinger equation *Theor. Math. Phys.* **69** 1089–1093

[7] E. Belokolos and V. Enol'skii 1994 Reduction of theta functions and elliptic finite-gap potentials, *Acta Appl. Math.* **36**, 87–117

[8] C.M. Bender and S. Böttcher 1998 Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$-symmetry, *Phys. Rev. Lett.* **80**, 5243–5246

[9] D.C. Brody 2017 PT-symmetry, indefinite metric, and nonlinear quantum mechanics, *J. Phys. A* **50**, Paper 485202,

[10] Kui Chen and Da-jun Zhang 2018 Solutions of the nonlocal nonlinear Schrödinger hierarchy via reduction, *Applied Mathematics Letters* **75**, 82–88

[11] J.L. Cieslinski 2009 Algebraic construction of the Darboux matrix revisited, *J. Phys. A* **42**, Paper 404003

[12] P.A. Deift, Applications of a commutation formula 1978 *Duke Math. J.* **45** 267–310

[13] L.D. Faddeev and L.A. Takhtajan 1987 *Hamiltonian methods in the theory of solitons.* (Berlin etc.: Springer)

[14] F.M. Fernandez, R. Guardiola, J. Ros, and M. Znojil 1999 A family of complex potentials with real spectrum, *J. Phys. A* **32**, 3105–3116

[15] A.S. Fokas 2016 Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation, *Nonlinearity* **29**(2), 319–324

[16] B. Fritzsche, B. Kirstein, I.Ya. Roitberg, and A.L. Sakhnovich 2018 Continuous and discrete dynamical Schrödinger systems: explicit solutions, *J. Phys. A: Math. Theor.* **51**, Paper 015202

[17] B. Fritzsche, B. Kirstein, and A.L. Sakhnovich 2006 Completion problems and scattering problems for Dirac type differential equations with singularities, *J. Math. Anal. Appl.* **317**, 510–525

[18] J. Eckhardt, F. Gesztesy, H. Holden, A. Kostenko, and G. Teschl 2017 Real-valued algebro-geometric solutions of the two-component Camassa-Holm hierarchy, *Ann. Inst. Fourier (Grenoble)* **67**, 1185–1230

[19] T.A. Gadzhimuradov and A. M. Agalarov 2016 Towards a gauge-equivalentmagnetic structure of the nonlocal nonlinear Schrödinger equation, *Phys Rev A* **93**, Paper 062124

[20] V.S. Gerdjikov, G.G. Grahovski, and R.I. Ivanov 2017 On integrable wave interactions and Lax pairs on symmetric spaces, *Wave Motion* **71**, 53–70

[21] F. Gesztesy and H. Holden 2003 *Soliton equations and their algebro-geometric solutions. Vol. I. (1+1)-dimensional continuous models* (Cambridge: Cambridge University Press)

[22] F. Gesztesy, H. Holden, J. Michor, and G. Teschl 2008 *Soliton equations and their algebro-geometric solutions. Vol. II. (1+1)-dimensional discrete models* (Cambridge: Cambridge University Press)

[23] F. Gesztesy and R. Ratnaseelan 1998 An alternative approach to algebro-geometric solutions of the AKNS hierarchy, *Rev. Math. Phys.* **10**, 345–391

[24] F. Gesztesy and G. Teschl 1996 On the double commutation method, *Proc. Amer. Math. Soc.* **124**, 1831–1840

[25] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich 1998 Pseudo-canonical systems with rational Weyl functions: explicit formulas and applications, *J. Differential Equations* **146**(2), 375–398

[26] C. Gu, H. Hu, and Z. Zhou 2005 *Darboux transformations in integrable systems* (Dordrecht: Springer)

[27] Gürses M. and Pekcan A. 2018 Nonlocal nonlinear Schrödinger equations and their soliton solutions *J. Math. Phys.* **59** 051501

[28] R. Hermann and C. Martin 1977 *Algebro-geometric and Lie-theoretic techniques in systems theory. Part A.* Interdisciplinary Mathematics Vol. XIII (Brookline, Mass. Math. Science Press)

[29] A. Its and V. Kotlyarov 1976 *Explicit formulas for solutions of a nonlinear Schrödinger equation*, Dokl. Akad. Nauk Ukrain. SSR Ser. A, 965–968, 1051 (in Russian)

[30] C. Kalla and C. Klein 2012 New construction of algebro-geometric solutions to the Camassa-Holm equation and their numerical evaluation, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **468**, 1371–1390

[31] Kuznetsov E.A. 1977 Solitons in a parametrically unstable plasma. *Sov. Phys. Dokl* **22** 507–508.

