A new proof of the alternating sign matrix theorem

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Alternating sign matrices

Square matrices of 0s, 1s, -1s for which

- the sum of entries in each row and in each column is 1
- the non-zero entries of each row and of each column alternate in sign.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

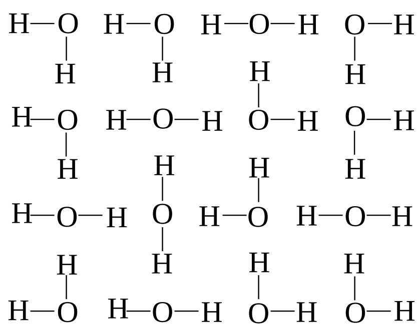
Proofs so far

- Doron Zeilberger 1996: using constant term identities he shows that the number of $n \times n$ alternating sign matrices equals the number of $2n \times 2n \times 2n$ totally symmetric self-complementary plane partitions. 84 pages!

- Greg Kuperberg 1996: using the equivalence of alternating sign matrices with the six-vertex model for "square ice".

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating Sign Matrix



Square Ice

ASMs $\Leftrightarrow$ Monotone triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles

are triangular arrays of integers of the form

$$\begin{array}{ccccccccc} & & & & a_{1,1} & & & & \\ & & & a_{2,1} & & a_{2,2} & & & \\ & & a_{3,1} & & a_{3,2} & & a_{3,3} & & \\ & a_{4,1} & & a_{4,2} & & a_{4,3} & & a_{4,4} & \\ a_{5,1} & & a_{5,2} & & a_{5,3} & & a_{5,4} & & a_{5,5} \end{array}$$

which are

- weakly increasing in northeast direction and in southeast direction and
- strictly increasing along rows.

Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices.

An operator formula for the number of monotone triangles

Theorem (F. 2004). The number of monotone triangles with n rows and bottom row $(k_1, k_2, \ldots, k_n)$ is

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_p} \Delta_{k_q}) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where $E_x p(x) = p(x+1)$ and $\Delta_x := E_x - \mathrm{id}$.

Sketch of proof of the operator formula

$\alpha(n; k_1, \ldots, k_n)$ = Number of monotone triangles with $(k_1, \ldots, k_n)$ as bottom row.

Recursion

Deletion of the last row $(k_1, \ldots, k_n)$ of a monotone triangle gives a monotone triangle with last row $(l_1, \ldots, l_{n-1})$ such that $k_1 \leq l_1 \leq k_2 \leq \ldots \leq k_{n-1} \leq l_{n-1} \leq k_n$ and $l_i \neq l_{i+1}$.

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{(l_1, \ldots, l_{n-1}), l_i \neq l_{i+1} \\ k_1 \leq l_1 \leq k_2 \leq \ldots \leq k_{n-1} \leq l_{n-1} \leq k_n}} \alpha(n-1; l_1, \ldots, l_{n-1})$$

$$=: \sum_{(l_1, \ldots, l_{n-1})}^{(k_1, \ldots, k_n)} \alpha(n-1; l_1, \ldots, l_{n-1})$$

$$\alpha(1; k_1) = 1$$

$$\alpha(2; k_1, k_2) = 1 - k_1 + k_2$$

$$\begin{aligned}\alpha(3; k_1, k_2, k_3) = \frac{1}{2}(&-3k_1+k_1^2+2k_1k_2-k_1^2k_2-2k_2^2+k_1k_2^2+3k_3-4k_1k_3\\ &+ k_1^2k_3 + 2k_2k_3 - k_2^2k_3 + k_3^2 - k_1k_3^2 + k_2k_3^2)\end{aligned}$$

