Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity

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(joint work with Lukas Riegler)

Consider the following rational function P

$$\prod_{1 \le i < j \le n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j^{-1}}$$

and let R denote the function we obtain after symmetrizing it, that is $R = \mathbf{Sym}\, P$ with $\mathbf{Sym}\, f(z_1, \ldots, z_n) = \sum_{\sigma \in \mathcal{S}_n} f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Since $P(z_1, \ldots, z_n) = P(z_n^{-1}, \ldots, z_1^{-1})$, it is obvious that $R(z_1, \ldots, z_n) = R(z_1^{-1}, \ldots, z_n^{-1})$, however, computer experiment suggest that also

$$R(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) = R(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_n).$$

This is the special case $s = 0$ of the following conjecture.

Conjecture 1 (Fischer, Riegler). *For integers $s, t \ge 0$, consider the following rational function $P_{s,t}$*

$$\prod_{i=1}^{s} z_i^{2s-2i-t+1} (1 - z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s-t} (1 - z_i^{-1})^{s} \prod_{1 \le p < q \le s+t-1} \frac{1 - z_p + z_p z_q}{z_q - z_p}$$

and let $R_{s,t} = \mathbf{Sym}\, P_{s,t}$. If $s \le t$ then

$$R_{s,t}(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_{s+t-1})$$

for all $i \in \{1, 2, \ldots, s+t-1\}$.

In the talk I first explained how we came up with this conjecture in an attempt to prove a conjecture on a refined enumeration of *vertically symmetric alternating sign matrices*. An *alternating sign matrix* is a quadratic $0, 1, -1$ matrix such that the non-zero entries alternate and sum up to 1 in each row and column. Next we give an example of such an object

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is in fact symmetric with respect to the vertically axis. Vertically symmetric alternating sign matrices have been enumerated by Kuperberg [3]. In [1], I presented the following conjecture on a refined enumeration of vertically symmetric alternating sign matrices.

Conjecture 2. *The number of* $(2n+1)\times(2n+1)$ *vertically symmetric alternating sign matrices where the first* 1 *in the second row is in column* i *is*

$$\frac{\binom{2n+i-2}{2n-1}\binom{4n-i-1}{2n-1}}{\binom{4n-2}{2n-1}}\prod_{j=1}^{n-1}\frac{(3j-1)(2j-1)!(6j-3)!}{(4j-2)!(4j-1)!}.$$

In [2], this was shown that a consequence of Conjecture 1 implies Conjecture 2.

Theorem 1. *If* $R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1})$ *for all* $1 \le s \le t$ *then Conjecture 2 is true.*

In the talk, I have also sketched the proof of the following partial result towards proving Conjecture 1:

Theorem 2. *Suppose*

$$R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1}) \tag{1}$$

if $t = s$ *and* $t = s+1$, $s \ge 1$. *Then* (1) *holds for all* s,t *with* $1 \le s \le t$.

Coming back to the special case mentioned in the beginning: another result we have obtained is the following.

Theorem 3. *The coefficient of* z^i *in* $R(z,1,\ldots,1)$ *is the number of* $(2n+1)\times(2n+1)$ *vertically symmetric alternating sign matrices where the unique* 1 *in the first column is in row* $n+i+1$.

Conjecture 1 implies $R(z,1,\ldots,1) = R(z^{-1},1,\ldots,1)$, which has from the point of view of Theorem 3 the explanation that reflecting a $(2n+1)\times(2n+1)$ vertically symmetric alternating sign matrix $A = (a_{i,j})$ with $a_{n+i+1,1} = 1$ along the vertically axis transforms it into a matrix with $a_{n+1-i,1} = 1$. This makes it plausible that $R(z_1,\ldots,z_n)$ is a certain generating function of vertically symmetric alternating sign matrices, which, once the weight is identified, could also imply the fact that R is invariant under replacing z_i by z_i^{-1}.

References

[1] I. Fischer, *An operator formula for the number of halved monotone triangles with prescribed bottom row*, J. Combin. Theory Ser. A **116** (2009), 515 – 538.

[2] I. Fischer and L. Riegler, *Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity*, `arXiv:1403.0535`, preprint.

[3] G. Kuperberg, *Symmetry classes of alternating–sign matrices under one roof*, Ann. of Math. (2) **156** (2002), 835 – 866.