Enumeration of perfect matchings: Rhombus tilings and Pfaffian graphs

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Contents

Introduction: Connections between the two chapters of the thesis

At first sight this thesis consists of two independent chapters. In the first chapter I enumerate the rhombus tilings of a hexagon with side lengths a, b, c, a, b, c, which contain a fixed rhombus in the centre. The second chapter deals with the characterisation of Pfaffian graphs. Is there a connection between these two topics and if the answer is yes, what is this connection? The answer is indeed yes and, roughly speaking, both topics can be seen as a contribution to the theory of enumeration of perfect matchings.

A *matching* of a graph is a subset of edges of the graph, such that each vertex belongs to at most one chosen edge. A *perfect matching* is a matching, such that each vertex belongs to exactly one chosen edge. Thus every graph permits a matching but there are graphs which do not have a perfect matching.

The rhombus tiling enumeration problem in the first chapter of the thesis can obviously be seen as a perfect matching enumeration problem. Namely, there exists a correspondance between the rhombus tilings of a hexagon with side length a, b, c, a, b, c and the perfect matchings of a hexagonal grid. In order to see this, consider the graph G_1 in Figure 1.1 and the subgraph G_2 of its dual in Figure 0.1. Deletion of the vertex in the dual graph of G_1 corresponding to the exterior face of G_1 yields G_2. A rhombus tiling of the hexagon with side length 3, 5, 4, 3, 5, 4, is a 'pairing' of adjacent triangles in G_1, and thus corresponds to a perfect matching of G_2. In Figure 1.2 an example of a rhombus tiling is given and in Figure 0.2 the corresponding perfect matching of the hexagonal grid (the edges belonging to the perfect matching are thickened in Figure 0.2). From this point of view in the first chapter of my thesis I compute the number of perfect matchings of the hexagonal grid with 'side lengths' a, b, c, a, b, c which contain a fixed edge in the centre of the grid.

We turn our attention to the second chapter of my thesis and investigate its connection with the enumeration of perfect matchings. In 1967, Kasteleyn [**9**] introduced a powerful method for enumerating perfect matchings. The physicist developed this method in order to enumerate the perfect matchings of the quadratic lattice. However it turned out that his method can be extended to a quite big class of graphs which includes, e.g., the planar graphs. The graphs to which Kasteleyn's method can be extended are called *Pfaffian* graphs and as far as I know the problem of characterising Pfaffian graphs is still open.

In 1975, Little [**14**] published a characterisation of Pfaffian bipartite graphs in terms of forbidden subgraphs. The purpose of the second chapter of my thesis is to extend this characterisation to *near bipartite* graphs. Here we need to explain the term 'near bipartite'. A graph is 1-*extendible* if every edge has a perfect matching containing it. A 1-extendible non-bipartite graph G is said to be near bipartite if there exist edges e_1 and e_2 such that $G - \{e_1, e_2\}$ is 1-extendible and bipartite.

Now we want to explain Kasteleyn's method for enumerating perfect matchings and, thus, motivate the study of Pfaffian graphs. We first describe his method for bipartite graphs.

Consider a simple bipartite graph G with bipartition (U, W), where $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. (Without loss of generality we assume that $|U| = |W|$ for otherwise G does not have a perfect matching). Define the *biadjacency matrix* $A = (a_{ij})$ of G as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } \{u_i, w_j\} \in EG, \\ 0 & \text{otherwise,} \end{cases}$$

where EG denotes the edge set of G. Furthermore we define the permanent of the $n \times n$ matrix A by

$$\operatorname{per}(A) = \sum_{\pi \in \mathcal{S}_n} a_{1\pi(1)} a_{2\pi(2)} \ldots a_{n\pi(n)}.$$

Note that the only difference between this and the definition of the determinant is that in the definition of the permanent all terms are taken with the same sign. The significance of the permanent lies in the fact that the permanent of the biadjacency matrix counts the number of perfect matchings in the bipartite graph G: Each perfect matching corresponds to a bijection between U and W and therefore corresponds to a permutation π of $\{1, 2, \dots, n\}$ with $a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)} = 1$. The main problem with this technique is that the permanent, similar as it may seem to the determinant, is much more difficult to handle. The determinant is 'well-behaved' under certain row and column operations and consequently, by Gaussian algorithm, it is possible to evaluate a determinant in polynomial time. But if we want to evaluate a permanent, we have a problem which is at least as hard as any NP-complete problem (see [**23, 24**]).

But still the permanent is not too far away from the determinant and we therefore ask whether it is possible to reduce the calculation of the permanent to the calculation of a determinant in our special graph theoretical setting. This is indeed possible for a big class of graphs — which includes, e.g., all planar graphs.

The main idea is to compensate $\operatorname{sgn}\pi$ in the definition of the determinant by changing some 1's to -1's in the biadjacency matrix A of G. Phrased differently, we find an orientation of G such that the determinant of the biadjaceny matrix (or its negative) of the directed graph G^* is equal to the number of perfect matchings of G, where the *biadjacency matrix* $A' = (a'_{ij})$ of a directed graph G^* with bipartition (U, W) is defined as

$$a'_{ij} = \begin{cases} 1 & \text{if } (u_i, w_j) \in EG^*, \\ -1 & \text{if } (w_j, u_i) \in EG^*, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, the sign of $a'_{1\pi(1)} a'_{2\pi(2)}, \dots, a'_{n\pi(n)}$ is

$$(-1)^{\# i : (w_{\pi(i)}, u_i) \in EG^*} \tag{1}$$

if $\pi \in \mathcal{S}_n$ corresponds to a perfect matching of G. Thus $\operatorname{per}(A) = \det(A')$ if and only if the expression in (1) is equal to $\operatorname{sgn}\pi$ for all π that induce a perfect matching of G. Similarly $\operatorname{per}(A) = -\det(A')$ if and only if the expression in (1) is equal to $-\operatorname{sgn}\pi$ for all such π. This condition can also be formulated more elegantly as follows. Let $\{f_1, f_2, \dots, f_k\}$ be the set of perfect matchings of G^*. For every i write

$$f_i = \{(x_{i1}, y_{i1}), (x_{i2}, y_{i2}), \dots, (x_{in}, y_{in})\},$$

where $x_{ij}, y_{ij} \in VG^*$ and VG^* denotes the vertex set of G. Associate with f_i a plus sign if

$$x_{i1} y_{i1} x_{i2} y_{i2} \dots x_{in} y_{in}$$

is an even permutation of

$$x_{11} y_{11} x_{12} y_{12} \dots x_{1n} y_{1n}$$

and a minus sign otherwise. With this definition $\operatorname{per}(A) = |\det(A')|$ if and only if every perfect matching of G is associated to a plus sign. Graphs that permit such an orientation under which every perfect matching is associated with a plus sign are called *Pfaffian* and the orientation is called a *Pfaffian orientation* of G. It is easy to see that for example $K_{3,3}$ is non-Pfaffian and thus Kasteleyn's method does not work for all graphs.

We now extend this method to non-bipartite graphs. Let G be a simple graph with $VG = \{v_1, v_2, \dots, v_{2n}\}$. The number of perfect matchings of G is given by

$$Hf(B) = \sum_m \prod_{\substack{i<j,\\ \{v_i,v_j\}\in m}} b_{ij}$$

where the sum is taken over all perfect matchings of K_{2n} and $B = (b_{ij})$ is the adjacency matrix of G. Similar to the permanent of a matrix this expression above is not easy to calculate. But the *Pfaffian form*, similar to the determinant, is:

$$Pf(A) = \sum_m \operatorname{sgn} m \prod_{\substack{i<j,\\ \{v_i,v_j\}\in m}} b_{ij}.$$

Again the sum is taken over all perfect matchings of K_{2n} and the sign $\operatorname{sgn} m$ of a matching is defined as above. The Pfaffian form $Pf(A)$ is easy to calculate since it can be proved that

$$Pf(A')^2 = det(A') \tag{2}$$

if A' is a skew symmetric matrix (see [**18, 19**]). Again we want to compensate $\operatorname{sgn} m$ in $Pf(A)$ by changing some 1's in A to -1's. The skew adjacency matrix $B' = (b'_{ij})$ of the directed graph G^* is defined as follows:

$$b'_{ij} = \begin{cases} 1 & \text{if}(v_i, v_j) \in EG^*, \\ -1 & \text{if}(v_j, v_i) \in EG^*, \\ 0 & \text{otherwise.} \end{cases}$$

Note that B' is a skew symmetric matrix. Then $Hf(B) = Pf(B')$ holds if and only if every perfect matching of G is associated to the same sign under the orientation of G^*.

We saw that we needed some more linear algebra for the non-bipartite case, namely the identity (2). For bipartite graphs this identity is trivial, since

$$B = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix},$$

if we assume without loss of generality that $(\{v_1, \dots, v_n\}, \{v_{n+1}, \dots, v_{2n}\})$ is a bipartition of G.

So Kasteleyn's method reduces the problem of enumerating the perfect matchings of a graph to first finding a Pfaffian orientation and then evaluating a determinant. The perfect matching enumeration problem in Chapter 1 could be solved with Kasteleyn's technique too, since the hexgonal grid is planar and therefore Pfaffian. However we solved the problem in Chapter 1 differently – but mysteriously also ended up with a determinant (different to the one we would get from Kasteleyn's method however) which we subsequently had to evaluate.

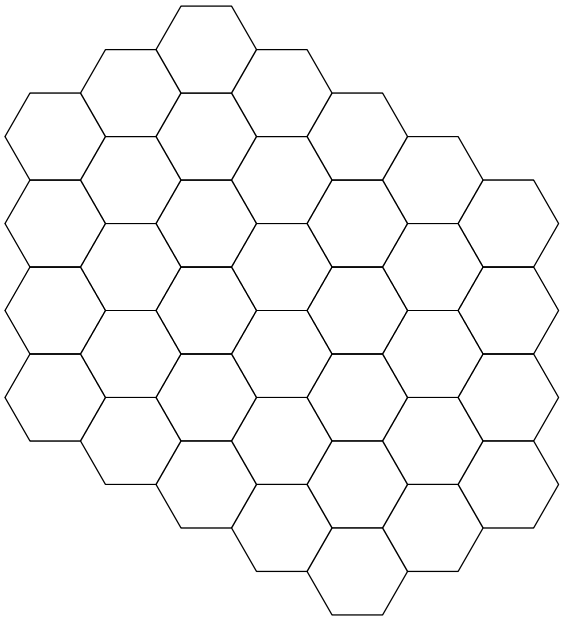


FIGURE 0.1. The dual graph of Figure 1.1.

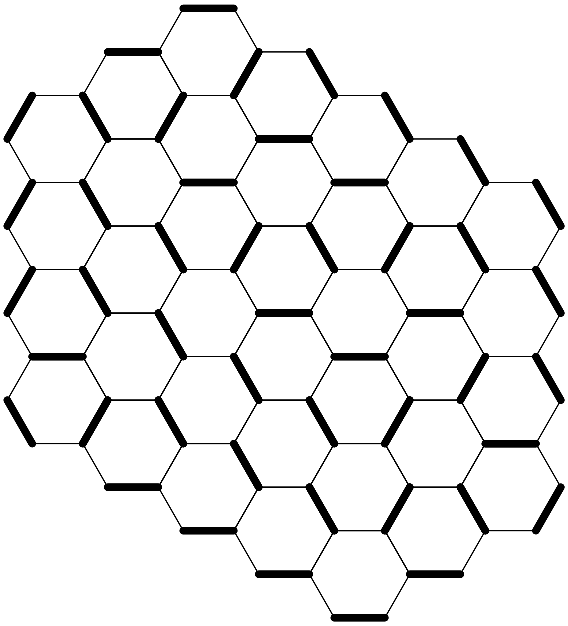


FIGURE 0.2. The perfect matching corresponding to the rhombus tiling in Figure 1.2.

CHAPTER 1

Enumeration of rhombus tilings of a hexagon which contain a fixed rhombus in the centre

1. Introduction

Let a, b and c be positive integers and consider a hexagon with side lengths a,b,c,a,b,c whose angles are 120° (see Figure 1.1).

The subject of our interest is rhombus tilings of such a hexagon using rhombi with all sides of length 1 and angles 60° and 120°. Figure 1.2 shows an example of a rhombus tiling of a hexagon with $a=3$, $b=5$ and $c=4$.

A first natural question to be asked is how many rhombus tilings of a fixed hexagon exist. A well known bijection between such rhombus tilings and plane partitions contained in an $a\times b\times c$ box [**4**] and MacMahon's enumeration of plane partitions [**17**, Sec. 429, $q\to 1$; proof in Sec. 494] give the following answer: *The number of all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c equals*

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{1.1}$$

On a next level, one may ask for the number of rhombus tilings with special properties. In this thesis, we address this question. We study rhombus tilings of a hexagon which contain the central rhombus and rhombus tilings of a hexagon which contain the 'almost central' rhombus above the centre. By the 'central rhombus' we mean the rhombus whose centre is equal to the centre of the hexagon (see Figure 1.3, where the central rhombus is marked; furthermore, the example of a rhombus tiling in Figure 1.2 contains the central rhombus). By the 'almost central' rhombus above the centre we mean the horizontal rhombus whose lowest vertex is the centre of the hexagon (see Figure 1.4).

The main results of this part of my thesis are the following two theorems.

THEOREM 1.1. *Let a,b,c be positive integers with $a\equiv b\ (mod\,2)$ and $a\not\equiv c\ (mod\,2)$. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the rhombus in the centre is*

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-1}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\sum_{k=0}^{(a-1)/2}\Bigg[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}$$
$$\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\Bigg] \tag{1.2}$$

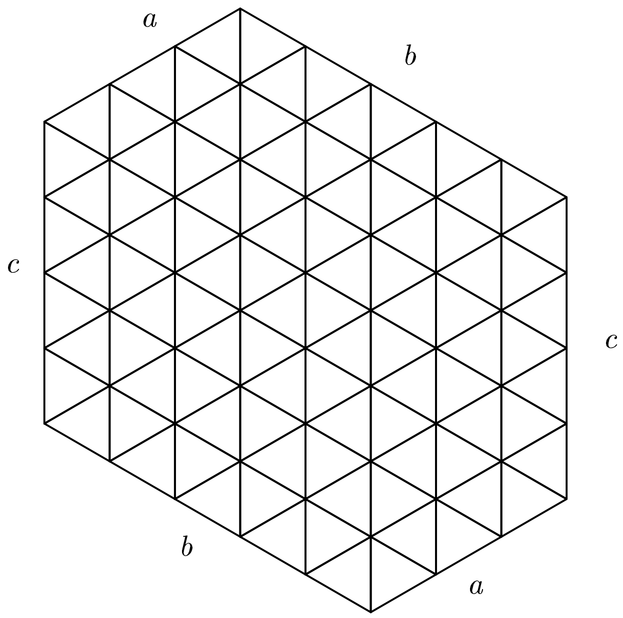


FIGURE 1.1. A hexagon with side lengths a, b, c, a, b, c, where $a = 3$, $b = 5$, $c = 4$.

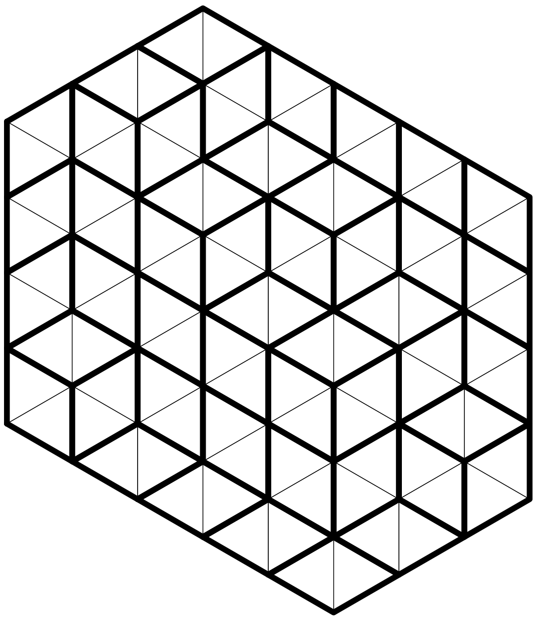


FIGURE 1.2. A rhombus tiling of a hexagon with side lengths a, b, c, a, b, c.

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{b(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-1}{2}}{\frac{b}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b}{2}}2^{a-2}$$
$$\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right] \qquad (1.3)$$

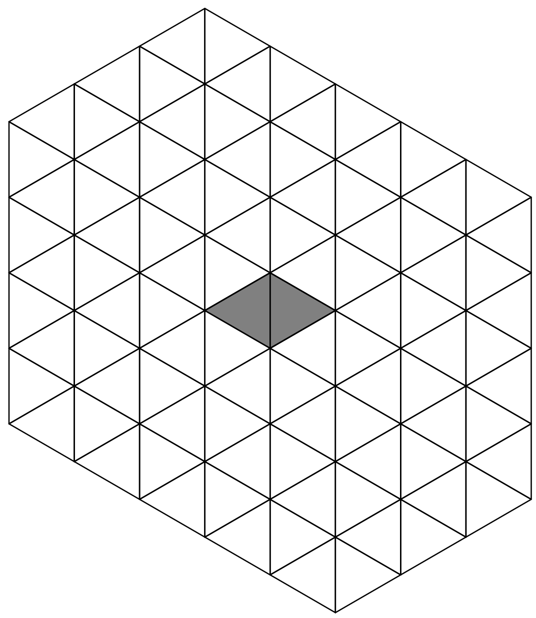


FIGURE 1.3. A hexagon with side lengths $a, b, c, a, b, c,$ where the central rhombus is marked.

in case that a is even, where the Pochhammer symbol $(a)_k$ *is defined by* $(a)_k := a(a+1)\dots(a+k-1)$ *if* $k \geq 1$ *and* $(a)_0 := 1$.

The special case $a = b$ in Theorem 1.1 was previously derived in [**2**, Theorem 1, 2]. (In order to see that the two sums in Theorem 1.1 are equal to two sums in Theorem 1 and Theorem 2 from [**2**] in this special case, one has to apply Bailey's transformation (5.3) for balanced ${}_4F_3$–series.)

The assumption about the parity of the side lengths a,b,c, i.e., that not all side lengths of the hexagon have the same parity, comes from the fact that this condition is necessary for the existence of a central rhombus in a rhombus tiling of a hexagon with side lengths a,b,c,a,b,c. There is also a result similar to Theorem 1.1 for a hexagon whose side lengths have the same parity. In this case we choose for the fixed rhombus the horizontal rhombus whose lowest vertex is the centre of the hexagon. (We could have also chosen the horizontal rhombus whose uppermost vertex is the centre of the hexagon). The number of these rhombus tilings is given by formulas quite similar to those in Theorem 1.1, and, furthermore, the proofs of these formulas are analogous to the proofs of the formulas in Theorem 1.1. We obtain the following:

THEOREM 1.2. *Let* a,b,c *be positive integers with* $a \equiv b \equiv c \pmod 2$. *Then the number of rhombus tilings of hexagon with side lengths* a,b,c,a,b,c *which contain the horizontal rhombus*

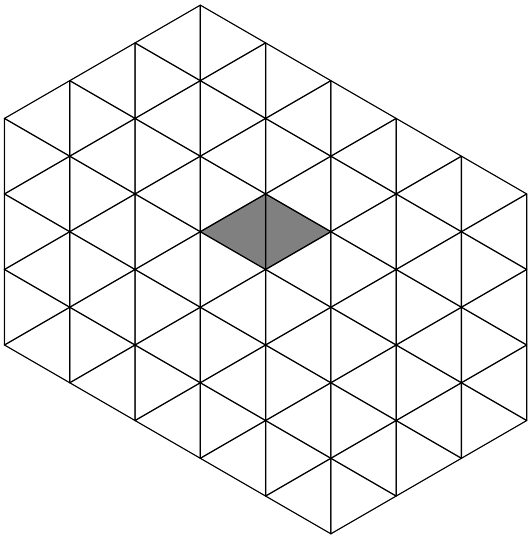


FIGURE 1.4. A hexagon with side lengths a, b, c, a, b, c, where the 'almost central' rhombus above the center is marked.

whose lowest vertex is the centre of the hexagon is equal to

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right) \tag{1.4}$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_{c+a}}\binom{\frac{b+c-2}{2}}{\frac{b-2}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-2}{2}}2^{a}$$
$$\times\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right) \tag{1.5}$$

in case that a is even.

The special case $a = b$ in Theorem 1.2 was previously derived in [**5**, Theorem 1, 2]. (In order to see that the two sums in Theorem 1.2 are equal to the two sums in Theorem 1 and Theorem 2 from [**5**] for this special case, one first has to apply a certain contiguous relation on the balanced ${}_4F_3$–series in Theorem 1.2. In each case we obtain the sum of a balanced ${}_3F_2$–series, which can be evaluated by Saalschütz's summation formula (see [**22**, (2.3.1.3)]), and another balanced ${}_4F_3$–series. An application of Bailey's transformation to the latter ${}_4F_3$–series finally shows the equivalence of the result in Theorem 1.2 and the result in [**5**, Theorem 1, 2] for the special case.)

These enumeration results for rhombus tilings with a fixed rhombus are not only interesting because they add more results to the growing set of results on the enumeration of rhombus tilings with special properties. These enumerations also contribute to the interesting question of what a 'typical' rhombus tiling of a hexagon looks like. Cohen, Larsen und Propp address this question in [**3**]. We are able to deduce the following two theorems from Theorem 1.1 and Theorem 1.2. They confirm Conjecture 1 in [**3**] for a special case.

THEOREM 1.3. *Let α, β and γ be non-negative real numbers. Then the probability that a rhombus tiling of the hexagon with side lengths a, b, c, a, b, c, where $a \sim \alpha N$, $b \sim \beta N$, $c \sim \gamma N$ and $a \equiv b \not\equiv c \pmod 2$, contains the rhombus in the centre is asymptotically*

$$\frac{2}{\pi}\arcsin\left(\sqrt{\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}}\right) \tag{1.6}$$

as N tends to infinity.

THEOREM 1.4. *Let α, β and γ be non-negative real numbers. Then the probability that a rhombus tiling of the hexagon with side lengths a, b, c, a, b, c, where $a \sim \alpha N$, $b \sim \beta N$, $c \sim \gamma N$ and $a \equiv b \equiv c \pmod 2$, contains the 'almost central' rhombus above the centre is asymptotically*

$$\frac{2}{\pi}\arcsin\left(\sqrt{\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}}\right) \tag{1.7}$$

as N tends to infinity.

Roughly speaking, Conjecture 1 in [**3**] predicts the following: Fix an arbitrary point (x, y) in the hexagon with side lengths *a*,*b*,*c*,*a*,*b*,*c*. Then the probability that a random rhombus tiling of a hexagon contains a vertical rhombus at this point tends to $\mathcal{P}_{\alpha,\beta,\gamma}(x, y)$ as the hexagon becomes large. Here, $\mathcal{P}_{\alpha,\beta,\gamma}$ is a certain function (defined in [**3**, Theorem 1]) that depends on the proportions α,β,γ of the side lengths of the hexagon (see Theorem 1.3). Theorem 1.3 and Theorem 1.4 confirm this conjecture for the centre of the hexagon. (The reader should know that Cohen, Larsen and Propp use another coordinate system in [**3**] than we do and therefore they are in the position to speak of vertical rhombi. But of course, their result can easily be translated into our coordinate system where we fix horizontal rhombi instead of vertical rhombi.)

We want to direct the reader's attention to a side result of the present work, given in Lemma 1.2. It expresses, for *any* fixed rhombus, the number of rhombus tilings containing that fixed rhombus as a triple sum. I am currently pursuing an asymptotic analysis of this triple sum with the ultimate goal of effectively proving Cohn, Larsen and Propp's conjecture.

In order to prove Theorem 1.1 and Theorem 1.2 we make use of a bijection between rhombus tilings which contain the central rhombus, respectively rhombus tilings which contain the 'almost central' rhombus above the centre, and non-intersecting lattice paths. This bijection actually works for arbitrary rhombi, i.e., the fixed rhombus need not be placed in the centre or next to the centre. The description of this bijection is the subject of Section 2. Since the number of non-intersecting lattice paths is given by a determinant due to Lindström, Gessel and Viennot (see Theorem 1.5), this bijection reduces the problem to evaluating a certain determinant with binomial entries, see Lemma 1.1. In case that the fixed rhombus is situated in the centre or next to the centre we are able to evaluate the determinant, see Lemma 1.3 and Lemma 1.4 in Section 4. We partly make us of a method that has already produced evaluations of other binomial determinants (see e.g. [**5**], [**11**]). In the course of evaluating the determinant we need an alternative expression for the number of rhombus tilings that we are interested in, in form of the aforementioned triple sum. This is given in Lemma 1.2 in Section 3. Finally, in Section 5 we provide the proofs of Theorem 1.3 and Theorem 1.4.

2. From rhombus tilings to non-intersecting lattice paths and determinants

As mentioned before, the subject of this section is the bijection between non-intersecting lattice paths and rhombus tilings. In our context, non-intersecting lattice paths are disjoint paths in the lattice $\mathbb{Z}^2$ with steps in direction $(1,0)$ and $(0,-1)$. Figure 1.6 shows an example of such a family of non-intersecting lattice paths. It consists of three paths, each one connecting an initial point A_i with a destination point E_i, $i=1,2,3$.

Figure 1.5 and Figure 1.6 illustrate the bijection between rhombus tilings of a hexagon with side lengths a,b,c,a,b,c and non-intersecting lattice paths with initial points

$$A_i = (i-1, c+i-1) \tag{2.1}$$

and destination points

$$E_i = (b+i-1, i-1), \tag{2.2}$$

$i=1,2,\dots,a$. Figure 1.5 shows the rhombus tiling from Figure 1.2 and an indication of the corresponding non-intersecting lattice paths. The initial points A'_i and the destination points E'_i of the paths in this figure are the centres of the sides of the rhombi that form the two sides of the hexagon with side lengths a. Roughly speaking these paths just describe 'the way down' from A'_i to E'_i on the three dimensional pile of cubes — elsewhere called *plane partitions* — which the rhombus tiling, when interpreted as three-dimensional object, gives.

In order to obtain the corresponding family of non-intersecting lattice paths in Figure 1.6, we only have to change the $120°$ angles of the paths in Figure 1.5 into right angles.

Therefore we already have a bijection between all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c and all non-intersecting lattice paths starting in A_i and stopping in E_i. However, we are interested in special rhombus tilings — namely in those with one fixed rhombus. Because of that we have to look for a simple description of the non-intersecting lattice paths that correspond to the rhombus tilings with a fixed rhombus. Studying Figure 1.5 and Figure 1.6 again, we see that the marked rhombus in Figure 1.5 corresponds to the edge $(x-1,y) \to (x,y)$ in Figure 1.6 (in our example $x=4$ and $y=3$). Therefore the rhombus tilings which contain a fixed horizontal rhombus correspond to the families of non-intersecting lattice paths which contain a fixed horizontal edge. Thus, we have found a bijection between rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain a fixed horizontal rhombus and non-intersecting lattice paths with initial points A_i and destination points E_i, $i=1,2,\dots,a$, which contain a fixed horizontal edge. As we will see in a moment, the latter can be counted with the help of the following main theorem on non-intersecting lattice paths:

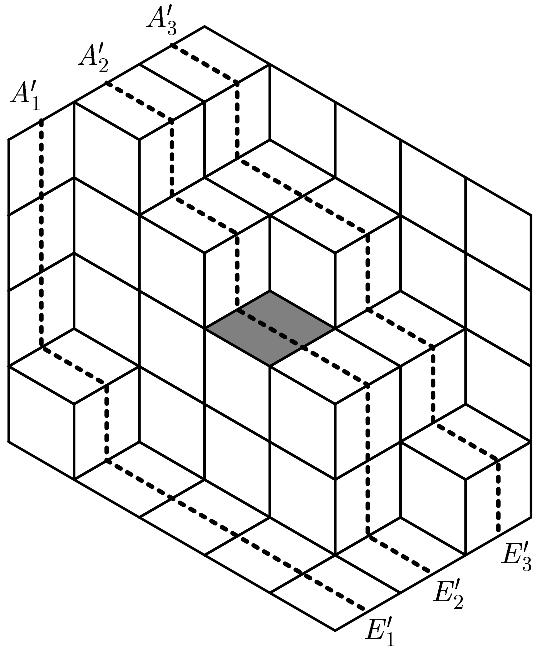


FIGURE 1.5. A rhombus tiling of a hexagon with an indication of the corresponding family of non-intersecting lattice paths.

