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On definable matchings in o-minimal bipartite graphs

by

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Abstract. We consider bipartite graphs definable in o-minimal structures, in which the edge relation G is a finite union of graphs of certain measure-preserving maps.

We establish the existence of definable matchings with few short augmenting paths. Under the additional assumptions of $G \subseteq [0,1]^n$ and 2-regularity, this yields the existence of definable matchings covering all vertices outside of a set of arbitrarily small positive measure (Lebesgue measure of the standard part). As an application we obtain an approximate 2-cancellation result for the semigroup of definable subsets of $[0,1]^n$ modulo an equivalence relation induced by measure-preserving maps.

1. Introduction. This paper is a first step towards understanding definable matchings in definable bipartite graphs in o-minimal structures. Matchings play an important role in many areas of mathematics, such as the theory of equidecompositions, which is in turn closely related to the existence of measures. In the o-minimal setting, equidecompositions constitute a possible approach to a long-standing open question on the existence of certain invariant measures on definable sets (see for instance p. 576 in Hrushovski, Peterzil, and Pillay [6] for a possible formulation of this question).

We shall assume knowledge of the basics of o-minimality. A standard reference is van den Dries [3]. A reference for graph theory is for instance Diestel [2], though knowledge of Appendix C in Tomkowicz and Wagon [12] should largely suffice.

Here, a *graph* consists of a nonempty set of *vertices* V and a symmetric, antireflexive relation $E \subseteq V^2$ whose elements are called *edges*. So graphs have no loops, no multiple edges, and edges are not oriented. A *bipartite graph* is a graph whose set of vertices can be partitioned into two disjoint sets A and B so that each edge has one vertex in A and the other in B.

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A *matching* in a bipartite graph $(A \dot\cup B, E)$ is a subset of E which is the graph of a bijection between a subset of A and a subset of B. A matching is *perfect* if it covers all vertices, i.e., if it is a bijection of A onto B.

Throughout, we let R be an o-minimal expansion of an ordered field. Definable shall mean definable in R. A graph (V, E) is *definable* if both $V \subseteq R^n$ and $E \subseteq R^{2n}$ are definable. A *definable bipartite graph* $(A \dot\cup B, E)$ is a definable graph with a definable bipartition, i.e., both A and B are definable.

What is the situation like for perfect matchings in bipartite graphs without any definability assumptions? In the finite case, we have König's Theorem [7], which ensures the existence of a perfect matching under the additional assumption of k-regularity (that is, every vertex has degree k). This is a special case of the (finite) Hall–Rado Theorem (also known as the Marriage Theorem; a proof can be found for instance in [2]), according to which a graph admits a perfect matching if for each k, every k-element set has at least k neighbors (this is the so-called finite marriage condition). Both these theorems have infinite counterparts. In particular, the infinite two-sided Hall–Rado–Hall Theorem says that a locally finite bipartite graph admits a perfect matching if it satisfies the marriage condition for finite sets in either part – see Hall [5] for the original proof, or [12] for a quick proof using Tikhonov's Theorem.

However, the definable versions of these theorems for infinite graphs fail, as evidenced by an example by Laczkovich [8]. Laczkovich defines a semilinear graph whose edge relation E is a closed subset of the unit square and which consists of finitely many line segments with slopes ± 1 (in fact, E is just the unit circle, when considered as a space with normalized linear measure, up to a measure-preserving homeomorphism). While E contains a perfect matching by König's Theorem, Laczkovich shows that it does not contain a Borel matching nor a Lebesgue measurable matching. This is roughly due to the fact that, while the normalized linear measure of a matching M in E would be $1/2$, M would also have to be fixed by a certain map which is essentially an irrational rotation of the circle, hence ergodic. Given that E is in particular definable in an o-minimal structure, this dashes the hope of a definable analogue of König or Hall–Rado–Hall.

One way around this, in the presence of a measure, is to relax the requirement of the matching being perfect to being perfect only outside of a small set. This has been done in the Borel case by Lyons and Nazarov [9, p. 8, Remark 2.6] (for a detailed exposition of the proof see Wang [13]). Before stating their theorem, we introduce a couple definitions. A set of vertices is *independent* if no two vertices in it are neighbors, i.e., no two vertices are incident with the same edge. For a set of vertices Y and edge relation G, we set

$$N_G(Y) = \{x \colon \exists y \in Y \ (x, y) \in G\}.$$

Finally, a locally finite Borel graph (X, G) on a standard Borel space with Borel probability measure ν is *ν-preserving* if for every Borel automorphism $f\colon X \to X$ whose graph is a subset of G, $\nu Y = \nu f^{-1}(Y)$ for every measurable $Y \subseteq X$.

THEOREM 1.1 ([9]). *Let $\mathcal{G} = (X, G)$ be a Borel graph on a standard Borel space with a Borel probability measure ν, which is locally finite, ν-preserving, bipartite, and satisfies the following expansion condition:*

$$\exists c > 1 : \quad \textit{for all independent } Y \subseteq X, \quad \nu N_G(Y) \geq c \cdot \nu Y.$$

Then $\mathcal{G}$ has a Borel perfect matching ν-a.e.

We prove an approximate version of Theorem 1.1 (as Theorem 1.2 below), when the measure under consideration is the Lebesgue measure of the standard part and when we restrict ourselves to 2-regular graphs. The assumption of 2-regularity replaces the expansion condition in Theorem 1.1, which is never satisfied in the bounded definable setting, given that the bipartition of a definable bipartite graph is assumed to be definable. The condition of being μ-preserving corresponds, in our setting, to the edge relation being a finite union of graphs of isomorphisms (roughly, definable C^1-diffeomorphisms with Jacobian determinant equal to ± 1).

THEOREM 1.2. *Let $\mathcal{G} = (A \dot{\cup} B, G)$ be a definable bipartite μ-preserving graph which is 2-regular and such that $A, B \subseteq [0,1]^n$. Then for every $\epsilon \in \mathbb{R}^{>0}$ there is a definable matching $M \subseteq G$ covering all vertices of $\mathcal{G}$ outside of a set of μ-measure $< \epsilon$.*

Theorem 1.2 is Corollary 4.3 below, which is a special case of the slightly more general Theorem 4.2. The proof follows the general outline of the proof in [13]. In particular, we first prove the existence of matchings with few short augmenting paths, adapting an argument by Elek and Lippner [4]. We start with the archimedean case. This is Proposition 2.6 – the measure under consideration is Lebesgue measure and there is no need to use 2-regularity. The general case (Theorem 3.6) is then derived using results from Maříková [10, 11] concerning the structure induced on the residue field by the standard part map. Theorem 1.2 is then derived by an argument similar to the one in [13], but with 2-regularity yielding an expansion condition.

We remark that Theorem 1.2 cannot be improved to yield a definable matching μ-a.e. due to the example in [8].

We use Theorem 1.2 to obtain a cancellation result for certain semigroups. Here is some context. Let $B[n]$ be the lattice of bounded definable subsets of R^n, and let $\sim$ be the equivalence relation induced on $B[n]$ by isomorphisms (see Definition 5.1). Then $\mathcal{T}_n = B[n]/{\sim}$ is a semigroup with addition given by disjoint union. Tarski's Theorem (see [12, p. 194]) links

$\mathcal{T}_n$ to the above-mentioned open question about the existence of invariant measures on definable sets in o-minimal structures:

Theorem 1.3. *Let* $(\mathcal{T}, +, 0, \epsilon)$ *be a commutative semigroup with identity* 0 *and a specified element* ϵ. *Then the following are equivalent:*

(1) *For all* $n \in \mathbb{N}$, $(n+1)\epsilon \not\leq n\epsilon$.
(2) *There is a homomorphism of semigroups* $\mu\colon \mathcal{T} \to [0, \infty]$ *such that* $\mu(\epsilon) = 1$ *and* $\mu(\alpha + \beta) = \mu\alpha + \mu\beta$.

