A bijective proof of the hook-length formula for shifted standard tableaux

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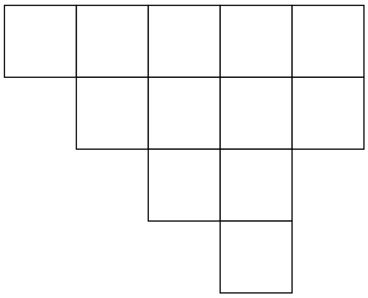
Shifted standard tableaux

Strict partition: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_r \geq 0$.

Shifted Ferrers diagram of shape $(5, 4, 2, 1)$.



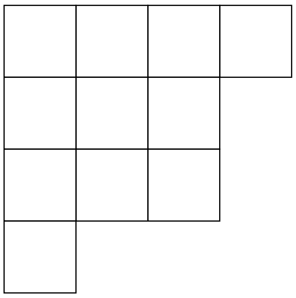
Shifted standard tableau of shape $(5, 4, 2, 1)$.

1	2	4	5	7
	3	6	9	12
		8	10	
			11	

Ordinary standard tableaux

Partition: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$.

Ferrers diagram of shape $(4, 3, 3, 1)$.



Standard (Young) tableau of shape $(4, 3, 3, 1)$.

1	3	7	11
2	4	9	
5	8	10	
6			

Roots of tableaux

Representation of a group G = Vector space V over $\mathbb{C}$ with a multiplication $g\,\mathbf{v}$ such that $g \to g\,\mathbf{v}$ is a group homomorphism from G into the invertible linear transformations from V to itself.

What are the representations of the symmetric group $\mathcal{S}_n$?

Irreducible representation: Representation without non-trivial subrepresentations.

Maschke's Decomposition Theorem: Every representation can be decomposed into irreducible representations.

Symmetric group $\mathcal{S}_n$:

- Number of irreducible representations = Number of partitions of n

- Dimension of the irreducible representation corresponding to the partition $\lambda =$ Number of tableaux of shape λ ($= f_\lambda$)

Well-known:

Degree formula $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

Has a proof in representation theory and a combinatorial proof (Robinson-Schensted correspondence).

Roots of shifted tableaux

Projective representation of G = Vector space V over $\mathbb{C}$ with is a multiplication $g\,\mathbf{v}$ such that $g \to g\,\mathbf{v}$ is a group homomorphism from G into the invertible projective linear transformations from V to itself.

- Number of irreducible projective representations of $\mathcal{S}_n =$
  $=$ Number of partitions of n
  $+$ Number of even strict partitions of n (i.e. $2|(n-r)$)
  $+$ 2$\cdot$ Number of odd strict partitions of n

- Dimension of an irreducible projective representation corresponding to a strict partition $\lambda =$ $2^{[(n-r)/2]}g_\lambda$, where $g_\lambda :=$ Number of shifted tableaux of shape λ

Degree formula $\sum_{\lambda \models n} 2^{n-r} g_\lambda^2 = n!$

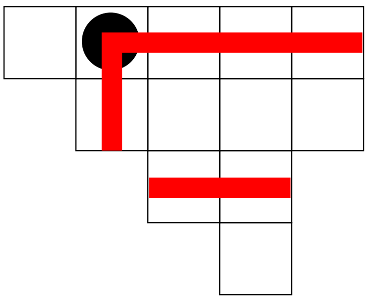
Hook-length formula for shifted tableaux

The number of shifted tableaux of shape λ is

$$\frac{n!}{\prod_{(i,j)} h_{i,j}},$$

where the product in the denominator is taken over all cells in the shifted Ferrers diagram of shape λ.

Hook:



Hook-lengths:

9	7	6	5	2
	6	5	4	1
		3	2	
			1	

$\Rightarrow$ The number of shifted tableaux of shape $(5, 4, 2, 1)$ is

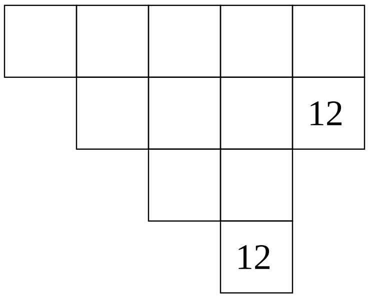
$$\frac{12!}{9 \cdot 7 \cdot 6 \cdot 5 \cdot 2 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 176.$$

Inductive proof of $g_\lambda = \frac{n!}{\prod_{(i,j)} h_{i,j}}$

- $\frac{n!}{\prod_{(i,j)} h_{i,j}} = \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_r}$

- $g_\lambda = \sum_{i=1}^{r} g_{\lambda - e_i}$

  Example:



$$g_{(5,4,2,1)} = g_{(5,3,2,1)} + g_{(5,4,2)}$$

- Remains to show

$$n = \sum_{i=1}^{r} \lambda_i \prod_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i + \lambda_j - 1} \frac{\lambda_i - \lambda_j - 1}{\lambda_i - \lambda_j}.$$

- For that purpose consider the expansion of

$$(2t - 1) \prod_{i=1}^{r} \frac{(t + \lambda_i)(t - \lambda_i - 1)}{(t + \lambda_i - 1)(t - \lambda_i)}$$

into partial fractions.

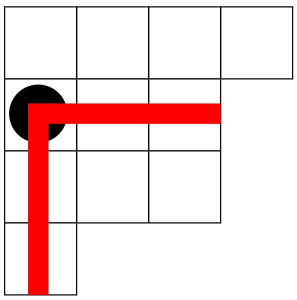
Hook-length formula for ordinary tableaux (Frame, Robinson and Thrall, 1953)

The number of tableaux of shape λ is

$$\frac{n!}{\prod_{(i,j)} h_{i,j}},$$

where the product in the denominator is taken over all cells in the Ferrers diagram of shape λ.

Hook:



Hook-lengths:

7	5	4	1
5	3	2	
4	2	1	
1			

$\Rightarrow$ The number of tableaux of shape $(4,3,3,1)$ is

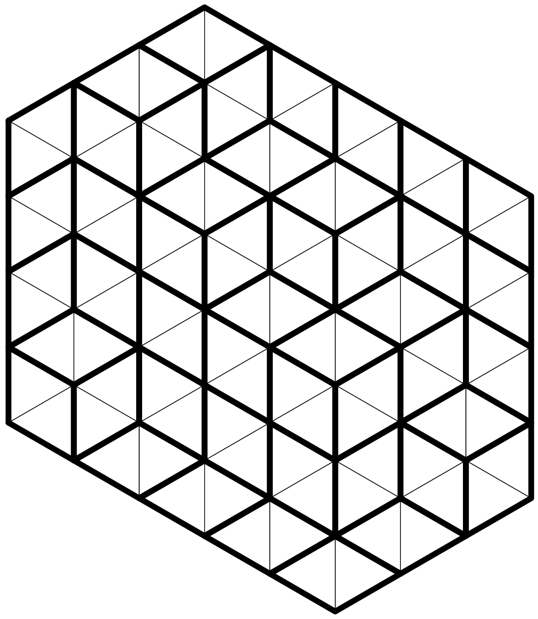
$$\frac{11!}{7\cdot 5\cdot 4\cdot 5\cdot 3\cdot 2\cdot 4\cdot 2} = 1188.$$

An enumeration result from my thesis

Theorem: Let a,b,c be positive integers, a, b odd and c even. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the rhombus in the center is

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)$$
$$\times\quad \frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-1}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\quad \sum_{k=0}^{(a-1)/2}\Bigg[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k$$
$$\times\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}$$
$$\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\Bigg],$$

where the Pochhammer symbol $(a)_k$ is defined by $(a)_k := \prod_{i=0}^{k-1}(a+i)$.