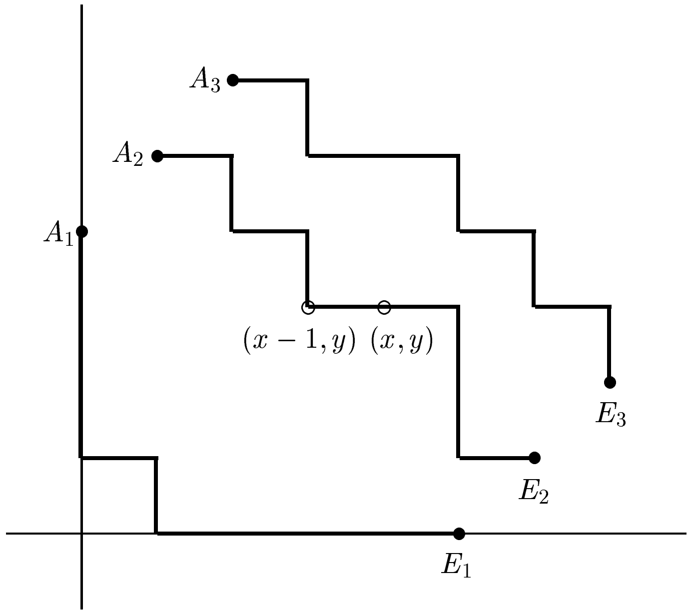


FIGURE 1.6. The familiy of non-intersecting lattice paths corresponding to the rhombus tiling in Figure 1.5.

THEOREM 1.5 (Lindström-Gessel-Viennot). *Let $A_1, A_2, \ldots, A_r$ and $E_1, E_2, \ldots, E_r$ be points in $\mathbb{Z}^2$. Then the determinant*

$$\det_{1\le i,j\le r}\left(\left|P\left(A_i \to E_j\right)\right|\right) \tag{2.3}$$

is equal to

$$\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \left| P^+ (A \to E_\sigma) \right| ,$$

where $|P(A_i \to E_j)|$ *denotes the number of lattice paths connecting* A_i *to* E_j, *and* $|P^+(A \to E_\sigma)|$ *denotes the number of families* $(P_1, P_2, \ldots, P_r)$ *of non-intersecting lattice paths, the* i*th path* P_i *connecting* A_i *to* $E_{\sigma(i)}$, $1 \le i \le r$.

This theorem can be found in [**7**, Theorem 1] or, alternatively, in [**12**, Lemma 1].

REMARK 1.1. Usually the theorem is applied to the following situation: There exists only one permutation σ (normally the identity permutation) such that the set of families of non-intersecting lattice paths connecting A_i to $E_{\sigma(i)}$, $1 \le i \le r$, is not empty (e.g., this is the case for the points A_i and E_i defined in (2.1) and (2.2)). Then Theorem 1.5 solves the enumeration problem of counting non-intersecting lattice paths — by means of the determinant (2.3) — totally. This is because $|P(A \to E)|$ is easy to determine: Let $A = (a_1, a_2)$ and $E = (e_1, e_2)$ be two points in $\mathbb{Z}^2$, such that A is located in the north-west of E (i.e., $e_1 \ge a_1$ and $a_2 \ge e_2$). Then the number of lattice paths with steps in direction $(1, 0)$ and $(0, -1)$ is

$$|P(A \to E)| = \binom{e_1 - a_1 + a_2 - e_2}{e_1 - a_1} = \binom{e_1 - a_1 + a_2 - e_2}{a_2 - e_2}, \tag{2.4}$$

since each lattice paths corresponds to a choice of $e_1 - a_1$ steps in direction $(1, 0)$ out of the total number of $e_1 - a_1 + a_2 - e_2$ steps.

Remark 1.1 shows that Theorem 1.5 can be useful in the enumeration of non-intersecting lattice paths with given initial and destination points. Since our enumeration problem of non-intersecting lattice paths not only involves fixed initial and destination points but also the additional condition about the fixed edge $(x-1, y) \to (x, y)$, Theorem 1.5 seems useless in our situation at first glance. The following 'trick' remedies this matter: We add the following pair of initial and destination points:

$$A_{a+1} = (x, y) \qquad \text{and} \qquad E_{a+1} = (x - 1, y) \tag{2.5}$$

(see Figure 1.6). Then, as is not difficult to see, the number of rhombus tilings with a fixed rhombus 'at' (x, y) equals the number of families of non-intersecting lattice paths with initial points A_i and destination points E_i, $i = 1, 2, \ldots, a + 1$.

Now we are ready to apply Theorem 1.5 to the points A_i and E_i, $i = 1, 2, \ldots, a + 1$, defined in (2.1), (2.2) and (2.5). In order to figure out what the determinant (2.3) actually gives, we have to find the permutations σ for which there exist non-intersecting lattice paths connecting A_i to $E_{\sigma(i)}$ with $1 \le i \le a+1$. It is quite easy to see that this is only accomplished by transpositions of the form $(i, a+1)$, where $i \ne a+1$. The fact that all transpositions have the same sign — namely -1 — implies that the determinant in (2.3) applied to our special points gives the number of families of non-intersecting lattice paths with initial points A_i and destination points E_i, $1 \le i \le a + 1$, with negative sign.

We use (2.4) to compute $|P(A_i \to E_j)|$ in (2.3), and finally obtain the following:

LEMMA 1.1. *Let a,b and c be positive integers and* (x, y) *be an integer point such that* $0 \le x \le b + a - 1$ *and* $1 \le y \le c + a - 1$. *Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the fixed horizontal rhombus that corresponds to the point* (x, y) *in the bijection described above equals*

$$
-\det_{1\le i,j\le a+1}
\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1}\\
\hline
\binom{c+x-y-1}{x-j} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ i=a+1 \end{array}. \tag{2.6}
$$

In Lemma 1.1 we refer to a correspondence between the integer points (x,y), $0\le x\le b+a-1$ and $1\le y\le c+a-1$, and the horizontal rhombi we use for the rhombus tilings of a hexagon with side lengths a,b,c,a,b,c. This correspondence is implicitly given by the bijection between rhombus tilings and non-intersecting lattice paths described above. The following remark makes the bijection between points and rhombi more explicit.

REMARK 1.2. Again we consider a hexagon with side lengths a,b,c,a,b,c. We introduce the following oblique angled coordinate system: Its origin is located in one of the two vertices, where sides of lengths b and c meet, and the axes are induced by those two sides (see Figure 1.7). The units are chosen such that the (Euclidean) side lengths of the considered hexagon are a,b,c,a,b,c in this coordinate system, too. (That is to say, the two triangles in Figure 1.7 with vertices in the origin form the unit 'square'). Thus, in this coordinate system, the points in Figure 1.5 have coordinates $A'_1=(1/2,9/2)$, $A'_2=(3/2,11/2)$, $A'_3=(5/2,13/2)$, $E'_1=(11/2,1/2)$, $E'_2=(13/2,3/2)$ and $E'_3=(15/2,5/2)$. Phrased differently, the coordinate system was chosen such that $A'_i=A_i+(1/2,1/2)$ and $E'_i=E_i+(1/2,1/2)$, where A'_i and E'_i are the initial and destination points of the paths in Figure 1.5, and A_i and E_i are defined as in (2.1), (2.2) and (2.5), $1\le i\le a+1$. From this point of view, the family of paths in Figure 1.5 is the family of paths in Figure 1.6 translated by $(1/2,1/2)$ and drawn in the oblique angled coordinate system. Furthermore, we see that the integer point (x,y) in the oblique angled coordinate system is just the lowest vertex of the corresponding rhombus under the aforementioned bijection between points and rhombi (see Figure 1.5 and Figure 1.6).

3. From the determinant to a triple sum

The aim of this section is the derivation of a triple sum that is equal to (2.6) and therefore gives the number of all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain a fixed rhombus with lowest vertex (x,y):

LEMMA 1.2. *Let a, b and c be positive integers, and let (x,y) be an integer point such that $0\le x\le b+a-1$ and $1\le y\le c+a-1$. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the fixed horizontal rhombus with lowest vertex (x,y) (in the oblique angled coordinate system; see Remark 1.2) equals*

$$
\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c}
$$
$$
\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\Bigg[(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b-x+y+s-1}{b-x+s-1}\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}
$$
$$
\binom{m-1}{s-1}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\Bigg]. \tag{3.1}
$$

Outline of the proof of Lemma 1.2. We derive the triple sum by starting from (2.6). The matrix underlying the determinant in (2.6) has a 'homogeneous' definition, except for

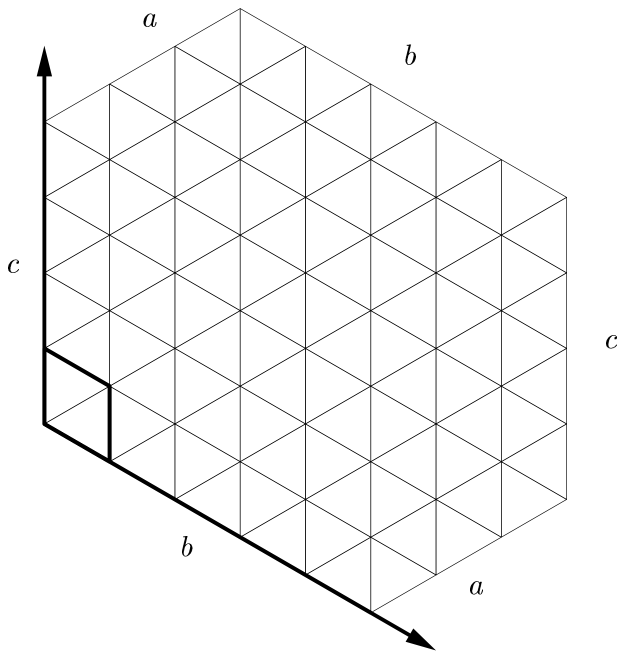


FIGURE 1.7. The oblique angled coordinate system.

the last row and the last column. Our proof starts with some elementary row and column operations that transform the 'homogeneous' submatrix into a matrix of triangular form (see (3.10)). Next we expand the determinant along the exceptional row and along the exceptional column (see (3.11)). The result is a triple sum with a summand that involves another determinant. But this determinant is the determinant of the aforementioned triangular matrix with one row and one column deleted (which row and which column depends on the summation index of the triple sum, see (3.13)) and therefore we are able to compute it: The triangle property allows an expansion of this remaining determinant along the first $(n-1)$ columns and the last $(a-m)$ rows, where m denotes the missing row and n denotes the missing column (see (3.14)). Finally the remaining $(m-n)\times(m-n)$ determinants can be reduced to Vandermonde's determinant (see (3.15)).

Proof of Lemma 1.2 – The details. As mentioned before, we first describe some elementary row and column operations that transform the 'homogeneous' $a \times a$ submatrix $\left(\binom{b+c}{c-i+j}\right)_{1\le i,j\le a}$ of the matrix underlying the determinant in (2.6) into an upper triangular matrix.

We begin with some elementary column operations: By using the symmetry of the binomial coefficient (i.e., the identity $\binom{n}{k}=\binom{n}{n-k}$) in the last row of the matrix underlying the determinant in (2.6) we observe that the determinant in (2.6) is equal to

$$
-\det_{1\le i,j\le a+1}
\begin{array}{c c}
\begin{array}{c|c} 1\le j\le a & j=a+1 \end{array} & \\
\left(\begin{array}{c|c}
\binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1} \\
\hline
\binom{c+x-y-1}{c-y+j-1} & 0
\end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}
\end{array}
. \qquad (3.2)
$$

We add the $(j-1)$th column to the jth column, $j = a, a-1, \dots, 2$, in that order. The entries of the changed matrix read as

$$\binom{b+c}{c-i+j} + \binom{b+c}{c-i+j-1} = \binom{b+c+1}{c-i+j}$$

for $i = 1, 2, \dots, a$ and $j = 2, 3, \dots, a$, and

$$\binom{c+x-y-1}{c-y+j-1} + \binom{c+x-y-1}{c-y+j-2} = \binom{c+x-y}{c-y+j-1}$$

for $i = a+1$ and $j = 2, 3, \dots, a$. The other entries do not change. Thus, the following determinant is equal to the determinant in (3.2):

$$\begin{array}{cc} \qquad\qquad \begin{array}{ccc} j=1 & 2\le j\le a & j=a+1 \end{array} & \\ -\det_{1\le i,j\le a+1}\left(\begin{array}{c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{c-y} & \binom{c+x-y}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.3}$$

Next we repeat the procedure, i.e., we add the $(j-1)$th column to the jth column, $j = a, a-1, \dots, 3$, in that order. Thus, we obtain that the following determinant is equal to the determinant in (3.3):

$$\begin{array}{cc} \qquad\qquad \begin{array}{cccc} j=1 & j=2 & 3\le j\le a & j=a+1 \end{array} & \\ -\det_{1\le i,j\le a+1}\left(\begin{array}{c|c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+2} & \binom{b+c+2}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{c-y} & \binom{c+x-y}{c-y+1} & \binom{c+x-y+1}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}.$$

We repeat this procedure of adding successive columns from right to left each time stopping one column earlier than before, as long as possible. This means that the procedure is performed $a-1$ times including the two steps described in detail. Since the upper parameter of the binomial entry increases by one every time it is involved and every entry in the jth column participates $j-1$ times exactly, $1 \le j \le a$, this procedure yields the following determinant:

$$\begin{array}{cc} \qquad\qquad \begin{array}{cc} 1\le j\le a & j=a+1 \end{array} & \\ -\det_{1\le i,j\le a+1}\left(\begin{array}{c|c} \binom{b+c+j-1}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.4}$$

Now we apply some elementary row operations to (3.4). In fact these row operations are analogous to the column operations we just applied to (3.2). In order to do so, we first use the symmetry of the binomial coefficient for the first a rows of the determinant in (3.4) and observe that the determinant

$$-\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c+j-1}{b+i-1} & \binom{b-x+y}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array} \tag{3.5}$$

is equal to the determinant in (3.4). As announced, we now add the $(i-1)$th row of the determinant in (3.5) to the ith row, starting at $i=a$ and stopping at $i=2$. Next we do the same with the resulting determinant, starting at $i=a$ but stopping at $i=3$. After repeating this procedure $a-1$ times we obtain that the following determinant is equal to the determinant in (3.5):

$$-\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c+i+j-2}{b+i-1} & \binom{b-x+y+i-1}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.6}$$

Now we take the factor $(-1)^{b+c+i-1}(b+c+i-1)!/(b+i-1)!$ out of the ith row of the determinant in (3.6), $i=1,2,\dots,a$. This yields

$$-\prod_{i=1}^{a}(-1)^{b+c+i-1}\frac{(b+c+i-1)!}{(b+i-1)!}$$
$$\times\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!}\binom{-j}{b+c+i-1} & (-1)^{b+c+i-1}\frac{(b+i-1)!}{(b+c+i-1)!}\binom{b-x+y+i-1}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.7}$$

Finally we want to apply the elementary row operations we just applied to the determinant in (3.5) once again. An analysis of the elementary row operations that we applied to the determinant in (3.5) yields that the determinant in (3.6) was produced from the determinant in (3.5) by replacing the ith row by

$$\sum_{s=1}^{i}\binom{i-1}{s-1}A(s),$$

$1\le i\le a$, where $A(s)$ denotes the sth row of the determinant in (3.5). We perform this replacement of the entries in the determinant in (3.7) and obtain

$$-\prod_{i=1}^{a}(-1)^{b+c+i-1}\frac{(b+c+i-1)!}{(b+i-1)!}$$
$$\times \det_{1\le i,j\le a+1}\left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!}\binom{i-j-1}{b+c+i-1} & \sum_{s=1}^{i}\binom{i-1}{s-1}(-1)^{b+c+s-1}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-1}{b+s-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \\ 1\le j\le a & j=a+1 \end{array}\right)\begin{array}{l}1\le i\le a\\ i=a+1\end{array}. \tag{3.8}$$

The entry in the ith row and jth column, $1 \le i \le a$ and $1 \le j \le a$, develops from the corresponding entry in (3.7) by using Vandermonde's summation formula:

$$\sum_{s=1}^{i}\binom{i-1}{s-1}\frac{(j-1)!}{(c+j-1)!}\binom{-j}{b+c+s-1}=\frac{(j-1)!}{(c+j-1)!}\binom{i-j-1}{b+c+i-1}.$$

Next we apply the elementary identity

$$\binom{n}{k}=\binom{-n+k-1}{k}(-1)^k \tag{3.9}$$

to this entry in the ith row and jth column of the matrix underlying the determinant in (3.8), $1 \le i \le a$ and $1 \le j \le a$, and then use the symmetry of the binomial coefficient. Finally we take the factor $(-1)^{b+c+i-1}$ out of the ith row, $1 \le i \le a$, and obtain that the expression in (3.8) is equal to

$$-\prod_{i=1}^{a}\frac{(b+c+i-1)!}{(b+i-1)!}$$
$$\times \det_{1\le i,j\le a+1}\left(\begin{array}{c|c} 1\le j\le a & j=a+1 \\ \frac{(j-1)!}{(c+j-1)!}\binom{b+c+j-1}{j-i} & \sum_{s=1}^{i}\binom{i-1}{s-1}(-1)^{i-s}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-1}{b+s-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right)\begin{array}{l}1\le i\le a\\ i=a+1\end{array}. \tag{3.10}$$

With pleasure we discover that the $a \times a$ submatrix induced by the first a rows and first a columns of the matrix underlying the determinant in (3.10) is an upper triangular matrix. This is because the binomial coefficient $\binom{\alpha}{k}$ is defined to be zero if $k < 0$ for any indeterminante α. Therefore the determinant is of that form we were looking for at the beginning of the proof.

Next we expand (3.10) along the last row and then along the last column, to obtain

$$-\prod_{i=1}^{a}\frac{(b+c+i-1)!}{(b+i-1)!}\sum_{n=1}^{a}(-1)^{a+1+n}\binom{c+x-y+n-2}{c-y+n-1}$$
$$\times\sum_{m=1}^{a}(-1)^{a+m}\sum_{s=1}^{m}\binom{m-1}{s-1}(-1)^{m-s}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-1}{b+s-x-1}$$
$$\times \det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\frac{(j-1)!}{(c+j-1)!}\binom{b+c+j-1}{j-i}\right). \tag{3.11}$$

Then we take the factor $(j-1)!/(c+j-1)!$ out of the jth column of the remaining determinant. This, together with some other manipulations, gives

$$\prod_{i=1}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!} \\ \times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b-x+y+s-1}{b+s-x-1}\binom{m-1}{s-1} \\ \times\frac{(b+s-1)!}{(b+c+s-1)!}\frac{(c+n-1)!}{(n-1)!}\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{3.12}$$

Now we have to compute

$$\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{3.13}$$

This is a determinant of an upper triangular matrix, where the mth row and nth column was deleted. Therefore it is only different from zero if $n\le m$. So, let us assume $n\le m$. Expansion of the determinant along the first $(n-1)$ columns and along the last $(a-m)$ rows yields

$$\det_{\substack{n+1\le j\le m\\ n\le i\le m-1}}\left(\binom{b+c+j-1}{j-i}\right)=\det_{\substack{n\le j\le m-1\\ n\le i\le m-1}}\left(\binom{b+c+j}{j-i+1}\right) \tag{3.14}$$

for the determinant in (3.13).

We are going to reduce this determinant to Vandermonde's determinant. In order to do so, we take $(b+c+j)!/(j-n+1)!$ out of the jth column, $1\le j\le a$, and $1/(b+c+i-1)!$ out of the ith row, $1\le i\le a$. Thus, we obtain that the determinant in (3.14) is equal to

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\det_{\substack{1\le j\le m-n\\ 1\le i\le m-n}}\left((j-i+2)_{i-1}\right). \tag{3.15}$$

The entries of this determinant are monic polynomials in j of degree $i-1$, where i and j denote as usual the index of the row and the column of the entry. It is now straightforward to reduce this determinant by appropriate row operations to Vandermonde's determinant,

$$\det_{\substack{1\le i\le m-n\\ 1\le j\le m-n}}\left((j-i+2)_{i-1}\right)=\det_{\substack{1\le i\le m-n\\ 1\le j\le m-n}}\left(j^{i-1}\right)=\prod_{i=1}^{m-n-1}i!. \tag{3.16}$$

The last equation was obtained by applying the well-known formula for Vandermonde's determinant to $\det_{1\le i\le m-n,1\le j\le m-n}\left(j^{i-1}\right)$.

Using (3.16) in (3.15), we obtain that the determinant in (3.13) equals

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\prod_{i=1}^{m-n-1}i!=\frac{(b+c+n)_{m-n}}{(m-n)!}.$$

Accordingly, we replace the determinant in (3.12) by its value $(b+c+n)_{m-n}/(m-n)!$. Thus, we obtain the following triple sum for the determinant in (2.6):

$$\left(\prod_{i=2}^{a} \frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}\right)$$
$$\times \sum_{n=1}^{a} \sum_{m=1}^{a} \sum_{s=1}^{m} \left[(-1)^{n+s} \binom{c+x-y+n-2}{x-1} \binom{b-x+y+s-1}{b-x+s-1} \binom{m-1}{s-1} \right.$$
$$\left. \times \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}} \frac{(c+1)_{n-1}}{(n-1)!} \frac{(b+c+n)_{m-n}}{(m-n)!}\right].$$

Therefore Lemma 1.2 is finally proved since

$$\prod_{i=2}^{a} \frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!} = \left(\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}\right) \frac{(1)_c}{(b+1)_c}.$$

4. Evaluation of the determinants

In this section we compute the determinant from Lemma 1.1 in case that the fixed rhombus is placed in the centre (see Lemma 1.3) and in case that the lowest vertex of the fixed rhombus is placed in the centre (see Lemma 1.4). The combination of these two lemmas and Lemma 1.1 then establish Theorem 1.1 and Theorem 1.2.

In order to do so, we first have to figure out which integer point (x, y) corresponds to the central rhombus, respectively the 'almost central' rhombus above the centre, via the bijection described in Section 2. In the oblique angled coordinate system introduced in Remark 1.2 the point $((a+b)/2, (a+c)/2)$ is the centre of the considered hexagon. Because of that, $((a+b)/2, (a+c-1)/2)$ is the lowest vertex of the central rhombus and therefore the integer point that corresponds to the central rhombus (see Remark 1.2). Accordingly, $((a+b)/2, (a+c)/2)$ is the lowest vertex of the 'almost central' rhombus above the centre.

The evaluations of both determinants, namely the determinant corresponding to the case that the fixed rhombus is placed in the centre, and the determinant corresponding to the case that the fixed rhombus is placed next to the centre on the other hand, are quite similar. Therefore we concentrate on the first case, see the following lemma. Lemma 1.4 at the end of this section is devoted to the second case, but there I only explain the major differences to the first case.

LEMMA 1.3. *Let a,b,c be integers. Then the determinant*

$$- \det_{1 \le i,j \le a+1} \begin{array}{c} \begin{array}{cc} \quad 1 \le j \le a \quad & \quad j = a+1 \quad \end{array} \\ \left(\begin{array}{c|c} \dbinom{b+c}{c-i+j} & \dbinom{\frac{b+c-1}{2}}{\frac{c}{2} - i + \frac{a+1}{2}} \\ \hline \dbinom{\frac{b+c-1}{2}}{\frac{c}{2} - \frac{a+1}{2} + j} & 0 \end{array}\right) \end{array} \begin{array}{l} 1 \le i \le a \\ \\ i = a+1 \end{array}$$

is equal to

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_c}\frac{1}{(b+c+1)_{a-1}}\binom{\frac{b+c-1}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right]$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_c}\frac{b}{(b+c+1)_{a-1}}\binom{\frac{b+c-1}{2}}{\frac{b}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b}{2}}2^{a-2}$$
$$\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right]$$

in case that a is even.

Outline of the proof of Lemma 1.3. The following procedure is used for computing the determinant: The determinant depends on the side lengths a, b and c of the hexagon. More precisely: The dimension of the matrix underlying the determinant is $a+1$, and the entries are certain binomial coefficients that depend on b, c and a. First, in Step 1, we reduce the problem to the computation of a determinant with entries that are polynomials in b and c if we fix a, see (4.6), and therefore a determinant that is itself a polynomial in those two variables. The comparison of the determinant in Lemma 1.3 and the determinant in (4.6), the latter will be denoted by $\det(D_a(b,c))$, yields that we have to show that the determinant $\det(D_a(b,c))$ is a product of certain linear factors and an irreducible polynomial (over $\mathbb{Z}$) in b and c, see (4.7), respectively (4.8).

In Step 2 of our proof we show that every linear factor on the right-hand side of (4.7), respectively (4.8), is indeed a linear factor of $\det(D_a(b,c))$. I explain the used procedure by an example: We have to show, e.g., that the linear factor $(c+1)$ is a factor of $\det(D_a(b,c))$ in case that a is even (see (4.8)). It is a fundamental algebraic fact that, in order to show that, it suffices to show that $\det(D_a(b,-1))=0$ if a is even. (Here we use the fact that $\det(D_a(b,c))$ is a polynomial in c if we fix a). Clearly, a determinant of a matrix is equal to zero if and only if there exists a linear combination of rows, or, equivalently, of columns. Therefore we find a linear combination of rows of $D_a(b,-1)$ which vanishes, and then prove it, see (4.13). Proving in this case means to establish hypergeometric identities [1].

At this point it is worth mentioning that the procedure described so far gives a complete solution for determinants that factorise completely into linear factors.

In Step 3 of the proof we finally compute the irreducible polynomial, which will be denoted by $P_a(b,c)$. In order to do so, we first look for special values of b, where $P_a(b,c)$ is 'nice'. Indeed, we discovered that $P_a(b,c)$ factors completely into linear factors for $b=-c-k$,

[1]At this point it is worth mentioning that I used C. Krattenthaler's *Mathematica* package HYP [**10**] to handle most of the hypergeometric identities within this thesis.

$k = 1, 3, \dots, 2\lfloor (a-1)/2 \rfloor + 1$. We work out these evaluations of $P_a(b,c)$ in (4.30) and (4.31), and subsequently prove them by making use of the triple sum derived in Section 3. As it turns out, the degree of $P_a(b,c)$ as a polynomial in b is exactly $\lfloor (a-1)/2 \rfloor$. Thus, the above $\lfloor (a-1)/2 \rfloor + 1$ evaluations suffice to compute $P_a(b,c)$ by using Lagrange interpolation, see (4.11) and (4.12).

Proof of Lemma 1.3 – The details.

Step 1: From our determinant to a determinant with polynomial entries.