Questions about $\mathcal{T}_n$ often boil down to questions about the existence of definable matchings in definable bipartite graphs whose edge relation is a finite union of graphs of isomorphisms. For instance, if we knew that $\mathcal{T}_n$ has cancellation, then in order to verify Condition 1, it would suffice to verify the case $n = 1$. And cancellation is proved – at least in the non-definable setting – by finding a perfect matching in a bipartite graph whose edge relation is a finite union of graphs of isomorphisms.

We use Theorem 1.2 to prove an approximate cancellation result for the semigroup of bounded definable sets modulo the equivalence relation induced by isomorphisms. More precisely, let

$$SB[n] = \{X \in B[n] \colon X \subseteq [-m, m]^n \text{ for some } m \in \mathbb{N}\}$$

be the lattice of strongly bounded definable subsets of R^n. For $X, Y \in SB[n]$ and $\epsilon \in \mathbb{R}^{>0}$, we write $X =_\epsilon Y$ iff $\mu(X \triangle Y) < \epsilon$, where μ is the standard part map composed with Lebesgue measure, and $\triangle$ is symmetric difference. We write $X =_a Y$ iff $X =_\epsilon Y$ for all $\epsilon \in \mathbb{R}^{>0}$. For $\alpha, \beta \in \mathcal{T}_n$, we write $\alpha =_a \beta$ iff there are $X \in \alpha$, $Y \in \beta$ such that $X, Y \in SB[n]$ and $X =_a Y$. Then Theorem 1.2 yields what is Theorem 5.2 below:

Theorem 1.4. *Suppose* $\alpha, \beta \in \mathcal{T}_n$ *have representatives in* $SB[n]$. *Then* $2\alpha =_a 2\beta$ *implies* $\alpha =_a \beta$.

Some further conventions and definitions. We let $\mathcal{O}$ be the convex hull of $\mathbb{Q}$ in R. Then $\mathcal{O}$ is a valuation ring in R with maximal ideal $\mathfrak{m}$ and residue map $\mathrm{st}\colon \mathcal{O} \to \mathbf{k}$, where $\mathbf{k} = \mathcal{O}/\mathfrak{m}$ is the ordered residue field. The residue map extends coordinate-wise to $\mathrm{st}\colon \mathcal{O}^n \to \mathbf{k}^n$. If R is sufficiently saturated, then $\mathcal{O}/\mathfrak{m} = \mathbb{R}$ and the residue map is called the *standard part map*. In this case, we denote by $\mathbb{R}_{\mathrm{ind}}$ the structure on $\mathbb{R}$ which is generated by the standard part map, i.e., the ordered field $\mathbb{R}$ expanded by the relations $\mathrm{st}\, X$, where $X \in \mathrm{Def}^n(R)$ and $\mathrm{st}\, X := \mathrm{st}(X \cap \mathcal{O}^n)$, for all n. It was observed in [6] that it follows from a theorem by Baisalov and Poizat [1] that the structure $\mathbb{R}_{\mathrm{ind}}$ is o-minimal.

If S is the underlying set of a structure and $1 \leq m \leq n$, then we denote by $\pi^n_m \colon S^n \to S^m$ the projection onto the first m coordinates. If f is a function, then $\mathit{\Gamma} f$ denotes its graph.

Definition 1.5. A *measure* on $SB[n]$ is a finitely additive map

$$\mu\colon SB[n] \to \mathbb{R}^{\geq 0},$$

with addition on $SB[n]$ being given by disjoint union, such that $\mu(\emptyset) = 0$.

Definition 1.6.

(1) An *n-isomorphism* is a definable C^1-diffeomorphism $f\colon U \to f(U)$, where $U \subseteq R^n$ is definable and open, and $|Jf(x)| = 1$ for all $x \in U$.
(2) A measure μ on $SB[n]$ is *invariant* if $\mu(X) = \mu(f(X))$ whenever X and $f(X)$ are in $SB[n]$ and f is an n-isomorphism.
(3) Let μ be an invariant measure on $SB[n]$. We say that a definable graph $\mathcal{G} = (V, G)$ is *μ-preserving* if $V \subseteq \mathcal{O}^n$, and there is a partition of V into cells such that for each open cell C in this partition, $G \cap (C \times R^n)$ is a finite union of graphs of n-isomorphisms.

For R sufficiently saturated, we define an invariant measure μ on $SB[n]$ by assigning to $X \in SB[n]$ the n-dimensional Lebesgue measure of its standard part (see [10, p. 18, proof of Lemma 6.4] for a proof of invariance).

Remark 1.7. It will be easy to see that Theorem 3.6, Corollary 4.3, and Theorem 5.2 remain valid if we replace Definition 1.6 by the following, perhaps more natural, definition.

Definition 1.8.

(1) An *n-isomorphism* is a definable C^1-diffeomorphism $f\colon R^n \to R^n$ with $|Jf(x)| = 1$ for all $x \in R^n$.
(2) A measure μ on $SB[n]$ is *invariant* if $\mu(X) = \mu(f(X))$ whenever X and $f(X)$ are in $SB[n]$ and f is an n-isomorphism.
(3) Let μ be an invariant measure on $SB[n]$. We say that a definable graph $\mathcal{G} = (V, G)$ with $V \subseteq \mathcal{O}^n$ is *μ-preserving* if there is a partition of V into cells such that for each open cell C in this partition, $G \cap (C \times R^n)$ is a finite union of graphs of n-isomorphisms restricted to C.

For $x, y \in R^n$ and definable, bounded $X \subseteq R^n$, we let $d(x, y)$ be the Euclidean distance between x and y, and we set

$$d(X, y) = \inf\{d(x, y)\colon x \in X\}.$$

For $x \in R^n$ and $r > 0$, we denote by $B_r(x)$ the open ball of radius r centered at x, i.e., the set $\{y \in R^n\colon d(x, y) < r\}$.

2. The archimedean case. In this section, we assume that the underlying set of R is $\mathbb{R}$. Then $SB[n] = B[n]$ and Lebesgue measure λ is an invariant measure on $B[n]$.

2.1. Colorings

Definition 2.1. Let $\mathcal{G} = (V, G)$ be a definable graph. We say that a definable map $c\colon V \to X$ is a *definable coloring* of $\mathcal{G}$ if X is a finite set, and whenever $(v, w) \in G$, then $c(v) \neq c(w)$.

Lemma 2.2. *Let $\mathcal{G} = (V, G)$ with $V \in B[n]$ be a definable graph such that every vertex has finite degree. Then there is a definable coloring of $\mathcal{G}$ outside of a definable subset of V of arbitrarily small positive λ-measure, i.e., for every $\epsilon > 0$ there is a definable $V' \subseteq V$ with $\lambda(V \setminus V') < \epsilon$ and a definable coloring of $\mathcal{G}' := (V', G \cap (V')^2)$.*

Proof. Let $\mathcal{C}$ be a decomposition of $\mathbb{R}^{2n}$ into cells partitioning G, and let

$$\mathcal{D} = \{\pi^{2n}_n C\colon C \in \mathcal{C} \,\&\, C \subseteq G \,\&\, \dim C = n\}.$$

Then, because every vertex of $\mathcal{G}$ has finite degree, we may assume that for each $D \in \mathcal{D}$, $G \cap (D \times \mathbb{R}^n)$ is a finite disjoint union of graphs of definable, continuous functions. Let $\mathcal{F}_D$ be the collection of these functions.

