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Hausdorff dimension of metric spaces definable in o-minimal expansions of the real field

by

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Abstract. Let R be an o-minimal expansion of the real field and (X,d) an R-definable metric space. We show that the Hausdorff dimension of (X,d) is an R-definable function of its defining parameters, an element of the field of powers of R, and is equal to the packing dimension of (X,d). The proof uses a basic topological dichotomy for definable metric spaces due to the second author, and the work of Shiota and the first author on measure theory over nonarchimedean o-minimal structures.

1. Introduction. Throughout, we let R be an o-minimal expansion of the real field $\overline{\mathbb{R}} = (\mathbb{R}, <, +, \times)$. By "definable" we shall mean "definable in R" (possibly with parameters), unless stated otherwise. If M is an expansion of an ordered abelian group, then we write $M^{>}$ instead of $M^{>0}$, and $M^{\geq}$ instead of $M^{\geq 0}$. We let Λ be the *field of powers* of $\mathbb{R}$, i.e. the set of $r \in \mathbb{R}$ such that t^r is a definable function of $t \in \mathbb{R}$. Given a definable set A we let $\dim(A)$ be the o-minimal dimension of A. A *definable metric space* is a definable set X equipped with a definable metric $d : X^2 \to \mathbb{R}^{\geq}$. Some basic facts about definable metric spaces were established in [16].

See Section 2 for definitions and comments on Hausdorff and packing dimensions. The Hausdorff, packing, and topological dimensions of a definable set agree. In fact, the following was shown in [5]:

Theorem 1.1. *Let $\mathcal{M}$ be an expansion of the real field which does not define the set of integers and let $X \subseteq \mathbb{R}^n$ be a closed $\mathcal{M}$-definable set. Then the Hausdorff, packing, and topological dimensions of X agree and equal the largest m for which there is a coordinate projection $\mathbb{R}^n \to \mathbb{R}^m$ which maps X onto a set with nonempty interior.*

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Thus the Hausdorff dimension of a closed $\mathcal{M}$-definable set A is a natural number and a definable function of the defining parameters of A. It also follows that the Hausdorff dimensions of the elements of an $\mathcal{M}$-definable family of closed sets take only finitely many values. In contrast, there are R-definable metric spaces whose Hausdorff dimension exceeds topological dimension. The simplest example is given by letting $0 < r < 1$ be in Λ and d_r be the definable metric on $[0,1]$ given by

$$d_r(x,y) = |x-y|^r \quad \text{for all } x,y \in [0,1].$$

Then $([0,1], d_r)$ has topological dimension 1 and Hausdorff dimension $1/r$. Note that $\{(X, d_r) : 0 < r < 1\}$ is an $\overline{\mathbb{R}}_{\mathrm{exp}}$-definable family of metric spaces whose elements have infinitely many distinct Hausdorff dimensions. Other examples are given by certain semialgebraic left-invarient metrics on certain nilpotent Lie groups, known as Carnot groups. These metrics are well known in geometry. The simplest example is the Heisenberg group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We equip H with a norm $\| \ \|_H$ by declaring the norm of the matrix above to be $[x^4 + y^4 + z^2]^{1/4}$ and define a left-invarient metric d_H by declaring $d_H(A,B) = \|A^{-1}B\|_H$ for all $A, B \in H$. Then (H, d_H) has topological dimension 3 and Hausdorff dimension 4. Subriemannian metrics give other examples of $\overline{\mathbb{R}}_{\mathrm{an}}$-definable metric spaces whose Hausdorff dimension is strictly larger than topological dimension [16, 5.9].

In this paper we prove the following:

THEOREM 1.2. *Let $\mathcal{X} = \{(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$ be a definable family of metric spaces. Then the Hausdorff dimension of (X_α, d_α) is a definable function of α taking values in $\Lambda \cup \{\infty\}$. If R is polynomially bounded, then the Hausdorff dimension of the elements of $\mathcal{X}$ takes only finitely many values. The Hausdorff dimension and packing dimension of each (X_α, d_α) agree.*

Theorem 1.2 implies that the Hausdorff dimension of an $\overline{\mathbb{R}}_{\mathrm{an}}$-definable metric space is a rational number. By a result of Howroyd [6] the coincidence of Hausdorff and packing dimensions implies:

COROLLARY 1.3. *The Hausdorff dimension of the product of two definable metric spaces $(X,d), (X',d')$ is $\dim_H(X,d) + \dim_H(X',d')$.*

Here is an outline of the proof of Theorem 1.2. In Section 2 we recall necessary definitions and results from metric geometry. The crucial metric result we use is the following easy corollary to well known results (Proposition 2.5 below).

Proposition 1.4. *Let (X,d) be a metric space and μ be a finite Borel measure on X which gives every nonempty open set positive measure. For $p \in X$ and $t \in \mathbb{R}^{>}$, let $B(p,t) \subseteq X$ be the closed ball of radius t centered at p. Suppose*

$$\phi(p) := \lim_{t \to 0^+} \frac{\log[\mu B(p,t)]}{\log(t)} \in \mathbb{R}_\infty \quad \textit{exists for all } p \in X,$$

and $\phi : X \to \mathbb{R} \cup \{\infty\}$ is continuous. Then

$$\dim_H(X,d) = \dim_P(X,d) = \sup\{\phi(p) : p \in X\}.$$

Let (X,d) be a definable metric space. To apply the previous proposition we need a measure μ. If the metric topology on (X,d) agrees with the usual topology on X then we take μ to be the $\dim(X)$-dimensional Lebesgue measure. The following theorem allows us to make this assumption:

Theorem 1.5 ([16, 9.0.1]). *Let (X,d) be a definable metric space. One of the following holds:*

(1) *There is an infinite definable subset $A \subseteq X$ such that (A,d) is discrete.*
(2) *There is a definable homeomorphism between (X,d) and a definable set equipped with its euclidean topology.*

If (1) holds, then the Hausdorff and packing dimensions of (X,d) are infinite, so we only consider case (2). Omitting technical details, we prove Theorem 1.2 by applying the following proposition, a consequence of mild extensions of results in [8], to the definable family $\{B(p,t) : (p,t) \in X \times \mathbb{R}^{>}\}$ of balls in (X,d).

Proposition 1.6. *Let λ be the k-dimensional Lebesgue measure. Let X be a definable set and $\{A_{p,t} : (p,t) \in X \times \mathbb{R}^{>}\}$ a definable family of k-dimensional sets such that $\lim_{t \to 0^+} \lambda(A_{p,t}) = 0$ for all $p \in \mathbb{R}^l$. Then*

$$\phi(p) := \lim_{t \to 0^+} \frac{\log \lambda(A_{p,t})}{\log t} \in \mathbb{R} \cup \{\infty\}$$

is a definable function of p taking values in $\Lambda \cup \{\infty\}$. If R is polynomially bounded, then ϕ takes only finitely many distinct values.

For $\bar{\mathbb{R}}_{\mathrm{an}}$, this is an easy consequence of the work of Comte, Lion and Rolin [1].

Notation and conventions. By k, l, m, n we shall always denote nonnegative integers.