Nice result $\Rightarrow$ nice proof

Bijective proof of the hook-length formula for tableaux by Novelli, Pak and Stoyanovskii (1997)

$S_\lambda =$ set of tableaux of shape λ

$$n! = |S_\lambda| \cdot \prod_{(i,j)} h_{i,j}$$

$T_\lambda =$ set of tabloids of shape λ

10	5	8	3
2	11	4	
9	7	1	
6			

tabloid = filling of the Ferrers diagram with $1, 2, \ldots, n$; rows and columns need not be increasing.

$\Rightarrow |T_\lambda| = n!$

H_λ = set of hook tabloids of shape λ

(2,1)	(1,4)	(3,3)	(1,4)
(4,1)	(2,2)	(3,3)	
(3,2)	(3,2)	(3,3)	
(4,1)			

hook tabloid = filling of the Ferrers diagram with coordinates pointing to a cell in the hook

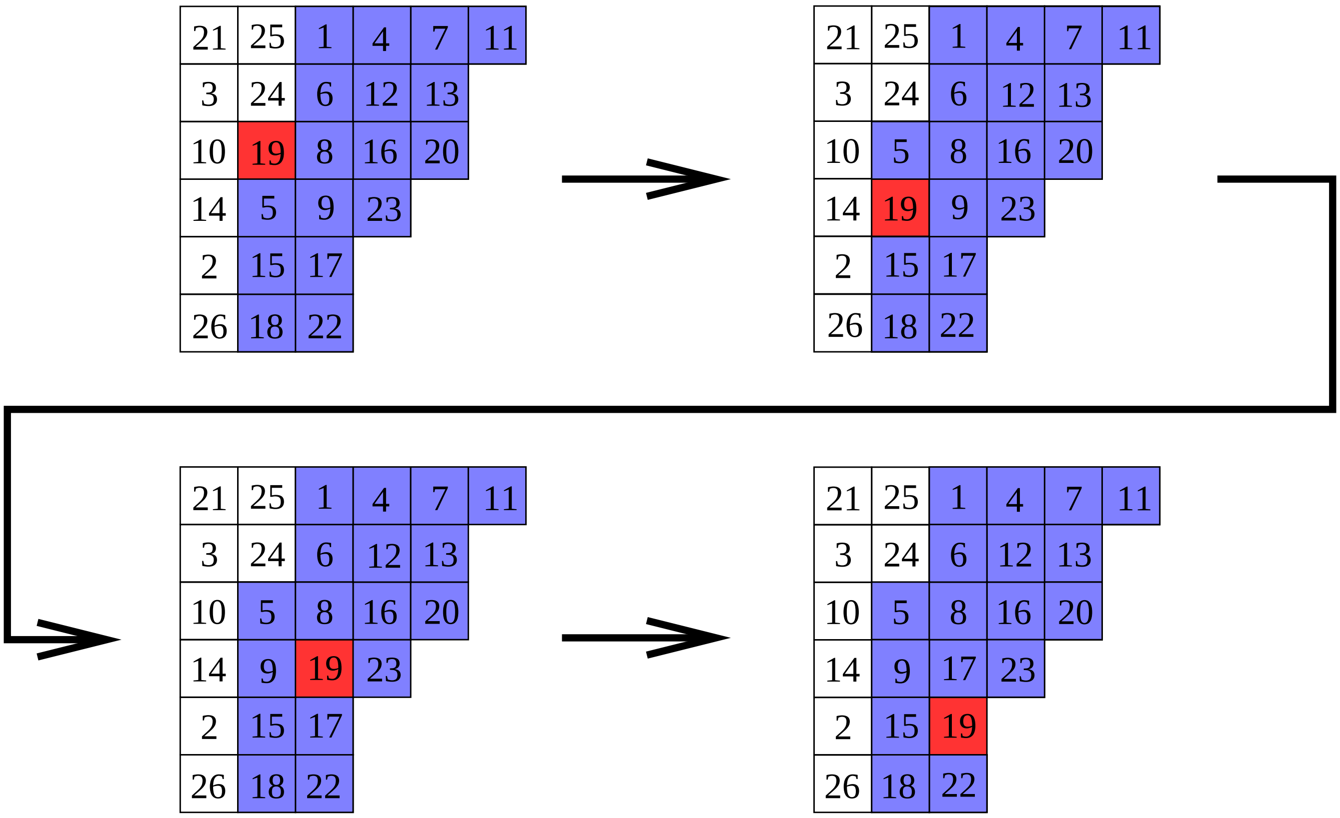
$\Rightarrow |H_\lambda| = \prod_{(i,j)} h_{i,j}$

It suffices to find:

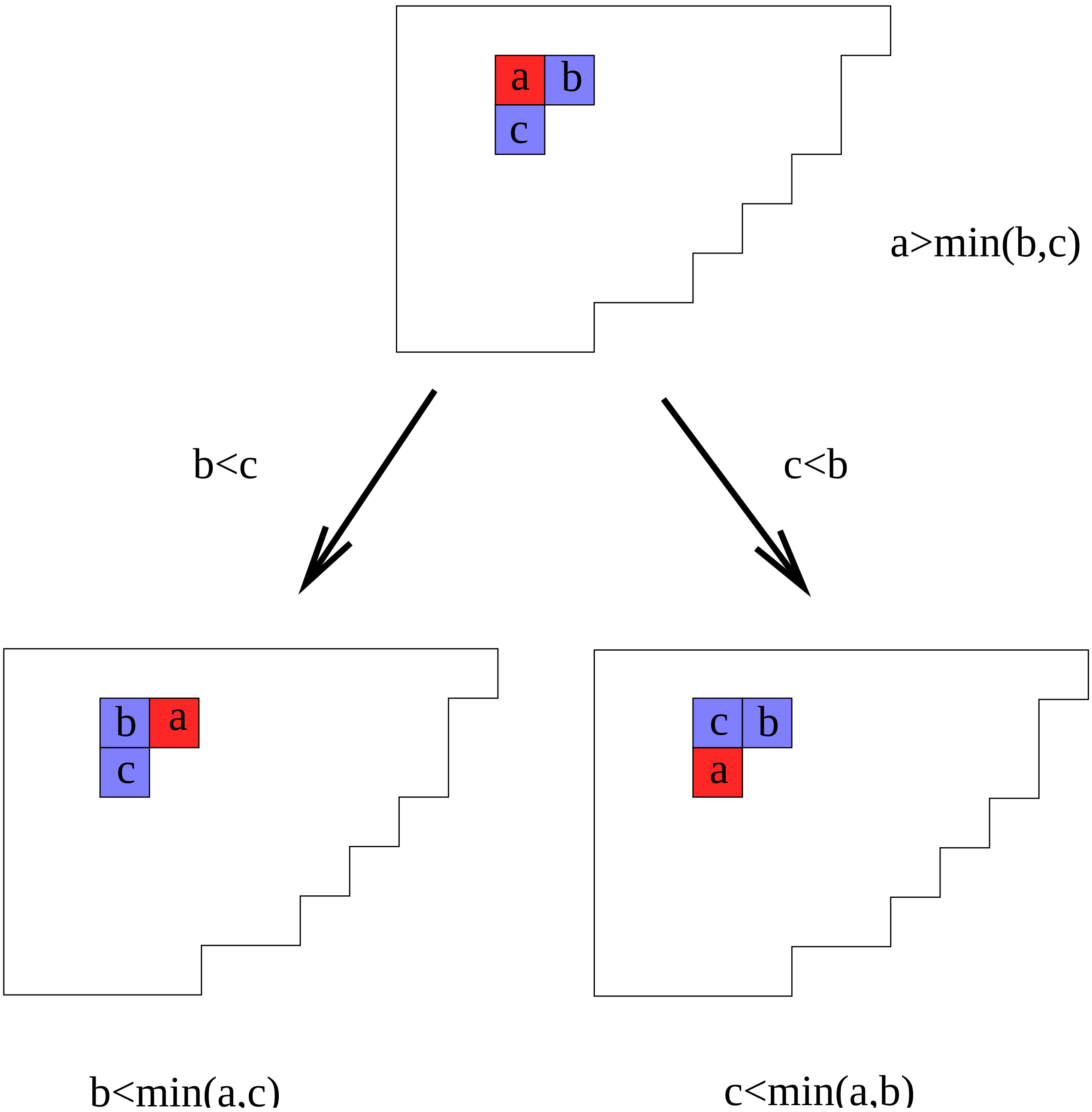
Bijection $T_\lambda \leftrightarrow S_\lambda \times H_\lambda$