Claim. *Let $\epsilon > 0$, $D \in \mathcal{D}$ and $f \in \mathcal{F}_D$. Denote by $\mathcal{G}_f$ the graph $(D \cup f(D), \Gamma f)$. Then there is a definable coloring of $\mathcal{G}_f$ outside of a definable subset of $D \cup f(D)$ of λ-measure $< \epsilon$.*

Proof of Claim. We set

$$D_\delta := \{x \in D\colon d(x, \partial D) \geq \delta\},$$

where $\partial D := \mathrm{cl}(D) \setminus \mathrm{int}(D)$, and $\delta > 0$ is such that $\lambda(D \setminus D_\delta) < \epsilon/2$ and $\lambda(f(D \setminus D_\delta)) < \epsilon/2$ (the existence of such a δ follows from the boundedness of the vertex set). Define

$$F\colon D_\delta \to \mathbb{R}^{\geq 0}\colon x \mapsto d(x, f(x)).$$

Then, because $\Gamma f|_{D_\delta} \subseteq G$, G is antireflexive, f is continuous, and D_δ is closed and bounded, F is bounded away from 0, say by $r > 0$. Since D_δ is compact, we can find a finite covering $\mathcal{B}$ of D_δ by open balls of radius $r/2$. For $x \in D_\delta$, define $c(x) = i$, where i is the smallest index of a ball from $\mathcal{B}$ containing x. If $x, y \in D_\delta$ are such that $c(x) = c(y)$, then $x, y \in B$ for some $B \in \mathcal{B}$, so $d(x, y) < r$ and x, y cannot be neighbors. $\blacksquare_{\text{Claim}}$

Let $\epsilon > 0$. We shall now define $V' \subseteq V$ with $\lambda(V' \setminus V) < \epsilon$, and find a definable coloring of the graph $(V', G \cap (V')^2)$.

Let $|\mathcal{D}| = N$, and let M be an upper bound for the degrees of the vertices of $\mathcal{G}$. Note that M exists by cell decomposition and the assumption that all degrees are finite. Since vertices of degree 0 may be colored by any color, we may assume that $\mathcal{F}_D \neq \emptyset$ for each $D \in \mathcal{D}$. For every $D \in \mathcal{D}$ and every $f \in \mathcal{F}_D$, use the claim to find a definable coloring c_f of $\mathcal{G}_f$ outside of a definable $S_f \subseteq D \cup f(D)$ of λ-measure $< \frac{\epsilon}{MN}$. Set $V' := V \setminus \bigcup_{D \in \mathcal{D}} \bigcup_{f \in \mathcal{F}_D} S_f$.

Note that

$$\lambda\Big(\bigcup_{D\in\mathcal{D}}\bigcup_{f\in\mathcal{F}_D} S_f\Big) < M\cdot N\cdot\frac{\epsilon}{MN} = \epsilon.$$

Define a map c with $\mathrm{domain}(c) = V'$ and range the power set of $\dot{\bigcup}\,\mathrm{rng}(c_f)$, where the union is taken over all $D \in \mathcal{D}$ and all $f \in \mathcal{F}_D$ as follows. For $x \in V'$ let $c(x)$ consist of all the $c_f(x)$ with $x \in \mathrm{domain}(c_f)$. Suppose $x, y \in V'$ and $(x, y) \in G \cap (V')^2$. Then $f(x) = y$ for some $f \in \mathcal{F}_D$ with $x \in D$. Hence $c_f(x) \neq c_f(y)$, and so $c(x) \neq c(y)$. It follows that c is as required. ∎

Given a definable graph $\mathcal{G} = (V, G)$ and a measure on V, we shall say that $\mathcal{G}$ is *definably almost-colorable* if for every $\epsilon \in \mathbb{R}^{>0}$, there is $V_\epsilon \subseteq V$ with measure of $V \setminus V_\epsilon$ less than ϵ, and a definable coloring of $(V_\epsilon, G \cap (V_\epsilon)^2)$.

Remark 2.3. Lemma 2.2 fails for R non-archimedean: Let ϵ be a positive infinitesimal in R, and consider the graph $\mathcal{G} = ([0, 1], G)$, where $(x, y) \in G$ iff $y = x + \epsilon$ and $0 \leq x \leq 1 - \epsilon$. Then $\mathcal{G}$ is not definably almost-colorable.

2.2. Matchings with few short augmenting paths. In the case of a finite graph, to enlarge the set of vertices covered by a matching M, one considers augmenting paths (or "augmenting paths for M", when the matching is not clear from the context). Those are paths $(a_0, a_1, \ldots, a_n)$ of odd length such that a_0, a_n are not covered by M, $(a_{2k}, a_{2k+1}) \notin M$ for $k \in \{0, \ldots, \frac{n-1}{2}\}$, and $(a_{2k+1}, a_{2k+2}) \in M$ for $k \in \{0, \ldots, \frac{n-3}{2}\}$. By flipping an augmenting path, i.e., by removing the edges (a_{2k+1}, a_{2k+2}) from M and placing the edges of the form (a_{2k}, a_{2k+1}) into M, one obtains a new matching that covers a strictly larger number of vertices. We want to use this idea in our setting, but we will need to handle infinitely many augmenting paths at the same time.

In this section, we let $\mathcal{G} = (A \dot{\cup} B, G)$ be a definable, λ-preserving, bipartite graph with $A, B \in B[n]$, and $M \subseteq G$ a definable matching. We fix $K \geq 0$.

We say that a finite set of subsets of a definable $X \subseteq R^n$ is an *open partition* of X if each of its members is an open cell contained in X, and its union covers X outside of a set of λ-measure 0. We say that an open partition $\{X_i\}$ of X partitions a definable $Y \subseteq X$ if a subset of $\{X_i\}$ constitutes an open partition of Y. An open partition $\{Y_j\}$ of X is a *refinement* of another open partition $\{X_i\}$ of X if $\{Y_j\}$ partitions each X_i.

Generating sequences of paths. Since the edge relation G is a finite union of graphs of functions, every path in $\mathcal{G}$ is determined by its starting vertex and a finite sequence of functions. The following construction serves the purpose of assigning a unique such sequence to every path of bounded length. Essentially, starting with a decomposition $\mathcal{C}$ of $\mathbb{R}^{2n}$ into cells, which partitions G and M, we shall eliminate all vertices that do not lie in an

n-dimensional cell $\pi^{2n}_n C \subseteq \pi^{2n}_n G$, where $C \in \mathcal{C}$ (i.e., all vertices in $A \setminus \bigcup \mathcal{A}_0$, where $\mathcal{A}_0$ is defined below), along with all vertices that lie on paths of length $\leq 2K+2$ and originate in a vertex in $A \setminus \bigcup \mathcal{A}_0$.

We shall define a sequence

$$\mathcal{A}_0, \mathcal{A}_2, \ldots, \mathcal{A}_{2K+2}$$

of open partitions of A, and a sequence

$$\mathcal{B}_1, \mathcal{B}_3, \ldots, \mathcal{B}_{2K+1}$$

of open partitions of B. In each sequence, every open partition will be a refinement of its predecessor.