We let $\mathcal{R}$ be a big elementary extension of R. The expansion of R by the logarithm, $R_{\log}$, is o-minimal [13]. By saturation we may view $\mathcal{R}$ as a substructure of an elementary expansion of $R_{\log}$, and in this sense we take the logarithm of elements of $\mathcal{R}$.

By $\mathcal{O}$ we denote the convex hull of $\mathbb{Q}$ in $\mathcal{R}$. Then $\mathcal{O}$ is a convex subring of $\mathcal{R}$, hence a valuation ring. We denote its maximal ideal by $\mathfrak{m}$. Its residue field is $\overline{\mathbb{R}}$ with residue map $\mathrm{st}\colon \mathcal{O} \to \mathbb{R}$.

Let M be an o-minimal field, and let $X, Y \subseteq M^n$ be definable. Then we write $X \subseteq_0 Y$ iff $\dim(X \setminus Y) < n$, and $X =_0 Y$ iff $X \subseteq_0 Y$ and $Y \subseteq_0 X$. A property holds for *almost all* elements of X if it holds for all elements of X outside of a definable subset of dimension $< n$. By a *box* in M^n we mean a definable set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq M^n$, where $a_i < b_i$ for all i. By $p^n_m\colon M^n \to M^m$ we denote the projection onto the first m coordinates. If (X, d) is a metric space and $p \in X$, $t \in \mathbb{R}^>$, then $B(p, t) \subseteq X$ is the closed ball of radius t centered at p.

We shall write $\mathbb{R}_\infty$ for $\mathbb{R} \cup \{\infty\}$, and Λ_∞ to mean $\Lambda \cup \{\infty\}$.

2. Dimensions. Throughout this section, (X, d) is a metric space. The *diameter* of $A \subseteq X$ is

$$\mathrm{Diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

We recall the definitions of the two metric dimensions used in this paper: Hausdorff dimension and packing dimension. We refer to Falconer [3] for more information about these dimensions.

We begin with Hausdorff dimension. Let $A \subseteq X$. Given $\delta > 0$ and $r \geq 0$ we let $\mathcal{H}^r_\delta(X)$ be the infimum of all sums $\sum_{i=0}^\infty s_i^r$ for which there is a sequence $\{A_i\}_{i\in\mathbb{N}}$ of subsets of A covering A such that each A_i has diameter $s_i \leq \delta$. The r-dimensional Hausdorff measure of A is

$$\mathcal{H}^r(A) = \lim_{\delta\to 0} \mathcal{H}^r_\delta(A) \in \mathbb{R}_\infty.$$

This gives a Borel measure on X. Let $0 \leq s < r$. If $\mathcal{H}^r(A) > 0$ then $\mathcal{H}^s(A) = \infty$ and if $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^r(A) = 0$. The *Hausdorff dimension* of (X, d) is the supremum of all $r \geq 0$ such that $\mathcal{H}^r(X) = \infty$. We denote the Hausdorff dimension of (X, d) by $\dim_H(X, d)$.

We now define the packing dimension of (X, d). Let $A \subseteq X$. Given $\delta > 0$ and $r \geq 0$, let $\mathcal{P}^r_\delta(A)$ be the supremum of all sums $\sum_{i=0}^\infty s_i^r$ for which there is a sequence $\{B_i\}_{i\in\mathbb{N}}$ of pairwise disjoint closed balls with centers in A such that each B_i has diameter $s_i \leq \delta$. The r-dimensional packing measure of A is

$$\mathcal{P}^r(A) = \lim_{\delta\to 0} \mathcal{P}^r_\delta(A) \in \mathbb{R}_\infty.$$

This gives a Borel measure on X. Let $0 \leq s < r$. If $\mathcal{P}^r(A) > 0$ then $\mathcal{P}^s(A) = \infty$ and if $\mathcal{P}^s(A) < \infty$ then $\mathcal{P}^r(A) = 0$. The *packing dimension* of (X, d) is the supremum of all $r \geq 0$ such that $\mathcal{P}^r(X) = \infty$. We denote the packing dimension of X by $\dim_P(X, d)$. It is well-known that

$$\dim_H(X, d) \leq \dim_P(X, d) \quad \text{for any metric space } (X, d).$$

The following facts follow directly from the definitions.

FACT 2.1. *If* $\{X_1, \ldots, X_n\}$ *is a partition of* X *into Borel sets then*

$$\dim_H(X,d) = \max\{\dim_H(X_1,d), \ldots, \dim_H(X_n,d)\},$$
$$\dim_P(X,d) = \max\{\dim_P(X_1,d), \ldots, \dim_P(X_n,d)\}.$$

FACT 2.2. *If* (X,d) *is not separable then* $\dim_H(X,d) = \dim_P(X,d) = \infty$.

Let (X', d') be a metric space. We equip $X \times X'$ with the metric $d_\Box$ given by

$$d_\Box((x,x'),(y,y')) = \max\{d(x,y), d'(x',y')\}.$$

See [6] for the following:

FACT 2.3. *We have*

$$\dim_H(X,d) + \dim_H(X',d') \le \dim_H(X \times X', d_\Box) \le \dim_P(X,d) + \dim_P(X',d').$$

So if $\dim_H(X,d) = \dim_P(X,d)$ *and* $\dim_H(X',d') = \dim_P(X',d')$ *then*

$$\dim_H(X \times X', d_\Box) = \dim_H(X,d) + \dim_H(X',d').$$

We now suppose μ is a Borel measure on (X,d) which assigns positive measure to every nonempty open subset of X. We define the *upper* and *lower dimensions* of (X,d) *at a point* $p \in X$, which depend on μ. The upper dimension at p is

$$\overline{\dim}_{\mathrm{loc}}(p) = \limsup_{t \to 0^+} \frac{\log \mu B(p,t)}{\log(t)} \in \mathbb{R}_\infty,$$

and the lower dimension at p is

$$\underline{\dim}_{\mathrm{loc}}(p) = \liminf_{t \to 0^+} \frac{\log \mu B(p,t)}{\log(t)} \in \mathbb{R}_\infty.$$

See Falconer [3, Proposition 2.2] for Proposition 2.4 below. The proof in [3] is given for subsets of Euclidean space but is easily seen to go through in general.

PROPOSITION 2.4. *Let* $r \in \mathbb{R}$.

(1) *If* $\overline{\dim}_{\mathrm{loc}}(p) \ge r$ *for all* $p \in X$ *then* $\dim_P(X,d) \ge r$.
(2) *If* $\overline{\dim}_{\mathrm{loc}}(p) \le r$ *for all* $p \in X$ *then* $\dim_P(X,d) \le r$.
(3) *If* $\underline{\dim}_{\mathrm{loc}}(p) \ge r$ *for all* $p \in X$ *then* $\dim_H(X,d) \ge r$.
(4) *If* $\underline{\dim}_{\mathrm{loc}}(p) \le r$ *for all* $p \in X$ *then* $\dim_H(X,d) \le r$.