In Section 2 it was finally shown that the number of rhombus tilings which contain a fixed horizontal rhombus with lowest vertex (x,y) is given by the following $(a+1) \times (a+1)$ determinant:

$$-\det_{1 \le i,j \le a+1} \begin{array}{c} \begin{array}{cc} 1 \le j \le a & j = a+1 \end{array} \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{x-j} & 0 \end{array} \right) \end{array} \begin{array}{l} 1 \le i \le a. \\ i = a+1 \end{array} \tag{4.1}$$

With displeasure we observe that the matrix underlying this determinant has an exceptional row and an exceptional column, namely the $(a+1)$th in both cases. In the following I describe some elementary row and column operations which lead to a matrix having the same determinant but a 'homogeneous' definition:

1. First we want the entries in the $(a+1)$th row to be zero — except for the first entry. Since $\binom{c+x-y-1}{x-j}$ is the jth entry in the $(a+1)$th row, we have to subtract $\binom{c+x-y-1}{x-j} / \binom{c+x-y-1}{x-j+1} = (x-j+1)/(c-y+j-1)$ times the $(j-1)$th column from the jth, starting at $j = a$ and stopping at $j = 2$. We obtain a new matrix with the desired behaviour in the $(a+1)$th row. The (i,j) entry of this new matrix is

$$\binom{b+c}{c-i+j} - \frac{x-j+1}{c-y+j-1} \binom{b+c}{c-i+j-1},$$

$i,j = 2,3,\dots,a$, it is $\binom{c+x-y-1}{x-1}$ for $i = a+1$ and $j = 1$, and it is 0 for $i = a+1$ and $j \neq 1$. The other entries do not change.

2. Next we want to do the same for the $(a+1)$th column: Accordingly, since $\binom{b-x+y}{y-i+1}$ is the ith entry in the $(a+1)$th column, we subtract $\binom{b-x+y}{y-i+1} / \binom{b-x+y}{y-i+2} = (y-i+2)/(b+i-x-1)$ times the $(j-1)$th row from the jth row, again starting at the bottom — that is to say, $j = a, a-1, \dots, 2$. Therefore the (i,j) entry of our new matrix is given as follows:

$$\begin{aligned} &\binom{b+c}{c-i+j} - \frac{x-j+1}{c-y+j-1} \binom{b+c}{c-i+j-1} \\ &- \frac{y-i+2}{b+i-x-1} \left(\binom{b+c}{c-(i-1)+j} - \frac{x-j+1}{c-y+j-1} \binom{b+c}{c-(i-1)+j-1} \right) \end{aligned} \tag{4.2}$$

if $i,j = 2,3\dots,a$, it is $\binom{b-x+y}{y}$ for $j = a+1$ and $i = 1$, and it is 0 for $j = a+1$ and $i \neq 1$. Again, the other entries do not change.

3. In summary, we obtain the following determinant,

$$\det_{1\le i,j\le a+1}\begin{pmatrix} * & * & * & * & \binom{b-x+y}{y} \\ * & & & & 0 \\ * & & a_{ij} & & \vdots \\ * & & & & \vdots \\ \binom{c+x-y-1}{x-1} & 0 & \dots & \dots & 0 \end{pmatrix},$$

where $*$ describes an entry which is of no interest for us, and where a_{ij} denotes the expression in (4.2). This form suggests an expansion of the determinant along the last row and along the last column. Therefore (4.1) is equal to

$$-\binom{c+x-y-1}{x-1}\binom{b-x+y}{y}\det_{2\le i,j\le a}(a_{ij}). \tag{4.3}$$

We have reduced our problem to computing $\det_{2\le i,j\le a}(a_{ij})$ and therefore the job to compute the determinant of a 'homogeneous' matrix, even if the entries are now more complex. The advantage is that we can now take several factors out of the determinant so that the remaining entries are polynomials: A simple calculation shows that

$$a_{ij}=\frac{(b+c)!(b+i-j+2)_{j-2}(c-i+j+2)_{a-j}}{(b+i-1)!(c-j+a+1)!(c-y+j-1)(b+i-x-1)}\times H,$$

where H is the polynomial

$$\begin{aligned} H=(b+i-j+1)&(c-i+j+1)(c-y+j-1)(b+i-x-1)\\ &-(c-i+j)(c-i+j+1)(x-j+1)(b+i-x-1)\\ &-(b+i-j)(b+i-j+1)(y-i+2)(c-y+j-1)\\ &+(b+i-j+1)(c-i+j+1)(x-j+1)(y-i+2). \end{aligned} \tag{4.4}$$

We take $(b+c)!/((b+i-1)!(b+i-x-1))$ out of the ith row, and $1/((c-j+a-1)!(c-y+j-1))$ out of the jth column of the matrix $(a_{ij})_{2\le i,j\le a}$. This gives

$$\begin{aligned}\det_{2\le i,j\le a}(a_{ij})=\left(\prod_{i=2}^{a}\frac{(b+c)!}{(b+i-1)!(c-i+a+1)!(c-y+i-1)(b+i-x-1)}\right)&\\ \times\det_{2\le i,j\le a}\left((b+i-j+2)_{j-2}(c-i+j+2)_{a-j}\cdot H\right).&\end{aligned} \tag{4.5}$$

Again we have reduced our problem to the evaluation of another 'homogeneous' determinant, namely

$$\det_{2\le i,j\le a}\left((b+i-j+2)_{j-2}(c-i+j+2)_{a-j}\cdot H\right). \tag{4.6}$$

But this one has the pleasing property to be a polynomial in b and c if we fix a (since the entries of the underlying matrix are). In the following, $D_a(b,c)$ denotes the matrix underlying the determinant in (4.6) evaluated at $x=(a+b)/2$ and $y=(a+c-1)/2$.

When I computed this determinant for small values of a, I was led to the following conjecture [2]: There holds

$$\begin{aligned}\det_{2\le i,j\le a}\left(D_a(b,c)\right) &= \left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)2^{a-1}\\ &\quad \left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+2}{2}\right)_{(a-1)/2}\left(\frac{1+b+c}{2}\right)_{(a-1)/2}\\ &\quad\times\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.\\ &\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right] \end{aligned}\tag{4.7}$$

if a is odd, and

$$\begin{aligned}\det_{2\le i,j\le a}\left(D_a(b,c)\right) &= \left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)2^{a-2}\\ &\quad b\left(\frac{b+2}{2}\right)^2_{(a-2)/2}\left(\frac{c+1}{2}\right)_{a/2}\left(\frac{1+b+c}{2}\right)_{a/2}\\ &\quad\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.\\ &\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right] \end{aligned}\tag{4.8}$$

if a is even. If we remember the assertion in Lemma 1.3 and the factors we have taken out of the determinant in (4.3) and (4.5), and if we then specialise $x = (a+b)/2$ and $y = (a+c-1)/2$, a tedious but straightforward calculation yields that in order to complete a proof of Lemma 1.3 it remains to show (4.7) and (4.8).

As described in the outline of the proof, we first prove that the claimed linear factors, which are

$$\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+2}{2}\right)_{(a-1)/2}\left(\frac{1+b+c}{2}\right)_{(a-1)/2}2^{2a-2}\tag{4.9}$$

if a is odd, and, respectively,

$$\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)b\left(\frac{b+2}{2}\right)^2_{(a-2)/2}\left(\frac{c+1}{2}\right)_{a/2}\left(\frac{1+b+c}{2}\right)_{a/2}2^{2a-2}\tag{4.10}$$

[2] In fact, only the linear factors in front of the sums in (4.7) and (4.8) can immediately be guessed, by computing $\det(D_a(b,c))$ for $a = 2,3,\dots,8$. In order to work out an conjecture concerning the two sums themselves I first computed them for $a = 2,3,\dots,23$. On the basis of these data, I worked out (guesses for) 'nice' evaluations of these polynomials at special values of b, for *all* values of a. (See the Outline of the proof of Lemma 1.3 and Step 3 of the proof of Lemma 1.3 for more details, and, in particular, for an explicit listing of these special values of b). Thereby I was helped by Krattenthaler's *Mathematica* 'guessing machine' RATE (which is available via Internet at `http://radon.mat.univie.ac.at/People/kratt/rate/rate.html`; see also [**11**, Appendix A]). Once I had got thus far, I computed the polynomials in (4.7) and (4.8) by Lagrange interpolation.

if a is even, divide $\det(D_a(b,c))$ (see Step 2). In Step 3 we finally calculate the remaining irreducible polynomial in b and c (over $\mathbb{Z}$), which reads

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-1}\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right] \tag{4.11}$$

if a is odd, and respectively,

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a}\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right] \tag{4.12}$$

if a is even.

Step 2: The linear factors in (4.9), respectively in (4.10), divide $\det(D_a(b,c))$ as a polynomial in b and c.

Essentially we have four different types of linear factors:

1. Factors of the form $(c+k)$: If a is odd, $k=2,4,\dots,a-1$. Otherwise $k=1,3,\dots,a-1$.
2. Factors of the form $(b+k)^2$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$. The factor b occurs once if a even.
3. Factors of the form $(b+c+k)^{a-k}$ for $k=1,3,\dots,2\lceil(a-1)/2\rceil-1$.
4. Factors of the form $(b+c+k)^{a-k-1}$ for $k=2,4,\dots,2\lfloor(a-1)/2\rfloor$.

re 1. — The factors of the form $(c+k)$ divide $\det(D_a(b,c))$: In order to show that $(c+k)$ is a linear factor of $\det(D_a(b,c))$, where $1\le k\le a$ and $k\not\equiv a \pmod 2$, it suffices to show that $\det(D_a(b,-k))=0$ for those special k's. Therefore we search for linearly dependent rows or columns in the matrices $D_a(b,-k)$.

By computer experiments [3], I found the following non-trivial linear combination of rows:

$$\sum_{i=(a-k+3)/2}^{a-k+1}\frac{(-1)^i(b+i)_{a-k+1-i}\left(\frac{-a+k-1+2i}{2}\right)_{a-k+1-i}}{(1)_{a-k+1-i}\left(\frac{b-a+2i-2}{2}\right)_{a-k+1-i}}D_a(b,-k)_{(i,j)}=0, \tag{4.13}$$

for $j=2,3,\dots,a$.

In order to complete the proof that $(c+k)$ divides the determinant $\det(D_a(b,c))$, we just have to prove this identity. *Gosper's algorithm* [**8**] for hypergeometric sums (which are sums, where the quotient of two successive summands is a rational function in the summation index) helps us to recognize that this is actually a telescoping sum: First we examine the factor $((-a+k-1+2i)/2)_{a-k+1-i}$ in the linear combination. I claim that this factor is equal to zero if $i<(a-k+3)/2$ and because of that we can omit the lower bound on the summation index on the left-hand side of (4.13): In this case $(-a+k-1+2i)/2\le 0$ and

[3]Again, I computed these linear combinations for small values of a, i.e., I solved the following system of linear equations in $c_i(a,b,k)$: $\sum_{i=2}^{a}c_i(a,b,k)D_a(b,-k)_{(i,j)}=0$, $j=2,3,\dots,a$, for $a=2,3,\dots,15$. I was led to an conjecture for the coefficient $c_i(a,b,k)$ for all values of a by using Krattenthaler's *Mathematica* 'guessing machine' RATE (which is available via Internet at `http://radon.mat.univie.ac.at/People/kratt/rate/rate.html`).

$(a-k-1)/2 \geq 0$. Because of that, and since $(-a+k-1+2i)/2$ is an integer, one of the factors of the Pochhammer symbol

$$\left(\frac{-a+k-1+2i}{2}\right)_{a-k+1-i} = \left(\frac{-a+k-1+2i}{2}\right)\left(\frac{-a+k-1+2i}{2}+1\right)\dots\left(\frac{a-k-1}{2}\right)$$

is equal to zero. If we then reverse the order of summation, we obtain that we must show the following identity:

$$\sum_{i=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}-\frac{a}{2}+\frac{k}{2}\right)_i (-a-b+k)_{i+j-2}}{(1)_{-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_i} H\,\Big|_{x=(a+b)/2,y=(a+c-1)/2,c=-k,i\to a-k+1-i} = 0. \tag{4.14}$$

Let $f(i)$ denote the summand of the sum in (4.14). The magnificent algorithm due to Gosper decides whether there exists a hypergeometric $g(i)$ with

$$g(i+1) - g(i) = f(i).$$

And in case of its existence the algorithm also computes $g(i)$. If such an $g(i)$ was found we would have

$$\sum_{i=0}^{\infty} f(i) = \sum_{i=0}^{\infty} (g(i+1) - g(i)) = \lim_{i\to\infty} (g(i) - g(0)).$$

($\lim_{i\to\infty} g(i)$ exists in our cases, since $f(i) = 0$ for all but a finite number of i and therefore $g(i)$ is finally constant.)

A computer implementation of Gosper's algorithm [**20**] prints out

$$\frac{(-1)^j (-a-b+k)_{i+j-2}\left(\frac{1}{2}-\frac{a}{2}+\frac{k}{2}\right)_i (-1-b+k)(i-j+k)}{2(1)_{-1-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_{i-1}}$$

to be a suitable $g(i)$ for our $f(i)$. One may check the identity $g(i+1)-g(i) = f(i)$ by dividing the left-hand side by the right-hand side and simplifying the resulting rational function to 1.

This implies $\sum_{i=0}^{\infty} f(i) = 0$, i.e., the truth of (4.14), since $g(0) = 0$, caused by the factor $1/(1)_{-1-a+j}$, and $g(i) = 0$ for $i \geq (1+a-k)/2$, caused by $((1+a-k-2i)/2)_i$ (again we use the fact that $1 \leq k \leq a$ and $k \not\equiv a \pmod 2$). Thus we have proved that $c+k$ is a factor of $\det(D_a(b,c))$.

re 2. — The factors of the form $(b+k)$ *divide* $\det(D_a(b,c))$*:* Next we have to prove that $(b+k)^2$ is a factor of $\det(D_a(b,c))$ if $0 < k < a$ and $k \equiv a \pmod 2$, and, in addition, that b is a factor of $\det(D_a(b,c))$ if a is even.

In case that $k = a-2$, this is easy: If we examine the matrix $D_a(-a+2,c)$ carefully, we observe that the $(a-1)$th and ath row vanish. (For $j > 2$ this is caused by $(b+i-j+2)_{j-2}$, and for $j = 2$ by the nasty polynomial H.) Therefore $(b+a-2)$ is a factor of the $(a-1)$th row and a factor of the ath row of $D_a(b,c)$. Thus, we have proved that $(b+a-2)^2$ is a factor of the determinant $\det(D_a(b,c))$.

In case that $k < a-2$ we use the same method as for the factors of form $(c+k)$ above. But since we are now dealing with factors of higher multiplicity, it is not enough to find just one linear combination of rows or columns of $D_a(-k,c)$. In fact we have to find two linearly independent combinations if we want to prove that $(b+k)^2$ is a factor.

We claim that

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)_{i-k-2}^2} \\ \times p_{i-k-1}(c+a-k-i+1)D_a(-k,c)_{(i,j)} = 0 \quad (4.15)$$

if $0 \le k < a-2$ and $k \equiv a \pmod 2$, and that

$$\sum_{i=k+3}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-1}\left(\frac{a+k-2i+4}{2}\right)_{i-k-1}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)_{i-k-2}^2} \\ \times p_{i-k-2}(c+a-k-i)D_a(-k,c)_{(i,j)} = -D_a(-k,c)_{(k+1,j)} \quad (4.16)$$

if $0 < k < a-2$ and $k \equiv a \pmod 2$ are such linearly independent linear combinations of rows, $p_n(c)$ being the sequence of polynomials given by

$$p_n(c) = \sum_{h=0}^{n-1} \left(\frac{1+c-n}{2}\right)_{n-h-1} \left(\frac{1+c-2h+n}{2}\right)_h . \quad (4.17)$$

Notice that the exceptional factor b of $\det(D_a(b,c))$ for a is even, is included in the first linear combination (4.15).

In order to see that the two linear combinations in (4.15) and (4.16) are linearly independent we remark that the first linear combination (4.15) involves the $(k+1)$th row of $D_a(-k,c)$ whereas the second linear combination (4.16) does not (here we use $k < a-2$).

It remains to show the two hypergeometric identities (4.15) and (4.16). The situation is a bit more complicated this time compared to the hypergeometric identity (4.13) that proved that $(c+k)$ divides $\det(D_a(b,c))$: Actually these two identities are double sum identities, since they involve the polynomials $p_n(c)$. Luckily the proofs of the two identities are quite similar. We start with (4.15).

To begin with, we split the sum into four smaller sums — each one corresponding to one of the four summands of H (according to the representation of H in (4.4)):

$$\begin{aligned}
&\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} \\
&\qquad\qquad \times p_{i-k-1}(c+a-k-i+1)D_a(-k,c)_{(i,j)} = \\
&\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1) \\
&\times(-k+i-j+1)_{j-1}(c-i+j+1)_{a-j+1}((-1-a+c+2j)/2)((-2-a+2i-k)/2) \\
&+\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1) \\
&\times(-k+i-j+2)_{j-2}(c-i+j)_{a-j+2}((-2-a+2i-k)/2)((2+a-2j-k)/2) \\
&+\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1) \\
&\times(-k+i-j)_{j}(c-i+j+2)_{a-j}((3+a+c-2i)/2)((-1-a+c+2j)/2) \\
&+\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1) \\
&\times(-k+i-j+1)_{j-1}(c-i+j+1)_{a-j+1}((3+a+c-2i)/2)((2+a-2j-k)/2). \qquad (4.18)
\end{aligned}$$

Next we examine the sum of the first and the third sum of the right-hand side in (4.18). After some simplifications and interchange of summations one gets

$$\begin{aligned}
&\sum_{h=0}^{(a-k-2)/2}\left[\sum_{i=h+k+2}^{(a+k+2)/2}(-1)^{a-j+1}\frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}\right. \\
&\qquad\qquad\left.\times\frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-1+i-k}(1-a-c+k)_{-1+a+i-j-k}}{(1)_{i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-2+i-k}}\right] \\
&\quad-\sum_{h=0}^{(a-k-2)/2}\left[\sum_{i=h+k+2}^{(a+k+2)/2}(-1)^{a-j+1}\frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}\right. \\
&\qquad\qquad\qquad\left.\times\frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}}\right].
\end{aligned}$$

Notice that these two double sums are quite similar. If we combine the exterior sums, take some common factors out of the two inner sum and shift the index of the first inner sum by one, we obtain

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\left[\sum_{i=h+k+3}^{(a+k+4)/2} \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}}\right.$$
$$\left.-\sum_{i=h+k+2}^{(a+k+2)/2} \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}}\right].$$

This makes us discover, that the two inner sums nearly cancel out each other, because the two summands are exactly the same. Therefore the latter expression simplifies further to

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_h(1-a-c+k)_{a+h-j}}{(1)_{1+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-1+h}}.$$

Thus we have already reduced the sum of the two double sums to the following single sum

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h(1-a-c+k)_{a+h-j}}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h(1)_{1+h-j}}. \tag{4.19}$$

Analogously one can show that the sum of the second und the fourth sum in (4.18), which reads

$$\sum_{h=0}^{(a-k-2)/2}\sum_{i=2+h+k}^{(a+k+2)/2} (-1)^{-1+a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-1+i-k}(1-a-c+k)_{a+i-j-k}}{(1)_{1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-2+i-k}}$$
$$-\sum_{h=0}^{(a-k-2)/2}\sum_{i=2+h+k}^{(a+k+2)/2} (-1)^{-1+a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-1+a+i-j-k}}{(1)_{i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}},$$

after interchange of summation and some cancellations, simplifies to

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h(1-a-c+k)_{1+a+h-j}}{(1)_{2+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}. \tag{4.20}$$

Now it remains to show that the two (single) sums (4.19) and (4.20) sum up to zero. Because of the factor $1/(1)_{1+h-j}$, we are able to change the lower bound on the summation index of the sum in (4.19) to $j-1$, since $j \geq 2$ and $1/(1)_n = 0$ if n is a negative integer.

Analogously we change the lower bound of the summation index of the sum in (4.20) to $j-2$ according to the factor $1/(1)_{2+h-j}$. Therefore the sum of (4.19) and (4.20) is

$$\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)$$
$$\times\left[\sum_{h=j-1}^{(a-k-2)/2}(-1)^{a-j}\frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h(1-a-c+k)_{a+h-j}}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h(1)_{1+h-j}}\right.$$
$$\left.+\sum_{h=j-2}^{(a-k-2)/2}(-1)^{a-j}\frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h(1-a-c+k)_{1+a+h-j}}{(1)_{2+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}\right].$$

Using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{k!(b_1)_k\dots(b_s)_k}z^k,$$

and, after canceling the factor $\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)$, it remains to show that

$$\frac{(1-\frac{a}{2}+\frac{k}{2})_{-1+j}\,(1-a-c+k)_{-1+a}}{(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k)_{-1+j}}$$
$$\times\left({}_2F_1\left[\begin{matrix}-\frac{a}{2}+j+\frac{k}{2},-c+k\\ \frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\right.$$
$$\left.-{}_2F_1\left[\begin{matrix}-1-\frac{a}{2}+j+\frac{k}{2},-c+k\\ -\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\right)\right)=0. \tag{4.21}$$

The factor $((2-a+k)/2)_{-1+j}$ tells us that this is true at least for $j\geq(-2+a-k)/2$. Otherwise, we use Vandermonde's summation formula (see [**22**, (1.7.7), Appendix (III.4)])

$${}_2F_1\left[\begin{matrix}a,-n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n}, \tag{4.22}$$

which is valid if n is a nonnegative integer. Namely, if we apply this formula to the left-hand side of (4.21) with $n=(a-2j-k)/2$, respectively $n=(2+a-2j-k)/2$ (here we use $j<(-2+a-k)/2$), we see that the two ${}_2F_1$–series in (4.21) sum up to zero. Thus, we have proved the first linear combination, (4.15), for the factors of form $(b+k)$.

The proof of the second linear combination (4.16) is quite similar: Again we split the double sum into four smaller sums according to the summands of the polynomial H. After some simplifications and interchange of summation we have reduced our problem to the following hypergeometric identity:

$$\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{1+a+h-j}\,(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j)_1\,(\frac{-a}{2}+\frac{k}{2})_{i-k}\,(-a-c+k)_{a+i-j-k}}{(1)_{i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-2-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{1+a+h-j}\,(1+\frac{a}{2}-j-\frac{k}{2})_1\,(\frac{-a}{2}+\frac{k}{2})_{i-k}\,(-a-c+k)_{1+a+i-j-k}}{(1)_{1+i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-2-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{a+h-j}\,(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j)_1\,(\frac{-a}{2}+\frac{k}{2})_{-1+i-k}\,(-a-c+k)_{-1+a+i-j-k}}{(1)_{-1+i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-3-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{a+h-j}\,(1+\frac{a}{2}-j-\frac{k}{2})_1\,(\frac{-a}{2}+\frac{k}{2})_{-1+i-k}\,(-a-c+k)_{a+i-j-k}}{(1)_{i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-3-h+i-k}}$$
$$=-D_a(-k,c)_{(k+1,j)}. \quad (4.23)$$

As before, the sum of the first and the third summand on the left-hand side of (4.23) simplifies to a single sum:

$$\sum_{h=0}^{(-4+a-k)/2}(-1)^{a+h-j}\frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)_1\left(-\frac{a}{2}+\frac{k}{2}\right)_{2+h}(-a-c+k)_{2+a+h-j}}{\left(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k\right)_{1+h}(1)_{2+h-j}}. \quad (4.24)$$

Analogously, the second and fourth summand of (4.23) give

$$\sum_{h=0}^{(-4+a-k)/2}(-1)^{a+h-j}\frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)_1\left(-\frac{a}{2}+\frac{k}{2}\right)_{2+h}(-a-c+k)_{3+a+h-j}}{\left(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k\right)_{1+h}(1)_{3+h-j}}. \quad (4.25)$$

Again we want to change the lower bound of the remaining single sums in (4.24) and (4.25) to $j-2$, respectively to $j-3$, according to the factor $1/(1)_{2+h-j}$, respectively $1/(1)_{3+h-j}$. This is because then the two sums are ${}_2F_1$–series and this makes it possible to apply Vandermonde's summation formula (4.22) again. In case of (4.25) and $j=2$ this change is problematic since $j-3<0$ for $j=2$. But the right-hand side of (4.16) compensates this missing term.

In terms of hypergeometric notation, it remains to show that

$$\frac{(-1)^a\,(\frac{-a}{2}+\frac{k}{2})_j\,(-a-c+k)_a}{(\frac{1}{2}+\frac{a}{2}+\frac{c}{2}-j-k)\,(\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-j-k)_{-2+j}}$$
$$\times\left[{}_2F_1\left[\begin{matrix}-\frac{a}{2}+j+\frac{k}{2},-c+k\\ \frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\right.$$
$$\left.+\,{}_2F_1\left[\begin{matrix}-1-\frac{a}{2}+j+\frac{k}{2},-c+k\\ -\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(\frac{1}{2}+\frac{a}{2}+\frac{c}{2}-j-k\right)\right]=0.$$

For $j\geq(a-k+1)/2$ this is obvious because of the factor $((-a+k)/2)_j$. Otherwise, we use again Vandermonde's summation formula (4.22) with $n=(a-2j-k)/2$, respectively $n=(2+a-2j-k)/2$.

Finally we have completely proved that $(b+k)^2$ is a factor of $\det(D_a(b,c))$.

re 3., 4. — $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+2)/2)_{\lceil(a-2)/2\rceil}$ *is a factor of* $\det(D_a(b,c))$*:* In order to prove that the third and the fourth type of factors (see the beginning of Step 2) divide $\det(D_a(b,c))$, we could use the same procedure as for the factors of form $(c+k)$ and the factors of form $(b+k)^2$. But, fortunately, we will see that the triple sum that is the subject of

Section 3 provides an easier proof. 'Fortunately,' since in this case the procedure leads to quite complicated hypergeometric identities. Furthermore, the underlying linear combinations are more difficult to guess because an additional parameter originating in the higher multiplicity of the factors is involved.

If we look at (4.3) and (4.5), we see that (4.6) is equal to (3.1) evaluated at $x = (a+b)/2$ and $y = (a+c-1)/2$ modulo some factors. Therefore a combination of (4.3), (4.5) and Lemma 1.2 gives

$$\det(D_a(b,c)) = \left(\prod_{i=2}^{a}(b+c+1)_{i-1}(i-1)!\right)\left(\frac{c-a+3}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1} \times \sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\frac{\left(\frac{1-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{3-a+c}{2}\right)_{(a+b-2)/2}}\frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c-1)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c-1)/2}}\binom{m-1}{s-1} \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right]. \tag{4.26}$$

We are aiming to show that $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+2)/2)_{\lceil (a-2)/2\rceil}$ is a factor of (4.6). Using (4.26), it remains to show that the following sum is equal to a polynomial in b and c for all integers b and c if we fix a:

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{b+c+2}{2}\right)_{\lceil (a-2)/2\rceil}\left(\frac{c-a+3}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1} \times \sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\frac{\left(\frac{1-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{3-a+c}{2}\right)_{(a+b-2)/2}}\frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c-1)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c-1)/2}}\binom{m-1}{s-1} \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right]. \tag{4.27}$$

As it stands, this does not appear as a polynomial in b and c, since b and c occur in the second arguments of some Pochhammer symbols. But by using

$$\frac{(r)_n}{(s)_n} = \frac{(s+n)_{r-s}}{(s)_{r-s}}, \tag{4.28}$$

which is valid for all integers n, we are able to ban b and c from the second arguments of the Pochhammer symbols in (4.27). Then, after some cancellations, we obtain

$$\prod_{i=2}^{a}(i-1)!\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\left(\frac{1-a+c+2n}{2}\right)_{a-n}\left(\frac{b+c+1}{2}\right)_{n-1}\left(\frac{-a+b+2s}{2}\right)_{a-s} \times \binom{m-1}{s-1}\frac{\left(\frac{b+c+2}{2}\right)_{\lceil (a-2)/2\rceil}\left(\frac{b+c+1}{2}\right)_{s-1}}{(b+c+1)_{s-1}}\frac{(b+1)_{s-1}(c+1)_{n-1}(b+c+n)_{m-n}}{(n-1)!(m-n)!}$$

for (4.27), and this is manifestly a polynomial in b and c. Indeed, because of $s \le m \le a$, thanks to the binomial coefficient, the term $(b+c+1)_{s-1}$ cancels with the two terms on top of the fraction.

