Extreme DADASMs

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Jointly with Arvind Ayyer and Matjaž Konvalinka

Permutation matrices

Binary matrices s.t. each row/column contains precisely one 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are $n!$ permutation matrices of size n.

Permutation triangles

Triangular binary arrays with n rows such that

- each row contains precisely one 1,
- each column contains at most one 1.

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 1 & 0 & & \\
 & & & 1 & 0 & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

There are $n!$ permutation triangles of size n.

ASM=Alternating Sign Matrix

Quadratic $0, 1, -1$ matrix such that in each row and each column

- the non–zero entries appear with alternating signs and
- the sum of entries is 1, that is the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Generalize permutation matrices!

AST = Alternating Sign Triangle

Triangular $0, 1, -1$ array such that

- in each row and column the non-zero elements alternate,
- the sum of entries in each row is 1,
- the first non-zero entry of each column is 1 ($\Rightarrow$ c-sums$= 0, 1$).

$$
\begin{array}{ccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & \\
 & & 0 & 0 & 1 & -1 & 1 & & \\
 & & & 1 & -1 & 1 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Generalize permutation triangles!

This is a good generalization in the following sense:

Conjecture (A.-F. 2012). There is the same number of $n \times n$ ASMs as there is of ASTs with n rows.

Refinement by Matjaž:

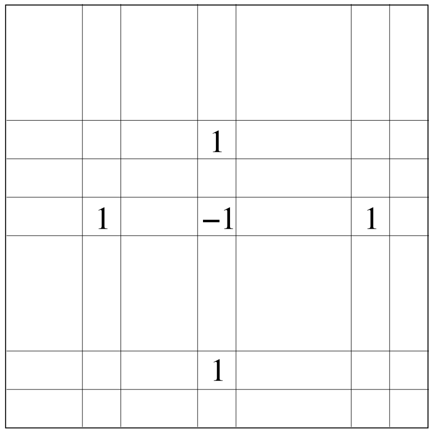
Conjecture (K. 2014). Let n, k be non-negative integers. There is the same number of $n \times n$ ASMs with k occurrences of -1 as there is of ASTs with n rows and k occurrences of -1.

We started the talk by proving the conjecture for $k = 0$.

$k = 1$: ASMs

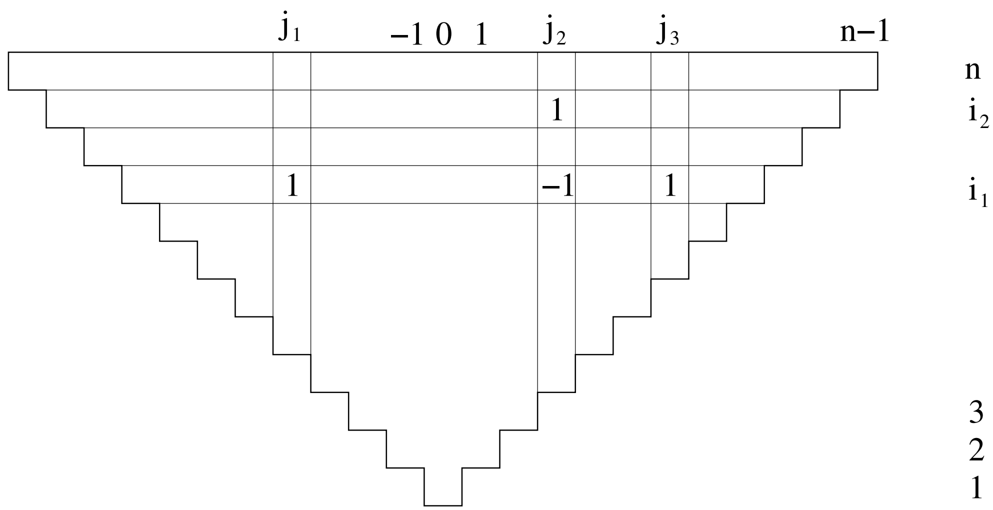
Number of $n \times n$ ASMs with precisely one -1:

$$\binom{n}{3}^2 (n-3)!$$



• $\binom{n}{3}$ choices for the rows of -1 and the two 1s in the same column.
• $\binom{n}{3}$ choices for the columns of -1 and the two 1s in the same row.
• $(n-3)!$ choices for the permutation matrix obtained after deleting the three rows and the three columns.

$k = 1$: ASTs



Notation:

$$p(a,b) = \begin{cases} a(a+1)\cdots b & \text{if } a \le b, \\ 1 & \text{otherwise.} \end{cases}$$

$$A(j_1, j_3, i_1) = p(1, m)p(m, M-1)p(M-1, i_1-3)$$

$$\begin{aligned} B(j_1, j_2, j_3, i_1) &= p(1, \mathrm{min})p(\mathrm{min}, \mathrm{mid}-1) \\ &\quad \times p(\mathrm{mid}-1, \mathrm{max}-2)p(\mathrm{max}-2, i_1-4) \end{aligned}$$

where $m = \min(|j_1|, |j_3|)$, $M = \max(|j_1|, |j_3|)$, $\mathrm{min} = \min(|j_1|, |j_2|, |j_3|)$, $\mathrm{max} = \max(|j_1|, |j_2|, |j_3|)$, $\mathrm{mid} = |j_1| + |j_2| + |j_3| - \mathrm{min} - \mathrm{max}$.