We now prove

PROPOSITION 2.5. *Suppose*

$$\phi(p) := \lim_{t \to 0^+} \frac{\log[\mu B(p,t)]}{\log(t)} \in \mathbb{R}_\infty \quad \textit{exists for all } p \in X,$$

and $\phi : X \to \mathbb{R}_\infty$ is continuous. Then

$$\dim_H(X,d) = \dim_P(X,d) = \sup\{\phi(p) : p \in X\}.$$

Proof. Note that

$$\overline{\dim}_{\mathrm{loc}}(p) = \phi(p) = \underline{\dim}_{\mathrm{loc}}(p) \quad \text{for all } p \in X.$$

Let s be the supremum of the range of ϕ. As $\phi(p) \leq s$ for all $p \in X$, we have $\dim_H(X,d), \dim_P(X,d) \leq s$ by Proposition 2.4. Let $\epsilon > 0$. As ϕ is continuous there is an open $U \subseteq X$ such that $\phi(p) > s - \epsilon$ for all $p \in U$. Applying Proposition 2.4 to (U,d) and the restriction of μ to U we have $\dim_H(U,d), \dim_P(U,d) \geq s - \epsilon$. So

$$s - \epsilon \leq \dim_H(X,d), \dim_P(X,d) \leq s \quad \text{for all } \epsilon > 0. \blacksquare$$

3. A consequence of Miller's dichotomy. We prove

Proposition 3.1. *If $f : \mathbb{R}^n \times \mathbb{R}^> \to \mathbb{R}^>$ is definable, then*

$$F(p) := \lim_{t \to 0^+} \frac{\log f(p,t)}{\log(t)} \in \mathbb{R}_\infty$$

is a definable function of p taking values in Λ_∞. If $\mathbb{R}$ is polynomially bounded, then F takes only finitely many distinct values.

Proposition 3.1 is a corollary of two results of Miller. The first is the fundamental dichotomy [9]:

Theorem 3.2. *If R is not polynomially bounded, then the exponential function is definable.*

The second is a slight variation of [10, Proposition 5.2, p. 92].

Fact 3.3. *Suppose R is polynomially bounded, and $f : \mathbb{R}^n \times \mathbb{R}^> \to \mathbb{R}^>$ is definable. Then there are $r_1, \ldots, r_k \in \Lambda$, a partition $\{B_1, \ldots, B_k\}$ of $\mathbb{R}^n$ into definable sets, and a definable $c : \mathbb{R}^n \to \mathbb{R}^>$ such that*

$$\lim_{t \to \infty} \frac{f(p,t)}{t^{r_i}} = c(p) \quad \textit{for all } p \in B_i.$$

Proof of Proposition 3.1. Suppose the exponential function is definable, in which case $\Lambda = \mathbb{R}$. Then the function

$$F_0(p,t) := \frac{\log f(p,t)}{\log(t)}$$

is definable. It follows from the o-minimal monotonicity theorem that for any p, $\lim_{t \to 0^+} F_0(p,t)$ exists in $\mathbb{R}_\infty$. Now suppose R is polynomially bounded. Let $r_1, \ldots, r_k, \{B_1, \ldots, B_k\}$, and c be as in Fact 3.3 above. Fix p in B_i and let $\epsilon > 0$ be such that $c(p) - \epsilon > 0$. Then

$$[c(p) - \epsilon]t^{r_i} \leq f(p,t) \leq [c(p) + \epsilon]t^{r_i} \quad \text{for sufficiently small } t > 0.$$

Taking logs and dividing through by $\log(t)$, we have

$$\frac{\log[c(p)-\epsilon]}{\log(t)}+r_i \leq \frac{\log f(p,t)}{\log(t)} \leq \frac{\log[c(p)+\epsilon]}{\log(t)}+r_i \quad \text{for sufficiently small } t>0.$$

Letting $t \to 0^+$ we have

$$\frac{\log f(p,t)}{\log(t)} = r_i \quad \text{for all } p \in B_i.$$

Thus $F(p) = r_i$ when $p \in B_i$. The proposition follows. ■

4. Measures on definable sets. In this section we prove Proposition 4.16. We shall use the following terminology. We say that a definable $X \subseteq \mathcal{O}^n$ is *d-thin* if $\dim(X) \leq d$ and $\operatorname{int}(\operatorname{st}(\pi X)) = \emptyset$ for all orthogonal projections $\pi : \mathcal{R}^n \to \mathcal{R}^d$. In the terminology of Valette [15] a thin set is "$\mathcal{O}$-thin". If a definable $X \subseteq \mathcal{O}^n$ is not d-thin then we say that it is *d-fat*. Note that a definable subset of $\mathcal{O}^n$ is n-fat if and only if $\operatorname{int}(\operatorname{st} X) \neq \emptyset$.

In [8], the authors define a finitely additive measure ν (see Definition 4.2) on the definable subsets of $\mathcal{O}^n$ which takes values in an ordered semiring (a quotient of $\mathcal{O}^{\geq}$) and agrees with the Lebesgue measure of $\operatorname{st} X$ in the case when X is n-fat. When X is n-thin, the measure of an open cell X agrees with the supremum of the measure of all boxes inscribed in a certain isomorphic image of X (an isomorphism $X \to \phi X$ here is, roughly, a C^1-diffeomorphism ϕ such that $|\det J\phi(x)| = 1$ for almost all $x \in X$). We first extend the definitions in [8] to d-dimensional measure. We note that while we assume for simplicity that $\mathcal{R}$ is sufficiently saturated, this assumption is not needed for the definition of ν in [8], nor is it needed for its d-dimensional version. The only adjustment one needs to make when dropping the saturation assumption is to replace $\widetilde{\mathcal{O}}$ in Definition 4.2 by its Dedekind completion.

Definitions 4.1 and 4.2 below are from [8] (stated in slightly weaker form). Lemma 4.13 is a consequence of results from [8]. It will yield the desired result on limits of families of open sets (Corollary 4.15), which will in turn imply the result in full generality.

We shall use the following convention. Suppose M is an o-minimal field and $X \subseteq M^n$ is an open cell with $p^n_k X = (f_k, g_k)$ for $k = 1, \ldots, n$. Let $x \in X$ and $t \in (0,1)^n$ be such that

$$x_i = (1-t_i)f_i(x_1, \ldots, x_{i-1}) + t_i g_i(x_1, \ldots, x_{i-1}).$$

Then $\tau_X \colon X \to \tau_X X \subseteq M^n$ is the map $\tau_X(x) = y$, where

$$y_i = t_i(g_i(x_1, \ldots, x_{i-1}) - f_i(x_1, \ldots, x_{i-1})).$$

We define an equivalence relation $\sim$ on $\mathcal{O}^{\geq}$:

Definition 4.1. Let $x, y \in \mathfrak{m}^{\geq}$. Then $x \sim y$ iff $y^q \leq x \leq y^p$ for all $p, q \in \mathbb{Q}^{>}$ such that $p < 1 < q$. If $x, y > \mathfrak{m}$, then $x \sim y$ iff $\operatorname{st} x = \operatorname{st} y$. We let $\widetilde{\mathcal{O}}$ be the quotient $\mathcal{O}^{\geq}/{\sim}$.