Therefore we have finally proved that the linear factors in (4.9), respectively in (4.10), divide the determinant $\det(D_a(b,c))$ as a polynomial in b and c, and we may now turn to the computation of the remaining irreducible polynomial.

Step 3: Computation of the irreducible polynomial. We emphasize once more that Step 1 and Step 2 show that

$$\mathit{det}\,(D_a(b,c)) = (\text{Linear factors in (4.9), respectively (4.10)}) \times P_a(b,c), \tag{4.29}$$

where $P_a(b,c)$ is a certain polynomial in b and c if we fix a. In this final step we prove that $P_a(b,c)$ equals (4.11), respectively (4.12).

The computation of the irreducible polynomial, which will be denoted by $P_a(b,c)$, is done in the following way: We consider the polynomial $P_a(b,c)$ as a polynomial in b over $\mathbb{Z}[c]$ and find that it has 'nice' evaluations at $b=-c-k$, k odd and $1 \le k \le a$, see (4.30) and (4.31). We are able to prove these 'nice' evaluations with the help of Lemma 1.2. Furthermore, we will see that $P_a(b,c)$ is a polynomial in b with a degree smaller or equal to $\lfloor (a-1)/2 \rfloor$. Therefore these 'nice' evaluations are just enough so that we can compute $P_a(b,c)$ by using Lagrange's interpolation formula, see (4.32), respectively (4.33).

First we will convince ourselves that the assertion about the degree of $P_a(b,c)$ in b is true: The entry in the ith row and the jth column of the matrix $D_a(b,c)$, see (4.6), is a polynomial in b of degree $j+1$, since H is a polynomial in b of degree 2. Hence, in the defining expansion of the determinant, each summand has degree $2+3+\cdots+a=(a+2)(a-1)/2$ in b and therefore the degree of the determinant itself is at most $(a+2)(a-1)/2$. One easily checks that the product of the linear factors is a polynomial in b with degree $a^2/2$ in case of a is even and $(a+1)(a-1)/2$ otherwise. Because of that $(a+2)(a-1)/2-a^2/2=\lfloor (a-1)/2 \rfloor$ respectively $(a+2)(a-1)/2-(a+1)(a-1)/2=\lfloor (a-1)/2 \rfloor$ is an upper estimation for the degree of $P_a(b,c)$ in b. Later we will see that this is in fact the exact degree.

We claim that

$$P_a(-c-k,c) = \left(\frac{1}{2}\right)^{a-1} \left(\prod_{i=2}^{a}(i-1)!\right) \left(\frac{1}{2}\right)_{(a-1)/2} \frac{\left(\frac{k-1}{2}\right)!(-1)^{(k-1)/2}}{\left(\frac{a-k+1}{2}\right)_{(k-1)/2}} \times \left(\frac{c+k+1}{2}\right)_{(a-k)/2} \left(\frac{c+1}{2}\right)_{(k-1)/2} \tag{4.30}$$

if a is odd and $k=1,3,\dots,a$, and

$$P_a(-c-k,c) = \left(\frac{1}{2}\right)^{a} \left(\prod_{i=2}^{a}(i-1)!\right) \left(\frac{1}{2}\right)_{(a-2)/2} \frac{\left(\frac{k-1}{2}\right)!(-1)^{(k-1)/2}}{\left(\frac{a-k}{2}\right)_{(k-1)/2}} \times \left(\frac{c+k+2}{2}\right)_{(a-k-1)/2} \left(\frac{c+2}{2}\right)_{(k-1)/2} \tag{4.31}$$

if a is even and $k=1,3,\dots,a-1$.

Assuming the truth of the claim, we would have

$$P_a(b,c) = \left(\frac{1}{2}\right)^{a-1} \prod_{i=2}^{a}(i-1)! \sum_{k=0}^{(a-1)/2} \left[\left(\frac{c+1}{2}\right)_k \left(\frac{1+b+c}{2}\right)_k \left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2} \times \left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2} \frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}} \right] \tag{4.32}$$

if a is odd, and, respectively,

$$P_a(b,c)=\left(\frac{1}{2}\right)^a\prod_{i=2}^{a}(i-1)!\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right]\qquad(4.33)$$

if a is even, by Lagrange interpolation.

Thus, it remains to show (4.30) and (4.31). By definition, $P_a(b,c)$ is the quotient of $\det(D_a(b,c))$ and the linear factors in (4.9), respectively in (4.10) (see (4.29)). This, together with a tedious but straightforward calculation shows that the claims in (4.30) and (4.31) conflate to the following

$$\left.\frac{\det(D_a(b,c))\binom{\frac{b+c-1}{2}}{\frac{a+b-2}{2}}\binom{\frac{b+c-1}{2}}{\frac{a+c-1}{2}}}{\left(\prod_{i=2}^{a}(1+b+c)_{i-1}(i-1)!\right)\left(\frac{c-a+3}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}}\right|_{b=-c-k}=(-1)^a\frac{(c+1)_{k-1}}{(k-1)!}.\qquad(4.34)$$

The reader should notice that we are not able to directly set $b=-c-k$ on the left-hand side of (4.34), because the denominator vanishes for $b=-c-k$. As mentioned before we want to replace the determinant on the left-hand sides of (4.34) by the triple sum in Lemma 1.2. Still even after cancellations we are not able to directly set $b=-c-k$ in this triple sum, since then the denominator of the summand becomes zero if $k>n$.

In order to avoid an indefinite expression we act as follows: If we reexamine the proof of Lemma 1.2, we obtain the following generalization of Lemma 1.2:

Let a,b,c be positive integers and b',x,x be integers. Then the determinant

$$-\det_{1\le i,j\le a+1}\begin{array}{c}\begin{array}{cc}1\le j\le a & j=a+1\end{array}\\ \left(\begin{array}{c|c}\binom{b+c}{c-i+j} & \binom{b'-x+y}{y-i+1}\\ \hline \binom{c+x-y-1}{x-j} & 0\end{array}\right)\end{array}\begin{array}{l}1\le i\le a\\ i=a+1\end{array}.\qquad(4.35)$$

equals

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c}$$
$$\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b'-x+y+s-1}{b'-x+s-1}\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\right.$$
$$\left.\binom{m-1}{s-1}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right].\qquad(4.36)$$

We set $b=-c-k+\varepsilon$ and $b'=-c-k$ in (4.36). In view of the fact that (4.35) is continuous in b, and if we compare the determinant in (4.6) with the determinant in (2.6) evaluated at $x = (a+b)/2$ and $y = (a+c-1)/2$, we obtain that the left-hand side of (4.34) is equal to

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\Bigg[(-1)^{n+s}\binom{\frac{-k+2n-3}{2}}{\frac{a-c-k-2}{2}}\binom{\frac{-k+2s-3}{2}}{\frac{a+c-1}{2}}\binom{m-1}{s-1} \\ \times\frac{(-c-k+\varepsilon+1)_{s-1}}{(-k+\varepsilon+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(-k+\varepsilon+n)_{m-n}}{(m-n)!}\Bigg]. \qquad (4.37)$$

Now it remains to show that the right-hand side of (4.34) equals (4.37) for $k = 1, 3, \dots, 2\lfloor a - 1/2\rfloor$.

Therefore let us consider the triple sum (4.37). First of all, we are allowed to extend the sum over s to all positive integers, since $\binom{m-1}{s-1} = 0$ if $s > m$. Using hypergeometric notation for this innermost sum, we get

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\Bigg[(-1)^{n+1}\,{}_3F_2\left[\begin{matrix}1-m, 1-c-k+\varepsilon, \frac{1}{2}-\frac{k}{2}\\ 1-k+\varepsilon, 1-\frac{a}{2}-\frac{c}{2}-\frac{k}{2}\end{matrix};1\right] \\ \frac{(1+c)_{-1+n}\,(1-\frac{a}{2}-\frac{c}{2}-\frac{k}{2})_{-\frac{1}{2}+\frac{a}{2}+\frac{c}{2}}\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{m-n}}{(1)_{-\frac{1}{2}+\frac{a}{2}+\frac{c}{2}}\,(1)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(1)_{m-n}\,(1)_{-1+n}}\Bigg].$$

We apply the following transformation formula due to Thomae [**6**, (3.1.1)] to this hypergeometric sum,

$$ {}_3F_2\left[\begin{matrix}a, b, -n\\ d, e\end{matrix};1\right] = \frac{(-b+e)_n}{(e)_n}\,{}_3F_2\left[\begin{matrix}-n, b, -a+d\\ d, 1+b-e-n\end{matrix};1\right] \qquad (4.38)$$

with $a = 1-m$, $b = 1-c-k+\varepsilon$, $n = (k-1)/2$, $d = 1-k+\varepsilon$ and $e = 1-a/2-c/2-k/2$.

Writing the resulting ${}_3F_2$–series as a sum over s, after some simplifications and cancellations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right) \\ \times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{m-n+s}}{(1)_{m-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

Next we interchange the two inner sums and reverse the order of summation in the innermost sum. We obtain

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\sum_{m=0}^{-1+a}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right) \\ \times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{a-m-n+s}}{(1)_{a-m-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

The reason why reversing was so helpful, is that we now can extend the summation with respect to m to all nonnegative integers (since $1/(1)_{a-m-n} = 0$ if $m > a-n$). If we use hypergeometric notation this yields

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\,{}_2F_1\left[\begin{matrix}1,-a+n\\ 1-a+k+\varepsilon-s\end{matrix};1\right]$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2})\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{a-n+s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

We apply Vandermonde's summation formula (4.22) to this hypergeometric sum. After some manipulations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{-\frac{1}{2}+\frac{k}{2}}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right)\frac{(-a+k-\varepsilon-s)_{a-n}\,(1+k-\varepsilon-n-s)_s}{(1)_{\frac{1}{2}+\frac{k}{2}-n}(1)_{-\frac{1}{2}+\frac{k}{2}-s}}$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{3}{2}+\frac{a}{2}-\frac{c}{2}-k+s)_{-1+k-n-s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_s\,(1-k+\varepsilon)_s}\Bigg].$$

Now we are able to perform the limit $\varepsilon\to 0$, since, because of the factor $1/(1)_{-1/2+k/2-s}$, we have $k>s$. 'Performing the limit' means that we simply set $\varepsilon=0$.

With pleasure we notice that our triple sum has simplified to a double sum. Big surprise arises when I finally claim that the summand of the sum is only different from zero if and only if $n=k/2+1/2$ and $s=k/2-1/2$, and we therefore get rid of all sums: The factor $1/(1)_{1/2+k/2-n}$ implies $n\le 1/2+k/2$, and the factor $1/(1)_{-1/2+k/2-s}$ implies $s\le -1/2+k/2$. Next we notice that

$$(-a+k-s)_{a-n}(1+k-n-s)_s=\frac{(-a+k-s)_{1+a-n-s}}{(k-n-s)}.$$

The numerator on the right-hand side is zero since $-a+k-s\le -a+k\le 0$ and $k-n\ge 0$. Therefore the summand is only different from zero if $k-n-s=0$. This together with our first observation gives my assertion.

It remains to compute the summand in the last double sum for $n=k/2+1/2$ and $s=k/2-1/2$ (and, of course, $\varepsilon=0$). After some simplifications one does indeed get the right-hand side of (4.34) and Lemma 1.3 is finally proved.

Now we turn to the case that the side lengths a,b,c of the hexagon have the same parity and therefore to the evaluation of the determinant in Lemma 1.1 with $x=(a+b)/2$ and $y=(a+c)/2$ (see the beginning of this section).

LEMMA 1.4. *Let a,b,c be integers and $a\equiv c \pmod 2$. Then the determinant*

$$-\det_{1\le i,j\le a+1}\begin{array}{c}\begin{array}{cc}\quad 1\le j\le a\quad & \quad j=a+1\quad\end{array}\\ \left(\begin{array}{c|c}\dbinom{b+c}{c-i+j} & \dbinom{\frac{b+c}{2}}{\frac{c}{2}-i+\frac{a+2}{2}}\\ \hline \dbinom{\frac{b+c-2}{2}}{\frac{c}{2}-\frac{a+2}{2}+j} & 0\end{array}\right)\end{array}\begin{array}{l}1\le i\le a\\ \\ i=a+1\end{array}$$

is equal to

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\Bigg(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\Bigg)$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_{c+a}}\binom{\frac{b+c-2}{2}}{\frac{b-2}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-2}{2}}2^{a}$$
$$\times\Bigg(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\Bigg)$$

in case that a is even.

As already mentioned at the beginning of this section the proofs of Lemma 1.3 and Lemma 1.4 are quite similar. Therefore we restrict ourselves to just pointing out the differences to the proof of Lemma 1.3, but we omit the details. The only major difference to Lemma 1.3 is that in the present case the irreducible polynomial $P_a(b,c)$ (see, e.g., (4.29)) has one 'exceptional evaluation' which has to be treated in a different way than the rest. By the way, this exceptional evaluation also causes the formulas for the number of rhombus tilings which contain the 'almost central' rhombus above the centre to be not as simple compared to the formulas for the number of rhombus tilings which contain the central rhombus.

Proof of Lemma 1.4.

Step 1: From our determinant to a determinant with polynomial entries.

We can straightforwardly adopt Step 1 from the proof of Lemma 1.3, where we have reduced the problem to the evaluation a polynomial determinant, since we have not even specialised the coordinates of the fixed rhombus (x,y) there. In the following $\hat{D}_a(b,c)$ denotes the matrix underlying the determinant in (4.6) evaluated for $x=(a+b)/2$ and $y=(a+c)/2$.

Again I was led to a conjecture concerning the determinant $\det(\hat{D}_a(b,c))$ by computing it for small values of a:

$$\det_{2\le i,j\le a}\left(\hat{D}_a(b,c)\right)=\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)$$
$$\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{2+b+c}{2}\right)_{(a-1)/2}2^{a-1}$$
$$\times\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_k\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right)\tag{4.39}$$

if a is odd, and

$$\det_{2\le i,j\le a}\left(\hat{D}_a(b,c)\right)=\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)$$
$$\left(\frac{b}{2}\right)_{a/2}\left(\frac{b+2}{2}\right)_{(a-2)/2}\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{2+b+c}{2}\right)_{(a-2)/2}2^{a-1}$$
$$\times\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_k\left(\frac{b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right)\tag{4.40}$$

if a is even. Just as in the proof of Lemma 1.3 the two assertions (4.39) and (4.40) are equivalent to the assertion in Lemma 1.4.

The linear factors of $\det(\hat{D}_a(b,c))$ are

$$\prod_{i=2}^{a-1}(1+b+c)_{i-1}\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{2+b+c}{2}\right)_{(a-1)/2}2^{2a-2}\tag{4.41}$$

if a is odd, and, respectively,

$$\prod_{i=2}^{a-1}(1+b+c)_{i-1}\left(\frac{b}{2}\right)_{a/2}\left(\frac{b+2}{2}\right)_{(a-2)/2}\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{2+b+c}{2}\right)_{(a-2)/2}2^{2a-3}\tag{4.42}$$

if a is even. The irreducible polynomial, denoted by $\hat{P}_a(b,c)$, reads

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-1}\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{(\frac{1}{2})_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right) \tag{4.43}$$

if a is odd, and respectively,

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-2}\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{(\frac{1}{2})_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{(\frac{1}{2})_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right) \tag{4.44}$$

if a is even.

Step 2: The linear factors in (4.41), respectively in (4.42), divide $\det(\hat{D}_a(b,c))$ as a polynomial in b and c.

Again we have four different types of linear factors:

1. Factors of the form $(c+k)$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$.
2. Factors of the form $(b+k)^2$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$. The factor b occurs once if a even.
3. Factors of the form $(b+c+k)^{a-k-1}$ for $k=1,3,\dots,2\lceil(a-2)/2\rceil-1$.
4. Factors of the form $(b+c+k)^{a-k}$ for $k=2,4,\dots,2\lfloor(a-2)/2\rfloor$.

re 1. — The factors of the form $(c+k)$ divide $\det(\hat{D}_a(b,c))$: As described in the analogous passage of the proof of Lemma 1.3, each linear factor of $\det(\hat{D}_a(b,c))$ corresponds to a linear combination of rows of a certain matrix. The linear combinations for the factors of the form $(c+k)$ are

$$\sum_{i=(a-k+2)/2}^{a-k+1}\frac{(-1)^{i-1}(b+i)_{a-k+1-i}\left(\frac{-a+k-2+2i}{2}\right)_{a-k+1-i}}{(1)_{a-k+1-i}\left(\frac{b-a+2i-2}{2}\right)_{a-k+1-i}}D_a(b,-k)_{(i,j)}=0 \tag{4.45}$$

for $j=2,3,\dots,a$, $1\le k\le a-1$ and $k\equiv a \pmod 2$. In order to prove that $(c+k)$ divides the determinant $\det(\hat{D}_a(b,c))$ we just have to prove the identity (4.45).

The comparison of the identity in (4.13) and the identity in (4.45) shows that these two identities are quite similar. And in fact the proofs do not differ essentially from each other, either: Again we are able to apply *Gosper's algorithm* [**8**] for hypergeometric sums to the left-hand side of (4.45) and recognize that this is actually a telescoping sum.

The expression analogous to (4.14) is

$$\sum_{i=0}^{\infty}\frac{(-1)^j\left(1-\frac{a}{2}+\frac{k}{2}\right)_i(-a-b+k)_{i+j-2}}{(1)_{-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_i}\,H|_{x=(a+b)/2,y=(a+c)/2,c=-k,i\to a-k+1-i}=0, \tag{4.46}$$

which is the identity in (4.45) after reversing the summation order. If $\hat{f}(i)$ denotes the summand in the previous sum and $\hat{g}(i)$ denotes the following expression

$$\frac{(-1)^j \left(1-\frac{a}{2}+\frac{k}{2}\right)_i (-a-b+k)_{i+j-2}}{(1)_{-1-a+i+j} \left(1-\frac{a}{2}-\frac{b}{2}+k\right)_{i-1}} \times \left(\frac{-2-a-b-bi+2j+bj-bk+ik-jk+k^2}{2}\right),$$

then we have

$$\hat{f}(i) = \hat{g}(i+1) - \hat{g}(i).$$

By using this identity it is easy to compute the left-hand side of (4.46) and to show that it is equal to zero.

re 2. — The factors of the form $(b+k)$ divide $\det(\hat{D}_a(b,c))$*:*

The two linearly independent linear combination for the factors of the form are $(b+k)$ are

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k} \left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1} \left(\frac{c+a-2i+4}{2}\right)^2_{i-k-2}} \times p_{i-k-1}(c+a-k-i+2)\hat{D}_a(-k,c)_{(i,j)} = 0 \quad (4.47)$$

if $0 \le k < a-2$ and $k \equiv a \pmod 2$, and

$$\sum_{i=k+3}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-1} \left(\frac{a+k-2i+4}{2}\right)_{i-k-1}}{(1)_{i-k-1} \left(\frac{c+a-2i+4}{2}\right)^2_{i-k-2}} \times p_{i-k-2}(c+a-k-i+1)\hat{D}_a(-k,c)_{(i,j)} = -\hat{D}_a(-k,c)_{(k+1,j)} \quad (4.48)$$

if $0 < k < a-2$ and $k \equiv a \pmod 2$, where $p_n(c)$ is the same sequence of polynomials as in Lemma 1.3, see (4.17). The first identity, (4.47), corresponds to (4.15), and the second identity, (4.48), corresponds to (4.16). Again these two linear combinations do not cover the case that $k = a-2$. But similar to the situation in Lemma 1.3 it can be verified directly that $(b+a-2)^2$ is a factor of the determinant $\det(\hat{D}_a(b,c))$.

The proofs of the identities (4.47) and (4.48) are analogous to the proofs of their corresponding identity: Again we first have to interchange the summation and then split the double sum into four smaller sums according to the polynomial H. Just as in Lemma 1.3 the inner sums of the first and the third double sum nearly cancel out each other, and so do the inner sums of the second and the fourth summand. This reduces the problem to identities only involving ${}_2F_1$–series and therefore Vandermonde's summation formula (4.22) finishes the proof of these identities.

re 3., 4. — $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+1)/2)_{\lceil (a-1)/2\rceil}$ *is a factor of* $\det(\hat{D}_a(b,c))$*:* Just as in Lemma 1.3 the triple sum from Lemma 1.2 provides an easy proof that the factors of type 3 and type 4 divide the determinant $\det(\hat{D}_a(b,c))$ as a polynomial in b and c. Again a combination of (4.3), (4.5) and Lemma 1.2 gives the equation analogous to (4.26), namely

$$\det(\hat{D}_a(b,c)) = \left(\prod_{i=2}^{a}(b+c+1)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1} \\ \times \sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\Bigg[(-1)^{n+s}\frac{\left(\frac{-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{2-a+c}{2}\right)_{(a+b-2)/2}}\frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c)/2}}\binom{m-1}{s-1} \\ \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\Bigg]. \tag{4.49}$$

By using (4.28) we obtain that the quotient of (4.49) and the product of the linear factors $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+1)/2)_{\lceil (a-1)/2\rceil}$ is the following polynomial in b and c if we fix a:

$$\prod_{i=2}^{a}(i-1)!\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\left(\frac{-a+c+2n}{2}\right)_{a-n}\left(\frac{b+c}{2}\right)_{n-1}\left(\frac{-a+b+2s}{2}\right)_{a-s} \\ \times\binom{m-1}{s-1}\frac{\left(\frac{b+c+1}{2}\right)_{\lceil (a-1)/2\rceil}\left(\frac{b+c+2}{2}\right)_{s-1}}{(b+c+1)_{s-1}}\frac{(b+1)_{s-1}(c+1)_{n-1}(b+c+n)_{m-n}}{(n-1)!(m-n)!}.$$

Step 3: Computation of the irreducible polynomial. Analogously to the situation in Lemma 1.2 we now evaluate the quotient of the linear factors in (4.41), respectively (4.42), and the determinant $\det(\hat{D}_a(b,c))$. This quotient, which will be denoted by $\hat{P}_a(b,c)$, is a polynomial in b and c if we fix a. Again we will find that the polynomial $\hat{P}_a(b,c)$ has enough 'nice' evaluations so that we can compute the polynomial by using Lagrange's interpolation formula. Namely, these 'nice' evaluations arise for $b=-c-k$, $0\le k\le a$ and k is even, and the degree of $\hat{P}_a(b,c)$ as a polynomial in b happens to be $\lfloor a/2\rfloor$.

The assertion about the degree of $\hat{P}_a(b,c)$ as a polynomial in b can be checked routinely, just as in Lemma 1.3. Concerning the computation of 'nice' evaluations of the polynomial $\hat{P}_a(b,c)$ there arises the one and only major difference to Lemma 1.3: For $k=2,4,\dots,\lfloor a/2\rfloor$ the evaluations $\hat{P}_a(-c-k,c)$ can be proved in the same way as in Lemma 1.3, but for $k=0$ the situation is different, and therefore this exceptional evaluation needs a separate proof. This fact already shows up if we look at the following conjecture for the 'nice' evaluations:

We claim that

$$\hat{P}_a(-c,c) = \left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2} \tag{4.50}$$

if a is odd, and

$$\hat{P}_a(-c,c) = \left(\frac{1}{2}\right)^{a-2}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{a}{2}\right)\left(\frac{1}{2}\right)_{a/2}\left(\frac{c+2}{2}\right)_{(a-2)/2} \tag{4.51}$$

if a is even, and, furthermore,

$$\hat{P}_a(-c-k,c) = \left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)_{(a-1)/2}\frac{\left(\frac{k}{2}\right)!(-1)^{k/2}}{\left(\frac{a-k}{2}\right)_{k/2}} \\ \times\left(\frac{c+k+1}{2}\right)_{(a-k+1)/2}\left(\frac{c+2}{2}\right)_{(k-2)/2} \tag{4.52}$$

if a is odd and $k=2,4,\dots,a-1$, and

$$\hat{P}_a(-c-k,c)=\left(\frac{1}{2}\right)^{a-2}\prod_{i=1}^{a-1}i!\left(\frac{1}{2}\right)_{(a-2)/2}\frac{\left(\frac{k}{2}\right)!(-1)^{k/2}}{\left(\frac{a-k+1}{2}\right)_{(k-2)/2}}\\ \times\left(\frac{c+k+2}{2}\right)_{(a-k)/2}\left(\frac{c+1}{2}\right)_{k/2}\qquad(4.53)$$

if a is even and $k=2,4,\dots,a$.

Assuming the truth of the claim, we would have

$$\hat{P}_a(b,c)=\left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}\right.\\ +\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}\\ \left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right)$$

if a is odd, and, respectively

$$\hat{P}_a(b,c)=\left(\frac{1}{2}\right)^{a-2}\prod_{i=2}^{a}(i-1)!\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}\right.\\ +\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}\\ \left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right)$$

if a is even, by Lagrange interpolation.

Thus, it remains to show (4.50), (4.51), (4.53) and (4.52). In these claims we replace the polynomial $\hat{P}_a(b,c)$ by the quotient of $\det(\hat{D}_a(b,c))$ and the linear factors in (4.41), respectively (4.42), in order to see that the four claims conflate to the following two:

$$\left.\frac{2\det(\hat{D}_a(b,c))\binom{\frac{b+c-2}{2}}{\frac{a+b-2}{2}}\binom{\frac{b+c}{2}}{\frac{a+c}{2}}}{\left(\prod_{i=2}^{a}(1+b+c)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}(b+c)}\right|_{b=-c}\\ =\frac{-a+(-1)^{a+1}\,a+c+(-1)^{a+1}\,c}{c\,(a+c)},\qquad(4.54)$$

and

$$\left.\frac{\det(\hat{D}_a(b,c))\binom{\frac{b+c-2}{2}}{\frac{a+b-2}{2}}\binom{\frac{b+c}{2}}{\frac{a+c}{2}}}{\left(\prod_{i=2}^{a}(1+b+c)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}}\right|_{b=-c-k} = (-1)^{a-1}\frac{(c+1)_{k-1}}{(k-1)!} \tag{4.55}$$

for $k=2,4,\dots,\lfloor a/2\rfloor$. Again the reader should know that we are not able to directly set $b=-c-k$ on the left hand sides of (4.55) and (4.54), because the denominator vanishes for $b=-c-k$.

First we consider the case that $k\neq 0$. As already mentioned, the situation in this case is quite the same as in Lemma 1.3. Therefore I only explain the essential steps and omit details.

For the left-hand side of (4.55) we use again the modification of Lemma 1.2, see (4.35) and (4.36). Thus, this left-hand side is equal to

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{\frac{-k+2n-4}{2}}{\frac{a-c-k-2}{2}}\binom{\frac{-k+2s-2}{2}}{\frac{a+c}{2}}\binom{m-1}{s-1}\right.$$
$$\left.\times\frac{(-c-k+\varepsilon+1)_{s-1}}{(-k+\varepsilon+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(-k+\varepsilon+n)_{m-n}}{(m-n)!}\right]. \tag{4.56}$$

Hence, it remains to show that the right hand side of (4.55) is equal to (4.56) for $k=2,4,\dots,\lfloor a/2\rfloor$.