$$\begin{aligned}\alpha(4; k_1, k_2, k_3, k_4) = \frac{1}{12}&(20k_2+11k_1k_2-16k_1^2k_2+3k_1^3k_2+4k_1k_2^2+3k_1^2k_2^2\\ &-k_1^3k_2^2+4k_2^3-5k_1k_2^3+k_1^2k_2^3-20k_3+16k_1k_3-4k_1^2k_3-27k_2k_3+9k_1^2k_2k_3\\ &-2k_1^3k_2k_3-3k_1^2k_2^2k_3+k_1^3k_2^2k_3-3k_2^3k_3+4k_1k_2^3k_3-k_1^2k_2^3k_3+16k_1k_3^2-12k_1^2k_3^2\\ &+ 2k_1^3k_3^2 - 9k_1k_2k_3^2 + 6k_1^2k_2k_3^2 - k_1^3k_2k_3^2 + 9k_2^2k_3^2 - 3k_1k_2^2k_3^2 - 3k_2^3k_3^2\\ &+ k_1k_2^3k_3^2 - 4k_3^3 + 8k_1k_3^3 - 2k_1^2k_3^3 - 3k_2k_3^3 - 2k_1k_2k_3^3 + k_1^2k_2k_3^3 + 3k_2^2k_3^3\\ &-k_1k_2^2k_3^3-27k_1k_4+20k_1^2k_4-3k_1^3k_4+16k_2k_4+24k_1k_2k_4-24k_1^2k_2k_4+4k_1^3k_2k_4\\ &-16k_2^2k_4+9k_1k_2^2k_4+3k_1^2k_2^2k_4-k_1^3k_2^2k_4+8k_2^3k_4-6k_1k_2^3k_4+k_1^2k_2^3k_4+11k_3k_4\\ &-24k_1k_3k_4+15k_1^2k_3k_4-2k_1^3k_3k_4-9k_2^2k_3k_4+2k_2^3k_3k_4-4k_3^2k_4+9k_1k_3^2k_4\\ &- 6k_1^2k_3^2k_4 + k_1^3k_3^2k_4 + 3k_2^2k_3^2k_4 - k_2^3k_3^2k_4 - 5k_3^3k_4 + 6k_1k_3^3k_4 - k_1^2k_3^3k_4\\ &- 4k_2k_3^3k_4 + k_2^2k_3^3k_4 - 20k_1k_4^2 + 9k_1^2k_4^2 - k_1^3k_4^2 + 4k_2k_4^2 + 15k_1k_2k_4^2\\ &-9k_1^2k_2k_4^2+k_1^3k_2k_4^2-12k_2^2k_4^2+6k_1k_2^2k_4^2+2k_2^3k_4^2-k_1k_2^3k_4^2+16k_3k_4^2-24k_1k_3k_4^2\\ &+9k_1^2k_3k_4^2-k_1^3k_3k_4^2+9k_2k_3k_4^2-6k_2^2k_3k_4^2+k_2^3k_3k_4^2+3k_3^2k_4^2-3k_1k_3^2k_4^2+3k_2k_3^2k_4^2\\ &-k_3^3k_4^2+k_1k_3^3k_4^2-k_2k_3^3k_4^2-3k_1k_4^3+k_1^2k_4^3+2k_1k_2k_4^3-k_1^2k_2k_4^3-2k_2^2k_4^3+k_1k_2^2k_4^3\\ &+3k_3k_4^3-4k_1k_3k_4^3+k_1^2k_3k_4^3+2k_2k_3k_4^3-k_2^2k_3k_4^3+k_3^2k_4^3-k_1k_3^2k_4^3+k_2k_3^2k_4^3)\end{aligned}$$

A surprising observation.

The degree of $\alpha(n; k_1, \dots, k_n)$ in k_i is $n-1$.

Why is this surprising ?

In general, the degree of

$$\sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{n+1})} a(l_1, \dots, l_n)$$

in k_i is

$$\deg_{l_{i-1}} a(l_1, \dots, l_n) + \deg_{l_i} a(l_1, \dots, l_n) + 2.$$

Thus, one would expect exponential growth of the degree rather than linear growth.

An Operator and ...

$$
\begin{aligned}
T_{x,y} &= (\mathrm{id} + E_y E_x^{-1} S_{x,y}) \\
&\times \frac{E_x^{-1}(\mathrm{id} + E_y \Delta_x)}{E_x^{-1}(\mathrm{id} + E_y \Delta_x) + E_y^{-1}(\mathrm{id} + E_x \Delta_y)}
\end{aligned}
$$

where $S_{x,y} a(x,y) = a(y,x)$.