Note that $\mathbb{R} \subseteq \widetilde{\mathcal{O}}$ by the saturation assumption on $\mathcal{R}$. The quotient $\widetilde{\mathcal{O}}$ can be made into an ordered semiring, where the ordering is induced by the ordering on $\mathcal{O}$, and $\widetilde{x} + \widetilde{y} = \max\{\widetilde{x}, \widetilde{y}\}$ if $x \in \mathfrak{m}^{>}$ or $y \in \mathfrak{m}^{>}$, and $\widetilde{x} + \widetilde{y} = \widetilde{x + y}$ otherwise. For $x, y \in \mathcal{O}^{\geq}$, we set $\widetilde{x} \cdot \widetilde{y} = \widetilde{x \cdot y}$.

For the remainder of this section, λX is the n-dimensional Lebesgue measure of $X \subseteq \mathbb{R}^n$, and $\lambda_k X$, $k < n$, is the k-dimensional Lebesgue measure of $X \subseteq \mathbb{R}^n$.

Definition 4.2.

(a) Let $X \subseteq \mathcal{O}^n$ be a cell. If X is n-fat, then $\nu X = \lambda \operatorname{st}(X)$. If X is n-thin, then $\nu X = \widetilde{a}$, where

$$a = \max\Big\{\prod_{i=1}^{n}(y_i - x_i) \colon [x_1, y_1] \times \cdots \times [x_n, y_n] \subseteq \operatorname{cl}(\tau_X X)\Big\}.$$

(b) Suppose $X \subseteq \mathcal{O}^n$ is a definable set. Then $\nu X = \sum_{i=1}^{d} \nu C_i$, where $\{C_i\}$ is a finite partition of X into cells.

It is shown in [8] that the above definition is independent of the decomposition of X into cells, and that $\nu X > 0$ iff the interior of X in $\mathcal{R}^n$ is nonempty.

We now define a d-dimensional measure ν_d on the d-thin definable subsets of $\mathcal{O}^n$ such that $\nu_d X > 0$ whenever $\dim X \geq d$. On d-fat sets one can define a d-dimensional measure as Fornasiero and Vasquez do in [4]. While the proofs in [4] work only in a sufficiently saturated o-minimal expansion of a real closed field, using [2, Theorem 3.3, p. 244], the results from [4] can be transferred to the general case.

We use the following definitions and theorem of Pawłucki [12] (with slightly modified terminology). We state these for R, but they hold equally well for $\mathcal{R}$ (and we shall use the same terminology for subsets of $\mathcal{R}^n$ as for subsets of R^n).

Definition 4.3. Let $f \colon X \to R$, where $X \subseteq R^n$ is open, be R-definable and C^1, and let $L \in R^{>}$. Then we say that f is *L-controlled* if $\left|\frac{\partial f}{\partial x_i}(a)\right| \leq L$ for all $i \in \{1, \ldots, n\}$ and almost all $a \in X$.

Definition 4.4. An open cell $C = (f, g) \subseteq R^n$ will be called a *standard L-cell* if it is an open interval in the case $n = 1$ and, for $n > 1$, if $p^n_{n-1} C$ is a standard L-cell and whenever f or g is finite, then it is L-controlled.

THEOREM 4.5. *Let $X \subseteq R^n$ be open and definable. Then we can find $S_1, \ldots, S_k \subseteq R^n$ such that*

$$X =_0 S_1 \dot\cup \cdots \dot\cup S_k,$$

where each S_i is a standard L-cell after a permutation of coordinates and $L \in R^>$ depends only on n.

It is an exercise left to the reader to derive the following version of the above theorem for definable subsets of dimension $< n$:

COROLLARY 4.6. *Let $X \subseteq \mathcal{R}^n$ be definable of dimension $d < n$. Then there are $S_1, \ldots, S_k \subseteq \mathcal{R}^n$ such that $X =_0 S_1 \dot\cup \cdots \dot\cup S_k$, and there are permutations of coordinates $\tau_1, \ldots, \tau_k \colon \mathcal{R}^n \to \mathcal{R}^n$ such that each $\tau_i S_i$ is an $(i_1, \ldots, i_n)$-cell with $i_j = 1$ when $j \leq d$ and $i_j = 0$ when $j > d$, and each $p^n_d \tau_i S_i$ is a standard L-cell, and each $p^n_m \tau_i S_i$ with $m > d$ is the graph of an L-controlled function. Furthermore, $L \in \mathcal{O}^>$ depends only on n.*

From now on we shall always assume that $L \in \mathcal{O}^>$. We will refer to $(i_1, \ldots, i_n)$-cells C such that $i_j = 1$ for all $j \leq d$ and $i_j = 0$ for $j > d$ and such that $p^n_d C$ is a standard L-cell and each $p^n_m C$ with $m > d$ is the graph of an L-function, as *L-cells* of dimension d. For definable $X, S_i \subseteq \mathcal{R}^n$ (or R^n) we say that $\{S_i\}$ is an *almost partition* of X if $\{S_i\}$ is finite and $X =_0 \dot\bigcup_i S_i$.

DEFINITION 4.7. Let $X \subseteq \mathcal{O}^n$ be $\mathcal{R}$-definable and d-thin. Let $\{S_1, \ldots, S_k\}$ be an almost partition of X and let $\tau_1, \ldots, \tau_k \colon \mathcal{R}^n \to \mathcal{R}^n$ be permutations of coordinates such that each $\tau_i S_i$ is an L-cell. Then we set

$$\nu_d X := \max\{\nu p^n_d \tau_i S_i : 1 \leq i \leq k\}.$$

To show that the definition of ν_d makes sense, we must show that it does not depend on the choice of S_i and τ_i. Note that if $\dim(X) < d$ then $\nu_d(X) = 0$.

Below, $\det J\phi(x)$ is the Jacobian determinant of ϕ at x.

DEFINITION 4.8. Let X, Y be n-dimensional definable subsets of $\mathcal{R}^n$. We say that a map ϕ is a *weak isomorphism* $X \to Y$ if $\phi \colon U \to V$, where $U, V \subseteq \mathcal{R}^n$ are open, is a definable C^1-diffeomorphism, $X \subseteq_0 U$, $Y \subseteq_0 V$, $\phi(X \cap U) =_0 Y$, and $1/L \leq |\det J\phi(x)| \leq L$ for almost all $x \in U$, where $L \in \mathcal{O}^>$. We say that X and Y are *weakly isomorphic* if there is a definable weak isomorphism $X \to Y$.

We now show that ν is invariant under weak isomorphisms on thin sets. The proof is a slight variant of the proof of [8, Theorem 5.4]. For the sake of completeness we outline the argument here; see [8] for more details.

LEMMA 4.9. *Suppose $C, D \subseteq \mathcal{R}^n$ are definable, n-dimensional and n-thin. If C and D are weakly isomorphic, then $\nu(C) = \nu(D)$.*

Proof. Let $\phi : C \to D$ be a weak isomorphism. We assume towards a contradiction that $\nu C > \nu D$. Let $a \in \mathfrak{m}^{>}$ be such that $\nu D < \widetilde{a} < \nu C$. We first reduce to the case when D is an open cell and $C = \phi^{-1}(D)$. The next reduction is to the case when C is a cell and $\tau_D \circ \phi(C) \subseteq B$, where B is a box with $\nu B < \widetilde{a}$. Next, we find a box $P \subseteq \tau_C C$ with $\nu P > \widetilde{a}$. We define

$$\varPhi = \tau_D \circ \phi \circ \tau_C^{-1}|_P \colon P \to \varPhi(P) \subseteq B.$$

For some $L \in \mathcal{O}^{>}$, $1/L \leq |\det J\varPhi(x)| \leq L$ for almost all $x \in P$.