Just as in Lemma 1.2 we apply the transformation formula due to Thomae, see (4.38), to the innermost sum of the triple sum, with slightly changed parameters, i.e., $a=1-m$, $b=1-c-k+\varepsilon$, $n=(k-2)/2$, $d=1-k+\varepsilon$ and $e=1-a/2-c/2-k/2$, compared to the proof of Lemma 1.2. This yields

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=0}^{\infty}(-1)^{\frac{a}{2}+\frac{c}{2}+n+s}\left(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2}\right)$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k)_s\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(\varepsilon-k+n)_{m-n+s}}{(1)_{m-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1)_{1+\frac{a}{2}-\frac{c}{2}-k+s}\,(1+\varepsilon-k)_s}.$$

Next we interchange the two inner sums, reverse the order of the summation in the new innermost sum and use hypergeometric notation for this innermost sum. We obtain

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{m}\sum_{s=0}^{\infty}(-1)^{\frac{a}{2}+\frac{c}{2}-n+s}{}_2F_1\left[\begin{matrix}1,-a+n\\1-a-\varepsilon+k-s\end{matrix};1\right]$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k)_s\,(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2})\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(\varepsilon-k+n)_{a-n+s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1)_{1+\frac{a}{2}-\frac{c}{2}-k+s}\,(1+\varepsilon-k)_s}.$$

Then we apply Vandermonde's summation formula (4.22) to this hypergeometric sum. After some manipulations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}(-1)^{-1-\frac{k}{2}}\left(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2}\right)(1+c)_{-1+n}\,(1-c-k)_s \\ \times\frac{(-a-\varepsilon+k-s)_{a-n}\,(1-\varepsilon+k-n-s)_s\,(2+\frac{a}{2}-\frac{c}{2}-k+s)_{-1+k-n-s}}{(1)_{a-n}\,(1)_{1+\frac{k}{2}-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1+\varepsilon-k)_s}.$$

Now we are able to perform the limit $\varepsilon\to 0$ and then observe that the summand of the double sum is only different from zero if $n=k/2+1$ and $s=k/2-1$. The evaluation of the summand of the double sum for these special values of n and s finishes the proof of (4.55) for $k=2,4,\dots\lfloor a/2\rfloor$.

Finally we consider the case that $k=0$, see (4.54). By Lemma 1.2, (4.3) and (4.5) the left-hand side of (4.54) is equal to

$$\frac{2}{b+c}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\frac{(-1)^{n+s}\,(1+b)_{-1+s}\,(1+c)_{-1+n}}{(1)_{-1+s}\,(1)_{-1+n}} \\ \times\frac{(1+\frac{a}{2}+\frac{c}{2})_{-1-\frac{a}{2}+\frac{b}{2}+s}\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}+\frac{b}{2}}\,(b+c+n)_{m-n}\,(1+m-s)_{-1+s}}{(1)_{-1+\frac{a}{2}+\frac{b}{2}}\,(1)_{m-n}\,(1)_{-1-\frac{a}{2}+\frac{b}{2}+s}\,(1+b+c)_{-1+s}}.$$

before evaluating it at $b=-c$. We take the factor $(b+c)/2$ out of the Pochhammer symbol

$$\left(\frac{2+a+c}{2}\right)_{(-2-a+b+2s)/2}=\left(\frac{2+a+c}{2}\right)_{(-2+a+b)/2}\left(\frac{b}{2}+\frac{c}{2}\right)\left(\frac{b+c+2}{2}\right)_{s-1}$$

and set $b=-c$. After some cancellations and simplifications concerning the summand of this triple sum, this yields

$$\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\frac{(-1)^{-1+\frac{a}{2}-\frac{c}{2}+n+s}\,(1-c)_{-1+s}\,(1+c)_{-1+n}}{(1)_{-1+n}\,(1)_{-1+s}} \\ \times\frac{(\frac{a}{2}-\frac{c}{2})_{1-n}\,(1+\frac{a}{2}+\frac{c}{2})_{-1-\frac{a}{2}-\frac{c}{2}}\,(n)_{m-n}\,(1+m-s)_{-1+s}}{(1)_{1-n}\,(1)_{m-n}\,(1)_{-1-\frac{a}{2}-\frac{c}{2}+s}}. \tag{4.57}$$

The factor $1/(1)_{1-n}$ shows that the summand of the triple sum is only different from zero if $n=1$. Therefore we get rid of the sum over n. We use hypergeometric notation for the inner sum and obtain

$$\sum_{m=1}^{a}\frac{(-1)^{1+c}\,{}_2F_1\left[\begin{matrix}1-c,1-m\\ 1-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]}{\frac{a}{2}+\frac{c}{2}}.$$

Next we apply Vandermonde's summation formula (4.22) to the ${}_2F_1$-series. This yields

$$\sum_{m=1}^{a}\frac{(-1)^{c}\,(\frac{-a}{2}+\frac{c}{2})_{-1+m}}{(\frac{-a}{2}-\frac{c}{2})_m}. \tag{4.58}$$

Once more, Gosper's algorithm [**8**] can be used to see that this single sum is actually telescoping. Namely, there holds

$$(-1)^c \frac{\left(-\frac{a}{2}+\frac{c}{2}\right)_{-1+m}}{\left(-\frac{a}{2}-\frac{c}{2}\right)_m}$$
$$= \frac{(-1)^{c-1}}{\left(1+\frac{a}{2}+\frac{c}{2}\right)c}\left(\frac{\left(-\frac{a}{2}-\frac{c}{2}+m\right)\left(-1-\frac{a}{2}+\frac{c}{2}\right)_{m+1}}{\left(-\frac{a}{2}-\frac{c}{2}\right)_{m+1}} - \frac{\left(-\frac{a}{2}-\frac{c}{2}+m-1\right)\left(-1-\frac{a}{2}+\frac{c}{2}\right)_m}{\left(-\frac{a}{2}-\frac{c}{2}\right)_m}\right).$$

Because of that, and because a and c have the same parity, the single sum (4.58) is equal to

$$\frac{-a+(-1)^{a+1}\,a+c+(-1)^{a+1}\,c}{c\,(a+c)}.$$

If we compare this to the right-hand side of (4.54) we see that this is just what we claimed. This finally completes the proof of Lemma 1.4.

5. The Proofs of Theorem 1.3 and Theorem 1.4

We end this article with the proofs of Theorem 1.3 and Theorem 1.4. We start with Theorem 1.3. In Theorem 1.1 we showed that the probability to choose a rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contains a rhombus in the centre is, using hypergeometric notation,

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2}-\frac{a}{2},\frac{1}{2}+\frac{c}{2},\frac{1}{2}+\frac{b}{2}+\frac{c}{2},1\\ 1-\frac{a}{2},1+\frac{c}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\end{matrix};1\right]$$
$$\times\frac{\left(\frac{1}{2}\right)_{-\frac{1}{2}+\frac{a}{2}}(1)_c\left(1+\frac{c}{2}\right)_{-\frac{1}{2}+\frac{b}{2}}\left(1+\frac{c}{2}\right)_{-1+\frac{a}{2}+\frac{b}{2}}\left(\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\right)_{-\frac{1}{2}+\frac{a}{2}}2^{a-1}}{(1)_{-\frac{1}{2}+\frac{a}{2}}\,(1)_{-\frac{1}{2}+\frac{b}{2}}^{\,2}\,(1+b)_{-1+a+c}} \tag{5.1}$$

in case that a is odd, and

$$ {}_4F_3\left[\begin{matrix}1-\frac{a}{2},1+\frac{c}{2},\frac{1}{2}+\frac{b}{2}+\frac{c}{2},1\\ \frac{3}{2}-\frac{a}{2},\frac{3}{2}+\frac{c}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\end{matrix};1\right]$$
$$\times\frac{\left(\frac{1}{2}\right)_{-1+\frac{a}{2}}(1)_c\,(b)\left(\frac{1}{2}+\frac{c}{2}\right)_{\frac{b}{2}}\left(\frac{3}{2}+\frac{c}{2}\right)_{-1+\frac{a}{2}+\frac{b}{2}}\left(\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\right)_{-1+\frac{a}{2}}2^{a-2}}{(1)_{-1+\frac{a}{2}}\,(1)_{\frac{b}{2}}^{\,2}\,(1+b)_{-1+a+c}} \tag{5.2}$$

in case that a is even. In order to see this, divide (1.2), respectively (1.3), by MacMahon's formula for the total number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c given in (1.1).

Next we apply Bailey's transformation formula (see [**22**, (4.3.5.1)]) between two balanced ${}_4F_3$–series,

$$ {}_4F_3\left[\begin{matrix}a,b,c,-n\\ e,f,1+a+b+c-e-f-n\end{matrix};1\right] = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\times$$
$$ {}_4F_3\left[\begin{matrix}-n,a,1+a+c-e-f-n,1+a+b-e-f-n\\ 1+a+b+c-e-f-n,1+a-e-n,1+a-f-n\end{matrix};1\right], \tag{5.3}$$

which is valid if n is a positive integer, with $a=1, b=(1+c)/2, c=(1+b+c)/2, n=(a-1)/2, e=(2-a)/2, f=(c+2)/2$ in case of a is odd, and with $a=1, b=(2+c)/2, c=$

$(1+b+c)/2, n=(a-2)/2, e=(3-a)/2, f=(c+3)/2$ in case of a is even. This gives

$$
{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{a}{2},1+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]
$$
$$
\times\frac{a!b!c!((a+b+c-2)/2)!((a+b+c)/2)!}{((a-1)/2)!^2((b-1)/2)!^2(c/2)!((c-2)/2)!(a+b+c-1)!((-1+a+c)/2)((1+b+c)/2)}
$$

if a is odd, and

$$
{}_4F_3\left[\begin{matrix}1,1,1-\frac{a}{2},\frac{1}{2}+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]
$$
$$
\times\frac{(a-1)!b!c!((a+b+c-1)/2)!^2}{((a-2)/2)!^2(b/2)!^2((c-1)/2)!^2(a+b+c-1)!((-1+a+c)/2)((1+b+c)/2)}
$$

if a is even, after transforming the Pochhammer symbol $(a)_n=(a+n-1)!/(a-1)!$ to factorials.

Now we substitute $a\sim\alpha N$, $b\sim\beta N$ and $c\sim\gamma N$ and perform the limit $N\to\infty$. We use Stirling's formula to determine the limit for the quotient of the factorials in the second line as $\sqrt{\alpha}\sqrt{\beta}\sqrt{\gamma}\sqrt{\alpha+\beta+\gamma}/(2\pi(\alpha+\gamma)(\beta+\gamma))$. For the ${}_4F_3$–series, we may exchange limit and summation by uniform convergence:

$$
\lim_{N\to\infty}{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{a}{2},1+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]={}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}\right].
$$

A combination of these results and the use of the identity

$$
{}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};z\right]=\frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}}
$$

establishes Theorem 1.3.

The proof of Theorem 1.4 is analogous: Again we divide the formulas in Theorem 1.2, see (1.4) and (1.3), by MacMahon's formula for the total number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c given in (1.1), and obtain the probability to choose a rhombus tiling of a hexagon with side lengths a, b, c, a, b, c which contain the 'almost central' rhombus above the centre. Next we apply Bailey's transformation formula (5.3) on the hypergeometric sum and transform the Pochhammer symbols to factorials. We then substitute $a\sim\alpha N$, $b\sim\beta N$ and $c\sim\gamma N$ and perform the limit $N\to\infty$ in the same way as before. The additional summands (caused by the 'exceptional' evaluation of the irreducible polynomial $\hat{P}_a(b,c)$) that appear in the formulas in Theorem 1.2 (when compared to the formulas in Theorem 1.1) vanish when this limit is performed.

CHAPTER 2

A characterisation of Pfaffian near bipartite graphs

1. Introduction

The graphs considered in this part of the thesis have no loops, are finite and undirected unless an indication to the contrary is given. If v and w are vertices in a directed graph, then (v, w) denotes an edge joining v and w and directed from v to w. If G is any graph, then we denote its vertex set by VG and its edge set by EG. A *1-factor* (or *perfect matching*) of G is a subset f of EG such that every vertex has a unique edge of f incident on it.

Let G^* be a directed graph with an even number $2n$ of vertices and let F be the set $\{f_1, f_2, \ldots, f_k\}$ of 1-factors of G^*. For all i write

$$f_i = \{(u_{i1}, w_{i1}), (u_{i2}, w_{i2}), \ldots, (u_{in}, w_{in})\},$$

where $u_{ij}, w_{ij} \in VG^*$ for all j. Associate with f_i a plus sign if

$$u_{i1}w_{i1}u_{i2}w_{i2} \ldots u_{in}w_{in}$$

is an even permutation of

$$u_{11}w_{11}u_{12}w_{12} \ldots u_{1n}w_{1n},$$

and a minus sign otherwise. Note that the signs of the 1-factors are independent of the order in which their edges have been written. They are dependent on the choice of f_1, but the resulting partition of F into two complementary subsets is not. If G is an undirected graph, we say that G is a *Pfaffian* graph if there exists an orientation such that all the 1-factors of G have the same sign. We say that this orientation is a *Pfaffian* orientation of G. Pfaffian orientations have been used by Kasteleyn [**9**] to enumerate 1-factors in planar graphs. In fact his method can be used precisely for those graphs that are Pfaffian. It is therefore of interest to know which graphs are Pfaffian, but this question is open.

Pfaffian bipartite graphs have been characterised by Little [**14**], who proved the following theorem.

THEOREM 2.1. *A bipartite graph G is non-Pfaffian if and only if it contains an even subdivision J of $K_{3,3}$ such that $G - VJ$ has a 1-factor.*

Here we need to explain the term 'even subdivision'. An *edge subdivision* of a graph G is defined as a graph obtained from G by replacing an edge joining vertices v and w with a path P joining v and w but having no other vertices in common with G. The edge subdivision is *even* if P has odd length. A graph H is a *subdivision* of G if for some positive integer k there exist graphs $G_0, G_1, \ldots, G_k$ such that $G_0 = G$, $G_k = H$ and, for all $i > 0$, G_i is an edge subdivision of G_{i-1}. If $G_1, G_2, \ldots, G_k$ can be chosen so that in addition G_i is an even edge subdivision of G_{i-1} for all $i > 0$, then H is said to be an *even subdivision* of G. It is easy to see that G is Pfaffian if and only if H is Pfaffian. A more general result is proved in Lemma 2.3.

At this point it is worth mentioning that Robertson, Seymour and Thomas [**21**] have recently found a polynomial-time algorithm which decides whether a bipartite graph is Pfaffian or not.

A graph is *1-extendible* if every edge has a 1-factor containing it. Such graphs are the only graphs of interest in the study of the Pfaffian property, as any edge belonging to no 1-factor is irrelevant. A 1-extendible non-bipartite graph G is said to be *near bipartite* if there exist edges e_1, e_2 such that $G-\{e_1,e_2\}$ is 1-extendible and bipartite. The aim of the present paper is to extend Theorem 2.1 to a characterisation of Pfaffian near bipartite graphs in terms of forbidden subgraphs. If G were a 1-extendible graph and $G-\{e\}$ were bipartite for some edge e, then G would also be bipartite. This observation explains why we remove two edges from G, rather than one, in the definition of a near bipartite graph.

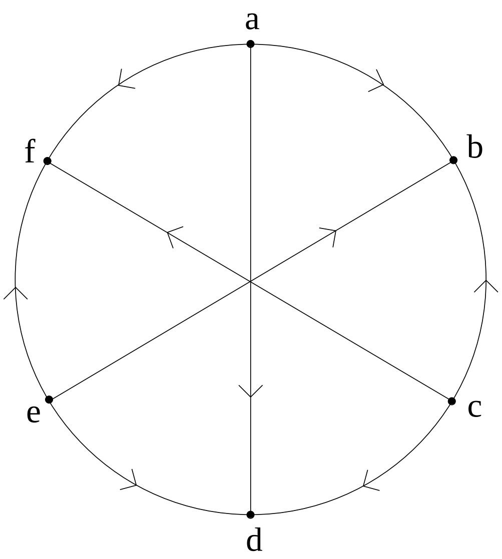
In the statement of our main theorem below, Γ_1 and Γ_2 refer to the graphs drawn in Figures 2.2 and 2.3 respectively, where the arrows are to be ignored. Both graphs are near bipartite, since $\Gamma_1 - \{(f,l),(i,c)\}$ and $\Gamma_2 - \{(f,e),(i,j)\}$ are 1-extendible and bipartite. Note that Γ_2 may be obtained from the Petersen graph by subdividing two fixed edges at a maximum distance apart and then joining the vertices of degree 2 by an edge. These graphs, like $K_{3,3}$, can easily be shown to be non-Pfaffian. Indeed, each graph in Figures 2.1–2.3 is accompanied by a set S of 1-factors such that each edge belongs to just two members of S and S contains an odd number of 1-factors of each kind of sign under the given orientation. The former property of S implies that the latter is still valid if we change the orientation of a single edge. Thus this property of S is independent of the orientation and therefore the graphs cannot be Pfaffian.

It follows that no even subdivision of these graphs is Pfaffian. It is shown in [**15**] that a graph G is non-Pfaffian if it has a circuit X, of odd length, such that the graph obtained from G by contracting VX is non-Pfaffian (see Lemma 2.17). In general, let us say that a graph G is *simply reducible* to a graph H if G has a circuit X, of odd length, such that H is obtained from G by contracting VX. More generally, we say that G is *reducible* to a graph H if for some positive integer k there exist graphs $G_0, G_1, \ldots, G_k$ such that $G_0 = G$, $G_k = H$ and, for all $i > 0$, G_{i-1} is simply reducible to G_i. Thus any graph that is reducible to an even subdivision of $K_{3,3}$, Γ_1 or Γ_2 is non-Pfaffian. In fact, a graph G must be non-Pfaffian if it has a subgraph J that is reducible to an even subdivision of $K_{3,3}$, Γ_1 or Γ_2 and has the property that $G - VJ$ has a 1-factor. The purpose of this part of the thesis is to show that the converse of this statement holds for near bipartite graphs.

THEOREM 2.2. *A near bipartite graph G is non-Pfaffian if and only if G contains a subgraph J, reducible to an even subdivision of $K_{3,3}$, Γ_1 or Γ_2, such that $G - VJ$ has a 1-factor.*

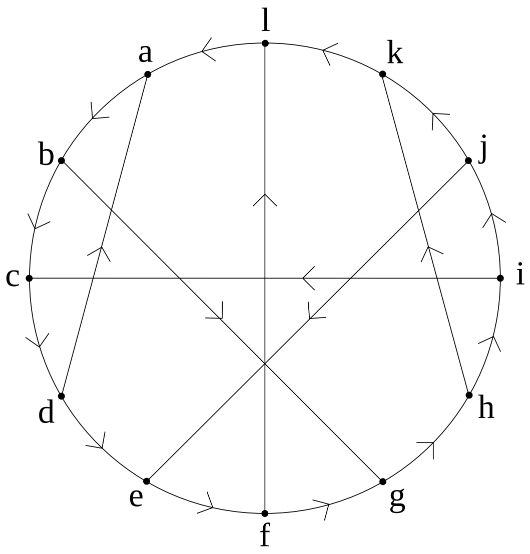
Definitions and Notation. The following definitions and notation are fundamental for this part of the thesis. Circuits, non-empty paths and, more generally, subgraphs with no isolated vertices are determined by their edge sets, and are therefore identified with them in this thesis. If X is a path or circuit in a graph G, then we denote by VX the set of vertices of X. If P is a path and $u, v \in VP$, then we denote by $P[u,v]$ the subpath of P joining u to v. If $P[u,v]$ is directed from u to v, then we also write $P(u,v) = P[u,v]$. If C is a circuit which includes a unique directed path from vertex u to vertex v, then that path is denoted by $C(u,v)$. From time to time we may perform a reorientation of C, that is to say we change the orientation of every oriented edge in C. This directed path is then denoted by $C(v,u)$.

A circuit is *alternating* with respect to each of two given 1-factors if it is included in their symmetric difference. A circuit that is alternating with respect to a 1-factor f is also said to be *f-alternating*, or *consanguineous* (with respect to f). Note that a graph with more than one edge and a 1-factor f is 1-extendible if and only if every edge has an f-alternating circuit



$$
\begin{array}{ll}
(a,b)(c,d)(e,f) & + \\
(a,b)(c,f)(e,d) & - \\
(a,d)(c,f)(e,b) & + \\
(a,d)(c,b)(e,f) & - \\
(a,f)(c,b)(e,d) & + \\
(a,f)(c,d)(e,b) & -
\end{array}
$$

FIGURE 2.1. The graph $K_{3,3}$.



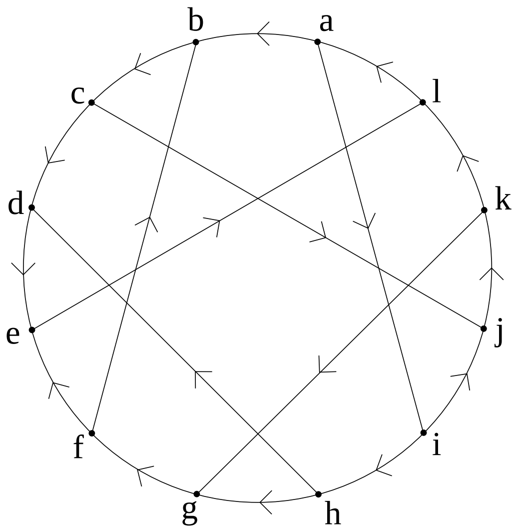
$$
\begin{array}{ll}
(a,b)(c,d)(e,f)(g,h)(i,j)(k,l) & + \\
(b,c)(d,e)(f,g)(h,i)(j,k)(l,a) & - \\
(d,a)(j,e)(b,c)(f,g)(h,i)(k,l) & + \\
(b,g)(h,k)(c,d)(e,f)(i,j)(l,a) & - \\
(i,c)(f,l)(a,b)(d,e)(g,h)(j,k) & + \\
(d,a)(b,g)(i,c)(j,e)(h,k)(f,l) & -
\end{array}
$$

FIGURE 2.2. The graph Γ_1.

containing it. A path P is *alternating* if every internal vertex of P is incident with an edge of $P \cap f$. An *ear* is a path of odd cardinality.

Let A and B be sets of edges in a graph G. Then an *AB-arc* is a non-empty maximal subpath of $A \cap B$, and an *$A\overline{B}$-arc* (or a *$\overline{B}A$-arc*) is a non-empty maximal subpath of $A - B$. A $G\overline{B}$-arc is also called a $\overline{B}$-arc.

How to attack a general characterisation of Pfaffian graphs and the significance of our theorem in this context. Next, let A be an alternating circuit in G and let H be a subgraph of G. If there are n $A\overline{H}$-arcs, and each is an ear, then we say that $G[EH \cup A]$ is obtained from H by an *n-ear adjunction.* An *ear decomposition* of a 1-extendible graph G is a sequence $G_0, G_1, \ldots, G_t$ of 1-extendible subgraphs of G such that G_0 is isomorphic to K_2, $G_t = G$ and, for each $i > 0$, G_i is obtained from G_{i-1} by a 1-ear or 2-ear adjunction. A



$$\begin{array}{ll}
(a,b)(c,d)(f,e)(h,g)(i,j)(k,l) & + \\
(b,c)(d,e)(g,f)(i,h)(j,k)(l,a) & - \\
(e,l)(a,b)(c,d)(g,f)(i,h)(j,k) & + \\
(h,d)(k,g)(b,c)(f,e)(i,j)(l,a) & - \\
(a,i)(f,b)(c,j)(d,e)(h,g)(k,l) & + \\
(a,i)(f,b)(c,j)(h,d)(e,l)(k,g) & -
\end{array}$$

FIGURE 2.3. The graph Γ_2.

theorem of Lovász and Plummer [**16**, Theorem 5.4.6] asserts that every 1-extendible graph has an ear decomposition. It can be stated as follows.

THEOREM 2.3. *Let f be a 1-factor in a 1-extendible graph G. Let H be a 1-extendible proper subgraph of G such that $EH \neq \emptyset$ and $f \cap EH$ is a 1-factor of H. Then G contains an f-alternating circuit A that admits just one or two $A\overline{H}$-arcs.*

In fact if G is bipartite then it can be shown that only 1-ear adjunctions are necessary.

The idea behind the proof of Theorem 2.1 runs as follows. Clearly we may assume that G is 1-extendible. Suppose that G is non-Pfaffian. We construct an ear decomposition $G_0, G_1, \ldots, G_t$ of G. Since G is bipartite, we may assume that, for each $i > 0$, G_i is obtained from G_{i-1} by the adjunction of a single ear. As G_0 is Pfaffian but G is not, there exists a smallest positive integer j such that G_j is non-Pfaffian. The graph G_j is studied in detail and eventually shown to contain J.

In [**15**] Little and Rendl explain how Theorem 2.3 provides a possible way to generalise this argument and with it they propose the following approach towards attacking the question of characterising Pfaffian graphs: If we drop the assumption that G is bipartite then, for each j, G_j is obtained from G_{j-1} by the adjunction of one or two ears. Moreover G_{j-1} might or might not be bipartite. Therefore the following four cases have to be looked at:

(a) G_{j-1} is bipartite and G_j is obtained from G_{j-1} by the adjunction of a single ear.
(b) G_{j-1} is bipartite and G_j is obtained from G_j by the adjunction of two ears.
(c) G_{j-1} is non-bipartite and G_j is obtained from G_j by the adjunction of a single ear.
(d) G_{j-1} is non-bipartite and G_j is obtained from G_j by the adjunction of two ears.

In each case the goal is to find a member of a class of forbidden (non-Pfaffian) graphs as a subgraph of G_j. In Case (a) this goal was achieved in [**14**], by showing that G_j contains an even subdivision of $K_{3,3}$. Case (b) is equivalent to characterising Pfaffian near bipartite graphs and is solved in this thesis. The remaining cases are still open.

Surprisingly, by a result of Carvalho, Lucchesi and Murty [**1**], one can restrict the study to graphs G that have an ear decomposition, which uses exactly one two ear adjunction in the ear decomposition, by adding the (non-Pfaffian) Petersen graph to the class of forbidden subgraphs. (See [**15**].) Thus it turns out unnecessary to investigate Case (d).

But it also turns out unnecessary to investigate Case (b), if Case (c) was solved: Carvalho, Lucchesi and Murty [**1**] show that one can choose an ear decompostition of G such that the unique two ear adjunction is at the beginnig of the ear decompostion. (See also [**16**, Theorem 5.4.10].) To be more accurate we can assume that either G_2 is obtained from G_1 by the adjunction of two ears and G_2 is still Pfaffian, or that G_3 is obtained from G_2 by the adjunction of two ears and G_3 is still Pfaffian. Thus we can avoid Case (b) — or better — reduce it to the unsolved Case (c).

But Case (c) turns out to be a lot harder and definitely requires new techniques. The lack of the bipartitness of graph G_{j-1} causes serious difficulties. Maybe it is no coincidence that the characterisation of Pfaffian graphs is still open whereas the characterisation of Pfaffian *bipartite* graphs was already achieved in the 1970's. Therefore I am not really convinced that an ear decomposition with the unique two ear adjunction at the beginning is the appropriate ear decomposition for the characterisation of Paffian graphs, since it destroys the bipartiteness as soon as possible. More desireable is an ear decomposition of G which destroys the bipartitness as late as possible. Therefore a possible way for attacking Case (c) could be first finding an ear decomposition of G where the (unique) two ear adjunction is made as late as possible.