Modified jeu de taquin

Ordering procedure: tabloid $\rightarrow$ tableau

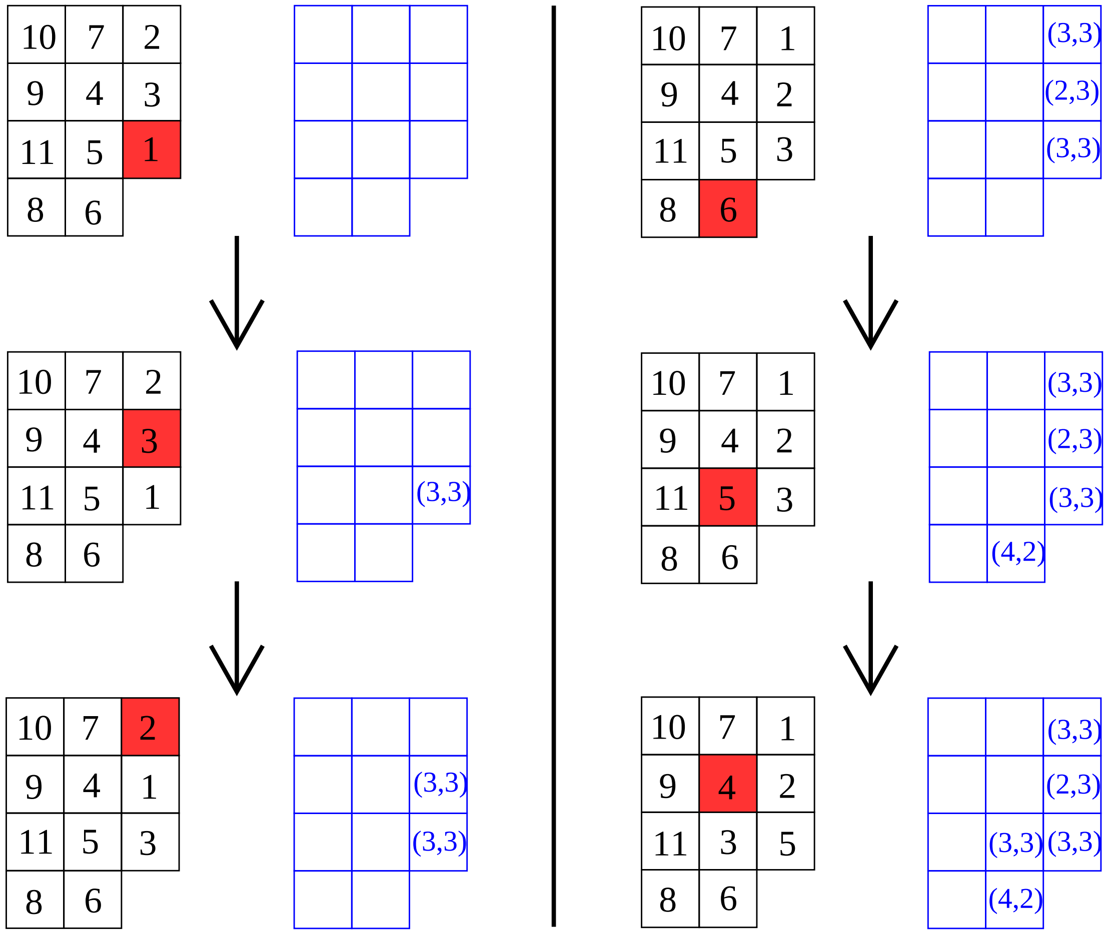


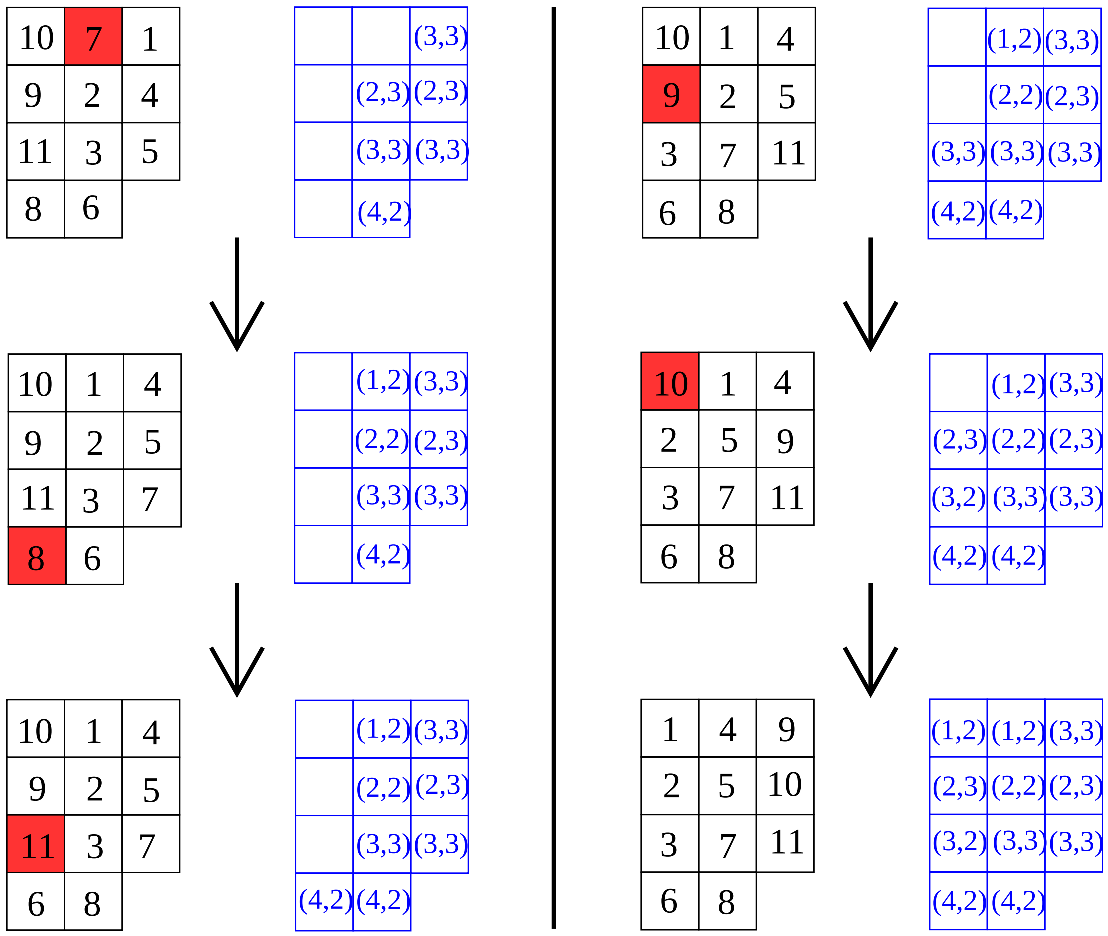
Modified jeu de taquin



The NPS-Algorithm

tabloid $\rightarrow$ (tableau, hook tabloid)





10 7 1
9 2 4
11 3 5
8 6
(3,3)
(2,3) (2,3)
(3,3) (3,3)
(4,2)
10 1 4
9 2 5
11 3 7
8 6
(1,2) (3,3)
(2,2) (2,3)
(3,3) (3,3)
(4,2)
10 1 4
9 2 5
11 3 7
6 8
(1,2) (3,3)
(2,2) (2,3)
(3,3) (3,3)
(4,2) (4,2)
10 1 4
9 2 5
3 7 11
6 8
(1,2) (3,3)
(2,2) (2,3)
(3,3) (3,3) (3,3)
(4,2) (4,2)
10 1 4
2 5 9
3 7 11
6 8
(1,2) (3,3)
(2,3) (2,2) (2,3)
(3,2) (3,3) (3,3)
(4,2) (4,2)
1 4 9
2 5 10
3 7 11
6 8
(1,2) (1,2) (3,3)
(2,3) (2,2) (2,3)
(3,2) (3,3) (3,3)
(4,2) (4,2)

Which additional information is necessary to reverse modified jeu de taquin?

$\Rightarrow$ Endcells of the paths of modified jeu de taquin suffice to reverse the procedure.

'Storage tabloid':

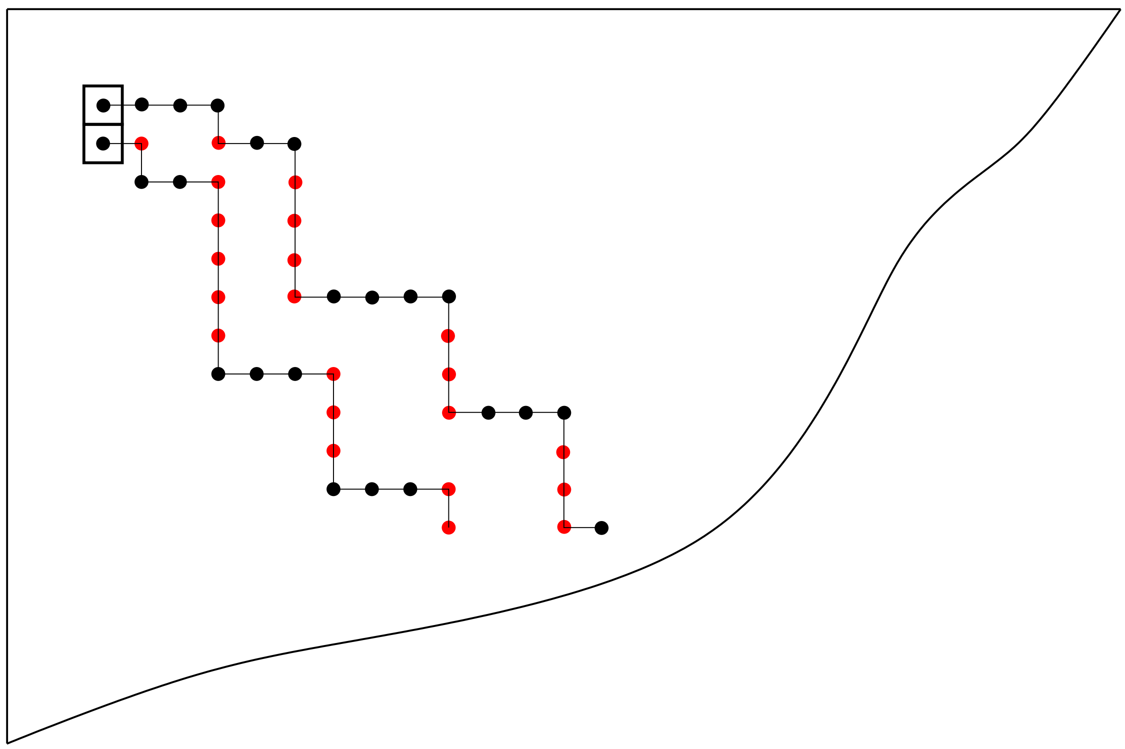
11	7	4	1
7	4	2	
4	2	1	
1			

$\Rightarrow$ But the hook tabloid

7	5	4	1
5	3	2	
4	2	1	
1			

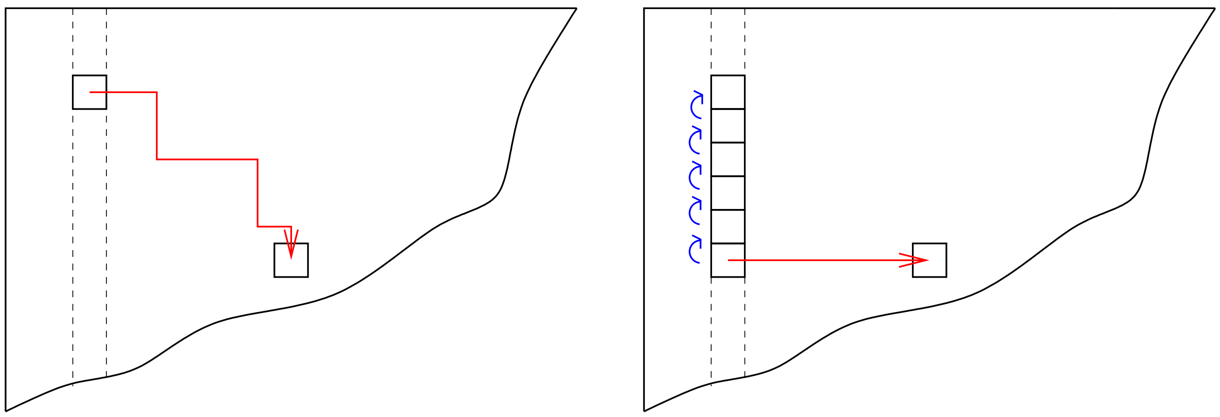
suffices for the storage

Reason: The forward paths are dependent.