Let $\theta\colon (0,1)^n \to \operatorname{int}(P)$ be a C^1-diffeomorphism with $\det J\theta(x) = b$ for all $x \in (0,1)^n$, where $\widetilde{b} = \nu P$. Let $\hat{\theta}\colon B \to P'$ be a C^1-diffeomorphism onto a box P' such that $\det J\hat{\theta}(x) = 1/b$ for all $x \in B$. Then

$$\hat{\theta} \circ \varPhi \circ \theta\colon (0,1)^n \to \hat{\theta} \circ \varPhi \circ \theta(0,1)^n \subseteq P'$$

is such that

$$1/L \leq |\det(J(\hat{\theta} \circ \varPhi \circ \theta)(x))| \leq L$$

for almost all $x \in [0,1]^n$. By [7, Corollary 6.2, p. 17], $\hat{\theta} \circ \varPhi \circ \theta$ induces a C^1-diffeomorphism $\operatorname{st}[0,1]^n \to \operatorname{st} P'$ outside of a subset of $\operatorname{st}[0,1]^n$ of dimension $< n$ with Jacobian determinant bounded between $1/\operatorname{st} L$ and $\operatorname{st} L$. But $\operatorname{int}(\operatorname{st} P') = \emptyset$, which is impossible. ∎

We now show that ν_d is well-defined.

LEMMA 4.10. *Let $X \subseteq \mathcal{O}^n$ be definable and d-thin. Let $\{X_i\}$ and $\{Y_j\}$ be almost partitions of X, and let $\tau_i, \tau'_j\colon \mathcal{R}^n \to \mathcal{R}^n$ be permutations of coordinates such that all $\tau_i X_i$ and $\tau'_j Y_j$ are L-cells. Then*

$$\max_i \nu p^n_d \tau_i X_i = \max_j \nu p^n_d \tau'_j Y_j.$$

Proof. Let $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ be an almost partition of X containing an almost partition of each X_i and Y_j. For each i we have

$$\nu p^n_d \tau_i X_i = \max\{\nu p^n_d \tau_i Z_j \colon j \in \{1, \ldots, m\} \text{ and } Z_j \subseteq X_i\}.$$

Hence

$$\max_i \nu p^n_d \tau_i X_i = \max\{\nu p^n_d \tau_i Z_j : 1 \leq j \leq m\}$$

and it suffices to see that for $Z_k \subseteq X_i \cap Y_j$ of dimension d, $\nu p^n_d \tau_i Z_k = \nu p^n_d \tau'_j Z_k$. But this follows from the map $\phi\colon p^n_d \tau_i Z_k \to p^n_d \tau'_j Z_k$ given by $p^n_d \tau_i(x) \mapsto p^n_d \tau'_j(x)$, where $x \in Z_k$, being a weak isomorphism and from Lemma 4.9. ∎

DEFINITION 4.11. Let M be an o-minimal field and X a definable subset of M^n. Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a collection of pairwise disjoint boxes in M^n. We say that $\mathcal{B}$ is an *inner approximation of* X if $B_i \subseteq X$ for each i. We say that $\mathcal{B}$ is an *outer approximation of* X if $X \subseteq_0 \bigcup_{i=1}^k B_i$. The volume of $\mathcal{B}$ is $\sum_{i=1}^k \nu B_i$.

The lemma below is [7, Lemma 4.1]:

LEMMA 4.12. *Let $X \subseteq \mathcal{O}^n$ be definable and n-fat. Then there is a box*

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \operatorname{cl}(X)$$

with $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Q}$.

When dealing with R-definable families, we shall not make a notational distinction between their realization in R and in $\mathcal{R}$, as which one is meant will always be clear from the context.

LEMMA 4.13. *Let $\{A_t : t \in \mathbb{R}^>\}$ be a definable family of open subsets of $\mathbb{R}^n$ such that $\lambda(A_t) \to 0$ as $t \to 0^+$.*

(a) *There is a definable function $h\colon (0,a)_{\mathcal{R}} \to \mathcal{R}^>$, where $a \in \mathbb{R}^>$, such that each $h(t)$ is the volume of an inner approximation of A_t and $\widetilde{h(t)} = \nu(A_t)$ for all $t \in \mathfrak{m}^>$.*

(b) *If $G\colon \mathcal{R} \to \mathcal{R}$ is definable such that $\nu A_t < \widetilde{G(t)}$ when $t \in \mathfrak{m}^>$, then there is a definable function $H\colon (0,a)_{\mathcal{R}} \to \mathcal{R}^>$, where $a \in \mathbb{R}^>$, such that each $H(t)$ is the volume of an outer approximation of A_t and*

$$\nu(A_t) \leq \widetilde{H(t)} < \widetilde{G(t)} \quad \text{for all } t \in \mathfrak{m}^>.$$

Proof. Without loss of generality, we may assume that $A_t \subseteq [0,1]^n$ for each t. First note that $\operatorname{int}(\operatorname{st} A_t) = \emptyset$ for all $t \in \mathfrak{m}^>$: Suppose towards a contradiction that $t \in \mathfrak{m}^>$ is such that $\operatorname{int}(\operatorname{st} A_t) \neq \emptyset$. By Lemma 4.12 there is a box $B \subseteq A_t$ such that all vertices have rational coordinates. For all sufficiently small $s \in \mathbb{R}^>$, $B \subseteq A_s$, in contradiction with $\lambda A_t \to 0$ as $t \to 0^+$ and $\lambda B > 0$.

We set $A = \bigcup\{A_t : t \in \mathbb{R}^>\}$. To prove (a), we let $\mathcal{D}$ be a decomposition of $\mathbb{R}^{1+n}$ into cells that partitions A. Let $D \in \mathcal{D}$ be such that $D \subseteq A$ and $p_1^{n+1} D = (0,a)$ for some $a \in \mathbb{R}^>$. Let $D_1, \ldots, D_k$ be the open cells in $\mathcal{D}$ with $D_i \subseteq A$ and $p_1^{n+1} D_i = (0,a)$. Then $\bigcup_{i=1}^k (D_i)_t =_0 A_t$ for all $t \in \mathfrak{m}^>$, and there is $i \in \{1, \ldots, k\}$ such that $\nu A_t = \nu((D_i)_t)$ for all $t \in \mathfrak{m}^>$: Define $h_i\colon (0,a) \to [0,1]$ by

$$h_i(t) = \sup\Big\{ \prod_{j=2}^{n+1} (b_j - a_j) : [a_2, b_2] \times \cdots \times [a_{n+1}, b_{n+1}] \subseteq \operatorname{cl}((\tau_{D_i} D_i)_t) \Big\},$$

hence $\nu((D_i)_t) = \widetilde{h_i(t)}$ for all $t \in \mathfrak{m}^>$. Since the functions $h_i\colon (0,a) \to [0,1]$ are R-definable, if $h_i(t) < h_j(t)$ for some $t \in \mathfrak{m}^>$ then $h_i(t) < h_j(t)$ for all $t \in \mathfrak{m}^>$. It follows that for some $i \in \{1, \ldots, k\}$, $\nu A_t = \widetilde{h_i(t)}$ for all $t \in \mathfrak{m}^>$. This finishes part (a) of the lemma.