Another reason, why I think that investigating Case (b) was not useless, is, that it (in a way systematically) produced the two minimal non-Pfaffian graphs Γ_1 and Γ_2— which form together with $K_{3,3}$ and the Petersen graph the class of known minimal non-Pfaffian graphs.

Idea behind the proof of Theorem 2.2. We use alternating circuits in preference to 1-factors. Kasteleyn [**9**] has shown that the 1-factors of a directed graph all have equal sign if and only if all the alternating circuits are clockwise odd. (The clockwise parity of a circuit of even length is the parity of the number of its edges that are directed in agreement with a specified sense.) Let G be a near bipartite graph which is minimal with respect to the property of being non-Pfaffian. Let e_1 and e_2 be edges of G such that $G - \{e_1, e_2\}$ is bipartite and 1-extendible. By minimality $G - \{e_1, e_2\}$ has a Pfaffian orientation. Extend this orientation to an orientation of G by orienting e_1 and e_2 arbitrarily.

Since G is non-Pfaffian, there exist two alternating circuits A and B of opposite clockwise parity. In Lemma 2.5 we use Theorem 2.3 to show that A and B can be chosen there to be just one or two AB-arcs. We then divide the argument into cases according to whether or not it is possible to choose A and B so that there is just one AB-arc. The first case is dealth with in Section 2 and the second in Section 3. In case there is just one AB-arc we introduce a path R in order to construct alternating circuits in $G-\{e_1, e_2\}$ whose union is $(A\cup B\cup R)-\{e_1, e_2\}$ and whose sum (symmetric difference) is $A + B$ (Theorem 2.4). Similarly we introduce two paths S and T in order to construct alternating circuits in $G - \{e_1, e_2\}$ whose union is $(A \cup B \cup S \cup T) - \{e_1, e_2\}$ and whose sum is $A + B$, if there are two AB-arcs (Theorem 2.5). The minimality of G will then imply that G is induced by $A \cup B \cup R$ or $A \cup B \cup S \cup T$ respectively. The list of non-Pfaffian near bipartite graphs induced by the two circuits A and B and a path R, respectively by the two circuits and two paths S and T is still infinite. In Subsections 2.3, 2.4 and Subsections 3.3, 3.4, respectively, we are then able to reduce this list to a finite list by invoking the minimality of G again. In Subsection 2.5 and Subsection 3.5, respectively, we finally show that every graph in this list can be obtained from $K_{3,3}$, Γ_1 or Γ_2 by means of the operations of reduction and even subdivision.

In Section 4 we finally demonstrate that neither Γ_1 nor Γ_2 is reducible to an even subdivision of $K_{3,3}$.

2. The two alternating circuits A and B

For the rest of this thesis let G be a non-Pfaffian near bipartite graph. We have to show that G contains a subgraph J, which is reducible to an even subdivision of $K_{3,3}$, Γ_1 or Γ_2, such that $G - VJ$ has a 1-factor.

In this section we first introduce two properties which may be assumed for G throughout the whole argument, namely that G is minimal non-Pfaffian and does not have a vertex of degree two. Next we find two alternating circuits A and B of opposite clockwise parity with respect to an 'extended' Pfaffian orientation of G. In Lemma 2.5, which is based on Theorem 2.3, we choose A and B so that there are just one or two AB-arcs. Next we see that $G[A \cup B]$ is homeomorphic to the graph with two vertices and three edges, if there is just one AB-arc, and homeomorphic to K_4, if there are exactly two AB-arcs (see Lemma 2.6).

Since $G[(A \cup B) - \{e_1, e_2\}]$ is not 1-extendible it is clear that the graph cannot have been built up completely by the two circuits. Using the very simple yet important Lemma 2.7 we obtain new structure in G, namely a path R if there is just one AB-arc and two paths S and T if there are two AB-arcs. Then $G[(A \cup B \cup R) - \{e_1, e_2\}]$, respectively $G[(A \cup B \cup S \cup T) - \{e_1, e_2\}]$ is not only 1-extendible but in fact equal to $G - \{e_1, e_2\}$ as we see in the following.

Minimality. We may assume that G is minimal with respect to the property of being non-Pfaffian, i.e. for all edges e of G the graph $G - \{e\}$ is Pfaffian. To see this point, suppose that G has an edge e such that $G - \{e\}$ is non-Pfaffian and has a subgraph J, reducible to an even subdivision of $K_{3,3}, \Gamma_1$ or Γ_2, such that $(G - \{e\}) - VJ$ has a 1-factor f. Then f is also a 1-factor of $G - VJ$, and so Theorem 2.2 holds also for G.

From this we can deduce that G has no multiple edges. Assume the contrary and let u and v be vertices connected by a least two edges a_1 and a_2. Note that G is 1-extendible and connected for G is minimal non-Pfaffian. Since G would be Pfaffian if $VG = \{u, v\}$, there exists a 1-factor f that neither contains a_1 nor a_2. A graph H with 1-factor g is Pfaffian if and only if there exists an orientation of H such that the g-alternating circuits are clockwise odd. Find such an orientation for the Pfaffian graph $G - \{a_1\}$ and extend it to an orientation of G by orienting a_1 as a_2. This is a Pfaffian orientation of G in contradiction to the fact that G is non-Pfaffian.

No vertices of degree 2. A set S of alternating circuits in a directed graph H is called *intractable* if the sum of the circuits in S is empty and an odd number of the members of S are clockwise even. The following lemma shows that the former property of S implies that the latter is independent of the orientation of H. This lemma is fundamental for the study of Pfaffian graphs.

LEMMA 2.1. *Let $\mathcal{C}$ be a set of circuits of even length and empty sum in a directed graph G. Then the parity of the number of clockwise even members of $\mathcal{C}$ is independent of the orientation of G.*

Proof: A change of orientation can be effected by changing orientations of edges one at a time. Each such change leaves unaltered the parity of the number of clockwise even circuits in $\mathcal{C}$. $\square$

The following lemma is proved in [**13**].

LEMMA 2.2. *A graph is Pfaffian if and only if it has no intractable set of alternating circuits.*

From this result we show that we can assume there to be no vertices of degree 2 in G.

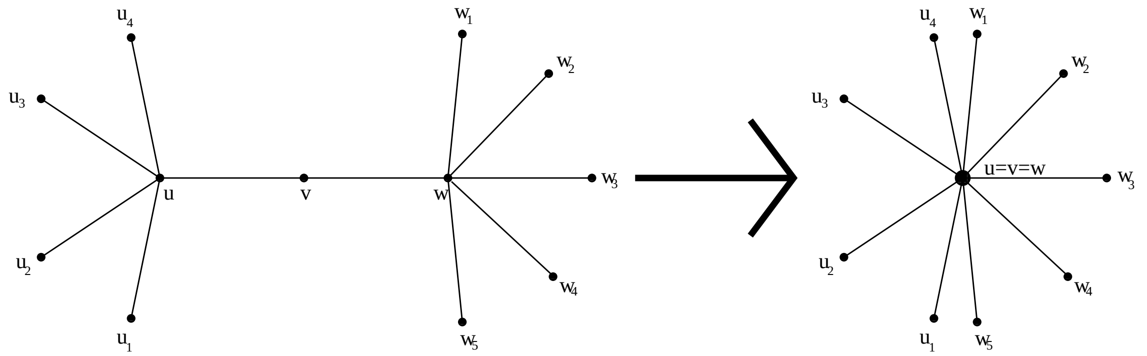


FIGURE 2.4. The contraction in Lemma 2.3.

LEMMA 2.3. *Let v be a vertex of degree 2 in G, and let G' be the graph obtained from G by contracting the edges incident on v. (See Figure 2.4.) Then G is Pfaffian if and only if G' is Pfaffian.*

Proof: Let a and b be the edges of G incident on v, and let u and w be the vertices adjacent to v.

Suppose there is an intractable set S of alternating circuits in G. Then the intersections of the circuits in S with $EG - \{a, b\}$ yield an intractable set in G'. Conversely, let S' be an intractable set of alternating circuits in G'. Let v' be the vertex in G' obtained by identifying u and w in G. Choose $C' \in S'$. If $v' \notin VC'$, or the edges of C' incident on v' in G' are both incident on u in G or both incident on w in G, then let $C = C'$; otherwise let $C = C' \cup \{a, b\}$. The set S of such circuits C forms an intractable set in G. (Note that the sum of the circuits in S is a subset of $\{a, b\}$ and therefore empty as it must be a cycle.) □

Let G be a graph with a vertex of degree 2 and let G' be the graph obtained from G by contracting the edges incident on it. Suppose that in G' there is a subgraph J, reducible to an even subdivision of $K_{3,3}, \Gamma_1$ or Γ_2, such that $G' - VJ$ has a 1-factor. Then the same is true for G, for $K_{3,3}$, Γ_1 and Γ_2 are cubic and so the converse of the reduction in the lemma gives an even subdivision of each of those graphs. Therefore we can assume that G has no vertex of degree 2.

The f-alternating circuits A and B. Since G is near bipartite, it is 1-extendible. Moreover there exist edges e_1 and e_2 such that $G - \{e_1, e_2\}$ is bipartite and 1-extendible. We call this graph H, and fix a 1-factor f of H. Note that $G - \{e_1\}$ is non-bipartite, for otherwise, since G is non-bipartite, every circuit containing e_1 would be of odd length, in contradiction to the fact that G has an alternating circuit containing e_1. Similarly $G - \{e_2\}$ is non-bipartite. Consequently any alternating circuit containing one of e_1 and e_2 must also contain the other.

Note that H is Pfaffian, by the minimality of G. Extend a Pfaffian orientation of H to an orientation of G by orienting e_1 and e_2 arbitrarily. We shall henceforth refer to this orientation as our extended Pfaffian orientation of G. As G is non-Pfaffian, it contains a clockwise even alternating circuit A. This circuit must contain e_1 and e_2. There must also be a clockwise odd alternating circuit B containing e_1 and e_2, for otherwise a Pfaffian orientation for G could be constructed by reorienting e_1 or e_2. Note that A and B are symmetric, since the clockwise parities of A and B can be changed be reorienting e_1. The following two lemmas show that it is possible to choose these circuits A and B such that $G[A \cup B]$ is not too complicated.

LEMMA 2.4. *Let A_1, A_2 be f-alternating circuits in a directed graph G with 1-factor f. Then A_1 and A_2 are of opposite clockwise parity if and only if A_1+A_2 includes an odd number of clockwise even alternating circuits.*

Proof: Let $A_1 = f + f_1$ and $A_2 = f + f_2$ for some 1-factors f_1 and f_2. Suppose that A_1 and A_2 are of opposite clockwise parity. Without loss of generality, let A_1 be clockwise odd and A_2 be clockwise even. Then f has the same sign as f_1 but the opposite sign from f_2. Hence f_1 and f_2 have opposite sign. Since $A_1 + A_2 = f_1 + f_2$. it follows that $A_1 + A_2$ includes an odd number of clockwise even alternating circuits.

Similarly if A_1 and A_2 have the same clockwise parity, then f_1 and f_2 have equal sign and $A_1 + A_2$ includes an even number of clockwise even alternating circuits. □

REMARK 2.1. *The previous lemma will be fundamental for the rest of the proof of Theorem 2.2. Phrased differently it states that the set of two alternating circuits A_1 and A_2, that are consanguineous with respect to the same 1-factor f, union the $f + A_1$-alternating circuits (or, equivalently, the $f + A_2$-alternating circuits) included in $A_1 + A_2$, do not form an intractable set of alternating circuits in G.*

LEMMA 2.5. *There exist two alternating circuits A and B in G of opposite clockwise parity and consanguineous with respect to some 1-factor f that contains neither e_1 nor e_2, such that there are just one or two AB-arcs, each AB-arc contains e_1 or e_2 and their union contains both.*

Proof: Choose two alternating circuits A and B of opposite clockwise parity, containing e_1 and e_2, that are consanguineous with respect to some 1-factor f that contains neither e_1 nor e_2, such that $A \cup B$ is minimal. Note that both graphs $G[A \cup B]$ and $G[A]$ are 1-extendible and that f induces a 1-factor on both graphs. By Theorem 2.3 there exists an f-alternating circuit C in $G[A \cup B]$ that admits just one or two $C\overline{A}$-arcs, or, equivalently, that admits just one or two CA-arcs.

Suppose that C contains neither e_1 nor e_2. Define $f' = f + C$, $A' = A + C$ and $B' = B + C$. Note that f' is a 1-factor that contains neither e_1 nor e_2. By Lemma 2.4 A' and B' are f'-alternating circuits of opposite clockwise parity since $A + B = A' + B'$. But

$$(A' \cup B') \cap (A \cap B \cap C) = \emptyset$$

in contradiction to the minimality of $A \cup B$, since

$$\emptyset \neq A \cap B \cap C \subseteq A \cup B.$$

Therefore C contains e_1 and e_2, since every alternating circuit in G that contains e_i also contains e_{3-i}. If A and C are of opposite clockwise parity we have $A \cup C = A \cup B$ by the minimality of $A \cup B$, and therefore $B = C$. Similarly we conclude that $A = C$ if B and C are of opposite clockwise parity, but this is a contradiction since C was chosen such that there is at least one $C\overline{A}$-arc. Thus there is just one or two AB-arc.

Since A and B both contain e_1 and e_2 we already know that the union of the AB-arcs contains e_1 and e_2. In case there are exactly two AB-arcs, it remains to show that every AB-arc contains e_1 or e_2.

Suppose the contrary, that is one AB-arc contains e_1 and e_2 and the other contains neither e_1 nor e_2. Let the former arc join vertices x_1 and x_2 and the latter vertices y_1 and y_2. Ajust the notation so that the vertices x_1, y_1, y_2, x_2 appear in that order as A is traversed in a specified sense. It is clear that one $B\overline{A}$-arc joins x_1 to y_i and the other joins x_2 to y_{3-i}, for some $i \in \{1, 2\}$. We show that $i = 1$. Suppose the contrary, that is one $B\overline{A}$-arc X joins x_1 to y_2. Let Y be the $A\overline{B}$-arc that joins x_1 to y_1 and Z be the AB-arc that joins y_1 to y_2.

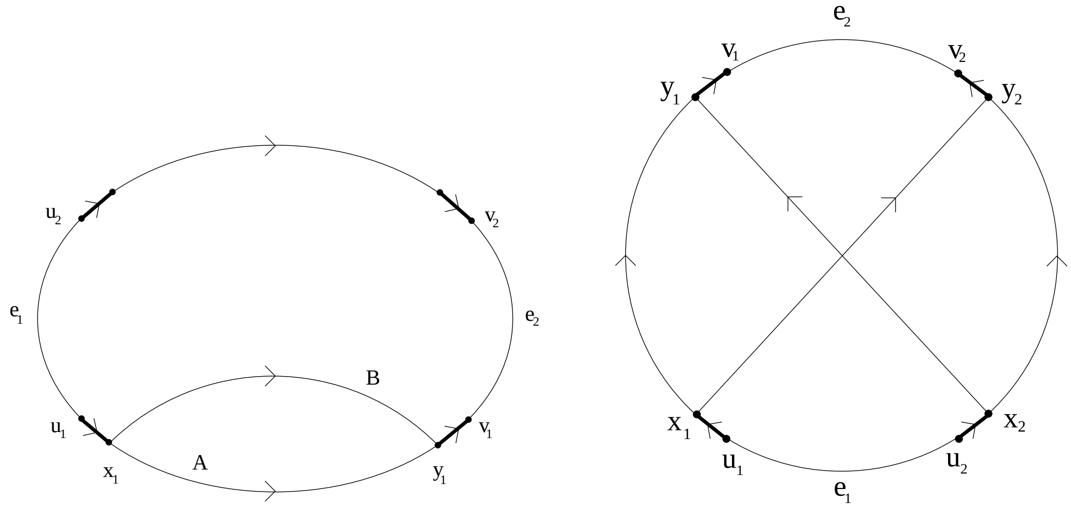


FIGURE 2.5. A homeomorph of $G[A \cup B]$.

Then $X \cup Y \cup Z$ is a circuit of odd length in $G - \{e_1, e_2\}$, in contradiction to the fact that $G - \{e_1, e_2\}$ is bipartite. Thus $i = 1$.

We define an f-alternating circuit C that contains e_1 and e_2 by replacing the $A\overline{B}$-arc joining x_1 to y_1 in A with the $B\overline{A}$-arc joining x_1 to y_1. If A and C have opposite clockwise parity we replace B with C, since there is a unique AC-arc. Otherwise B and C have opposite clockwise parity and we replace A with C, since there is a unique BC-arc too. In both cases we have a contradiction by the minimality of $A \cup B$. □

Thus A and B can be chosen so that there are at most two AB-arcs. We divide the argument into cases according to whether or not it is possible to choose A and B such that there is a unique AB-arc.

Let e_1 join vertices u_1 and u_2 and let e_2 join vertices v_1 and v_2. Define $A^* = A - \{e_1\}$. If there is a unique AB-arc, let x_1 and y_1 be its endvertices and adjust the notation so that the vertices u_1, x_1, y_1, v_1, v_2, u_2 appear in that order as A^* is traversed from u_1 to u_2. In this case the graph $G[A \cup B]$ is an even subdivision of that shown in Figure 2.5, left picture. The edges of f are thickened in this and subsequent figures, and in all subsequent figures the graph in question is an even subdivision of the one portrayed.

Now suppose that A and B are chosen so that there are exactly two AB-arcs, and therefore exactly two $\overline{A}B$-arcs and exactly two $A\overline{B}$-arcs. By Lemma 2.5 one of the AB-arcs contains e_1 and the other e_2. Let the former arc join vertices x_1 and x_2 and the latter join y_1 and y_2. Adjust the notation so that the vertices u_1, x_1, y_1, v_1, v_2, y_2, x_2, u_2 appear in that order as A^* is traversed from u_1 to u_2. The following lemma shows that $G[A \cup B]$ is homeomorphic to K_4 in this case.

LEMMA 2.6. *One of the $\overline{A}B$-arcs joins x_2 to y_1 and the other x_1 to y_2.*

Proof: Suppose the contrary. Note that the edges of f incident on x_1, x_2, y_1 and y_2, respectively, belong to $A^*[x_1, u_1] \cup A^*[u_2, x_2] \cup A^*[y_2, y_1]$, since e_1 and e_2 belong to AB-arcs. If an $\overline{A}B$-arc X were to join y_1 to y_2 then we should have the contradiction that the circuit $A^*[y_1, y_2] \cup X$ would be of even length yet contain e_2 but not e_1. On the other hand, suppose that an $\overline{A}B$-arc Y were to join x_1 to y_1. Let

$$C = Y \cup A^*[x_1, u_1] \cup \{e_1\} \cup A^*[u_2, y_1].$$

This circuit, an f-alternating circuit containing e_1 and e_2, has opposite clockwise parity from either A or B. Since there is a unique AC-arc and a unique BC-arc, we have a contradiction to the assumption that there is no choice for A and B such that there is a unique AB-arc. □

If there are two AB-arcs the graph $G[A \cup B]$ is an even subdivision of that shown in Figure 2.5, right picture. Since the structure of $G[A \cup B]$ is different in this case compared to the case where there is unique AB-arc, we will start the case distinction at this point. The next section deals with the first case and the following section deals with the second case. We conclude this section with some results which are important for both cases.

An orientation for 1-factor covered bipartite graphs. For a bipartite graph K with bipartition $\{M, N\}$ and 1-factor f there exists an orientation in which the directed paths and directed circuits are precisely the f-alternating paths and f-alternating circuits respectively: orient the edges of f from M to N and the remaining edges from N to M. Then every vertex has indegree 1 or outdegree 1, and every edge joins a vertex of indegree 1 to a vertex of outdegree 1. It follows that directed paths with an internal vertex in common meet in an edge incident on the vertex. We call this orientation the *reference* orientation for K with respect to (M, N, f).

Fix such a reference orientation for H with respect to f so that $A^*[u_1, v_1]$ is directed from u_1 to v_1. Henceforth the orientation associated with H will be this reference orientation unless an indication to the contrary is given.

If there is a unique $B\overline{A}$-arc, it is directed from x_1 to y_1 and $A^*[u_2, v_2]$ is directed from u_2 to v_2 (see Figure 2.5). Otherwise it follows that $B \cap EH$ includes a directed path from u_1 to v_2 and another from u_2 to v_1, and that $A^*[u_2, v_2]$ is directed from u_2 to v_2 (see Figure 2.5).

Let f' be another 1-factor of K. It is shown in [**14**] that the reference orientation for K with respect to (M, N, f') is obtained from the reference orientation with respect to (M, N, f) by reorienting the circuits included in $f + f'$.

The 'building-up' lemma. All we know about the structure of G up to now are the two circuits A and B. But, since $G[(A \cup B) - \{e_1, e_2\}]$ is not 1-extendible, the graph G is not equal to $G[A \cup B]$. With the help of the next lemma ([**13**]) we will be able to get additional structual information about G. Suprisingly, that is in a sense all structual information we need in order to construct all minimal non-Pfaffian near bipartite graph G as we will see in the following.

LEMMA 2.7. *Let G be a directed connected graph such that each edge has a directed circuit containing it. Then for every $a, b \in VG$, there exists a directed path from a to b.*

Proof: Since G is connected there exists a path P that joins a to b. For every edge e in P there exists a directed circuit C_e containing it. We replace an edge e in P with $C_e - \{e\}$ if e is not directed from a to b in P. We obtain a sequence of edges $(e_1, e_2, \ldots, e_n)$ so that for all $i > 1$ the origin of e_i is the terminus of e_{i-1}. Furthermore the origin of e_1 is a and that of e_n is b. From such a sequence it is easy to construct a directed path from a to b. (See Lemma 2.19.) □

We may apply this lemma to H, since every edge of the 1-extendible graph H must belong to a directed circuit under our reference orientation. Thus there is a directed path R from y_1 to x_1, if there is a unique AB-arc (see Figure 2.6, left picture; a dotted line in this and subsequent figures stands for a directed path, which can have intersections with the rest of the graph that are not indicated). Similarly there is a directed path S from y_1 to x_1 and a

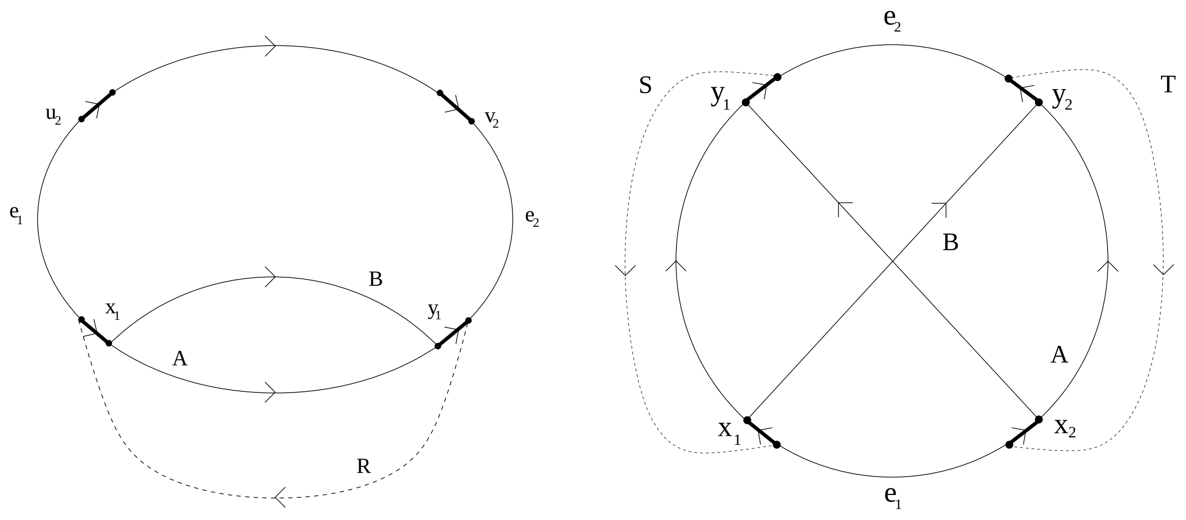


FIGURE 2.6. The directed paths R, S and T.

directed path T from y_2 to x_2, if there are two AB-arcs (see Figure 2.6, right picture). Our first aim in both cases is to show that G has been built up completely by the circuits A and B and the additional path(s). To be more accurate we will show that $G = G[A \cup B \cup R]$ and $G = G[A \cup B \cup S \cup T]$, respectively.

3. One AB-arc

In this section we assume that the circuits A and B were chosen such that there is a unique AB-arc.

3.1. The graph G is induced by $A \cup B \cup R$.

In this subsection we show that G is induced by the two alternating circuits A and B and the alternating path R (see Theorem 2.4). For that purpose we need a lemma.

The union of a directed path P with origin x and terminus y and a directed path Q with origin y and terminus x need not to be a single circuit. But, as we show in the following lemma, one can find a set of circuits, such that the symmetric difference of these circuits is $P + Q$.

LEMMA 2.8. *Let H be a directed graph, let P be a directed path from vertex x to vertex y and let Q be a directed path from y to x. Then there are directed circuits $C_1, C_2, \ldots, C_k$ such that*

$$\sum_{i=1}^{k} C_i = P + Q$$

and

$$\bigcup_{i=1}^{k} C_i = P \cup Q.$$

Proof: We use induction on $n = |VP \cap VQ| \geq 2$. If $n = 2$, then $\{P \cup Q\}$ is the required set of directed circuits.

Let $n > 2$ and suppose the lemma holds whenever $|VP \cap VQ| < n$. Let b be the last vertex of $VQ - \{x\}$ that is in VP, and let a be the last vertex of VQ incident with an edge of

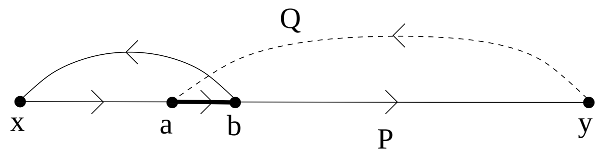


FIGURE 2.7. The situation in Lemma 2.8

$Q(y,b) - P$ (see Figure 2.7). By the inductive hypothesis there exist circuits $C_1, C_2, \ldots, C_{k-1}$ such that

$$\sum_{i=1}^{k-1} C_i = P(a,y) + Q(y,a).$$

The required set of circuits is

$$\{C_1, C_2, \ldots, C_{k-1}, P(x,b) \cup Q(b,x)\},$$

since

$$\begin{aligned} & P(a,y) + Q(y,a) + (P(x,b) \cup Q(b,x)) \\ = \ & P(a,y) + Q(y,a) + P(x,b) + Q(b,x) \\ = \ & P(a,y) + Q(y,a) + P(x,a) + P(a,b) + Q(b,x) \\ = \ & P(x,a) + P(a,y) + Q(y,a) + Q(a,b) + Q(b,x) \\ = \ & P + Q \end{aligned}$$

as $P(a,b) = Q(a,b)$. □

THEOREM 2.4. *Let G be a minimal non-Pfaffian near bipartite graph. Let e_1 and e_2 be edges such that $G - \{e_1, e_2\}$ is bipartite and 1-extendible. Let $H = G - \{e_1, e_2\}$, and let f be a 1-factor of H. Let A and B be f-alternating circuits in G containing e_1 and e_2, of opposite clockwise parity under an extended Pfaffian orientation. Suppose there is a unique AB-arc and let it join vertices x_1 and y_1. Let $A^* = A - \{e_1\}$ and let H be given its reference orientation with respect to f such that $A^*[x_1, y_1]$ is a directed path from x_1 to y_1. Let R' be a directed path from y_1 to x_1. Then G is induced by $A \cup B \cup R'$.*

Proof: We first show, by using Lemma 2.8, that we can write $A + B$ as a sum of directed circuits included in $A \cup B \cup R'$. In order to verify this claim, first we apply Lemma 2.8 to the directed paths $A(x_1, y_1)$ and R'. Let

$$A(x_1, y_1) + R' = \sum_{C \in \mathcal{C}} C, \tag{3.1}$$

where $\mathcal{C}$ is a set of directed circuits. Similarly we apply Lemma 2.8 to $B(x_1, y_1)$ and R'. Let

$$B(x_1, y_1) + R' = \sum_{D \in \mathcal{D}} D, \tag{3.2}$$

where $\mathcal{D}$ is a set of directed circuits. Let $\mathcal{E} = \mathcal{C} + \mathcal{D}$. Adding (3.1) and (3.2) we obtain

$$A(x_1, y_1) + B(x_1, y_1) = \sum_{C \in \mathcal{C}} C + \sum_{D \in \mathcal{D}} D = \sum_{E \in \mathcal{E}} E.$$

Since the left hand side is $A + B$, $\mathcal{E}$ is the required set of circuits.