Let $\mathcal{C}$ be a decomposition of $\mathbb{R}^{2n}$ into cells which partitions both G and M. Set

$$\mathcal{A}_0 = \{\pi^{2n}_n C \colon C \in \mathcal{C} \,\&\, \dim \pi^{2n}_n C = n \,\&\, \pi^{2n}_n C \subseteq A\}.$$

Since $\mathcal{G}$ is λ-preserving, we may assume that for each $A_{0,i} \in \mathcal{A}_0$,

$$G|_{A_{0,i}} := G \cap (A_{0,i} \times \mathbb{R}^n)$$

is a finite union of graphs of n-isomorphisms. We denote the set of these isomorphisms by $\mathcal{F}_{0,i}$. Note that for each $f \in \mathcal{F}_{0,i}$, the set $A_{0,i} \in \mathcal{A}_0$ consists of the starting vertices x of paths of length 1 with edge $(x, f(x))$.

Assuming that for each $A_{2m,i} \in \mathcal{A}_{2m}$, where $0 \leq m \leq K$, $\mathcal{A}_{2m}$ and $\mathcal{F}_{2m,i}$ have already been defined, we let $\mathcal{B}_{2m+1}$ be an open partition of B partitioning $f(A_{2m,i})$, for each $f \in \mathcal{F}_{2m,i}$ and each $A_{2m,i} \in \mathcal{A}_{2m}$. For $B_{2m+1,j} \in \mathcal{B}_{2m+1}$, let $\mathcal{F}_{2m+1,j}$ be the set of all $f^{-1}|_{B_{2m+1,j}}$ with $f \in \bigcup_i \mathcal{F}_{2m,i}$, where the union is taken over all i such that $A_{2m,i} \in \mathcal{A}_{2m}$, and $B_{2m+1,j} \subseteq \operatorname{rng}(f)$.

To define $\mathcal{A}_{2m+2}$, where $0 \leq m \leq K$, assume that $\mathcal{B}_{2m+1}$ and $\mathcal{F}_{2m+1}$ have been defined. Let $\mathcal{A}_{2m+2}$ be an open partition of A which partitions $f^{-1}(B_{2m+1,j})$ for each $f^{-1} \in \mathcal{F}_{2m+1,j}$ and each $B_{2m+1,j} \in \mathcal{B}_{2m+1}$.

Now set

$$\mathcal{C}_i = \begin{cases} \mathcal{A}_i & \text{if } i \text{ is even,} \\ \mathcal{B}_i & \text{if } i \text{ is odd.} \end{cases}$$

Let $\mathcal{P}$ be the set of paths p in $\mathcal{G}$ of length $l \leq 2K+1$ such that if $p_0 \in A$, then $p_i \in \bigcup \mathcal{C}_i$ for each $i = 0, \ldots, l$, and if $p_0 \in B$, then $p_i \in \bigcup \mathcal{C}_{i+1}$ for each $i = 0, \ldots, l$.

Definition 2.4. Given a path $p \in \mathcal{P}$ of length $l \leq 2K+1$, the *generating sequence* of p is the unique sequence $(g_0, \ldots, g_{l-1})$ of isomorphisms such that

(1) if $p_0 \in \bigcup \mathcal{A}_0$, then each g_i is in $\mathcal{F}_{i,j}$ for some j,
(2) if $p_0 \in \bigcup \mathcal{B}_1$, then each g_i is in $\mathcal{F}_{i+1,j}$ for some j,
(3) $p_{i+1} = g_i(p_i)$ for all $i \in \{0, \ldots, l-1\}$.

Let s be the generating sequence of a path $p \in \mathcal{P}$. We denote by $\mathcal{S}_s$ the set of all possible starting vertices of paths in $\mathcal{P}$ with generating sequence s.

Note that $\mathcal{S}_s \subseteq A_{0,j}$ or $\mathcal{S}_s \subseteq B_{1,j}$ for some j, and that we may identify $\mathcal{P}$ with $\dot{\bigcup}_s \mathcal{S}_s$, where the disjoint union is taken over all generating sequences s of paths in $\mathcal{P}$.

We now define a measure ν on the definable subsets $\mathcal{P}'$ of $\mathcal{P}$. We have $\mathcal{P}' = \dot{\bigcup}_s \mathcal{S}'_s$, where $\mathcal{S}'_s \subseteq \mathcal{S}_s$ and s ranges over the generating sequences of paths in $\mathcal{P}$. We set

$$\nu(\mathcal{P}') := \sum_s \lambda \mathcal{S}'_s,$$

so ν is just Lebesgue measure on $\dot{\bigcup}_s \mathcal{S}_s$.

Let $\mathcal{H} = (\mathcal{P}, H)$ be the definable graph with vertex set $\mathcal{P}$ and $(p,q) \in H$ iff $p \neq q$ and $p_k = q_l$ for some $0 \leq k, l \leq 2K+1$. Since every vertex of $\mathcal{H}$ has finite degree, by Lemma 2.2 we obtain the following.

LEMMA 2.5. *The graph $\mathcal{H}$ is definably almost-colorable* (*with respect to ν*)*.*

Augmenting paths. Our aim now is to define a matching M' which covers the vertices covered by M, but which has only few "short" augmenting paths.

PROPOSITION 2.6. *Let $\delta \in \mathbb{R}^{>0}$. There is a definable matching $M' \subseteq G$ covering the vertices covered by M, and not having any augmenting paths of length $\leq 2K+1$ outside of a definable subset of $\mathcal{P}$ of ν-measure $< \delta$.*

Proof. Since $\mathcal{G}$ is λ-preserving, $\dim A = \dim B$, and we may assume that $\dim A = n$. By Lemma 2.5, we can find a definable $\mathcal{P}' \subseteq \mathcal{P}$ such that $\nu(\mathcal{P} \setminus \mathcal{P}') < \delta/2$ and a definable coloring c of the graph $(\mathcal{P}', H \cap \mathcal{P}'^2)$ with $\operatorname{rng}(c) = \{0, 1, \dots, C-1\}$ for some $C \in \mathbb{N}$. Let $a = (a_k)$ be the sequence of remainders a_k of k modulo C.

We shall obtain the desired matching M' as a member of a sequence $M_0, M_1, \dots$ of definable matchings. We set $M_0 := M$. To obtain M_{k+1} from M_k, flip all augmenting paths for M_k that are contained in $c^{-1}(a_k)$. Given that we are only flipping paths with the same color, in each step we indeed obtain a new matching. Note that each M_{k+1} covers the vertices covered by M_k. It now suffices to establish the claim below.

CLAIM. *There is k such that M_k has no augmenting paths of length $\leq 2K+1$ outside of a definable subset of $\mathcal{P}$ of ν-measure $< \delta$.*

Proof of Claim. First note that an edge $(a,b) \in G$ belonging to M_k can flip for only finitely many k's: Set

$$R_a := \{x : x \text{ is reachable from } a \text{ in} \leq 2K+1 \text{ steps}\}.$$

Then $b \in R_a$, and R_a is finite because every vertex of $\mathcal{G}$ has finite degree. Every time we flip (a,b), this happens because (a,b) is part of an augmenting path of length $\leq 2K+1$ whose flipping results in an increase of the covered vertices of R_a. So the number of times (a, b) can flip in this process is bounded by $|R_a|$. (This argument can be found in [13, p. 12].)

Since for every vertex a, $|R_a| \leq d^{2K+1}$, where d is an upper bound on the degree in $\mathcal{G}$, there is a uniform bound on the number of times an edge can flip, say N.

It suffices to show that for $i \in \{0, \ldots, C-1\}$ there is k_i such that for every $m \geq k_i$, $c^{-1}(i)$ contains no augmenting paths of length $\leq 2K+1$ for M_{i+mC} outside of a definable subset of $\mathcal{P}$ of ν-measure $< \delta/(2C)$.