To prove (b), let $G\colon [0,1] \to [0,1]$ be R-definable with $\nu A_t < \widetilde{G(t)}$ for $t \in \mathfrak{m}^>$. Without loss of generality, we assume that A is an open cell of the form $(0, f)$. The proof is by induction on n.

If $A \subseteq R^{1+1}$, then $A_t = (0, f(t)) \subseteq R$ for each $t \in \mathfrak{m}^>$, so part (b) of the lemma is obvious. Now suppose the lemma holds for $n \geq 1$, and let $A \subseteq R^{1+(n+1)}$. Let h be as in part (a) of the lemma. Let $\epsilon \in \mathfrak{m}^>$. Then $\nu A_\epsilon = \widetilde{h(\epsilon)}$. Now $\widetilde{h(\epsilon)} < \widetilde{G(\epsilon)}$ implies that for some $p \in \mathbb{Q}^>$, $p < 1$,

$$\widetilde{h(\epsilon)} < \widetilde{h(\epsilon)}^p < \widetilde{G(\epsilon)}.$$

By the proof of the subclaim in [8, proof of Theorem 4.8, Case 1.1], there is l, depending only on p, and there are R-definable functions $y_0, \dots, y_l\colon (0, a) \to [0, 1]$, where $a \in \mathbb{R}^>$, such that

$$0 = y_0(t) < y_1(t) < \cdots < y_l(t) = 1$$

and

$$\widetilde{h(t)} \leq \sum_{i=1}^{l} \widetilde{y_i(t)} \cdot \nu(f_t^{-1}[y_{i-1}(t), y_i(t)]) < \widetilde{h(t)}^p$$

for all $t \in \mathfrak{m}^>$. (In the notation of [8, proof of Theorem 4.8], $y_i(t) = h(t)^{(l-i-1)q_3}$ where $i \in \{1, \dots, l-1\}$, and $q_3 \in \mathbb{Q}^>$ depends only on p.)

We first find, for each i, an R-definable function H_i such that

$$\widetilde{y_i(t)} \nu f_t^{-1}[y_{i-1}, y_i] \leq \widetilde{H_i(t)} < \widetilde{h(t)}^p$$

on $\mathfrak{m}^>$, and $H_i(t)$ is the volume of an outer approximation of

$$f_t^{-1}[y_{i-1}, y_i] \times [0, y_i]$$

on $(0, a)$, where $a \in \mathbb{R}^>$.

- Let $i = l$. Then for all $t \in \mathfrak{m}^>$,

$$\nu f_t^{-1}[y_{l-1}(t), y_l(t)] < \widetilde{h(t)}^p.$$

So, inductively, there is an R-definable function $H_i\colon (0, a) \to [0, 1]$, with $a \in \mathbb{R}^>$, such that

$$\nu f_t^{-1}[y_{l-1}(t), y_l(t)] \leq \widetilde{H_i(t)} < \widetilde{h(t)}^p \quad \text{for all } t \in \mathfrak{m}^>,$$

and such that each $H_i(t)$ is the volume of an outer approximation of $f_t^{-1}[y_{l-1}(t), y_l(t)]$. Then

$$\widetilde{y_i(t)} \cdot \nu f_t^{-1}[y_{i-1}(t), y_i(t)] \leq \widetilde{y_i(t)} \cdot \widetilde{H_i(t)} = \widetilde{H_i(t)} < \widetilde{h(t)}^p \quad \text{for all } t \in \mathfrak{m}^>.$$

Further, $y_i(t) \cdot H_i(t)$ is the volume of an outer approximation of $f_t^{-1}[y_{l-1}(t), y_l(t)] \times [0, y_i(t)]$ for all $t \in (0, a)$.

- Let i be such that $y_i(t) \in \mathfrak{m}^>$ and $\nu f_t^{-1}[y_{i-1}(t), y_i(t)] < \widetilde{\mathfrak{m}^>}$ for some (hence all) $t \in \mathfrak{m}^>$. Then the function assigning 1 to each $t \in (0, 1)$ is the volume of an outer approximation of $f_t^{-1}[y_{i-1}(t), y_i(t)]$ for all $t \in (0,1)$, and $H_i(t) = y_i(t)$ is as required.

• Suppose $y_i(t) \in \mathfrak{m}^>$ and $\nu f_i^{-1}[y_{i-1}(t), y_i(t)] \in \widetilde{\mathfrak{m}^>}$ for $t \in \mathfrak{m}^>$. Let $q \in \mathbb{Q}^>$ be such that

$$\widetilde{y_i(t)} \cdot \nu f_t^{-1}[y_{i-1}(t), y_i(t)] < \widetilde{h}^q < \widetilde{h}^p.$$

We shall find an R-definable function d with

$$\begin{aligned} &d(t) \in \mathfrak{m}^>, \quad \nu f_t^{-1}[y_{i-1}(t), y_i(t)] < \widetilde{d(t)}, \\ &\widetilde{y_i(t)} \cdot \nu f_t^{-1}[y_{i-1}(t), y_i(t)] \le \widetilde{y_i(t)} \cdot \widetilde{d(t)} \le \widetilde{h(t)}^q \end{aligned}$$

for all $t \in \mathfrak{m}^>$.

As mentioned above, for each $i \in \{1, \ldots, l-1\}$, $y_i(t) = h(t)^r$ for some $r \in \mathbb{Q}^>$. Note that either $\widetilde{h(t)}^{q-r} \in \mathfrak{m}^>$ for all $t \in \mathfrak{m}^>$, or $\widetilde{h(t)}^{q-r} > \widetilde{\mathfrak{m}^>}$ for all $t \in \mathfrak{m}^>$. In the first case, we may set $d(t) = h(t)^{q-r}$. In the second case, we set $d(t) = \sqrt{b(t)}$, where $\widetilde{b(t)} = \nu f_t^{-1}[y_{i-1}(t), y_i(t)]$ for all $t \in \mathfrak{m}^>$.

Inductively, we now obtain $H_i\colon (0,a) \to [0,1]$, an R-definable function, such that

$$\widetilde{y_i(t)} \cdot \nu f_t^{-1}[y_{i-1}(t), y_i(t)] \le \widetilde{y_i(t)} \cdot \widetilde{H_i(t)} < \widetilde{y_i(t)} \cdot \widetilde{d(t)} \le \widetilde{h(t)}^p$$

for all $t \in \mathfrak{m}^>$, and such that $y_i(t) \cdot H_i(t)$ is an upper approximation of

$$f_t^{-1}[y_{i-1}(t), y_i(t)] \times [y_{i-1}(t), y_i(t)]$$

for all $t \in (0,a)$, where $a \in \mathbb{R}^>$.