In our extended Pfaffian orientation of G, A is the only clockwise even circuit in $\mathcal{E} \cup \{A, B\}$. Therefore, by Lemma 2.1, an odd number of circuits in $\mathcal{E} \cup \{A, B\}$ are clockwise even for any orientation of G. But if there were a Pfaffian orientation of $G[A \cup B \cup \bigcup_{E \in \mathcal{E}} E]$ then

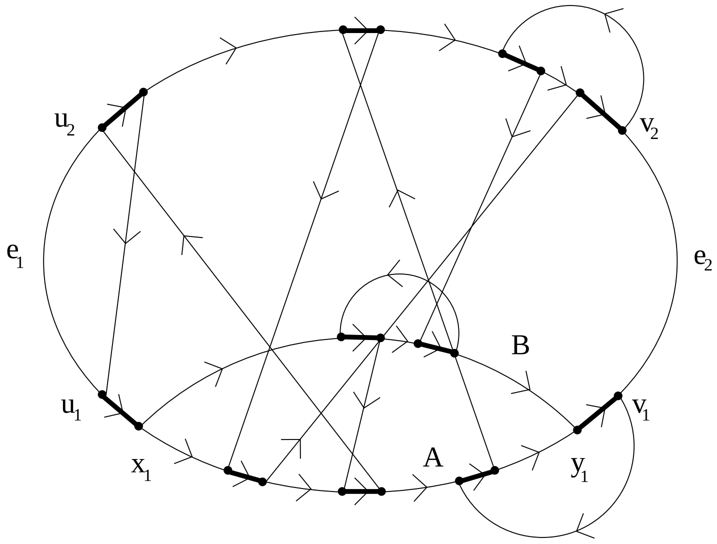


FIGURE 2.8. The trace of R is 020221102.

every circuit in $\mathcal{E} \cup \{A, B\}$ would be clockwise odd because they are f-alternating. Therefore the graph $G[A \cup B \cup \bigcup_{E \in \mathcal{E}} E]$ is non-Pfaffian, and so $G[A \cup B \cup R']$ is non-Pfaffian. By the minimality of G, we deduce that

$$G[A \cup B \cup R'] = G.$$

□

Application of this theorem to R we find that $G = G[A \cup B \cup R]$.

REMARK 2.2. *In the proof of Theorem 2.4 we showed that $\mathcal{E} \cup \{A, B\}$ is an intractable set of alternating circuits. Therefore, by Lemma 2.4, an odd number of circuits in $\mathcal{E} \cup \{A + B\}$ are clockwise even under any orientation of G. Under our reference orientation, $A + B$ is the only clockwise odd circuit in $\mathcal{E} \cup \{A + B\}$. Thus $|\mathcal{E}|$ is odd.*

Next we introduce a notation to describe the directed path R. Traversed from y_1 to x_1, R meets a succession of directed $(A \cup B)R$-arcs in H. The *trace* of R is the sequence obtained from R by recording:

0 for each $A(x_1, y_1)R$-arc,

1 for each $B(x_1, y_1)R$-arc,

2 for each $(A \cap B)(u_2, v_2)R$-arc.

In Figure 2.8 an example of a directed path R is given. Its trace is 020221102. Next we want to show that G is determined up to isomorphism by the trace of R. To this end we introduce the following notation.

Let P be a directed path from vertex x to vertex y. Let P_1 and P_2 be disjoint subpaths of P such that each edge of P_1 is closer to x in P than is any edge of P_2. In this case we write $P_1 <_P P_2$. If v_1 and v_2 are distinct vertices in P and v_1 is closer to x in P than v_2, we write $v_1 <_P v_2$.

The next lemma is obvious.

LEMMA 2.9. *Let u and v be vertices in $G[A \cup B]$. Under our reference orientation for H there exists at most one directed path in $G[A \cup B]$ from u to v.*

This lemma is used in the proof of the following lemma, where we show that R is unique.

LEMMA 2.10. *Let P be a directed path from y_1 to x_1 and let Q be a directed path included in $A \cup B$. Let P_1 and P_2 be distinct PQ-arcs. Then $P_1 <_P P_2$ if and ony if $P_2 <_Q P_1$ (see Figure 2.15, $i = j = 1$). Moreover $P = R$.*

Proof: For the first claim it suffices to show that if $P_1 <_P P_2$ then $P_2 <_Q P_1$. Assume the contrary, that is $P_1 <_P P_2$ and $P_1 <_Q P_2$. Let a be the terminus of P_1 and b be the origin of P_2. By assumption $P(a,b)$ and $Q(a,b)$ exist, but are not equal. By Lemma 2.9 there is an edge $e \in P(a,b) - (A \cup B)$. The set $P(y_1,a) \cup Q(a,b) \cup P(b,x_1)$ includes a directed path P' from y_1 to x_1. By Theorem 2.4 $G = G[A \cup B \cup P']$, in contradiction to $e \notin A \cup B \cup P'$. Thus $P_1 <_P P_2$ if and only if $P_2 <_Q P_1$.

By Theorem 2.4

$$G = G[A \cup B \cup R] = G[A \cup B \cup P].$$

Therefore we have

$$P - (A \cup B) = R - (A \cup B).$$

Again by Theorem 2.4 the components of $P - (A \cup B)$ are traversed in the same order in both paths P and R. Thus $P = R$ by Lemma 2.9. □

Now we can show that G is determined up to isomorphism by the trace of R. Firstly, there are a unique $A(y_1,v_1)R$-arc and a unique $A(u_1,x_1)R$-arc by Lemma 2.10. Secondly, the relative position of two different $(A \cup B)R$-arcs which are recorded by the same symbol in the trace of R, is determined by the property of R given in Lemma 2.10. Thus G is determined up to homeomorphism by the trace of R, and, consequently, determined up to isomorphism, since G has no vertices of degree 2.

We now investigate the possible traces R may have.

3.2. Forbidden $\overline{A \cup B}$–arcs

In the previous subsection we gained a strong structure theorem for G. We introduced three set of edges A, B, R, which together induce G. The remaining problem in order to complete this case of the characterisation of Pfaffian near bipartite graphs is to get an overview of the possible intersections of these three objects. Fortunately we know the relationship of A and B, but we have no idea about the intersections of the directed minimal path R with $A \cup B$. In this context it is helpful to rule out certain $\overline{A \cup B}$-arcs and with this reduce that number of possible traces.

To be more accurate we will show in this subsection that the symbols in the trace of R alternate between the sets $\{0,1\}$ and $\{2\}$. First we will investigate that any two consecutive digits in the trace of R are distinct.

LEMMA 2.11. *Let P be a directed path included in $A \cup B$ such that no internal vertex of P is in $\{x_1,y_1\}$. Then there does not exist a directed $\overline{A \cup B}$-arc joining vertices in VP.*

Proof: Suppose there exists a directed $\overline{A \cup B}$-arc Q from $x \in VP$ to $y \in VP$. There exist an RP-arc P_1 with terminus x and an RP-arc $P_2 \neq P_1$ with origin y. Let z_1 be the origin of P_1 and z_2 the terminus of P_2 (see Figure 2.9). Since $x <_R y$ we have $P_1 <_R P_2$, so that $P_2 <_P P_1$ by Lemma 2.10. Therefore $z_2 <_P z_1$.

Let C be the circuit $Q \cup P(y,x)$ and let f' be the 1-factor $f + C$. In our reference orientation of H we reorient C and define $A' = A + C$, $B' = B + C$, $R' = R + C$. Now we show that

$$G = G[A' \cup B' \cup R']. \tag{3.3}$$

By Lemma 2.4, A' and B' are f'-alternating circuits containing e_1 and e_2 of opposite clockwise parity, since A and B are of opposite clockwise parity and $A' + B' = A + B$. Moreover there is

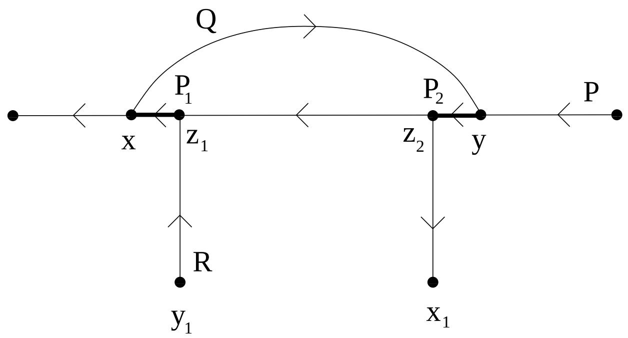


FIGURE 2.9. The situation in Lemma 2.11.

a unique $A'B'$-arc and the vertices of degree 3 in $G[A' \cup B']$ are the same as those in $G[A \cup B]$, since

$$\{x_1, y_1\} \cap (VC - \{x, y\}) = \emptyset.$$

In addition R' is a directed path from y_1 to x_1. Thus (3.3) holds, by Theorem 2.4 with f, A and B replaced by f', A' and B' respectively.

Now we observe that

$$(A' \cup B') \cap P(x, y) = \emptyset$$

since

$$\{x_1, y_1\} \cap (VP(x, y) - \{x, y\}) = \emptyset.$$

Note that

$$R' = R(y_1, z_1) \cup P(z_1, z_2) \cup R(z_2, x_1).$$

Therefore

$$R' \cap (P_1 \cup P_2) = \emptyset.$$

Thus $(A' \cup B' \cup R') \cap (P_1 \cup P_2) = \emptyset$, in contradiction to (3.3). $\square$

Next we show that the trace of R contains neither 01 nor 10.

LEMMA 2.12. *There is no directed $\overline{A \cup B}$-arc joining a vertex in the $A\overline{B}$-arc to a vertex in the $B\overline{A}$-arc.*

Proof: In view of the symmetry between A and B it suffices to prove that no $\overline{A \cup B}$-arc is directed from a vertex in $B(x_1, y_1)$ to a vertex in $A(x_1, y_1)$. Suppose such an arc P exists, joining a vertex $x_1' \in VB(x_1, y_1)$ to a vertex in $y_1' \in VA(x_1, y_1)$ (see Figure 2.10, left picture). Let

$$A' = B(u_1, x_1') \cup P \cup A(y_1', v_1) \cup \{e_2\} \cup (A \cap B)(u_2, v_2) \cup \{e_1\}.$$

This is an f-alternating circuits containing e_1 and e_2. Observe that $A(x_1, y_1')$ is the only $A\overline{A'}$-arc and that $B(x_1', y_1)$ is the only $B\overline{A'}$-arc. Since A and B are of opposite clockwise parity, we can assume without loss of generality that A and A' have the same clockwise parity. Henceforth A' will play the rôle previously assumed by A (see Figure 2.10, right picture, where A' is drawn as an ellipse). The two vertices of degree 3 in $G[A' \cup B]$ are x_1' and y_1. Moreover $R(y_1, x_1')$ is a directed path from y_1 to x_1'. Thus

$$G = G[A' \cup B \cup R(y_1, x_1')]$$

by Theorem 2.4. This is a contradiction since

$$R(x_1', x_1) - (A' \cup B) \supseteq A(x_1, y_1') \neq \emptyset.$$

$\square$

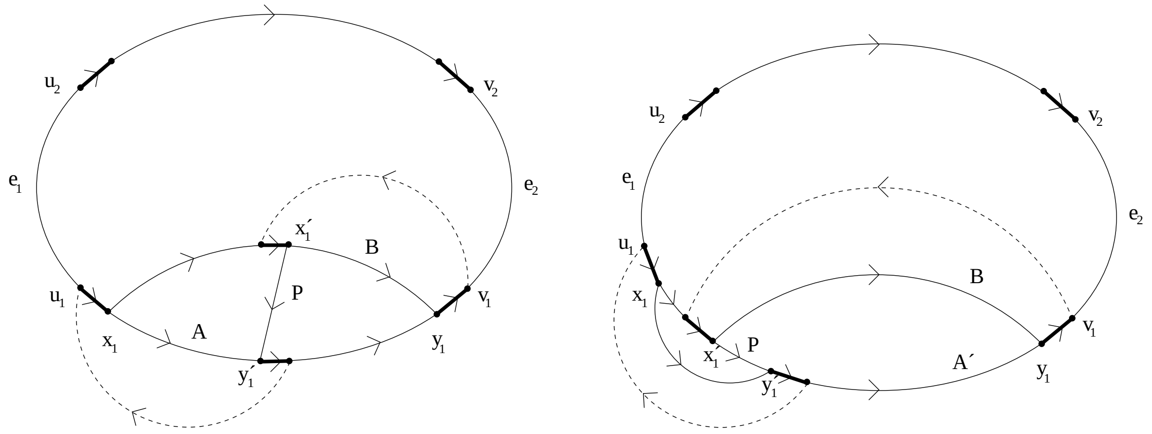


FIGURE 2.10. A directed arc joining a vertex x_1' in $B(x_1, y_1)$ to a vertex y_1' in $A(x_1, y_1)$.

By Lemma 2.11 and Lemma 2.12 every second symbol of the trace of R is 2, and the symbols in between are either 0 or 1. We obtain the *reduced* trace of R by removing each 2 from the trace.

3.3. Traces that produce even subdivisions of $K_{3,3}$

In this subsection we show that the trace of R contains none of 02020, 12121, $02*202*20$, $12*212*21$, where $*$ denotes a non-empty subsequence of the trace of R. This is done by showing that G contains an even subdivision of $K_{3,3}$ if the trace of R contains one of these subsequences.

LEMMA 2.13. *The trace of R contains neither* 02020 *nor* 12121.

Proof: By the symmetry between A and B it suffices to prove that the trace does not contain 02020. Therefore suppose the trace of R contains 02020, and fix one of these subsequences in the trace. Let z_1 be the terminus of the $A(x_1, y_1)R$-arc that corresponds to the first 0 in the fixed subsequence and z_8 the origin of the $A(x_1, y_1)R$-arc that corresponds to the last 0 in the fixed subsequence. Let $z_2, z_3, \ldots, z_7$ be the vertices in $VR(z_1, z_8) - \{z_1, z_8\}$ in the order in which the appear when R is traversed from z_1 to z_8 (see Figure 2.11). Then

$$G[(A \cup B \cup R(z_1, z_8)) - (A(x_1, z_8) \cup A(z_1, y_1))]$$

is an even subdivision of $K_{3,3}$. The vertices of degree 3 in this even subdivision are z_2, z_3, z_4, z_5, z_6, z_7, and the bipartition is $(\{z_2, z_4, z_6\}, \{z_3, z_5, z_7\})$. This is a contradiction, since G is minimal non-Pfaffian but yet contains a non-Pfaffian graph, which does not include all the edges of G. □

LEMMA 2.14. *The trace of R contains neither* $02 * 202 * 20$ *nor* $12 * 212 * 21$.

Proof: By the symmetry between A and B it suffices to prove that the trace does not contain $02*202*20$. Therefore suppose the trace of R contains $02*202*20$, and fix one of these subsequences in the trace. Let z_1 be the terminus of the $A(x_1, y_1)R$-arc that corresponds to the first 0 in the fixed subsequence and z_8 the origin of the $A(x_1, y_1)R$-arc that corresponds

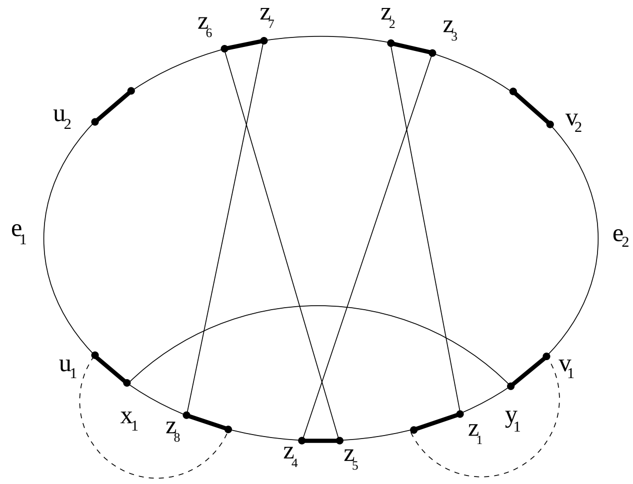


FIGURE 2.11. The situation in Lemma 2.13

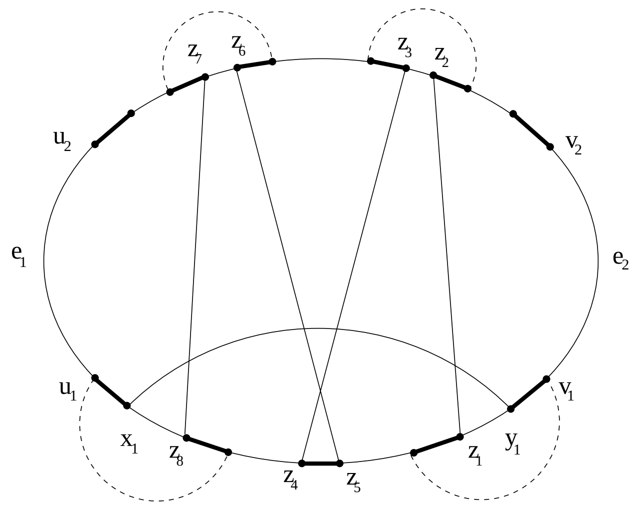


FIGURE 2.12. The situation in Lemma 2.14

to the last 0 in the fixed subsequence (see Figure 2.12). Let z_2, z_3, z_4, z_5, z_6, z_7 be the vertices as in Figure 2.12. Then

$$G[(A \cup B \cup R(z_1, z_2) \cup R(z_3, z_6) \cup R(z_7, z_8)) - (A(x_1, z_8) \cup A(z_1, y_1))]$$

contains an even subdivision of $K_{3,3}$. This is a contradiction, since G is minimal non-Pfaffian but still contains an even subdivision of $K_{3,3}$, which does not include all the edges of G. $\square$

3.4. The minimal non-Pfaffian near bipartite graphs I

In this subsection we conclude the case where there is a unique AB-arc. The following lemma is based on Remark 2.2.

LEMMA 2.15. *The reduced trace of R is of odd length.*

Proof: In the proof of Theorem 2.4 we constructed two sets of alternating circuits $\mathcal{C}$ and $\mathcal{D}$ with

$$A + B = \sum_{E \in \mathcal{E}} E$$

where $\mathcal{E} = \mathcal{C} + \mathcal{D}$. We observe that $|\mathcal{C}| - 1$ is the number of 0's in the trace of R and that $|\mathcal{D}| - 1$ is the number of 1's in the trace of R. Thus

$$|\mathcal{C}| + |\mathcal{D}| - 2 = |\mathcal{E}| - 2$$

is the length of the reduced trace, since $\mathcal{C}$ and $\mathcal{D}$ are disjoint. The result follows by Remark 2.2, where we showed that $|\mathcal{E}|$ is odd. $\square$

Next we show that we could assume that the trace of R does not start with 1 and does not end with 1.

LEMMA 2.16. *Let W be a string over $\{0, 1, 2\}$.*
(a) It is possible to choose A, B, f, x_1, y_1 and a directed path X from y_1 to x_1 in G such that the trace of X is $0W$ if and only if it is possible to choose A', B', f', x_1', y_1' and a directed path X' from y_1' to x_1' in G such that the trace of X' is $1W$.
(b) It is possible to choose A, B, f, x_1, y_1 and a directed path X from y_1 to x_1 in G such that the trace of X is $W0$ if and only if it is possible to choose A', B', f', x_1', y_1' and a directed path X' from y_1' to x_1' in G such that the trace of X' is $W1$.

Proof: By symmetry, it suffices to prove (a). Suppose the trace of X is $0W$. There is an $A(x_1, y_1)X$-arc that corresponds to the first 0 in the trace of X. Let v be its origin, and let

$$C = X(y_1, v) \cup A(v, y_1).$$

Let u be the terminus of the $A(y_1, v_1)X$-arc and w the terminus of the $A(x_1, y_1)X$-arc with origin v (see Figure 2.13). Define $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= C(u, v) \cup A(u_1, v) \cup \{e_1\} \cup A(u_2, v_2) \cup \{e_2\} \cup A(u, v_1), \end{aligned}$$

$$\begin{aligned} B' &= B + C \\ &= C(u, y_1) \cup B(u_1, y_1) \cup \{e_1\} \cup B(u_2, v_2) \cup \{e_2\} \cup B(u, v_1) \end{aligned}$$

and

$$\begin{aligned} X' &= X + A(v, y_1) \\ &= C(w, v) \cup X(w, x_1). \end{aligned}$$

Then A' and B' are f'-alternating circuits containing e_1 and e_2 and having opposite clockwise parity. Moreover there is a unique $A'B'$-arcs and the vertices of degree 3 in $G[A' \cup B']$ are x_1 and v. After reorientation of C, X' is a directed path from v to x_1. The trace of X' is $1W$, as $B'(y_1, w)$ is a $B'(x_1, v)X'$-arc which replaces the $A(x_1, y_1)X$-arc $A(v, w)$ (see Figure 2.13, right picture, where A' is drawn as an ellipse).

For the converse in (a) note that $f = f' + C$, $A = A' + C$, $B = B' + C$ and $X = X' + A(v, y_1)$. $\square$

Now we show that we can assume that neither the first nor the last symbol in the trace of R is 2. For this purpose we need the following lemma, which was already been proved in [**15**].

LEMMA 2.17. *Let G be a graph with a circuit C of odd length and let G^C be the graph obtained from G by contracting VC. If G^C is not Pfaffian, then neither is G.*

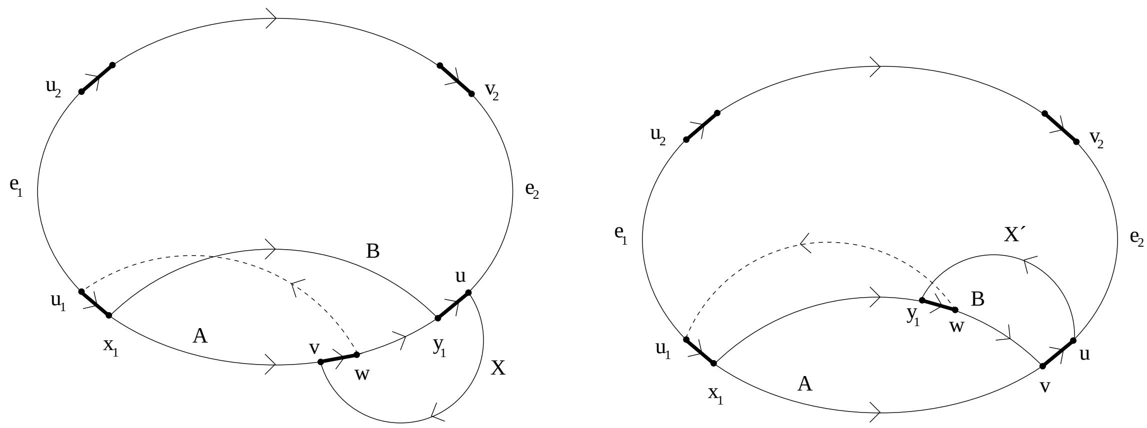


FIGURE 2.13. The situation in Lemma 2.16 before and after reorienting C.

LEMMA 2.18. *Let W be a string over $\{0, 1, 2\}$.*
(a) Suppose the first symbol in W is either 0 or 1. Then the minimal non-Pfaffian near bipartite graph G where R has trace $2W$ is reducible to the graph G' where the trace of R is W.
(b) Suppose the last symbol in W is either 0 or 1. Then the minimal non-Pfaffian near bipartite graph where the R has trace $W2$ is reducible to the graph where the trace of R is W.

Proof: By symmetry it suffices to show (a). Let v be the origin of the first $A(u_2, v_2)R$-arc in G. Contraction of the circuit $R(v_1, v) \cup A(v, v_2) \cup \{e_2\}$ yields G'. □

Now we are able to finish the case where we choose A and B such that there is a unique AB-arc. We show that the reduced trace of R has length 1. Suppose first that the reduced trace of R has length at least 5. Let $p \in \{0, 1\}$ be the third symbol in the reduced trace of R. By Lemma 2.16 and Lemma 2.18 we can assume that the trace of R starts and ends with p. This is a contradiction to Lemma 2.14. Therefore and by Lemma 2.15, the length of the reduced trace is either 1 or 3.

Suppose the reduced trace has length 3. Let $p \in \{0, 1\}$ be the second symbol in the reduced trace of R. Again by Lemma 2.16 and Lemma 2.18 we can assume that the trace of R starts and ends with p. This is a contradiction to Lemma 2.13. Therefore the length of the reduced trace is 1. By Lemma 2.18 we may assume that the trace of R has length 1, and therefore the trace of R is either 0 or 1. By symmetry we may assume that the trace of R is 0. Thus G is an even subdivision of $K_{3,3}$ (see Figure 2.14).

4. Two AB-arcs

Now we assume that for every choice of two alternating circuits A' and B' of opposite clockwise parity, containing e_1 and e_2 and consanguineous with respect to some 1-factor that contains neither e_1 nor e_2, there are at least two $A'B'$-arcs.

4.1. The graph G is induced by $A \cup B \cup S \cup T$.

In this subsection we show that G is induced by the two alternating circuits A and B and the two alternating paths S and T. Again this is done by constructing an intractable set of

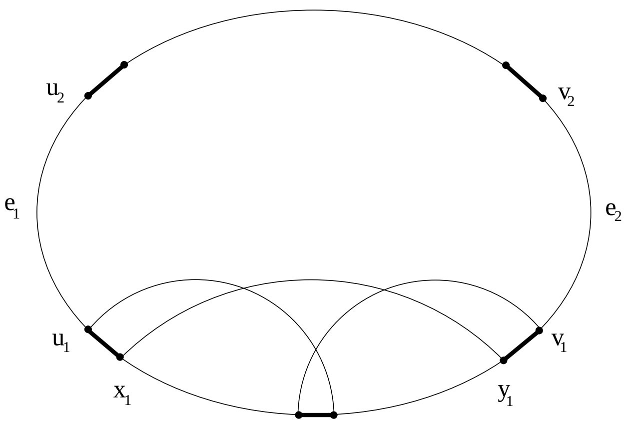


FIGURE 2.14. $K_{3,3}$

f-alternating circuits in $G[A \cup B \cup S \cup T]$. Similar to the previous case Lemma 2.8 will be fundamental. But in addition we need the following lemma.