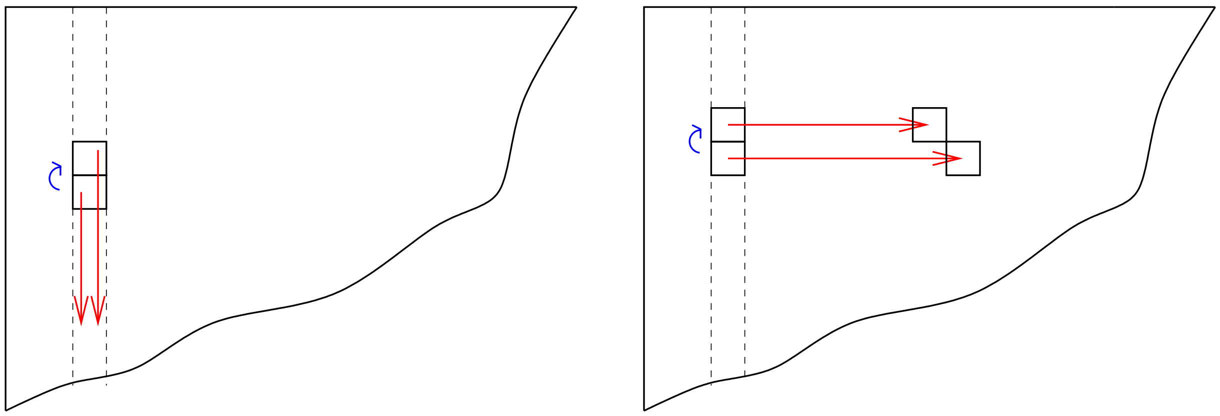


Consecutive paths do not intersect and touch at most at the red points.

Construction of the hook tabloid



Two types of shifts



Similar for shifted tableaux?

SS_λ = set of shifted tableaux of shape λ

ST_λ = set of shifted tabloids of shape λ

Example:

$$\begin{array}{ccccc} 11 & 4 & 9 & 8 & 1 \\ & 12 & 6 & 2 & 3 \\ & & 10 & 5 & \\ & & & 7 & \end{array}$$

SH_λ = set of shifted hook tabloids of shape λ

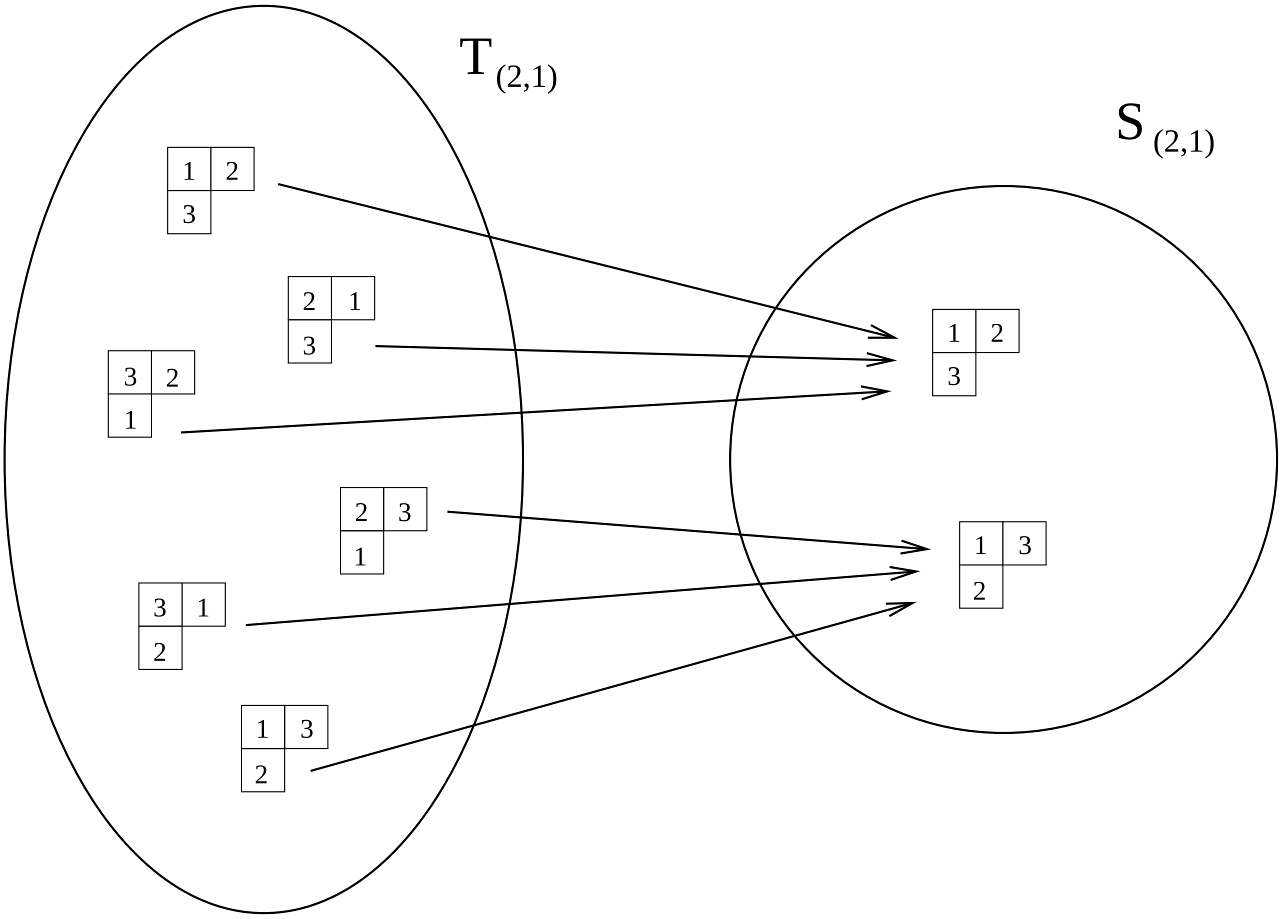
Example:

$$\begin{array}{ccccc} (2,2) & (1,2) & (1,5) & (4,4) & (2,5) \\ & (3,4) & (4,4) & (2,4) & (2,5) \\ & & (3,4) & (4,4) & \\ & & & (4,4) & \end{array}$$