Now $H(t) = \sum_{i=1}^l y_i(t) \cdot H_i(t)$ is an upper approximation of A_t for all $t \in (0,a)$, where $a \in \mathbb{R}^>$, and $\nu A_t \le \widetilde{H(t)} < \widetilde{G(t)}$ for all $t \in \mathfrak{m}^>$. ■

PROPOSITION 4.14. *Let $\{A_t\colon t \in \mathbb{R}^>\}$ be a definable family of open subsets of $\mathbb{R}^n$ such that $\lambda A_t \to 0$ as $t \to 0^+$. Let h be as in Lemma* 4.13(a). *Then*

$$\lim_{t \to 0^+} \frac{\log \lambda A_t}{\log h(t)} = 1.$$

Proof. Let $p, q \in \mathbb{Q}^>$ with $p < 1 < q$. Then, for $t \in \mathfrak{m}^>$,

$$\widetilde{h(t)}^q < \widetilde{h(t)} < \widetilde{h(t)}^p.$$

By Lemma 4.13(b), there is a definable $H\colon (0,a) \to [0,1]$, where $a \in \mathbb{R}^>$, such that, on $\mathfrak{m}^>$,

$$\widetilde{h(t)}^q < \widetilde{h(t)} \le \widetilde{H(t)} < \widetilde{h(t)}^p,$$

and $H(t)$ is the volume of an upper approximation of A_t for all $t \in (0,a)$. So, by Lemma 4.13,

$$h(t) \le \lambda A_t \le H(t) \quad \text{for all } t \in (0,a),$$

and hence

$$h(t)^q < \lambda A_t < h(t)^p \quad \text{for all sufficiently small } t \in \mathbb{R}^{>}.$$

It follows that

$$p < \lim_{t\to 0^+} \frac{\log \lambda A_t}{\log h(t)} < q. \ \blacksquare$$

Corollary 4.15. *Let $\{A_{p,t} : p \in \mathbb{R}^l,\, t \in \mathbb{R}^{>}\}$ be a definable family of open subsets of $\mathbb{R}^n$ such that $\lim_{t\to 0^+} \lambda A_{p,t} = 0$ for all $p \in \mathbb{R}^l$. Then*

$$F(p) := \lim_{t\to 0^+} \frac{\log \lambda A_{p,t}}{\log t} \in \mathbb{R}_\infty$$

is a definable function of p taking values in Λ_∞. If R is polynomially bounded then F takes only finitely many values.

Proof. For each $p \in \mathbb{R}^l$, let h_p be the function whose existence is guaranteed in Lemma 4.13(a), considered as a function $\mathbb{R}^{>} \to \mathbb{R}^{>}$. Note that h_p is uniformly definable in p. Proposition 4.14 shows that

$$\lim_{t\to 0^+} \frac{\log \lambda A_{p,t}}{\log h_p(t)} = 1 \quad \text{for all } p \in \mathbb{R}^l.$$

This implies

$$\lim_{t\to 0^+} \frac{\log \lambda A_{p,t}}{\log t} = \lim_{t\to 0^+} \frac{\log h_p(t)}{\log t} \quad \text{for all } p \in \mathbb{R}^l.$$

The corollary now follows by an application of Proposition 3.1. $\blacksquare$

Proposition 4.16. *Let $\mathcal{A} = \{A_{p,t} \colon (p,t) \in \mathbb{R}^k \times \mathbb{R}^{>}\}$ be a definable family of d-dimensional subsets of $\mathbb{R}^n$, where $d < n$ and $\lambda_d A_{p,t} \to 0$ as $t \to 0+$ for each $p \in \mathbb{R}^k$. Then*

$$\lim_{t\to 0+} \frac{\log \lambda_d A_{p,t}}{\log t}$$

is a definable function of p taking values in Λ_∞. If R is polynomially bounded, then this function takes only finitely many values.

Proof. By Proposition 3.1, it is enough to find $f \colon \mathbb{R}^k \times \mathbb{R}^{>} \to \mathbb{R}$, a definable function such that

$$\lim_{t\to 0+} \frac{\log \lambda_d A_{p,t}}{\log t} = \lim_{t\to 0+} \frac{\log f(p,t)}{\log t} \quad \text{for all } p \in \mathbb{R}^k.$$

Let L be the constant corresponding to n from Theorem 4.5. We shall say that a collection of definable sets $\{C_i\}$ and a collection of permutations of coordinates $\{\tau_i\}$ of $\mathbb{R}^n$ are *good* for a definable set $A \subseteq \mathbb{R}^n$ if $\{C_i\}$ is an almost partition of A and each $\tau_i C_i$ is an L-cell of dimension d.

Corollary 4.6 yields, for each $(p,t) \in \mathbb{R}^{k+1}$, a collection $\{C_i\}$ of definable subsets of $\mathbb{R}^{k+1+n}$, and a collection $\{\tau_i\}$ of permutations of coordinates of $\mathbb{R}^n$ such that $\{C_i(p,t)\}$ and $\{\tau_i\}$ are good for $A_{p,t}$ (here, we have $C_i(p,t) =$

$\{x \in \mathbb{R}^n \colon (p,t,x) \in C_i\})$. Note that being an almost partition of $A_{p,t}$ as well as being an L-cell are first-order properties. Furthermore, recall that Corollary 4.6 holds in any o-minimal field, so in particular in a saturated one. By model-theoretic compactness, we thus obtain finitely many collections

$$\{C_{1,1},\ldots,C_{1,l(1)}\},\ldots,\{C_{m,1},\ldots,C_{m,l(m)}\}$$

of finitely many families of sets $C_{ij} \subseteq \mathbb{R}^{k+1+n}$ such that for each $(p,t) \in R^{k+1}$ there is $i \in \{1,\ldots,m\}$ such that $\{C_{ij}(p,t)\}$ and some set $\{\tau_j\}$ of permutations of coordinates of $\mathbb{R}^n$ are good for $A_{p,t}$.

Let $\{\underline{\tau}_s\}$ be the set of all the tuples of permutations of coordinates $\mathcal{R}^n \to \mathcal{R}^n$ of length $\max\{l(1),\ldots,l(m)\}$ (note that $\{\underline{\tau}_s\}$ is finite), and let $(\underline{\tau_s})_j$ be the jth coordinate of $\underline{\tau}_s$. For each $i \in \{1,\ldots,m\}$ and each s, let $h_{is}\colon \mathcal{R}^{k+1} \to \mathcal{R}$ be the definable function such that

$$\widetilde{h_{is}(p,t)} = \max_j \nu p_d^n(\underline{\tau}_s)_j C_{ij}(p,t)$$

and whose existence was proved in Lemma 4.13.

Then the function $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$ which assigns to (p,t) the value of h_{is} at (p,t), where is is the smallest ordered pair in the lexicographic order such that $\{C_{ij}(p,t)\}$ and $\underline{\tau}_s$ are good for $A_{p,t}$, is as required. ∎

5. Proof of Theorem 1.2. Let $\mathcal{X} = \{(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$ be a definable family of metric spaces. We now prove Theorem 1.2.