Let $AP_{i+mC} \subseteq \mathcal{P}$ be the set of augmenting paths for M_{i+mC} in $c^{-1}(i)$, and assume to the contrary that there are arbitrarily large k such that $\nu(AP_{i+kC}) \geq \epsilon$. To create M_{i+kC+1}, the paths in AP_{i+mC} are flipped, but because each edge flips at most N times, this can only happen at most $\frac{\nu(c^{-1}(i))}{\epsilon} \cdot N$ times, a contradiction. $\blacksquare_{\text{Claim}}$

Set $k := \max\{k_0, \ldots, k_{C-1}\}$. Then the set of augmenting paths for M_k of length $\leq 2K+1$ has measure $< \frac{\delta}{2} + C \cdot \frac{\delta}{2C} = \delta$. $\blacksquare$

3. Reduction to the archimedean case. In this section, we drop the assumption that the underlying set of R is $\mathbb{R}$. Instead, we assume that R is $(2^{\aleph_0})^+$-saturated. We shall use the following definitions and lemmas from [10, 11].

By a $\mathbb{Q}$-*ball* in R^n we mean an open ball with rational radius, i.e., a ball of the form

$$B_r(x) = \{y \in R^n \colon d(x, y) < r,\, x \in R^n,\, r \in \mathbb{Q}^{>0}\}.$$

The lemma below is Lemma 4.1 in [10, p. 183] (here, we prefer to state it in terms of $\mathbb{Q}$-balls rather than $\mathbb{Q}$-boxes).

Lemma 3.1 ([10]). *Suppose $X \subseteq R^n$ is definable and* $\dim(\operatorname{st} X) = n$. *Then X contains a $\mathbb{Q}$-ball.*

The following definition [10, Definition 3.1, p. 179] and theorem (a slightly weaker version of [10, Corollary 6.2, p. 191]) will be crucial. For $Y \subseteq \mathbb{R}^n$, we set $Y^h := \operatorname{st}^{-1}(Y)$.

Definition 3.2 ([10]). Given functions $f \colon X \to R$, $X \subseteq R^n$, and $F \colon Y \to \mathbb{R}$ with $Y \subseteq \mathbb{R}^n$, we say that f *induces* F if f is definable (so X is definable), $Y^h \subseteq X$, $f|Y^h$ is continuous, $f(Y^h) \subseteq \mathcal{O}$ and $\Gamma F = \operatorname{st}(\Gamma f) \cap (Y \times \mathbb{R})$.

For $f \colon X \to R^n$ and $F \colon Y \to \mathbb{R}^n$, where $X \subseteq R^n$ and $Y \subseteq \mathbb{R}^n$, we say that f induces F if the coordinate functions of f induce the corresponding coordinate functions of F.

Theorem 3.3 ([10]). *If $f \colon Y \to R$, where $Y \subseteq R^n$ and $\Gamma f \subseteq \mathcal{O}^{n+1}$ is definable, then there is a decomposition $\mathcal{C}$ of $\mathbb{R}^n$ into cells that partitions*

$\operatorname{st} Y$ and is such that if $C \in \mathcal{C}$ is open and $C \subseteq \operatorname{st} Y$, then f is continuously differentiable on an open $X \subseteq Y$ containing $\operatorname{st}^{-1}(C)$ and $f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$, as functions on X, induce functions $g, g_1, \dots, g_n \colon C \to \mathbb{R}$ such that g is C^1 and $g_i = \frac{\partial g}{\partial x_i}$ for each i.

Part (1) of the next proposition is [11, Proposition 3.1, p. 126] and part (2) is extracted from the proof of [11, Lemma 2.15, p. 124]. For a definable, nonempty $Z \subseteq \mathcal{O}^n$ and $\epsilon \in R^{\geq 0}$ we set

$$Z^\epsilon = \{x \in R^n \colon d(x, Z) \leq \epsilon\}.$$

Proposition 3.4 ([11]).

(1) *If $C \in \operatorname{Def}^n(\mathbb{R}_{\text{ind}})$ is closed, then there is $Z \in \operatorname{Def}^n(R)$ such that $\operatorname{st} Z = C$.*
(2) *If $X, Y \in \operatorname{Def}^n(R)$ and $X, Y \subseteq \mathcal{O}^n$, then there is $\epsilon > 0$ such that*

$$\operatorname{st} X \cap \operatorname{st} Y = \operatorname{st}(X \cap Y^\epsilon).$$

Here is a useful consequence of Proposition 3.4.

Lemma 3.5. *Let $X, Y \subseteq \mathcal{O}^n$ be definable. If $\dim(\operatorname{st} X \cap \operatorname{st} Y) = n$, then $\dim(\operatorname{st}(X \cap Y)) = n$.*

Proof. By (2) in Proposition 3.4, there is $\rho \in \mathfrak{m}^{>0}$ such that

$$\operatorname{st} X \cap \operatorname{st} Y = \operatorname{st}(X \cap Y^\rho).$$

Now $Y^\rho = (\partial Y)^\rho \cup Y$, where $\partial Y = \operatorname{cl}(Y) \setminus \operatorname{int}(Y)$, so

$$\operatorname{st}(X \cap Y^\rho) = \operatorname{st}((X \cap (\partial Y)^\rho) \cup (X \cap Y)) = \operatorname{st}((X \cap (\partial Y)^\rho) \cup \operatorname{st}(X \cap Y).$$

But since $\operatorname{st}(\partial Y)^\rho = \operatorname{st} \partial Y$ and $\dim \operatorname{st} \partial Y < n$, $\dim \operatorname{st}(X \cap Y) = n$. ∎

Below, ν-measure is defined just as in the case when the underlying set of the structure is $\mathbb{R}$, except using μ rather than λ.

Theorem 3.6. *Let $\mathcal{G} = (A \dot\cup B, G)$ be a definable bipartite, μ-preserving graph, d an upper bound on the degrees of its vertices, and $M \subseteq G$ a definable matching. Further, let $K \in \mathbb{N}$, $\delta \in \mathbb{R}^{>0}$, and $\epsilon \in \mathbb{R}^{>0}$ subject to $\epsilon < \delta/d^{2K+2}$. Then there is a definable matching $X \subseteq G$ such that X covers all vertices covered by M outside of a definable set of μ-measure $< \epsilon$ and X has no augmenting paths of length $\leq 2K+1$ outside of a set of ν-measure $< \delta$.*

Proof. We may assume that $\dim(\operatorname{st} A) = n = \dim(\operatorname{st} B)$. Let $\mathcal{D}$ be a decomposition of R^{2n} into cells which partitions G and M. Set

$$\mathcal{D}_0 = \{\pi^{2n}_n D \colon D \in \mathcal{D} \,\&\, D \subseteq G \,\&\, \dim \pi^{2n}_n D = n\}.$$

We may assume that if $D \in \mathcal{D}_0$, then $G \cap (D \times R^n)$ is the union of finitely many cells in $\mathcal{D}$ of the form $\mathit{\Gamma} f$, where each f is an n-isomorphism, and we denote the collection of these n-isomorphisms on D by $\mathcal{F}_D$.