It suffices to find:

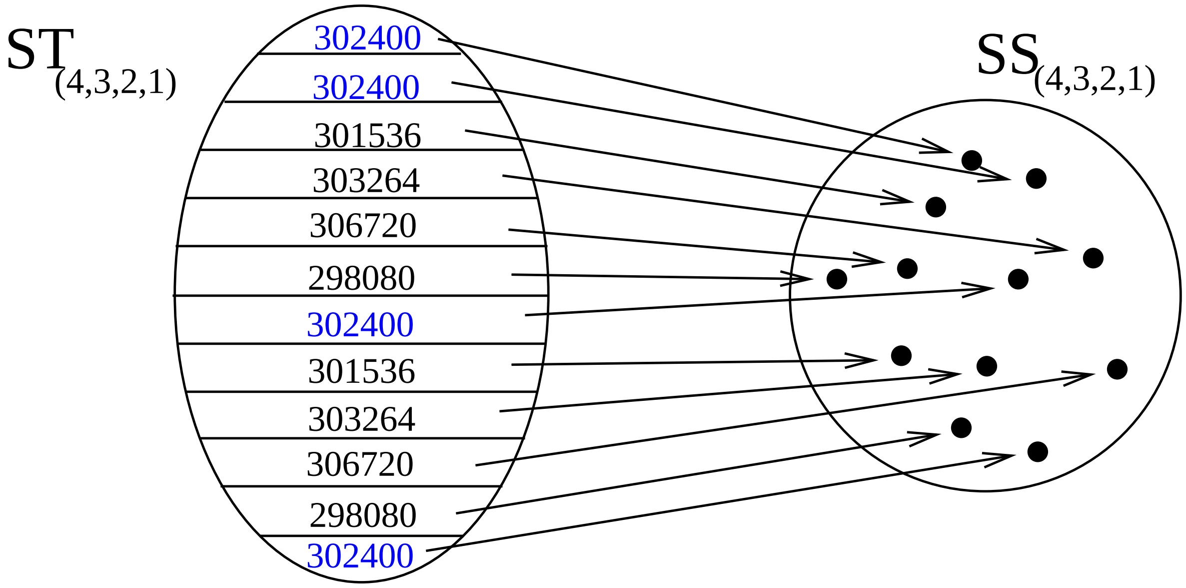
Bijection $ST_\lambda \leftrightarrow SS_\lambda \times SH_\lambda$

NPS-Algorithm shows: Performing modified jeu de taquin columnwise gives a $\prod_{(i,j)} h_{i,j}$ to 1 map from T_λ onto S_λ.



Computer experiments: Performing modified jeu de taquin columnwise to shifted tabloids does not give a $\prod_{(i,j)} h_{i,j}$ to 1 map from ST_λ onto SS_λ.

Counterexample $\lambda = (4,3,2,1)$.



(In this case $\prod_{(i,j)} h_{i,j} = 302400$.)

But: If we replace columnwise by rowwise the result is true for shifted tableaux!

Columnwise:

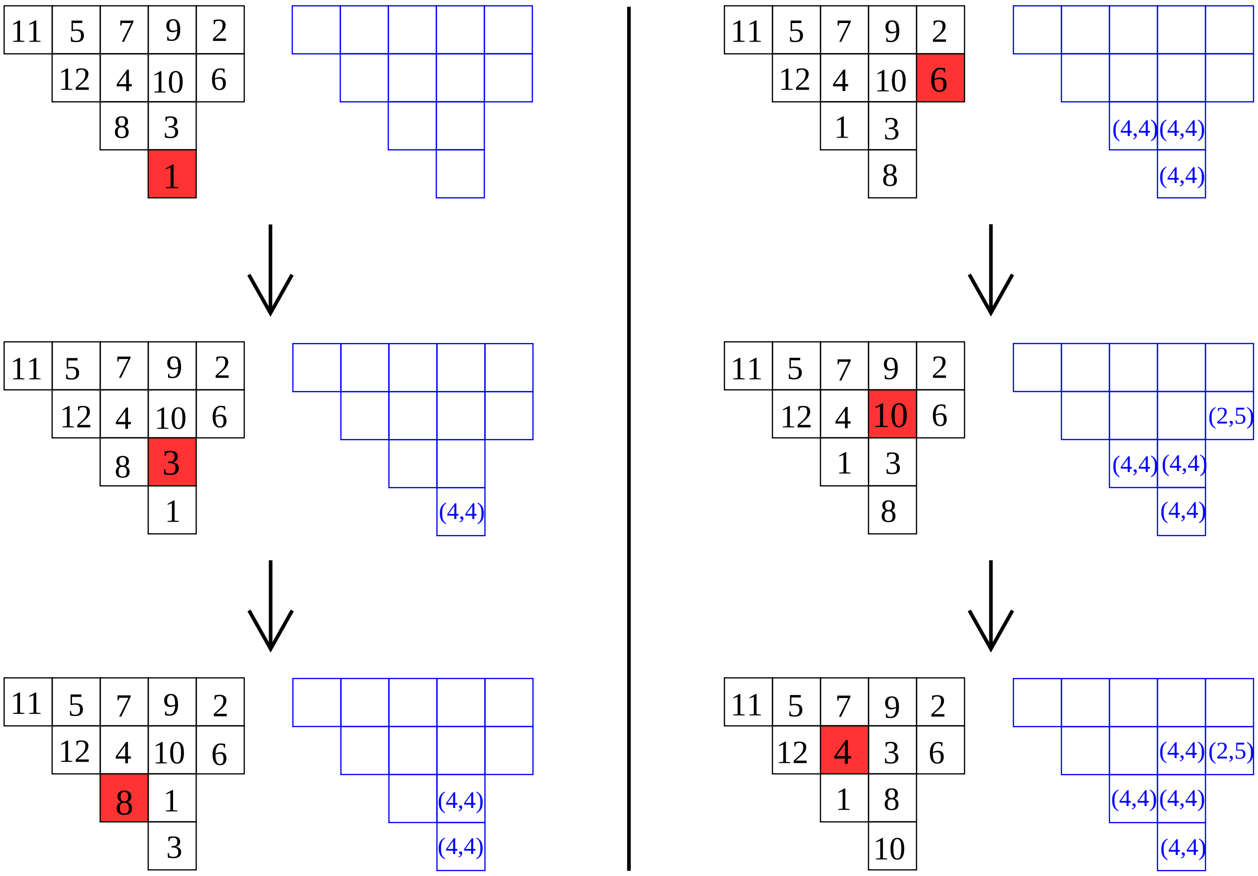
$$\begin{array}{ccccc} 12 & 11 & 9 & 6 & 2 \\ & 10 & 8 & 5 & 1 \\ & & 7 & 4 & \\ & & & 3 & \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccc} 1 & 2 & 4 & 7 & 11 \\ & 3 & 5 & 8 & 12 \\ & & 6 & 9 & \\ & & & 10 & \end{array}$$