THEOREM 5.1. *The Hausdorff dimension of (X_α, d_α) is a definable function of α which takes values in Λ_∞. If R is polynomially bounded, then the Hausdorff dimension of the elements of $\mathcal{X}$ takes only finitely many values. Furthermore,*

$$\dim_H(X_\alpha,d_\alpha) = \dim_P(X_\alpha,d_\alpha) \quad \textit{for all } \alpha \in \mathbb{R}^l.$$

Proof. By [16, Corollary 9.3.4] there is a partition of $\mathbb{R}^l$ into definable sets A, B, a definable family $\{Z_\alpha : \alpha \in A\}$ of sets and a definable family $\{h_\alpha : \alpha \in A\}$ of functions such that

(1) h_α is a homeomorphism $(X_\alpha, d_\alpha) \to (Z_\alpha, e)$, with e the euclidean metric, for all $\alpha \in A$,
(2) if $\beta \in B$, then there is an infinite definable $A \subseteq X_\beta$ such that (A, d_β) is discrete.

Any infinite definable set has cardinality $|\mathbb{R}|$. Thus if $\beta \in B$, then (X_β, d_β) contains a discrete subspace of cardinality $|\mathbb{R}|$ and is therefore not separable. Thus Fact 2.1 implies that if $\beta \in B$ then (X_β, d_β) has infinite Hausdorff and packing dimension. We therefore assume that B is empty. For all $\alpha \in \mathbb{R}^l$, we let d'_α be the metric on Z_α given by

$$d'_\alpha(h_\alpha(x), h_\alpha(y)) = d_\alpha(x,y) \quad \text{for all } x, y \in X_\alpha.$$

Then (X_α, d_α) is isometric to (Z_α, d'_α) for all $\alpha \in \mathbb{R}^l$. We also note that $\mathrm{id} : (Z_\alpha, d'_\alpha) \to (Z_\alpha, e)$ is a homeomorphism for all $\alpha \in \mathbb{R}^l$. It suffices to prove the theorem for the family $\{(Z_\alpha, d'_\alpha) : \alpha \in \mathbb{R}^l\}$. We therefore argue under the assumption that the topologies given by d_α and e agree on X_α for all $\alpha \in \mathbb{R}^l$. In view of the Trivialization Theorem there are a partition $\{F_1, \ldots, F_n\}$ of $\mathbb{R}^l$ into definable sets, definable sets $X_1, \ldots, X_n$, and a definable family of functions $\{g_\alpha : \alpha \in \mathbb{R}^l\}$ such that $g_\alpha : (X_\alpha, e) \to (X_i, e)$ is a homeomorphism for all $\alpha \in F_i$. For all $1 \le i \le n$ and $\alpha \in F_i$ we let d'_α be the metric on X_i given by

$$d'_\alpha(g_\alpha(x), g_\alpha(y)) = d_\alpha(x, y) \quad \text{for all } x, y \in X_\alpha.$$

It suffices to prove the theorem for each definable family $\{(X_i, d'_\alpha) : \alpha \in F_i\}$ separately. So we suppose $\mathcal{X} = \{(X, d_\alpha) : \alpha \in \mathbb{R}^l\}$ for some definable set X, and suppose the topology given by d_α agrees with the usual euclidean topology on X for all α. Let $k = \dim(X)$.

We apply induction to k. If $k = 0$, then X is finite and so (X, d_α) has Hausdorff and packing dimension zero for all α. Suppose $k \ge 1$. Let λ be the k-dimensional Lebesgue measure on X. We let $B_\alpha(p, t)$ be the open d_α-ball with center $p \in X$ and radius t, and $B_e(p, t)$ be the open euclidean ball in $\mathbb{R}^l$ with center p and radius t. Fix $\alpha \in \mathbb{R}^l$ and $p \in X$. For all $\delta \in \mathbb{R}^>$ there is an $\epsilon \in \mathbb{R}^>$ such that $B_\alpha(p, \epsilon) \subseteq B_e(p, \delta)$. Thus we have $\lambda[B_\alpha(p, t)] \to 0$ as $t \to 0^+$. Applying Proposition 4.16 we get a definable function $F : \mathbb{R}^l \times X \to \mathbb{R}_\infty$ such that

$$F(\alpha, p) = \lim_{t \to 0^+} \frac{\log \lambda[B_\alpha(p, t)]}{\log(t)} \quad \text{for all } \alpha \in \mathbb{R}^l,\ p \in X.$$

Proposition 4.16 also implies that F takes values in the field of powers of R and if R is polynomially bounded, then F takes only finitely many values. For all $\alpha \in \mathbb{R}^l$ we let $F_\alpha : X \to \mathbb{R}_\infty$ be given by $F_\alpha(p) = F(\alpha, p)$. We let $\{U_\alpha : \alpha \in \mathbb{R}^l\}$ be a definable family of open subsets of X such that for all α, the restriction of F_α to U_α is continuous, $U_\alpha \subseteq_0 X$, and every definable open subset of U_α has dimension k. By Fact 2.1,

$$\begin{aligned} \dim_H(X_\alpha, d_\alpha) &= \max\{\dim_H(U_\alpha, d_\alpha), \dim_H(X_\alpha \setminus U_\alpha, d_\alpha)\}, \\ \dim_P(X_\alpha, d_\alpha) &= \max\{\dim_P(U_\alpha, d_\alpha), \dim_P(X_\alpha \setminus U_\alpha, d_\alpha)\} \quad \text{for all } \alpha. \end{aligned}$$

The inductive assumption yields the theorem for $\{(X_\alpha \setminus U_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$. It therefore suffices to prove the theorem for $\{(U_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$. As every definable open subset of U_α has positive λ-measure, we have $\lambda[B_\alpha(p, t)] > 0$ for all $p \in U_\alpha$ and $t \in \mathbb{R}^>$. Thus Proposition 2.5 implies

$$\dim_H(U_\alpha, d_\alpha) = \dim_P(U_\alpha, d_\alpha) = \sup\{F_\alpha(p) : p \in U_\alpha\} \quad \text{for all } \alpha \in \mathbb{R}^l.$$

Therefore $\dim_H(U_\alpha, d_\alpha)$ is a definable function of α. If R is polynomially bounded, then F takes only finitely many values, all in Λ_∞, and so

$\dim_H(U_\alpha, d_\alpha)$ takes only finitely many values as α varies, and each value is an element of Λ_∞. ∎

Combining Theorem 1.2 with Fact 2.3 we directly obtain:

COROLLARY 5.2. *Suppose (X, d), (X', d') are definable metric spaces. Then*

$$\dim_H(X \times X', d_\square) = \dim_H(X, d) + \dim_H(X', d').$$

6. Bilipschitz equivalence. In [14] Valette classified definable sets equipped with their induced euclidean metrics and proved the following.

THEOREM 6.1. *There are $|\Lambda|$-many definable sets up to bilipschitz equivalence. If R is polynomially bounded, then a definable family of sets has only finitely many elements up to bilipschitz equivalence.*

One might speculate that the polynomially bounded case of Theorem 1.2 is a consequence of a generalization of Valette's theorem to definable metric spaces. This is not the case:

FACT 6.2. *There is a semialgebraic family of metric spaces which contains continuum many elements up to bilipschitz equivalence.*

The collection of Carnot metrics on $\mathbb{R}^k$ naturally forms a semialgebraic family of metric spaces. Pansu [11] proved that if two Carnot groups are not isomorphic as groups, then the associated Carnot metrics are not bilipschitz equivalent. It is known that if $k \geq 6$, then there are continuum many pairwise nonisomorphic Carnot group operations on $\mathbb{R}^k$. See [16, 5.5] for details and references.

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