Set

$$G_1 = G \cap \bigcup_{D \in \mathcal{D}_0} (D \times R^n) \quad \text{and} \quad M_1 = M \cap \bigcup_{D \in \mathcal{D}_0} (D \times R^n),$$

and let $\mathcal{C}$ be a decomposition of $\mathbb{R}^{2n}$ into cells partitioning $\operatorname{st} G_1$ and $\operatorname{st} M_1$ such that $\{\pi^{2n}_n C \colon C \in \mathcal{C}\}$ partitions each $\operatorname{st} D$ where $D \in \mathcal{D}_0$. Let

$$\mathcal{C}_0 = \{\pi^{2n}_n C \colon C \in \mathcal{C} \,\&\, C \subseteq \operatorname{st} G_1 \,\&\, \dim C = n\}.$$

Suppose $D \in \mathcal{D}_0$ and $f \in \mathcal{F}_D$, and let $C \in \mathcal{C}_0$ be such that $C \subseteq \operatorname{st} D$. Then by Theorem 3.3, we may assume that f induces an n-isomorphism $g \colon C \to g(C)$. For $C \in \mathcal{C}_0$, let $\mathcal{F}_C$ be the set of all $g \colon C \to \mathbb{R}^n$ that are induced by some $f \in \mathcal{F}_D$, where $D \in \mathcal{D}_0$ and $C \subseteq \operatorname{st} D$.

We set

$$G' = \operatorname{st} G_1 \cap \bigcup_{C \in \mathcal{C}_0} (C \times \mathbb{R}^n).$$

Then G' is the edge relation of the $\mathbb{R}_{\text{ind}}$-definable, λ-preserving, bipartite graph $\mathcal{G}'$ with bipartition A', B', where $A' = \pi^{2n}_n G'$ and B' is the projection of G' onto the last n coordinates.

Claim. *The relation $M' = \operatorname{st} M_1 \cap G'$ is a definable matching in G'.*

Proof of Claim. To see that M' is the graph of a function, assume towards a contradiction that $(X, Y), (X, Y') \in M'$ and $Y \neq Y'$. Then $X \in C$ for some $C \in \mathcal{C}_0$ and $C \subseteq \operatorname{st} D$ for some $D \in \mathcal{D}$. Since C is open, $\operatorname{st}^{-1}(X) \subseteq D$, and there are $x_1, x_2 \in D$ such that $\operatorname{st} x_1 = \operatorname{st} x_2 = X$, and $\operatorname{st} f(x_1) = Y$ and $\operatorname{st} f(x_2) = Y'$ for the unique $f \in \mathcal{F}_D$ with $\Gamma f \subseteq M_1$, contradicting that f induces a function $C \to \mathbb{R}^n$.

Suppose now that $(X, Y), (X', Y) \in M'$ with $X \neq X'$. If there is $C \in \mathcal{C}_0$ such that $X, X' \in C$, then $\operatorname{st}^{-1} X, \operatorname{st}^{-1} X' \subseteq D$ for some $D \in \mathcal{D}$, and for $f \in \mathcal{F}_D$ with $\Gamma f \subseteq M_1$, we have $\operatorname{st} f(x) = \operatorname{st} f(x')$ for some $x \in \operatorname{st}^{-1} X$, $x' \in \operatorname{st}^{-1} X'$, contradicting that f induces an isomorphism $C \to \mathbb{R}^n$.

So let $X \in C_1$, $X' \in C_2$, where $C_1, C_2 \in \mathcal{C}_0$, $C_1 \neq C_2$. Assume further that $F \colon C_1 \to \mathbb{R}^n$ and $G \colon C_2 \to \mathbb{R}^n$ are induced by $f \in \mathcal{F}_{D_1}$ and $g \in \mathcal{F}_{D_2}$, respectively, where $\Gamma f, \Gamma g \subseteq M_1$ and $F(X) = Y = G(X')$. Then there is $\delta > 0$ such that $B_\delta(X) \subseteq C_1$ and $B_\delta(X') \subseteq C_2$ and, since F, G are homeomorphisms, $F(B_\delta(X))$, $G(B_\delta(X'))$ are open subsets of $\mathbb{R}^n$.

Since $\operatorname{st}^{-1}(B_\delta(X)) \subseteq D_1$ and $\operatorname{st}^{-1}(B_\delta(X')) \subseteq D_2$, we have $B_{\delta/2}(x) \subseteq D_1$ and $B_{\delta/2}(x') \subseteq D_2$, where x, x' are such that $\operatorname{st} x = X$ and $\operatorname{st} x' = X'$. So $F(B_{\delta/2}(X)) \subseteq \operatorname{st} f(B_{\delta/2}(x))$ and $G(B_{\delta/2}(X')) \subseteq \operatorname{st} g(B_{\delta/2}(x'))$. Since

$$Y \in F(B_{\delta/2}(X)) \cap G(B_{\delta/2}(X')),$$

there is $\epsilon > 0$ such that

$$B_\epsilon(Y) \subseteq F(B_{\delta/2}(X)) \cap G(B_{\delta/2}(X')).$$

Hence $B_\epsilon(Y) \subseteq \operatorname{st} f(B_{\delta/2}(x)) \cap \operatorname{st} g(B_{\delta/2}(x'))$. But then, by Lemma 3.5,

$$\dim\bigl(\operatorname{st}(f(B_{\delta/2}(x)) \cap g(B_{\delta/2}(x')))\bigr) = n,$$

so by Lemma 3.1, $f(B_{\delta/2}(x)) \cap g(B_{\delta/2}(x'))$ contains a $\mathbb{Q}$-box, contradicting

$$f(B_{\delta/2}(x)) \cap g(B_{\delta/2}(x')) = \emptyset. \quad \blacksquare_{\text{Claim}}$$

By Proposition 2.6, we can find an $\mathbb{R}_{\text{ind}}$-definable matching $M'' \subseteq G'$ such that all augmenting paths outside of a definable $P \subseteq \mathcal{P}$ of ν-measure $< \delta/2$ are of length $> 2K+1$.

Let $\mathcal{C}'$ be a decomposition of $\mathbb{R}^{2n}$ into cells which is a refinement of $\mathcal{C}$ and partitions M'', and let $\mathcal{C}'_0$ consist of the cells $\pi^{2n}_n C$ of dimension n such that $C \in \mathcal{C}'$ and $C \subseteq M''$. Find $\alpha \in \mathbb{R}^{>0}$ such that

$$\sum_{C \in \mathcal{C}'_0} \lambda(C \setminus C_\alpha) < \frac{\delta}{4 \cdot d^{2K+1}} < \frac{\epsilon}{2},$$

where

$$C_\alpha = \{x \in C \colon d(\partial C, x) \geq \alpha\}.$$

By 3.4, we can find for each $C \in \mathcal{C}'_0$ and $D \in \mathcal{D}_0$ with $C \subseteq \operatorname{st} D$ a definable $D_C \subseteq D$ such that $\operatorname{st} D_C = \operatorname{cl} C_\alpha$. Note that

$$M''' := M'' \cap \bigcup_{C \in \mathcal{C}'_0} (C_\alpha \times \mathbb{R}^n)$$

covers the same vertices as M'' outside of a set of measure $< \epsilon$, hence M' does, too. Moreover, M''' has no augmenting paths of length $\leq 2K+1$ outside of a subset of $\mathcal{P}$ of ν-measure less than

$$d^{2K+1} \cdot \frac{\delta}{2 \cdot d^{2K+1}} + \frac{\delta}{2} = \delta.$$

We now define the desired matching $X \subseteq G$ as a subset of

$$\bigcup_{C \in \mathcal{C}'_0} (D_C \times R^n) \cap G.$$

For each $C \in \mathcal{C}'_0$ and D and D_C as above, let f_{D_C} be the restriction to D_C of the first function in $\mathcal{F}_D$ which induces the function with graph $M'' \cap (C \times \mathbb{R}^n)$. Then

$$X = \bigcup_{C \in \mathcal{C}'_0} \Gamma f_{D_C}.$$

It remains to check that X satisfies the desired properties.