The number of ASTs with n rows and one -1 is

$$\begin{aligned} \sum & B(j_1, j_2, j_3, i_1)p(i_1-1, i_2-3)p(i_2, n-1) \\ & + \left(A(j_1, j_3, i_1) - B(j_1, j_2, j_3, i_1)\right) p(i_1, i_2-2)p(i_2+1, n), \end{aligned}$$

where the sum is over all j_1, j_2, j_3 with $-n+1 \le j_1 < j_2 < j_3 < n-1$ and all i_1, i_2 with $\max(|j_1|, |j_2|, |j_3|) + 1 \le i_1 < i_2 \le n$.

Outline

- Symmetry classes of ASMs, DADASMs
- Behavior along diagonals and antidiagonals of DADASMs
- Conjectures on the numbers of extreme configurations
- Characterization of extreme configurations
- Operator expressions for the numbers
- Dimension-halving theorems (provide half of the information)

ASM-Theorem (Zeilberger, 1995)

The number of $n \times n$ ASMs is $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$.

David Robbins conjectured nice product formulas for many symmetry classes of ASMs.

Symmetry classes of ASMs

- Vertically symmetric ASMs: $a_{i,j} = a_{i,n+1-j}$
  n odd: Kuperberg (2002)

- Half-turn symmetric ASMs: $a_{i,j} = a_{n+1-i,n+1-j}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

- Diagonally symmetric ASMs: $a_{i,j} = a_{j,i}$
  no formula ?

- Quarter-turn symmetric ASMs: $a_{i,j} = a_{j,n+1-i}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

Symmetry classes of ASMs (Part 2)

- Horizontally and vertically symmetric ASMs: $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$
  n odd: Okada (2004)

- Diagonally and antidiagonally symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$
  n odd: Conjecture by Robbins (1980s)

- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$
  no formula ?

DADASMs(=DASASMs)

Example:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{d}(n)$ = number of $n \times n$ DADASMs

Conjecture (Robbins, 1980s): $\mathrm{d}(2n+1) = \prod_{i=1}^{n} \frac{\binom{3i}{i}}{\binom{2i-1}{i}}$

Sequence starts as follows: $1, 3, 15, 126, 1782, 42471, 1706562 \ldots$

Behavior along diagonals and antidiagonals

An $n \times n$ DADASM $A = (a_{i,j})$ is uniquely determined by its fundamental domain $\{a_{i,j} | 1 \leq i \leq (n+1)/2, i \leq j \leq n+1-i\}$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$\alpha \in \{-1, 0, 1\}$:
$\mathrm{n}_\alpha(A) =$ Number of α's along the diagonal and the antidiagonal of the fundamental domain

Bounds

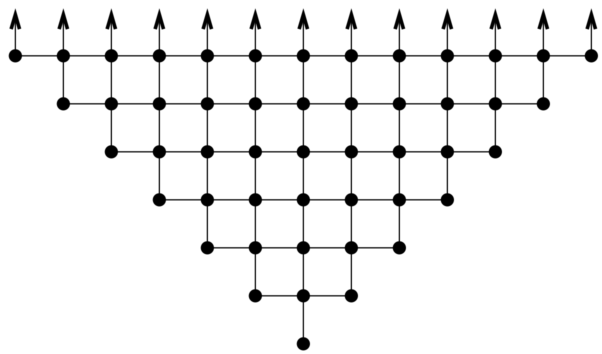
Computer experiments: suppose A is a $(2n+1) \times (2n+1)$ DADASM. Then the statistics $\mathsf{n}_\alpha(A)$ range in the following intervals.

- $n \le \mathsf{n}_0(A) \le 2n$
- $0 \le \mathsf{n}_1(A) \le n+1$
- $0 \le \mathsf{n}_{-1}(A) \le n$

All inequalities are sharp.

Why do there have to be so many zeros?

Translation into six vertex model:

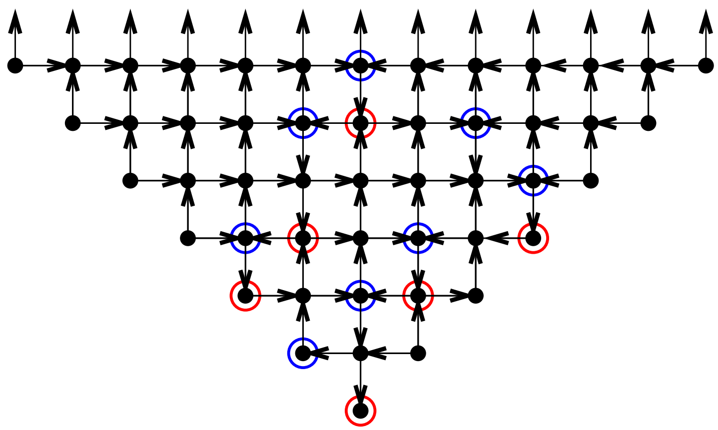


Orient edges such that

- all degree-4 vertices are "balanced", and
- all external edges at the top are oriented upward.