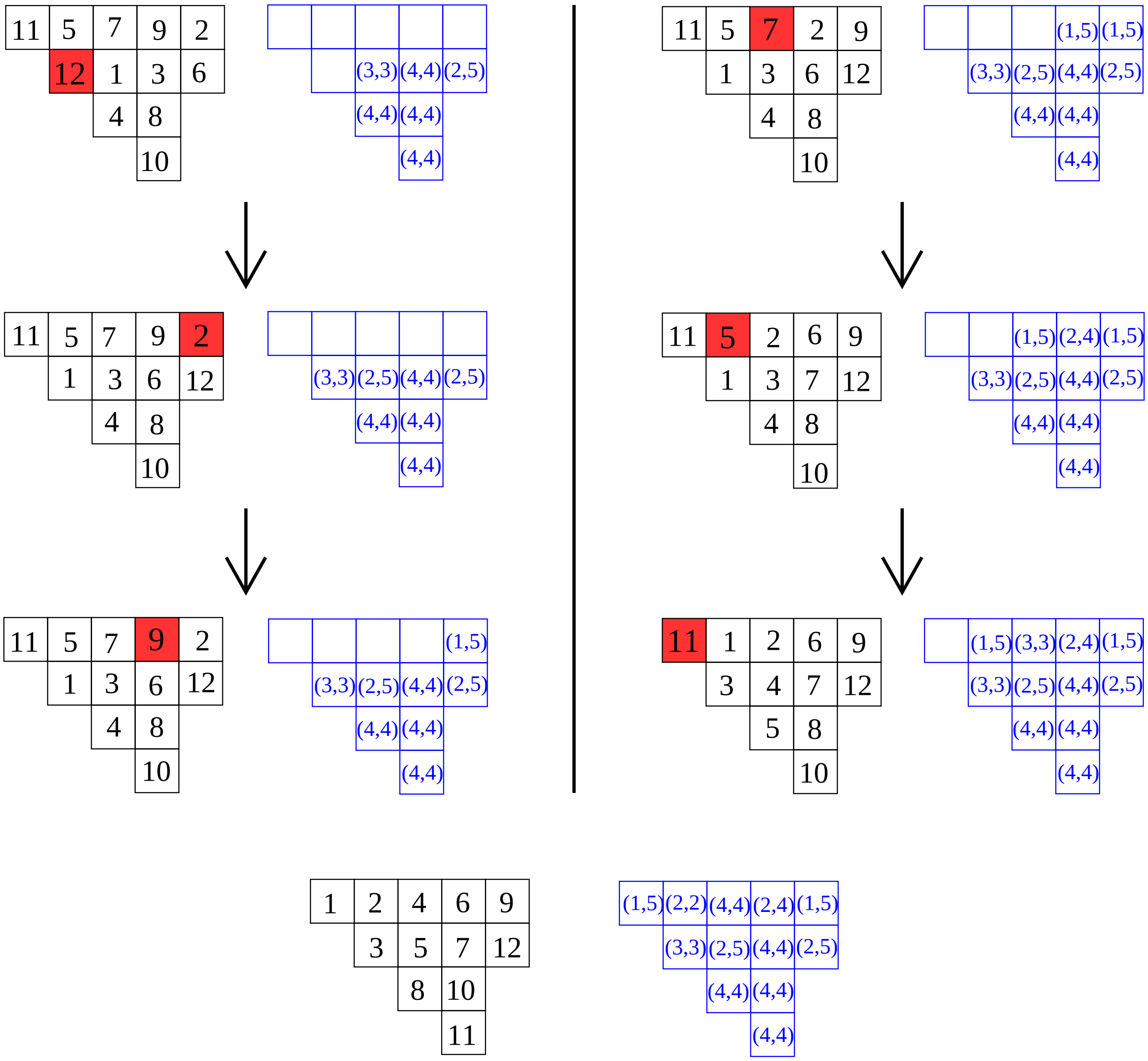
Rowwise:

$$\begin{array}{ccccc} 12 & 11 & 10 & 9 & 8 \\ & 7 & 6 & 5 & 4 \\ & & 3 & 2 & \\ & & & 1 & \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & 9 \\ & & 10 & 11 & \\ & & & 12 & \end{array}$$

Theorem: Modified jeu de taquin rowwise is a $\prod_{(i,j)} h_{i,j}$ to 1 map from the set of shifted tabloids onto the set of shifted tableaux.

Example

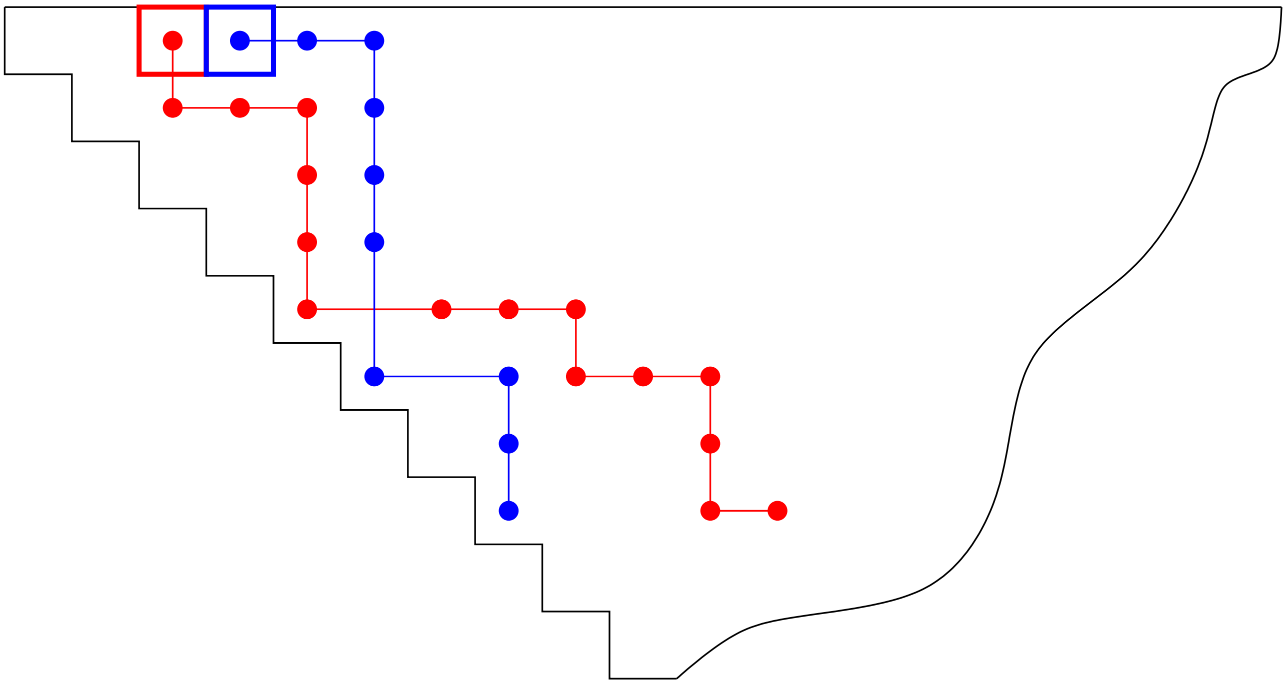

11 5 7 9 2
12 4 10 6
8 3
1
11 5 7 9 2
12 4 10 6
8 3
1
(4,4)
11 5 7 9 2
12 4 10 6
8 1
3
(4,4)
(4,4)
11 5 7 9 2
12 4 10 6
1 3
8
(4,4)(4,4)
(4,4)
11 5 7 9 2
12 4 10 6
1 3
8
(2,5)
(4,4)(4,4)
(4,4)
11 5 7 9 2
12 4 3 6
1 8
10
(4,4)(2,5)
(4,4)(4,4)
(4,4)