- $X \subseteq G$ *is a matching*: The only way X can fail to be a matching is if there are $x_1 \in D_{C_1}$ and $x_2 \in D_{C_2}$ with $C_1 \neq C_2$ and $(x_1, y), (x_2, y) \in X$. But then $(\operatorname{st} x_1, \operatorname{st} y), (\operatorname{st} x_2, \operatorname{st} y) \in M''$, so $\operatorname{st} x_1 = \operatorname{st} x_2$, a contradiction with $d(D_{C_1}, D_{C_2}) > \mathfrak{m}$.

- *The set of augmenting paths for X of length $\leq 2K+1$ has measure less than δ*: Let P be a set of augmenting paths for X of length $\leq 2K+1$ and generating sequence s. Then we may identify P with the set of starting vertices of the paths in P. Assume that $\mu(P_0) = \rho > 0$. It suffices to show that then P induces a set of augmenting paths for M''' of length $\leq 2K+1$ and a set of starting vertices of λ-measure $\geq \rho$. Note that we may assume that P is a set of paths in G_1. But then it follows straightforwardly from the definitions of G_1, G', X, M''' and from Lemma 3.1 that μ-a.e. on P, if $p = (p_0, \ldots, p_l) \in P$, then each $\operatorname{st} p_i$ is a vertex of G'; $\operatorname{st} p_0$, $\operatorname{st} p_l$ are not covered by M'''; $(\operatorname{st} p_i, \operatorname{st} p_{i+1}) \in G'$; and $(p_i, p_{i+1}) \in X$ if and only if $(\operatorname{st} p_i, \operatorname{st} p_{i+1}) \in M'''$.
- *X covers the vertices covered by M outside of a set of μ-measure $< \epsilon$*: Let $V \subseteq A \dot\cup B$ be definable and covered by M but not by X, and $\mu(V) \geq \epsilon$. Then V is μ-a.e. covered by M_1, and hence $\operatorname{st} V$ is covered by M' outside of a set of λ-measure 0. Since $\lambda(\operatorname{st} V) > \epsilon$, this yields a contradiction with $\operatorname{st} X = \operatorname{cl}(M''')$ and M''' covering the same vertices as M' outside of a set of measure $< \epsilon$. ■

4. Matchings in 2-regular bipartite graphs. Let R be as in Section 3. Unless indicated otherwise, we assume that $\mathcal{G} = (A \dot\cup B, G)$ is a definable bipartite graph which is μ-preserving and such that $1 \leq \deg x \leq 2$ for all $x \in A \dot\cup B$. We show that if

$$\mu\{x \in A \dot\cup B \colon \deg x < 2\} < \delta \in \mathbb{R}^{\geq 0},$$

then there is a definable matching in $\mathcal{G}$ covering all vertices outside of a set of arbitrarily small positive μ-measure $> \delta$.

We first consider the case in which we are given a definable matching without any short augmenting paths: Let K be an even integer and let $M \subseteq G$ be a definable matching without augmenting paths of length $\leq 2K+1$. Let $Y_0 \subseteq A \dot\cup B$ consist of the vertices not covered by M. For $Z_0 \subseteq Y_0$ and $k = 0, 1, \ldots, (K-2)/2$, we set $Z_{2k+1} := N_G(Z_{2k})$ and $Z_{2k+2} := N_M(Z_{2k+1})$.

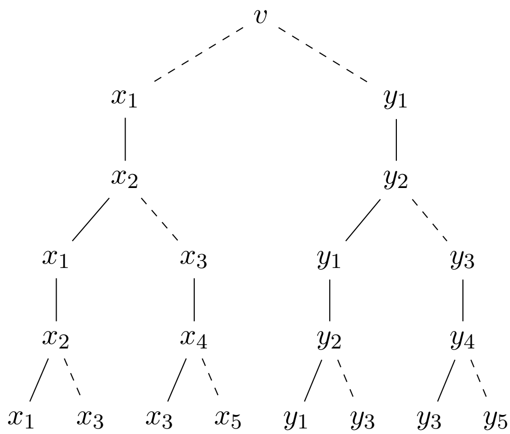
LEMMA 4.1. *Let $Z_0 \subseteq Y_0$ be definable and such that $\bigcup_{i=0}^{K-1} Z_i$ does not contain any vertices of degree < 2. Then $\mu Z_K = K \cdot \mu Z_0$.*

Proof. We sketch the proof; the details can be easily filled in by the reader, using induction and the absence of short augmenting paths.

Let $v \in Z_0$. We denote by T_v the following tree of depth $K-1$ rooted in v. From now on, we shall assume that $l \in \{0, 1, \ldots, (K-2)/2\}$. If x is a vertex of T_v at depth $2l$, then x has two children, labeled by the two vertices incident to x in $\mathcal{G}$. If x is at depth $2l+1$, then x has one child, labeled by the vertex incident with x via an edge in M (whose existence follows from

M not having any augmenting paths of length $\leq 2K+1$ and $\mathcal{G}$ not having any odd cycles). For simplicity, we assume that for $l > 0$, the left child of x at depth $2l$ is matched and the right child is unmatched. Furthermore, if x is a vertex of T_v, then we denote by T_x the maximal subtree of T_v rooted in x. We denote by X_l the set of labels of vertices in T_v at depth l, and by depth we shall always mean depth with respect to T_v (even when talking about a subtree). We write $d(x)$ for the depth of a vertex x. Note that we have $|X_{2l+1}| = |X_{2l+2}|$.

Let x_1 be the left and y_1 the right child of v – see the picture below.



Claim 1. *The set of labels of T_{x_1} and the set of labels of T_{y_1} are disjoint.*

Proof. Suppose $x_i = y_j$, where $d(x_i) \leq d(y_j)$, and there is no label in T_{x_1} appearing in T_{y_1} and being the label of a vertex at depth $\leq d(x_i)$. Because $\mathcal{G}$ has no odd cycles, both $d(x_i)$ and $d(y_j)$ are even or both are odd. We may assume that both are odd. Now x_i, y_j are each either a matched or an unmatched child of its respective parent, but any possible combination leads to a contradiction with the minimality of $d(x_i)$.

Claim 2. *Suppose $l > 0$. Then $|X_{2l}| = 2l$.*

Proof. Observe that in T_{x_1} and in T_{y_1} respectively, after identifying vertices with same labels at each depth (so a label can only repeat at different depths), at depth $2l+1$ there are exactly two vertices with indegree 1 (so all other vertices have indegree 2).

The lemma now follows from the next claim.

Claim 3. *Let $v, v' \in Z_0$, $v \neq v'$. Then the sets of labels at depth l of the trees T_v and $T_{v'}$ are disjoint.* ■

Theorem 4.2. *For every $\epsilon \in \mathbb{R}^{>\delta}$, $\mathcal{G}$ admits a definable matching covering all vertices outside of a set of measure $< \epsilon$.*

Proof. Let $\epsilon \in \mathbb{R}^{>\delta}$, and let $K \in \mathbb{N}^{>0}$ be such that $1/K < (\epsilon - \delta)/2$. Let M be a definable matching in $\mathcal{G}$ whose set of augmenting paths of length $\leq 2K + 1$ has measure less than $(\epsilon - \delta)/2$. Let Y_0 be the set of vertices not covered by M, and for $v \in Y_0$ let T_v be the tree of depth K with root v and such that if x is a vertex at even depth, then its children are the vertices incident with it in $\mathcal{G}$, and if x is at odd depth, then its child is the vertex incident with it via M, if there is one. Else x is a leaf.

CLAIM. *For distinct $v, w \in Y_0$, T_v and T_w have disjoint sets of vertices of degree* 1.