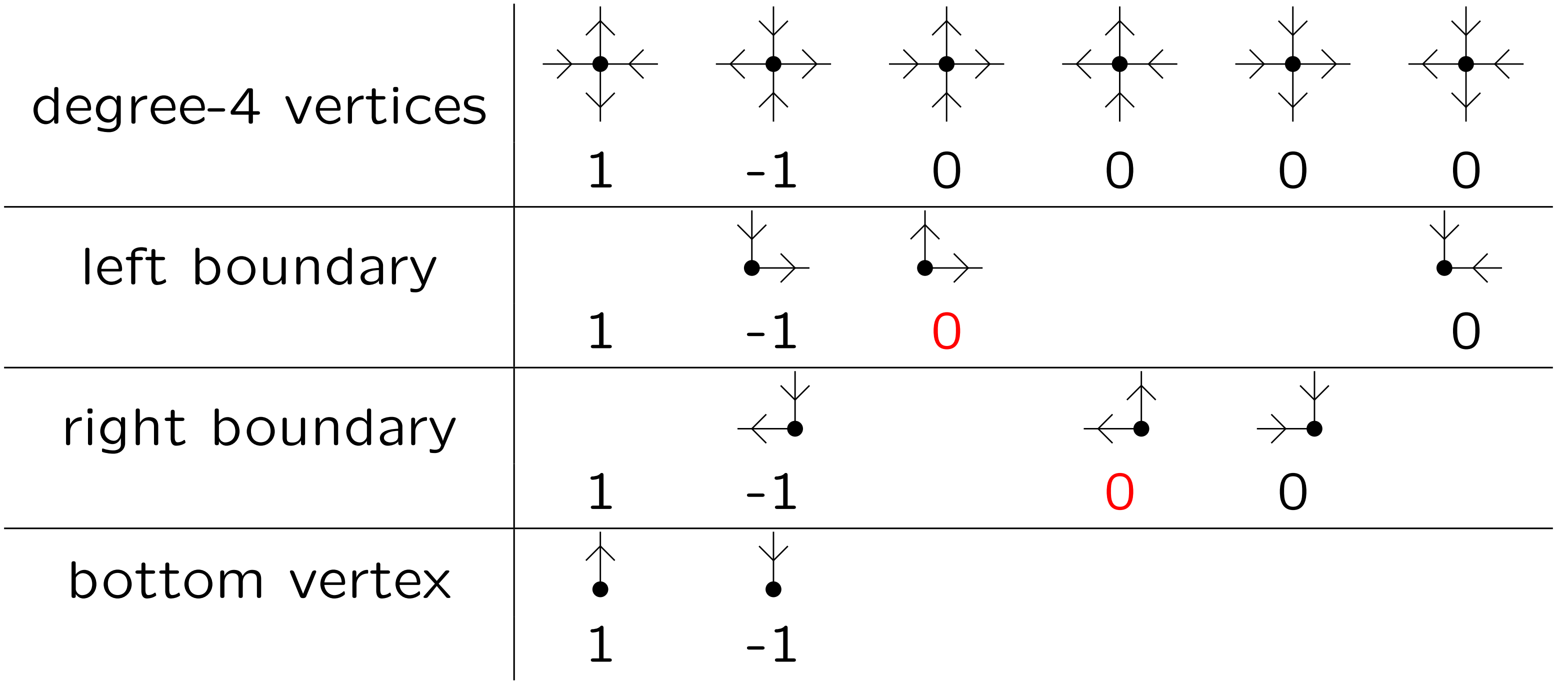
1-1 correspondence with fundamental domains of DADASMs

Example



0	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	1	-1	0	1	0	0	0	
		0	0	0	0	0	0	0	1	0		
			0	1	-1	0	1	0	-1			
				-1	0	1	-1	0				
					1	0	0					
						-1						

Dictionary



- Sum of indegrees = sum of outdegrees
- Inner vertex are balanced
- There have to be enough outdegree-2 zeros to compensate for the indegree-1 vertices on the top boundary.

Enumerating extreme configurations

Minimal number of zeros:

Conjecture 1. The number of $(2n+1) \times (2n+1)$ DADASMs A with $\mathrm{n}_0(A) = n$ is equal to the total number of $(n+1) \times (n+1)$ ASMs.