... two Lemmas about it

Lemma. Let $a(l_1, \ldots, l_n)$ be a polynomial with $T_{l_i,l_{i+1}} a(l_1, \ldots, l_n) = 0$ for all i. Then

$$\deg_{k_i} \sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} a(l_1, \ldots, l_n) \leq \max_{j=i-1,i} \deg_{l_j} a(l_1, \ldots, l_n) + 1.$$

Lemma. Let $a(l_1, \ldots, l_n)$ be a polynomial with $T_{l_i,l_{i+1}} a(l_1, \ldots, l_n) = 0$ for all i. Then

$$T_{k_i,k_{i+1}} \left(\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} a(l_1, \ldots, l_n) \right) = 0$$

for all i.

Therefore...

- $\deg_{k_i} \alpha(n; k_1, \ldots, k_n) = n - 1$ for all i

- $T_{k_i, k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0$ for all i

These two properties determine the polynomial

$$(k_1, \ldots, k_n) \to \alpha(n; k_1, \ldots, k_n)$$

uniquely and imply (after some further lemmas) the operator formula.

How to use the formula to give another proof of the refined alternating sign matrix theorem

Theorem (Zeilberger 1996). The number $A_{n,k}$ of $n \times n$ alternating sign matrices where the unique 1 in the first row is in the k-th column is

$$\binom{n+k-2}{n-1}\binom{2n-k-1}{n-1}(n-1)!\prod_{j=1}^{n-1}\frac{(3j-2)!}{(n+j-1)!}.$$

We need to evaluate

$$\left(\prod_{1\le p<q\le n}(\mathrm{id}+E_{k_p}\Delta_{k_q})\right)\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}$$

at

$$(k_1,k_2,\dots,k_n)=(1,2,\dots,k-1,k+1,\dots,n+1).$$

Fundamental lemma. Let $P(X_1,\ldots,X_n)$ be a symmetric polynomial in $(X_1,\ldots,X_n)$ over $\mathbb{C}$. Then

$$P(E_{k_1},\ldots,E_{k_n})\,\alpha(n;k_1,\ldots,k_n) = P(1,\ldots,1)\,\alpha(n;k_1,\ldots,k_n).$$

Proof. Operator formula $\square$

A tricky application of the fundamental lemma to the elementary symmetric functions

$$e_p(X_1,\ldots,X_n) = \sum_{1\le i_1<i_2<\ldots<i_p\le n} X_{i_1}X_{i_2}\ldots X_{i_p}$$

gives the following

Corollary. Let $0\le j\le p\le n-1$. Then

$$e_{p-j}(E_{k_1},\ldots,E_{k_{n-1}})\,\alpha(n;k_1,\ldots,k_n)|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)}$$
$$=(-1)^j\sum_{i=1}^{n} A_{n,i}\left(\binom{n-i}{p}+\sum_{l=0}^{j-1}\binom{n}{p-l}\binom{i+l-1}{l}(-1)^{l-1}\right).$$

In this corollary, we set $p = j = k - n$ and obtain

$$\alpha(n; 1, \ldots, n-1, k) = \sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1}$$

for all k. Another lemma implies that

$$k \to \alpha(n; 1, 2, \ldots, n-1, k)$$

is symmetric if n is odd and antisymmetric if n is even.

Lemma.

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

Proof. Operator formula $\square$

Consequently,

$$\sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1} = (-1)^{n-1} \sum_{i=1}^{n} A_{n,i} \binom{i-k-n-1}{i-1}$$

for all k. This is equivalent to the fact that $(A_{n,1}, A_{n,2}, \ldots, A_{n,n})$ is an eigenvector of

$$\left((-1)^{j+1} \binom{2n-i-1}{n-i-j+1} \right)_{1 \le i,j \le n}$$

with respect to the eigenvalue 1. This completes the proof since the eigenspace is one-dimensional.