[32] Ma Y.C. 1979 The perturbed plane-wave solutions of the cubic Schrödinger equation *Stud. Appl. Math.* **60** 43–58.

[33] V.A. Marchenko 1988 *Nonlinear equations and operator algebras* (Dordrecht: D. Reidel)

[34] V.B. Matveev and M.A. Salle 1991 *Darboux transformations and solitons* (Berlin: Springer)

[35] J. Nesemann 2011 *PT-symmetric Schrödinger operators with unbounded potentials* (Wiesbaden: Vieweg + Teubner)

[36] M. Plaza and J. Francisco 2015 Algebro-geometric solutions of the generalized Virasoro constraints, *SIGMA Symmetry Integrability Geom. Methods Appl.* **11**, Paper 052

[37] E. Previato 1985 Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation, *Duke Math. J* **52**, 329–377

[38] J. Rao, Y. Cheng, and J. He 2017 Rational and semirational solutions of the nonlocal Davey-Stewartson equations, *Stud. Appl. Math.* **139**(4), 568–598

[39] A.L. Sakhnovich 1994 Dressing procedure for solutions of nonlinear equations and the method of operator identities, *Inverse Problems* **10**, 699–710

[40] A.L. Sakhnovich 2001 Generalized Bäcklund–Darboux transformation: spectral properties and nonlinear equations, *J. Math. Anal. Appl.* **262**, 274–306

[41] A.L. Sakhnovich 2003 Non-Hermitian matrix Schrödinger equation: Bäcklund–Darboux transformation, Weyl functions, and $\mathcal{PT}$ symmetry, *J. Phys. A* **36**, 7789–7802

[42] A.L. Sakhnovich 2003 Matrix Kadomtsev–Petviashvili equation: matrix identities and explicit non-singular solutions, *J. Phys. A* **36**, 5023–5033

[43] A.L. Sakhnovich 2006 Non-self-adjoint Dirac-type systems and related nonlinear equations: wave functions, solutions, and explicit formulas, *Integral Equations Operator Theory* **55**(1), 127–143

[44] A.L. Sakhnovich 2012 On the compatibility condition for linear systems and a factorization formula for wave functions, *J. Differential Equations* **252**(5), 3658–3667

[45] A.L. Sakhnovich 2017 Dynamics of electrons and explicit solutions of Dirac-Weyl systems, *J. Phys. A: Math. Theor.* **50**, Paper 115201

[46] A.L. Sakhnovich, L.A. Sakhnovich, and I.Ya. Roitberg 2013 *Inverse problems and nonlinear evolution equations. Solutions, Darboux matrices and Weyl–Titchmarsh functions* (Berlin: De Gruyter)

[47] L.A. Sakhnovich 1999 *Spectral theory of canonical differential systems, method of operator identities*, Operator Theory Adv. Appl. **107** (Basel: Birkhäuser)

[48] The applicability of the generalized Bäcklund-Darboux transformation (GBDT), a program.
`http://www.mat.univie.ac.at/~sakhnov/BDT/description.html`

[49] V.E. Zakharov and A.V. Mikhailov 1978 Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method, *Soviet Phys. JETP* **74**, 1953–1973

[50] V.E. Zakharov and A.V. Mikhailov 1980 On the integrability of classical spinor models in two-dimensional space-time, *Comm. Math. Phys.* **74**, 21–40

[51] Zakharov V.E. and Shabat A.B. 1974 A scheme for integrating the non-linear equations of mathematical physics by the method of the inverse scattering problem. I, *Funkcional. Anal. i Prilozen.* **8**:3 43–53.

[52] M. Znojil and D.I. Borisov 2018 Two patterns of PT-symmetry break-down in a non-numerical six-state simulation, *Ann. Physics* **394**, 40–49

J. Michor,
Fakultät für Mathematik, Universität Wien,
Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
e-mail: Johanna.Michor@univie.ac.at

A.L. Sakhnovich,
Fakultät für Mathematik, Universität Wien,
Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
e-mail: oleksandr.sakhnovych@univie.ac.at