Maximal number of zeros:

Conjecture 2. The number of $(2n+1) \times (2n+1)$ DADASMs A with $\mathrm{n}_0(A) = 2n$ is equal to the number of $(2n+3) \times (2n+3)$ vertically and horizontally symmetric ASMs (VHASMs).

Are numbers round as well if $\mathrm{n}_0(A) = 2n - 1$ or $\mathrm{n}_0(A) = 2n - 2$?

Cases $\alpha = \pm 1$

Maximal number of 1s:

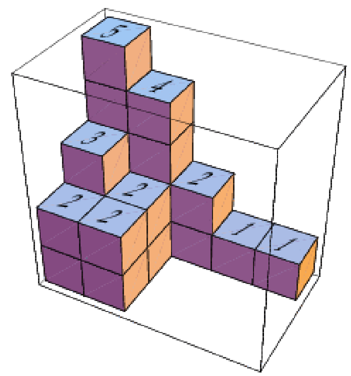
Conjecture 3. The number of $(2n+1) \times (2n+1)$ DADASMs with $\mathrm{n}_1(A) = n+1$ is equal to the number of cyclically symmetric plane partitions (CSPP) in an $n \times n \times n$ box.

Maximal number of -1s:

Conjecture 4. The number of $(2n+1) \times (2n+1)$ DADASMs with $\mathrm{n}_{-1}(A) = n$ is equal to the total number of $n \times n$ ASMs.

Conjecture 4 is equivalent to the conjecture on ASTs!

Plane Partitions



A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$

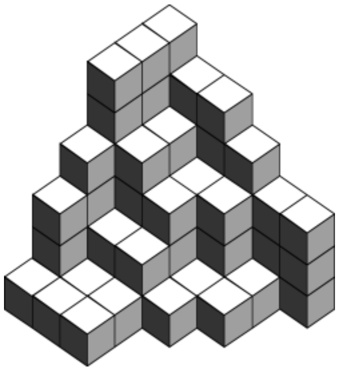
Cyclically symmetric plane partitions

An $n \times n \times n$ PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$

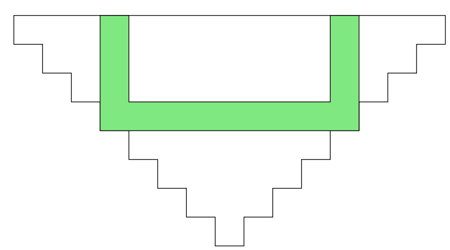


Characterization of extreme configurations

• s-alternating sequence = seq of 0/1/-1, non-zero elements alternate and elements sum to s

• DADASM-triangle = fundamental region of a DADASM

$$
\begin{array}{ccccccccccc}
a_{n,-n} & a_{n,-n+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n,n-1} & a_{n,n} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & & a_{2,-2} & a_{2,-1} & a_{2,0} & a_{2,1} & a_{2,2} & & & \\
 & & & & a_{1,-1} & a_{1,0} & a_{1,1} & & & & \\
 & & & & & a_{0,0} & & & & &
\end{array}
$$



Proposition. Let A be a DADASM-triangle of order n.

(1) $\mathrm{n}_1(A) = n + 1$ iff each row is 1-alternating.

(2) $\mathrm{n}_{-1}(A) = n$ iff, after deleting all diagonal and antidiagonal entries, each row is 1-alternating.

(3) $\mathrm{n}_0(A) = n$ iff $a_{1,1} = 1$ and, after deleting all diagonal and antidiagonal -1s, each row is 1-alternating.

Example for (3):

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & \\
 & & 1 & 0 & 0 & -1 & 0 & 1 & -1 & & \\
 & & & 0 & 0 & 1 & 0 & -1 & & & \\
 & & & & 1 & -1 & 1 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

Six-vertex model formulation:

(1): The leftmost and rightmost vertical edges of each row are oriented towards the top.

(2): The leftmost and rightmost horizontal edges of each row are oriented inwards.

Equivalence of Conj. 1 and Conj. 4

Theorem. There is the same number of DADASM-triangles of order n with $\mathsf{n}_0(A) = n$ as there is of DADASM-triangles of order $n+1$ with $\mathsf{n}_{-1}(A) = n+1$.

Proof by example: DADASM-triangle of order 5 with $\mathsf{n}_0(A) = 5$

$$
\begin{array}{ccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\
& & 0 & & 0 & & 0 & & 0 & & 1 & & -1 & & 0 & & 1 & & 0 & & \\
& & & & 1 & & 0 & & 0 & & -1 & & 0 & & 1 & & -1 & & & & \\
& & & & & & 0 & & 0 & & 1 & & 0 & & -1 & & & & & & \\
& & & & & & & & 1 & & -1 & & 1 & & & & & & & & \\
& & & & & & & & & & 1 & & & & & & & & & &
\end{array}
$$

Perform the following procedure:

- Replace the -1s among the diagonal-antidiagonal entries by 0s and put -1s below these zeros.