11 5 7 9 2
12 1 3 6
4 8
10
(3,3) (4,4) (2,5)
(4,4) (4,4)
(4,4)
11 5 7 9 2
1 3 6 12
4 8
10
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)
11 5 7 9 2
1 3 6 12
4 8
10
(1,5)
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)
11 5 7 2 9
1 3 6 12
4 8
10
(1,5) (1,5)
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)
11 5 2 6 9
1 3 7 12
4 8
10
(1,5) (2,4) (1,5)
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)
11 1 2 6 9
3 4 7 12
5 8
10
(1,5) (3,3) (2,4) (1,5)
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)
1 2 4 6 9
3 5 7 12
8 10
11
(1,5) (2,2) (4,4) (2,4) (1,5)
(3,3) (2,5) (4,4) (2,5)
(4,4) (4,4)
(4,4)

Difficulty in the shifted case

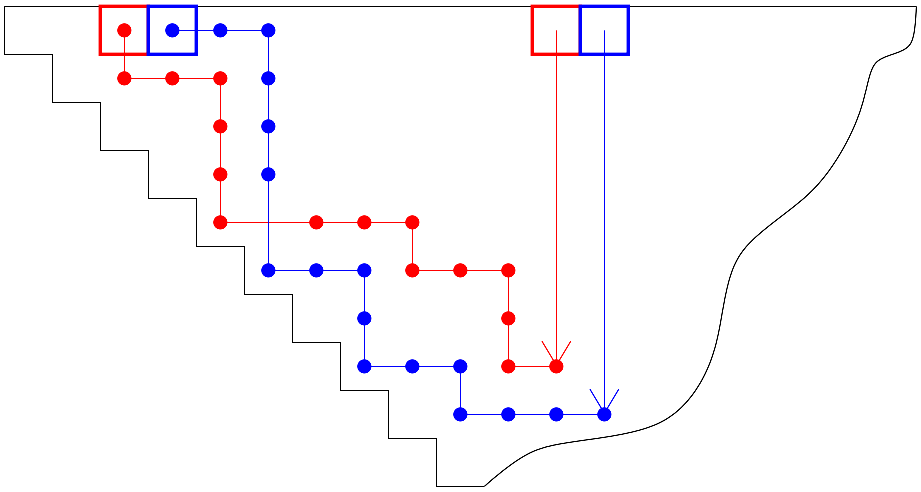
Paths of consecutive entries may intersect



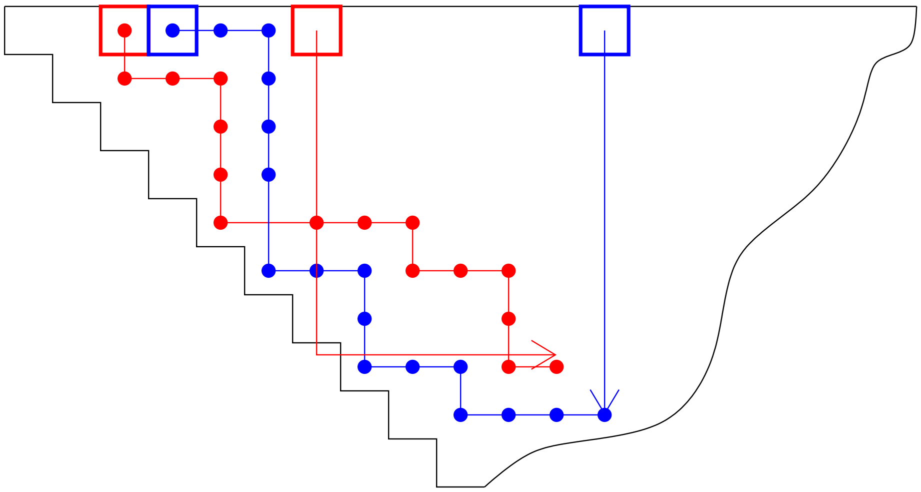
Intersection-cells: $(i, i+1)$

Construction of the shifted hook tabloid

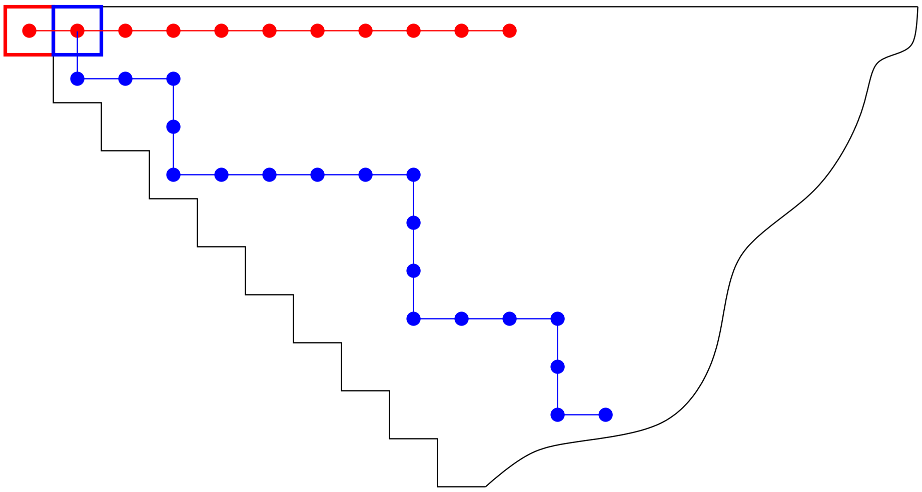
Not



but



Problem



A symmetry theorem

Modified jeu de taquin columnwise $\leftrightarrow$ $\begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{matrix}$

Modified jeu de taquin rowwise $\leftrightarrow$ $\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$

Every tableau induces an order for performing modified jeu de taquin.

P, Q tableaux of shape λ: $A_{P,Q}$ = number of tabloids which result in Q, when performing modified jeu de taquin in the order induced by P.

Example: $\lambda = (3, 2, 2)$

222222222222222222222
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222222222222222222222
222222222222222222222
222222222231313131222
222222222213131313222
222222222231313131222
222222222213131313222
222222222231313131222
222222222213131313222
222222222231313131222
222222222213131313222
222222222222222222222
222222222222222222222
222222222222222222222

with $1 \sim 232$, $2 \sim 240 = \prod_{(i,j)} h_{i,j}$ and $3 \sim 248$.

Theorem:

$$A_{P,Q} = A_{Q,P}$$