Proof. In the construction of T_v and T_w, vertices appear for the first time along a possible initial segment of an augmenting path starting in v. Suppose $i \geq 0$ is minimal such that x_i is a vertex of T_v of degree 1, appearing for the first time at depth i, and for some $j \geq i$ there is a vertex y_j of T_w, appearing for the first time at depth j, with $x_i = y_j$. If $i = 0$, then $j > 0$ and T_w contains an augmenting path of length $\leq K$. So suppose $i > 0$. Then $x_{i-1} = y_{j-1}$, contradicting the minimality of i. ∎

Let $Y' \subseteq Y_0$ be the set of roots of trees containing a vertex of degree 1. By the above claim, $\mu Y' < \delta$. Let $Y'' \subseteq Y_0$ be the set of starting vertices of augmenting paths of length $\leq 2K + 1$, so $\mu Y'' < (\epsilon - \delta)/2$. We set

$$Z_0 = \mu(Y \setminus (Y' \cup Y'')).$$

Then by Lemma 4.1, $\mu Z_0 \leq 1/K$, so $\mu Y_0 \leq 1/K + \delta + (\epsilon - \delta)/2 < \epsilon$. ∎

Here is a special case of the theorem above.

COROLLARY 4.3. *Let $\mathcal{G} = (A \dot\cup B, G)$ be a definable bipartite graph which is μ-preserving and 2-regular. Then for every $\epsilon \in \mathbb{R}^{>0}$ there is a definable matching $M \subseteq G$ that covers all vertices outside of a set of measure $< \epsilon$.*

5. A cancellation result. Let R be a sufficiently saturated expansion of a real closed field. We define an equivalence relation $\sim$ on $B[n]$ as follows.

DEFINITION 5.1. Let $X, Y \in B[n]$. Then $X \sim Y$ iff there are definable open partitions of $\{X\}_{1=1}^k$, $\{Y_i\}_{i=1}^k$ of X and Y respectively, and there are n-isomorphisms $f_1, \dots, f_k$ such that $Y_i = f(X_i)$ for each i.

We let $\mathcal{B}$ be the semigroup $(B[n]/\sim, +)$, where the binary operation $+$ is given by $a + b = c$, with c the equivalence class of the disjoint union of A and B, where the equivalence classes of A and B are a and b, respectively.

The proof of the next theorem is based on the proof of cancellation from Tomkowicz and Wagon [12, p. 177]. While [12] uses the Hall–Rado Infinite Marriage Theorem, we only have Corollary 4.3 at our disposal.

THEOREM 5.2. *Let $\alpha, \beta \in \mathcal{B}$ have strongly bounded representatives and suppose $\alpha + \alpha =_a \beta + \beta$ in $\mathcal{B}$. Then $\alpha =_a \beta$.*

Proof. Let $\epsilon \in \mathbb{R}^{>0}$, and let A, A' and B, B' be two pairs of disjoint copies of strongly bounded representatives of α and β, respectively. Let $\phi\colon A \to A'$, $\psi\colon B \to B'$ and $\theta\colon A \dot\cup A' \to B \dot\cup B'$ witness $A \sim A'$, $B \sim B'$ and $A \dot\cup A' \sim B \dot\cup B'$, respectively.

We define a bipartite graph $\mathcal{H}$ as follows. The bipartition consists of the two sets

$$\bar{A} = \{(a, \phi(a))\colon a \in A)\} \quad \text{and} \quad \bar{B} = \{(b, \psi(b))\colon b \in B\},$$

and we let $(a, \phi(a))$ be incident with $(b, \psi(b))$ iff $\theta(a) = b$ or $\theta(\phi(a)) = b$ or $\theta(a) = \psi(b)$ or $\theta(\phi(a)) = \psi(b)$. Then $\mathcal{H} = (\bar{A} \cup \bar{B}, H)$ is definable bipartite and μ-preserving, and every vertex is of degree ≤ 2. To construct a map witnessing $\alpha =_\epsilon \beta$, it will suffice to find a definable matching $M \subseteq H$ covering $\bar{A} \cup \bar{B}$ outside of a set of μ-measure $< \epsilon$.

Let $\bar{A}_0$ and $\bar{B}_0$ be the subsets of vertices of degree 0 of $\bar{A}$ and $\bar{B}$ respectively. Note that $\mu(\bar{A}_0 \cup \bar{B}_0) < \epsilon/2$, and replace $\bar{A}$ with $\bar{A} \setminus \bar{A}_0$ and $\bar{B}$ with $\bar{B} \setminus \bar{B}_0$ in $\mathcal{H}$.

Let $\bar{A}_1 \subseteq \bar{A}$ and $\bar{B}_1 \subseteq \bar{B}$ be the sets of vertices of degree 1 that are incident with another vertex of degree 1. Then $H \cap (\bar{A}_1 \times \bar{B}_1)$ is the graph of a bijection $\bar{A}_1 \to \bar{B}_1$, so there is no harm in replacing $\mathcal{H}$ with the graph

$$((\bar{A} \setminus \bar{A}_1) \dot\cup (\bar{B} \setminus \bar{B}_1)), H \cap ((\bar{A} \setminus \bar{A}_1) \times (\bar{B} \setminus \bar{B}_1)).$$

Then the remaining vertices of degree 1 in $\bar{A}$ are $(a, \phi(a))$ such that $\theta(a)$ is undefined and $\theta(\phi(a))$ is defined, or vice versa, and similarly for the vertices of degree 1 in $\bar{B}$. So $\mathcal{H}$ is 2-regular outside of a set of measure $< \epsilon/2$.

By Theorem 4.2, H contains a definable matching M covering all vertices outside of a set of μ-measure $< \epsilon/2$, hence covering all vertices of the original $\mathcal{H}$ outside of a set of μ-measure $< \epsilon$. ∎

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References

[1] Y. Baisalov et B. Poizat, *Paires de structures O-minimales*, J. Symbolic Logic 63 (1998), 570–578.

[2] R. Diestel, *Graph Theory*, Grad. Texts in Math. 173, Springer, Berlin, 2017.

[3] L. van den Dries, *Tame Topology and o-minimal Structures*, London Math. Soc. Lecture Note Ser. 248, Cambridge Univ. Press, 1998.

[4] G. Elek and G. Lippner, *Borel oracles. An analytical approach to constant-time algorithms*, Proc. Amer. Math. Soc. 138 (2010), 2939–2947.

[5] M. Hall Jr., *Distinct representatives of subsets*, Bull. Amer. Math. Soc. 54 (1948), 922–926.

[6] E. Hrushovski, Y. Peterzil and A. Pillay, *Groups, measures and the NIP*, J. Amer. Math. Soc. 21 (2008), 563–596.

[7] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. 77 (1916), 453–465.
[8] M. Laczkovich, *Closed sets without measurable matching*, Proc. Amer. Math. Soc. 103 (1988), 894–896.
[9] R. Lyons and F. Nazarov, *Perfect matchings as IID factors of non-amenable groups*, Eur. J. Combin. 32 (2011), 1115–1125.
[10] J. Maříková, *The structure on the real field generated by the standard part map on an o-minimal expansion of a real closed field*, Israel J. Math. 171 (2009), 175–195.
[11] J. Maříková, *o-minimal fields with standard part map*, Fund. Math. 209 (2010), 115–132.
[12] G. Tomkowicz and S. Wagon, *The Banach–Tarski Paradox*, 2nd ed., Encyclopedia Math. Appl. 163, Cambridge Univ. Press, 2016.
[13] A. Y. Wang, *Borel matchings and analogs of Hall's theorem*, Senior thesis (Major), California Institute of Technology, 2020.

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