- Put a 0 below each diagonal-antidiagonal 0.

- Put a -1 below each diagonal-antidiagonal 1.

- Put 0s at the beginning and end of the top row.

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & \\
 & & 1 & 0 & 0 & -1 & 0 & 1 & -1 & & \\
 & & & 0 & 0 & 1 & 0 & -1 & & & \\
 & & & & 1 & -1 & 1 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & & \\
 & & & -1 & 0 & 0 & 1 & 0 & 0 & -1 & & & \\
 & & & & 0 & 1 & -1 & 1 & -1 & & & & \\
 & & & & & -1 & 1 & -1 & & & & & \\
 & & & & & & -1 & & & & & &
\end{array}
$$

This is a DADASM-triangle of order 6 with $n_{-1}(A) = 6$. $\square$

Operator expressions

First explain the tool to derive the formulas.

A monotone triangle is a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers

$$
\begin{array}{ccccccccccccc}
 & & & & & & a_{1,1} & & & & & & \\
 & & & & & a_{2,1} & & a_{2,2} & & & & & \\
 & & & & \dots & & \dots & & \dots & & & & \\
 & & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & & \\
 & & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & & \\
 & a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} &
\end{array},
$$

monotone increasing in $\nearrow$ and $\searrow$ direction and strictly increasing along rows, except for the last row.

ASMs $\leftrightarrow$ Monotone Triangles

$n \times n$ ASMs and monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijective correspondence:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

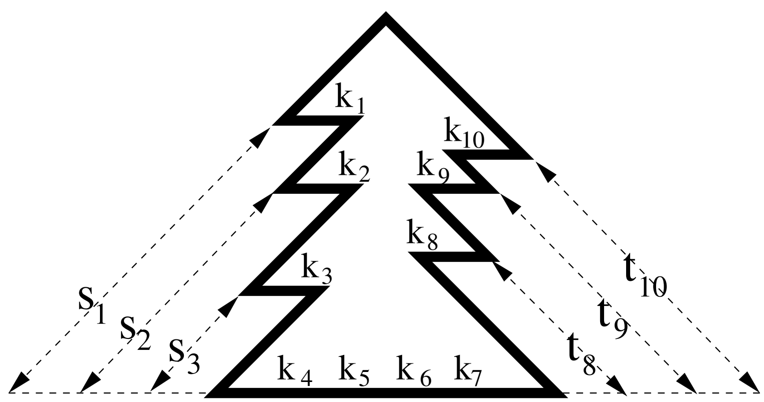
Partial columnsums!

Define

$$\alpha(n; k_1, \ldots, k_n) = \prod_{1 \le p < q \le n} (\mathrm{id} + \Delta_{k_p} \Delta_{k_q} + \Delta_{k_q}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where $\Delta_x p(x) = p(x+1) - p(x)$.
Let $\mathbf{s} = (s_1 \ge s_2 \ge \ldots \ge s_l \ge 0)$ and $\mathbf{t} = (0 \le t_{n-r+1} \le t_{n-r+2} \le \ldots \le t_n)$: An $(\mathbf{s}, \mathbf{t})$-tree of order n is a monotone triangle with n rows where

- the bottom s_i elements are deleted from the i-th NE-diagonal,
- the bottom t_i elements are deleted from the i-th SE-diagonal.



Backward difference: $\delta_x p(x) = p(x) - p(x-1)$

Theorem (F. 2009). Let $k_1, \ldots, k_n$ be an increasing sequence of integers. Then

$$(-\Delta_{k_1})^{s_1}(-\Delta_{k_2})^{s_2} \cdots (-\Delta_{k_l})^{s_l} \delta_{k_{n-r+1}}^{t_{n-r+1}} \delta_{k_{n-r+2}}^{t_{n-r+2}} \cdots \delta_{k_n}^{t_n} \alpha(n; k_1, \ldots, k_n)$$

is the number of $(\mathbf{s}, \mathbf{t})$-trees with the following properties:
- The bottom row is $k_{l+1}, \ldots, k_{n-r}$.
- For $1 \le i \le l$, the bottom entry of the i-th NE-diagonal is k_i. [*]
- For $n-r+1 \le i \le n$, the bottom entry of the i-th SE-diagonal is k_i. [†]

[*] This entry does not have to be strictly smaller than its right neighbor.

[†] This entry does not have to be strictly greater than its left neighbor.

How to apply this in our setting

DADASM-triangle with a maximal number of -1s on the diagonal/antidiagonal:

$$
\begin{array}{ccccccccccccc}
(0) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (0) \\
 & (0) & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & (0) & \\
 & & (-1) & 0 & 0 & 0 & 1 & 0 & -1 & 1 & (0) & & \\
 & & & (0) & 1 & 0 & -1 & 0 & 1 & (-1) & & & \\
 & & & & (-1) & 1 & 0 & 0 & (-1) & & & & \\
 & & & & & (-1) & 1 & (0) & & & & & \\
 & & & & & & (-1) & & & & & &
\end{array}
$$

Partial columnsums:

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\
 & & 0 & 0 & 0 & 1 & 0 & 0 & 1 & & \\
 & & & 1 & 0 & 0 & 0 & 1 & & & \\
 & & & & 1 & 0 & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

Corresponds to the following $((4,2,1),(2,3))$-tree:

$$\begin{array}{ccccccccccc}
 & & & & & -2 & & & & & \\
 & & & & -4 & & 2 & & & & \\
 & & & \bullet & & 0 & & 3 & & & \\
 & & \bullet & & -2 & & 2 & & \bullet & & \\
 & \bullet & & \bullet & & -1 & & \bullet & & \bullet & \\
\bullet & & \bullet & & \bullet & & 0 & & \bullet & & \bullet
\end{array}$$

Maximal number of -1s

The number of DADASM-triangles A of size n with $\mathsf{n}_{-1}(A) = n$ is

$$\sum_{m=1}^{n} \frac{\prod_{i=1}^{m-1}(-\delta_{k_i})^{m-i} \prod_{i=m+1}^{n} \Delta_{k_i}^{i-m}}{\prod_{i=1}^{m-1}(1 - \prod_{j=1}^{i}(-\delta_{k_j})) \prod_{i=0}^{n-m}(1 - \prod_{j=n-i}^{n} \Delta_{k_i})}$$

$$\times \prod_{1 \le p < q \le n} (\mathrm{id} + \Delta_{k_p}\Delta_{k_q} + \Delta_{k_q}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \Bigg|_{(k_1,\ldots,k_n)=(0,\ldots,0)}.$$

Maximal number of 1s

The number of DADASM-triangles A of size n with $\mathrm{n}_1(A) = n+1$ is

$$\sum_{m=0}^{n} \frac{\prod_{i=1}^{m} (-\delta_{k_i})^{m-i} \prod_{i=m+1}^{n} \Delta_{k_i}^{i-m-1}}{\prod_{i=1}^{m} (1 - \prod_{j=1}^{i} (-\delta_{k_j})) \prod_{i=0}^{n-m} (1 - \prod_{j=n-i}^{n} \Delta_{k_i})}$$

$$\times \prod_{1 \le p < q \le n} (\mathrm{id} + \Delta_{k_p} \Delta_{k_q} + \Delta_{k_q}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \Bigg|_{(k_1, \ldots, k_n) = (0, \ldots, 0)}.$$

Maximal number of 0s, even case

The number of DADASM-triangles A of size $2N$ with $\mathrm{n}_0(A) = 4N$ is

$$\sum_{m=0}^{N} (-1)^m \frac{\prod_{i=1}^{m} \delta_{k_{2i-1}}^{m-i+1} \delta_{k_i}^{m-i} \prod_{i=m+1}^{N} \Delta_{k_{2i-1}}^{N-m-1} \Delta_{k_{2i}}^{N-m}}{\prod_{i=1}^{m} (1 - \prod_{j=1}^{i} \delta_{k_{2j-1}} \delta_{k_{2j}}) \prod_{i=m+1}^{N} (1 - \prod_{j=i}^{N} \Delta_{k_{2j-1}} \Delta_{k_{2j}})}$$

$$\times \prod_{1 \le p < q \le 2N} (\mathrm{id} + \Delta_{k_p} \Delta_{k_q} + \Delta_{k_q}) \prod_{1 \le i < j \le 2N} \frac{k_j - k_i}{j - i} \Bigg|_{(k_1,\ldots,k_{2N})=(0,\ldots,0)} .$$

Maximal number of 0s, odd case

The number of DADASM-triangles A of size $2N+1$ with $\mathrm{n}_0(A) = 4N+2$ is

$$\sum_{m=0}^{N} (-1)^m \frac{\prod_{i=1}^{m} \delta_{k_{2i-1}}^{m-i+1} \delta_{k_i}^{m-i} \prod_{i=m+1}^{N} \Delta_{k_{2i}}^{N-m-1} \Delta_{k_{2i+1}}^{N-m}}{\prod_{i=1}^{m} (1 - \prod_{j=1}^{i} \delta_{k_{2j-1}} \delta_{k_{2j}}) \prod_{i=m+1}^{N} (1 - \prod_{j=i}^{N} \Delta_{k_{2j}} \Delta_{k_{2j+1}})}$$
$$\times \prod_{1 \le p < q \le 2N+1} (\mathrm{id} + \Delta_{k_p} \Delta_{k_q} + \Delta_{k_q}) \prod_{1 \le i < j \le 2N+1} \frac{k_j - k_i}{j - i} \Bigg|_{(k_1, \ldots, k_{2N+1}) = (0, \ldots, 0)} .$$

All four formulas are equivalent to constant term identities!

Dimension-halving theorems

What do I mean by that?

Another proof of the refined alternating sign matrix theorem:

• Consider $a_n = (A_{n,1}, A_{n,2}, \ldots, A_{n,n})^T$ where $A_{n,i}$ is the number of $n \times n$ ASMs with $a_{1,i} = 1$.

• Prove that a_n is an eigenvector of the upper triangular matrix $\left((-1)^{j+n}\binom{2n-1-i}{j-i}\right)_{1 \le i,j \le n}$ with respect to the eigenvalue 1.

• a_n lies in the $\lceil n/2 \rceil$-dimensional eigenspace S_n.

If we assume that we started with no information on the vector a_n, i.e. we only know that it has to lie in $\mathbb{Z}^n$, then this (approximately) halves the dimension of the subspace of $\mathbb{R}^n$ the vector a_n has to be contained in.

This is half of the information necessary:

- Symmetry $A_{n,i} = A_{n,n+1-i}$, i.e. a_n is also an eigenvector of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

with respect to the eigenvalue 1.

- a_n lies in the $\lceil n/2 \rceil$-dimensional eigenspace T_n of this matrix.

- We are lucky: $S_n \cap T_n$ is 1-dimensonal! a_n is determined up to a constant.

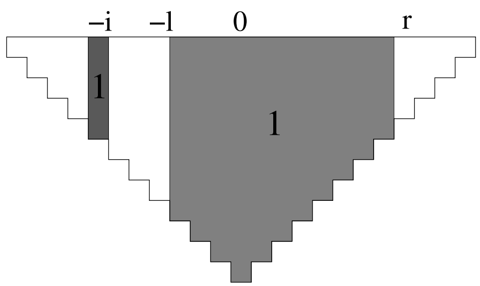
Conclusion: The fact that $a_n \in S_n$ gives half of the information necessary to compute a_n. The other half comes from the obvious symmetry.

We will consider two refined enumerations of ASTs and derive "dimension-halving" results.

Unfortunately we have no symmetry in this case (or something that is as useful) – so we lack the "other" half of the information necessary to compute these quantities.

First quantity

$\text{AST}_L(n; i, l, r)$ =Number ASTs of order n where
- $-i$ is the leftmost 1-alternating column,
- all other 1-alternating columns range between $-l+1$ and $r-1$.



The total number of ASTs of order n is

$$\sum_{i=0}^{n-1} \text{AST}_L(n; i, i, n).$$

$n = 5$

(l,r)	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	
$(1,4)$	14	**28**	**20**	**7**	**1**	$-7+8\cdot 1=1$
$(1,5)$	22	**41**	**26**	**8**	**1**	$-8+8\cdot 1=0$
$(2,3)$	-28	28	**72**	**44**	**9**	$-44+8\cdot 9=28$
$(2,4)$	-49	91	**156**	**79**	**14**	$-79+8\cdot 14=33$
$(2,5)$	-51	114	**177**	**84**	**14**	$-84+8\cdot 14=28$
$(3,2)$	2	-16	16	**28**	**9**	$-28+8\cdot 9=44$
$(3,3)$	-86	-44	116	**112**	**28**	$-112+8\cdot 28=112$
$(3,4)$	-177	-28	212	**161**	**35**	$-161+8\cdot 35=119$
$(3,5)$	-198	-17	235	**168**	**35**	$-168+8\cdot 35=112$
$(4,1)$	-1	1	-1	1	**1**	$-1+8\cdot 1=7$
$(4,2)$	51	-30	-5	33	**14**	$-33+8\cdot 14=79$
$(4,3)$	-42	-121	65	119	**35**	$-119+8\cdot 35=161$
$(4,4)$	-155	-141	149	168	**42**	$-168+8\cdot 42=168$
$(4,5)$	-180	-136	170	175	**42**	$-175+8\cdot 42=161$
$(5,1)$	-3	3	-2	0	1	$-0+8\cdot 1=8$
$(5,2)$	74	-32	-19	28	14	$-28+8\cdot 14=84$
$(5,3)$	-18	-141	39	112	35	$-112+8\cdot 35=168$
$(5,4)$	-135	-167	121	161	42	$-161+8\cdot 42=175$
$(5,5)$	-160	-162	142	168	42	$-168+8\cdot 42=168$

Theorem.

- $\mathsf{AST}_L(n;i,l,r) = \sum_{j=i}^{n-1} (-1)^{n+j+1}\binom{j+n-1}{j-i} \mathsf{AST}_L(n;j,r,l)$
- $\mathsf{AST}_L(n;n-1,l,r) = \sum_{p=1}^{l} \mathsf{AST}_L(n-1;p,p,r) + [r \geq n-1]$

Matrix: $M_n = \left((-1)^{n+j+1}\binom{j+n-1}{j-i}\right)_{0\leq i,j\leq n-1}$

Define

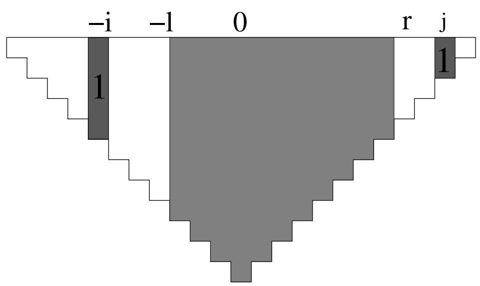
$$\mathsf{AST}_L^{+}(n;i,l,r) = \mathsf{AST}_L(n;i,l,r) + \mathsf{AST}_L(n;i,r,l),$$
$$\mathsf{AST}_L^{-}(n;i,l,r) = \mathsf{AST}_L(n;i,l,r) - \mathsf{AST}_L(n;i,r,l)$$

- $(\mathsf{AST}_L^{+}(n;i,l,r))_{0\leq i\leq n-1}$ is an eigenvector of M_n w.r.t to 1.
- $(\mathsf{AST}_L^{-}(n;i,l,r)_{0\leq i\leq n-1}$ is an eigenvector of M_n w.r.t to -1.

Again the dimensions of the eigenspaces are approximately $n/2$.

Second quantity

$\mathrm{AST}_{LR}(n;i,l,r,j)$ =Number ASTs of order n where
- $-i$ is the leftmost 1-alternating column,
- j is the rightmost 1-alternating column,
- all other 1-alternating columns range between $-l+1$ and $r-1$.



The total number of ASTs of order n is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathrm{AST}_{LR}(n;i,i,j,j).$$

Theorem.

- $\mathsf{AST}_{LR}(n; i, l, r, j) + \mathsf{AST}_{LR}(n; i+1, l, r, j) + \mathsf{AST}_{LR}(n; i, l, r, j+1)$
$$= (-1)^{i+j+1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \binom{-n-1-i}{p} \binom{-n-1-j}{q}$$
$(\mathsf{AST}_{LR}(n; j+q, l, r, i+p) + \mathsf{AST}_{LR}(n; j+q, l, r, i+p+1) + \mathsf{AST}_{LR}(n; j+q+1, l, r, i+p))$

- $\mathsf{AST}_{LR}(n; n-1, l, r, j) = \sum_{l_1=1}^{l} \mathsf{AST}_{LR}(n-1; l_1, l_1, r, j) + [r \geq n-2][j = n-2]$

Also this is a dimension-halving theorem in the above mentioned sense.

Refined ASM-Theorem

Observation: There is a unique 1 in the first row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in position $(1, i)$ is

$$\binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

Refined enumeration of DADASMs with respect to the position of the 1 in the first row

$\mathsf{d}(n,i) =$ Number of $n \times n$ DADASMs with a 1 in position $(1,i)$.

Observations:

- $\mathsf{d}(n,1) = \mathsf{d}(n-2) = \sum_{i=1}^{n-2} d(n-2,i)$
- $\mathsf{d}(n,2) = \mathsf{d}(n,1)$
- $\mathsf{d}(n,k) = \mathsf{d}(n,n+1-k)$

Conjecture (Ayyer, Fischer, 2012)

$$
\begin{aligned}
\mathrm{d}(2n+1,3) &= 2\,\mathrm{d}(2n-1) - 3\,\mathrm{d}(2n-3),\\
\mathrm{d}(2n+1,4) &= 4\,\mathrm{d}(2n-1) - 3(n+1)\,\mathrm{d}(2n-3)\\
\mathrm{d}(2n+1,5) &= 8\,\mathrm{d}(2n-1) - \frac{3(-36+11n+3n^2+4n^3)}{4(2n-3)}\,\mathrm{d}(2n-3)\\
\mathrm{d}(2n+1,6) &= 16\,\mathrm{d}(2n-1) - \frac{(n+2)(-90+51n+n^2+4n^3)}{4(2n-3)}\,\mathrm{d}(2n-3)
\end{aligned}
$$

General form ?

$$\mathrm{d}(2n+1,k) = 2^{k-2}\,\mathrm{d}(2n-1) + (\text{rational function in } n)\cdot \mathrm{d}(2n-3)$$

The center of DADASMs

The central entry $a_{n+1,n+1}$ of a $(2n+1) \times (2n+1)$ DADASM $(a_{i,j})$ is either 1 or -1.

$\mathrm{dc}^{+}(2n+1) =$ Number of DADASMs with $a_{n+1,n+1} = 1$

$\mathrm{dc}^{-}(2n+1) =$ Number of DADASMs with $a_{n+1,n+1} = -1$

Conjecture (Stroganov, 2008).

$$\frac{\mathrm{dc}^{+}(2n+1)}{\mathrm{dc}^{-}(2n+1)} = \frac{n+1}{n}$$

From Wikipedia on DADAISM

Dadaists expressed their rejection of the bourgeois capitalist society in artistic expression that appeared to reject logic and embrace chaos and irrationality.

I hope I did not embrace irrationality in this talk.

Thank you for your attention!