In the introduction we explain that we identify paths with their edge sets. If $P = \{e_1, e_2, \ldots, e_n\}$ is a directed path, and $e_1, e_2, \ldots, e_n$ is the order in which the edges appear when P is traversed from its origin to its terminus, P could also be identified by the sequence $(e_1, e_2, \ldots, e_n)$, although it does not make much sense. But this point of view is sensible if we are considering 'paths with repeated edges', as we do in the following simple lemma. We show that such a 'path with repeated edges' can be decomposed into a path and a set of circuits.

LEMMA 2.19. *Let $(e_1, e_2, \ldots, e_n)$ be a sequence of edges in a directed graph G in which each vertex has indegree 1 or outdegree 1. Suppose that for all $i > 1$ the origin of e_i is the terminus of e_{i-1}, and that the origin of e_1 has outdegree 1 and the terminus of e_n has indegree 1. Then there exist a directed path P from the origin of e_1 to the terminus of e_n and directed circuits $C_1, C_2, \ldots, C_k$ such that*

$$\sum_{i=1}^{n} \{e_i\} = P + \sum_{i=1}^{k} C_i$$

and

$$P \cup \bigcup_{i=1}^{k} C_i = \{e_1, e_2, \ldots, e_n\}.$$

Proof: We use induction on the number r of repetitions of edges in the sequence $L = (e_1, e_2, \ldots, e_n)$. If $r = 0$ then $P = \{e_1, e_2, \ldots, e_n\}$ and $k = 0$, since each vertex has indegree 1 or outdegree 1, the origin of a_1 has outdegree 1 and the terminus of a_n has indegree 1. Now suppose that $r > 0$ and that the lemma holds whenever the number of repetitions of edges is less than r. Let a be the edge in L that is repeated first. The part of L between the first two occurrences of a has no edges repeated within it. Therefore a and the edges between the first two occurrences of a form a directed circuit C. We now modify L by removing all the edges from the first occurrence of a to the edge immediately before the second occurrence of a. This modified sequence $L' = (e'_1, e'_2, \ldots, e'_m)$ has fewer repetitions of edges. Moreover the origin of e'_1 is that of e_1, the terminus of e'_m is that of e_n, and for all $i > 1$ the origin of e'_i is the terminus of e'_{i-1}. Therefore the inductive hypothesis may be applied to L', and the result follows from the equation

$$\sum_{i=1}^{n}\{e_i\} = \sum_{i=1}^{m}\{e_i'\} + C.$$

□

THEOREM 2.5. *Let G be a minimal non-Pfaffian near bipartite graph. Let e_1 and e_2 be edges such that $G-\{e_1,e_2\}$ is bipartite and 1-extendible. Let $H = G-\{e_1,e_2\}$, and let f be a 1-factor of H. Let A and B be f-alternating circuits in G of opposite clockwise parity under an extended Pfaffian orientation. Suppose that there are exactly two AB-arcs, one containing e_1 and the other e_2. Let the former arc join vertices x_1 and x_2 and the latter vertices y_1 and y_2. Let $A^* = A-\{e_1\}$, and suppose the vertices x_1, y_1, y_2, x_2 appear in that order when A^* is traced from x_1 to x_2. Let H be given its reference orientation with respect to f such that $A^*[x_1,y_1]$ is a directed path from x_1 to y_1. Let S' be a directed path from y_i to x_j and T' a directed path from y_{3-i} to x_{3-j}, where $i,j\in\{1,2\}$. Then G is induced by $A\cup B\cup S'\cup T'$.*

Proof: By the symmetry between A and B we take $i=j=1$. We first show, by using Lemmas 2.8 and Lemma 2.19, that we can write $A+B$ as a sum of directed circuits included in $A\cup B\cup S'\cup T'$. In order to verify this claim, first we apply Lemma 2.8 to the directed paths $A(x_1,y_1)$ and S'. Let

$$A(x_1,y_1) + S' = \sum_{V\in\mathcal{V}} V, \tag{4.1}$$

where $\mathcal{V}$ is a set of directed circuits. Similarly let

$$A(x_2,y_2) + T' = \sum_{W\in\mathcal{W}} W, \tag{4.2}$$

where $\mathcal{W}$ is a set of directed circuits. Now consider the sequence L of edges formed by the edges in the directed path S' followed by those in the directed path $B(x_1,y_2)$. We apply Lemma 2.19 to L to write

$$S' + B(x_1,y_2) = P + \sum_{X\in\mathcal{X}} X, \tag{4.3}$$

where P is a directed path from y_1 to y_2 and $\mathcal{X}$ is a set of directed circuits. Similarly we have

$$T' + B(x_2,y_1) = Q + \sum_{Y\in\mathcal{Y}} Y, \tag{4.4}$$

where Q is a directed path from y_2 to y_1 and $\mathcal{Y}$ is a set of directed circuits. Now we apply Lemma 2.8 to P and Q to obtain

$$P + Q = \sum_{Z\in\mathcal{Z}} Z, \tag{4.5}$$

where $\mathcal{Z}$ is a set of directed circuits. Let $\mathcal{C} = \mathcal{V}+\mathcal{W}+\mathcal{X}+\mathcal{Y}+\mathcal{Z}$. Adding equations (4.1)–(4.5) we obtain

$$\begin{aligned}
& A(x_1,y_1) + A(x_2,y_2) + B(x_1,y_2) + B(x_2,y_1) \\
= \ & \sum_{V\in\mathcal{V}} V + \sum_{W\in\mathcal{W}} W + \sum_{X\in\mathcal{X}} X + \sum_{Y\in\mathcal{Y}} Y + \sum_{Z\in\mathcal{Z}} Z \\
= \ & \sum_{C\in\mathcal{C}} C.
\end{aligned}$$

Since the left hand side is $A+B$, $\mathcal{C}$ is the required set of circuits.

In our extended Pfaffian orientation of G, A is the only clockwise even circuit in $\mathcal{C} \cup \{A, B\}$. Therefore by Lemma 2.1 an odd number of circuits in $\mathcal{C} \cup \{A, B\}$ are clockwise even for any orientation of G. But if there were a Pfaffian orientation of $G[A \cup B \cup \bigcup_{C \in \mathcal{C}} C]$ then every circuit in $\mathcal{C} \cup \{A, B\}$ would be clockwise odd because they are f-alternating. Therefore the graph $G[A \cup B \cup \bigcup_{C \in \mathcal{C}} C]$ is non-Pfaffian, and so $G[A \cup B \cup S' \cup T']$ is non-Pfaffian. By the minimality of G, we deduce that

$$G[A \cup B \cup S' \cup T'] = G.$$

□

Application of this theorem to S and T we find that $G = G[A \cup B \cup S \cup T]$.

Directed minimal paths. In fact we choose S and T to satisfy the following definition. The reason for that becomes clear at the beginning of Subsection 2.3, where we introduce the trace of a directed path P that satisfies the definition, and then show that $G[A \cup B \cup P]$ is determined up to homeomorphism by this trace.

DEFINITION 2.1. *Let $i, j \in \{1, 2\}$ and let P be a directed path from y_i to x_j under our reference orientation. We say that P is minimal if for every edge $e \in P - (A \cup B)$ there is no directed path from y_i to x_j included in $(A \cup B \cup P) - \{e\}$.*

It is clear that S and T may be assumed to be minimal. The next lemma is obvious.

LEMMA 2.20. *Let u and v be vertices in $G[A \cup B]$. Under our reference orientation for H there exists at most one directed path in $G[A \cup B]$ from u to v.*

This lemma is used in the proof of the following lemma, which characterises directed minimal paths.

LEMMA 2.21. *(a) Let P be a directed minimal path from y_i to x_j and let Q be a directed path included in $A \cup B$. Let P_1 and P_2 be distinct PQ-arcs. Then $P_1 <_P P_2$ if and only if $P_2 <_Q P_1$ (see Figure 2.15).*
(b) Conversely let P' be a directed path from y_i to x_j. If for every directed path Q' in $A \cup B$ and every pair of distinct $P'Q'$-arcs P'_1 and P'_2 we have $P'_1 <_{P'} P'_2$ if and only if $P'_2 <_{Q'} P'_1$, then P' is minimal.

Proof: (a) It suffices to show that if $P_1 <_P P_2$ then $P_2 <_Q P_1$. Assume the contrary, that is $P_1 <_P P_2$ and $P_1 <_Q P_2$. Let a be the terminus of P_1 and b the origin of P_2. By assumption $P(a, b)$ and $Q(a, b)$ exist, but are not equal. By Lemma 2.20 there is an edge $e \in P(a, b) - (A \cup B)$. The set $P(y_i, a) \cup Q(a, b) \cup P(b, x_j)$ includes a directed path from y_i to x_j. This path is included in $(A \cup B \cup P) - \{e\}$, in contradiction to the minimality of P.

(b) Conversely, assume that P' is not minimal. Choose $e \in P' - (A \cup B)$ so that there exists a directed path P from y_i to x_j in $(A \cup B \cup P') - \{e\}$. Let u be the last vertex in P that is also in P' and satisfies $P(y_i, u) = P'(y_i, u)$. Let v be the first vertex in $VP(u, x_j) - \{u\}$ that is also in P'. Then $P(u, v) \cap P' = \emptyset$, $u <_P v$ and $u <_{P'} v$. Hence $P(u, v) \subseteq A \cup B$ and so $P(u, v)$ is included in a maximal directed path Q' included in $A \cup B$ such that $u <_{Q'} v$. Let P'_1 be the $P'Q'$-arc that includes u and P'_2 the $P'Q'$-arc that includes v. Then P'_1 and P'_2 are distinct, $P'_1 <_{P'} P'_2$ but $P'_1 <_{Q'} P'_2$. □

4.2. Forbidden $\overline{A \cup B}$–arcs

Similar to the previous case in this characterisation we have to get an overview of the possible intersections of the four set of edges A, B, S and T. Again we know the relationship of A and B, but we have no idea about the intersections of the directed minimal paths S and

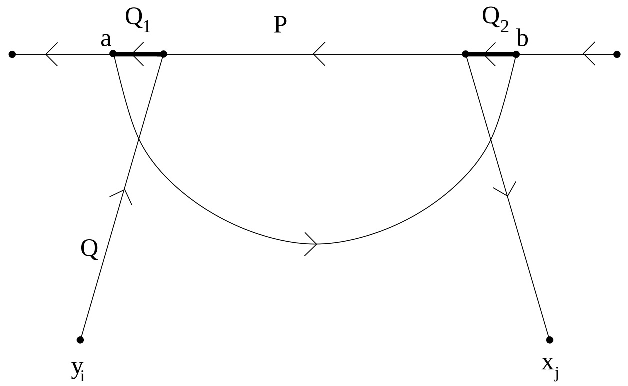


FIGURE 2.15. A directed minimal path P from y_i to x_j.

T with $A \cup B$ and with each other. A list of these 'forbidden $\overline{A \cup B}$-arcs' is presented in this subsection.

First we prove the following technical result, which will be useful for most of the lemmas in this subsection.

LEMMA 2.22. *Let P be a directed $\overline{A \cup B}$-arc. Then there exist $i, j \in \{1, 2\}$ such that P is included in a minimal path Q directed from y_i to x_j.*

Proof: If $S \cap T = \emptyset$, then S and T are vertex disjoint and, since $P \subseteq (S \cup T) - (A \cup B)$ by Theorem 2.5, it follows that $P \subseteq S$ or $P \subseteq T$. Assume therefore that $S \cap T \neq \emptyset$. Let a and b be, respectively, the first and last vertices of T that are also in S. It follows from Theorem 2.5 that

$$G = G[A \cup B \cup S \cup T(y_2, a) \cup T(b, x_2)], \tag{4.6}$$

as there exists a directed path from y_2 to x_2 included in

$$T(y_2, a) \cup S(a, x_1) \cup A(x_1, y_1) \cup S(y_1, b) \cup T(b, x_2).$$

We observe from (4.6) that any vertex of degree 3 and not in $VA \cup VB$ must be either a or b. It follows that if either a or b were not an internal vertex of P, then P would be included in S or T, since the edges of f incident on a and b are in $S \cap T$. In this case we could choose Q to be S or T. We therefore assume that a and b are the internal vertices of P, since G has no vertices of degree 2.

Let u and v be, respectively, the origin and terminus of P. If $b <_P a$, then $P(u, b) \cup P(a, v) \subseteq S \cap T$, and so $P \subseteq S$ or $P \subseteq T$ according to whether $P(b, a) \subseteq S$ or $P(b, a) \subseteq T$. Therefore we can assume that $a <_P b$. Then $S(a, b) = T(a, b)$. Moreover $P(u, a)$ is included in S or T, and similarly for $P(b, v)$. Without loss of generality we assume that $P(u, a) \subseteq S$. If $P(b, v) \subseteq S$, then we take $Q = S$. Suppose therefore that $P(b, v) \subseteq T$. In this case, we take $Q = S(y_1, b) \cup T(b, x_2)$.

It remains to show that Q is minimal. Suppose not. Then there exists $e \in Q - (A \cup B)$ such that there is a directed path Q' from y_1 to x_2 included in $(A \cup B \cup Q) - \{e\}$. Define $R = T(y_2, b) \cup S(b, x_1)$. Then by Theorem 2.5 we have $G = G[A \cup B \cup R \cup Q']$. If $e \in S(y_1, a) \cup T(b, x_2)$ then we have the contradiction that $e \notin A \cup B \cup R \cup Q'$. Therefore we suppose that $e \in Q(a, b)$. Then $P \cap Q' = \emptyset$, and we have the contradiction that

$$(S(u, a) \cup T(b, v)) \cap (A \cup B \cup R \cup Q') = \emptyset.$$

$\square$

Now we start our list of forbidden $\overline{A \cup B}$-arcs. In the first lemma we forbid directed $\overline{A \cup B}$-arcs, whose origin and terminus are both in one of the following directed paths in $G[A \cup B]$: $A(x_1, y_1)$, $A(x_2, y_2)$, $B(x_2, y_1)$, $B(x_1, y_2)$, $(A \cap B)(y_1, v_1)$, $(A \cap B)(y_2, v_2)$, $(A \cap B)(u_1, x_1)$ or $(A \cap B)(u_2, x_2)$. This lemma is similar to Lemma 2.11.

LEMMA 2.23. *Let P be a directed path included in $A \cup B$ such that no internal vertex of P is in $\{x_1, x_2, y_1, y_2\}$. Then there does not exist a directed $\overline{A \cup B}$-arc joining vertices in VP.*

Proof: Suppose there exists a directed $\overline{A \cup B}$-arc Q from $x \in VP$ to $y \in VP$. Then, by Lemma 2.22, for some $i, j \in \{1, 2\}$ there exists a directed minimal path Z from y_i to x_j that includes Q. By Lemma 2.7 we may choose a directed path W from y_{3-i} to x_{3-j}; thus $G = G[A \cup B \cup Z \cup W]$ by Theorem 2.5. There exist a ZP-arc P_1 with terminus x and a ZP-arc $P_2 \neq P_1$ with origin y. Let z_1 be the origin of P_1 and z_2 the terminus of P_2 (see Figure 2.16). Since $x <_Z y$ we have $P_1 <_Z P_2$, so that $P_2 <_P P_1$ by Lemma 2.21. Therefore $z_2 <_P z_1$.

Let C be the circuit $Q \cup P(y, x)$. First we show that we may assume there to be at most one CW-arc. Suppose there are two such arcs, W_1 and W_2, where $W_1 <_W W_2$. Let a be the terminus of W_1 and b be the origin of W_2. Note that we can choose W_1 and W_2 such that $C(a, b) \cap W = \emptyset$. (it is not sufficient to choose (W_1, W_2) as to be a pair of CW-arcs that are close together in C; additionally the following principle is used: Let D be a directed circuit with n vertices that are labeled with $\{1, 2, \ldots, n\}$. Then there exist two adjacent vertices x and y in D such that the edge in D incident on x and on y is oriented from x to y and that the label of y is greater than the label of x.) Then

$$W^* = W(y_{3-i}, a) \cup C(a, b) \cup W(b, x_{3-j})$$

is a directed path from y_{3-i} to x_{3-j} and has less CW-arc than W. We therefore may assume that there is at most one CW-arc. If such an arc exists, let its origin be w_1 and its terminus w_2 (see Figure 2.16). We also note that there is a unique CZ-arc, by the minimality of Z.

Let f' be the 1-factor $f + C$, and let $A' = A + C$, $B' = B + C$, $Z' = Z + C$ and $W' = W + C$. Now we show that

$$G = G[A' \cup B' \cup Z' \cup W']. \tag{4.7}$$

By Lemma 2.4, A' and B' are f'-alternating circuits containing e_1 and e_2 of opposite clockwise parity, since A and B are of opposite clockwise parity and $A' + B' = A + B$. Moreover there are exactly two $A'B'$-arcs and the vertices of degree 3 in $G[A' \cup B']$ are the same as those in $G[A \cup B]$, since

$$\{x_1, x_2, y_1, y_2\} \cap (VC - \{x, y\}) = \emptyset.$$

In addition Z' would become a directed path from y_i to x_j if C were reoriented, and a similar statement holds for W'. Thus (4.7) holds, by Theorem 2.5 with f, A and B replaced by f', A' and B' respectively.

Now we observe that

$$(A' \cup B') \cap P(y, x) = \emptyset$$

since

$$\{x_1, x_2, y_1, y_2\} \cap (VP(y, x) - \{x, y\}) = \emptyset.$$

Note that

$$Z' = Z(y_i, z_1) \cup P(z_2, z_1) \cup Z(z_2, x_j).$$

Therefore

$$Z' \cap (P_1 \cup P_2) = \emptyset.$$

Thus $(A' \cup B' \cup Z') \cap (P_1 \cup P_2) = \emptyset$, and so $P_1 \cup P_2 \subseteq W'$ by (4.7). We deduce that w_1 and w_2 exist.

Next we show that either $w_1, w_2 \in VP(z_2, z_1)$ and $z_2 <_P w_1 <_P w_2 <_P z_1$, or $w_1, w_2 \in VQ$ and $w_1 <_Q w_2$. First,

$$W' = W(y_{3-i}, w_1) \cup C(w_2, w_1) \cup W(w_2, x_{3-j}).$$

Hence

$$P_1 \cup P_2 \subseteq C(w_2, w_1),$$

and the desired conclusion follows.

Case 1: Suppose first that $w_1, w_2 \in VP(z_2, z_1)$ and $z_2 <_P w_1 <_P w_2 <_P z_1$. After reorientation of C, let X be a directed path from y_i to x_{3-j} included in

$$Z(y_i, z_1) \cup C(z_1, w_2) \cup W(w_2, x_{3-j})$$

and let Y be a directed path from y_{3-i} to x_j included in

$$W(y_{3-i}, w_1) \cup C(w_1, z_2) \cup Z(z_2, x_j).$$

Thus $G = G[A' \cup B' \cup X \cup Y]$. We now have the contradiction that

$$(C(x, z_1) \cup (C(w_2, w_1) \cup C(z_2, y)) \cap (A' \cup B' \cup X \cup Y) = \emptyset.$$

Case 2: Suppose on the other hand that $w_1, w_2 \in VQ$ and $w_1 <_Q w_2$. Without loss of generality we may assume that P is a maximal directed path in $A \cup B$ such that no internal vertex is in $\{x_1, x_2, y_1, y_2\}$. Let P be directed from vertex u to vertex v. Thus $u \in \{u_1, u_2, x_1, x_2, y_1, y_2\}$ and $v \in \{x_1, x_2, y_1, y_2, v_1, v_2\}$.

First we show that $u \neq y_i$. If $u = y_i$ then we observe that there is a directed path Z^* from y_i to x_j included in $P(u, z_2) \cup Z(z_2, x_j)$. Therefore

$$(A \cup B \cup Z^* \cup W) \cap (C(x, w_1) \cup C(w_2, y)) = \emptyset.$$

By Theorem 2.5 we have the contradiction that $G = G[A \cup B \cup Z^* \cup W]$. Thus $u \neq y_i$. A similar argument, with Z^* included in $Z(y_i, z_1) \cup P(z_1, v)$, shows that $v \neq x_j$.

Next we show that $u \neq y_{3-i}$. Otherwise we define W^* to be a directed path from y_{3-i} to x_{3-j} included in

$$P(u, x) \cup C(x, w_2) \cup W(w_2, x_{3-j}).$$

The union $A \cup B \cup Z \cup W^*$ does not contain the edge of $W - Q$ incident on w_1. This result contradicts Theorem 2.5, since $G = G[A \cup B \cup Z \cup W^*]$. Thus $u \neq y_{3-i}$. A similar argument, with W^* included in

$$W(y_{3-i}, w_1) \cup C(w_1, y) \cup P(y, v),$$

shows that $v \neq x_{3-j}$.

Now we show that $u \neq x_{3-j}$. Otherwise we define Z^* as a directed path from y_i to x_j included in

$$Z(y_i, w_2) \cup W(w_2, u) \cup P(u, z_2) \cup Z(z_2, x_j).$$

Then

$$(A \cup B \cup Z^* \cup W) \cap C(w_2, y) = \emptyset,$$

in contradiction to Theorem 2.5 since $G = G[A \cup B \cup Z^* \cup W]$. A similar argument, with Z^* included in

$$Z(y_i, z_1) \cup P(z_1, v) \cup W(v, w_2) \cup Z(w_2, x_j),$$

shows that $v \neq y_{3-i}$.

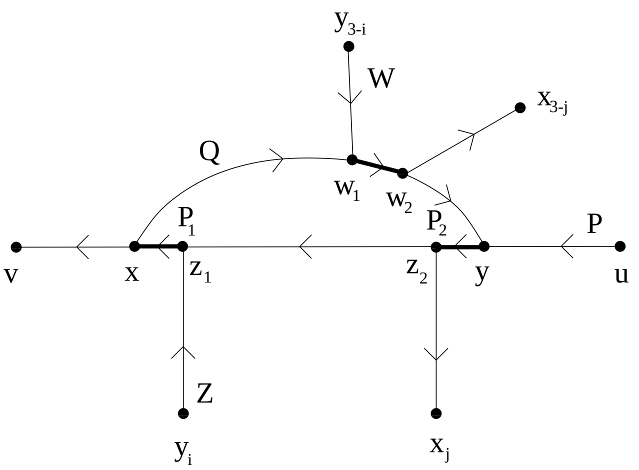


FIGURE 2.16. The situation in Lemma 2.23

Since $v \notin \{x_1, x_2\}$ we have $u \notin \{u_1, u_2\}$. We conclude that $u = x_j$, and similarly $v = y_i$. Define $P' = P + C$. Remember that W' is the only member of $\{A', B', W', Z'\}$ meeting $P_1 \cup P_2$. If C is reoriented then there is a directed path W'' from y_{3-i} to x_{3-j} included in

$$W'(y_{3-i}, w_1) \cup P'(w_1, v) \cup Z'(v, u) \cup P'(u, w_2) \cup W'(w_2, x_{3-j}).$$

Thus

$$W'' \cap (P_1 \cup P_2) = \emptyset.$$

We now have a contradiction, since $G = G[A' \cup B' \cup Z' \cup W'']$. □

In the following lemma we rule out directed $\overline{A \cup B}$-arcs with

- origin in $A(x_1, y_1)$ and terminus in $A(x_2, y_2)$,
- origin in $A(x_2, y_2)$ and terminus in $A(x_1, y_1)$,
- origin in $B(x_2, y_1)$ and terminus in $B(x_1, y_2)$ and
- origin in $B(x_1, y_2)$ and terminus in $B(x_2, y_1)$.

LEMMA 2.24. *There is no directed $\overline{A \cup B}$-arc joining vertices in distinct $A\overline{B}$-arcs, or in distinct $B\overline{A}$-arcs.*

Proof: In view of the symmetry between A and B it suffices to prove that no $\overline{A \cup B}$-arc is directed from a vertex in $A(x_1, y_1)$ to a vertex in $A(x_2, y_2)$. Suppose such an arc P exists, joining a vertex $x_1' \in VA(x_1, y_1)$ to a vertex $y_2' \in VA(x_2, y_2)$ (see Figure 2.17). Let

$$B' = A(u_1, x_1') \cup P \cup A(y_2', v_2) \cup \{e_2\} \cup B(u_2, v_1) \cup \{e_1\}.$$

This is an f-alternating circuit containing e_1 and e_2. Observe that $B(x_1, y_2)$ is the only $B\overline{B'}$-arc. It follows that B and B' have the same clockwise parity, for otherwise A and B could have been chosen to have a unique AB-arc. Henceforth B' will play the rôle previously assumed by B. The circuit A will play the same rôle as before, but we define $x_2' = x_2$ and $y_1' = y_1$. Note that there are exactly two AB'-arcs, one containing e_1 and the other containing e_2 and that $B(x_1, y_2)$ is an $\overline{A \cup B'}$-arc. Therefore, by Lemma 2.22, for some $i, j \in \{1, 2\}$ there exists a directed minimal path X from y_i' to x_j' including $B(x_1, y_2)$.

We now show that $j = 2$ and $i = 1$. Included in the set $X(y_i', x_1) \cup A(x_1, x_1')$ is a directed path W from y_i' to x_1'. This path is included in $(A \cup B \cup X) - B(x_1, y_2)$, in contradiction to the minimality of X if $j = 1$. Therefore $j = 2$. Similarly, included in the set $A(y_2', y_2) \cup X(y_2, x_2')$

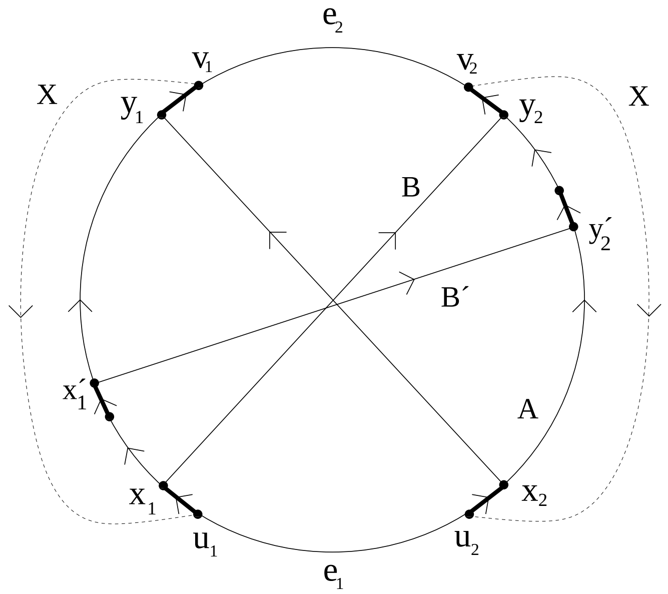


FIGURE 2.17. A directed arc joining a vertex in $A(x_1, y_1)$ to a vertex in $A(x_2, y_2)$.

is a directed path Z from y_2' to x_2'. This path is included in $(A \cup B \cup X) - B(x_1, y_2)$, in contradiction to the minimality of X if $i = 2$. Therefore $i = 1$.

By Theorem 2.5 we have $G = G[A \cup B' \cup W \cup Z]$, in contradiction to the fact that

$$B(x_1, y_2) \cap (A \cup B' \cup W \cup Z) = \emptyset.$$

□

In the next lemma we rule out directed $\overline{A \cup B}$-arcs with

- origin in $A(x_1, y_1)$ and terminus in $B(x_2, y_1)$ or in $B(x_1, y_2)$,
- origin in $A(x_2, y_2)$ and terminus in $B(x_1, y_2)$ or in $B(x_2, y_1)$,
- origin in $B(x_2, y_1)$ and terminus in $A(x_1, y_1)$ or in $A(x_2, y_2)$ and
- origin in $B(x_1, y_2)$ and terminus in $A(x_2, y_2)$ or in $A(x_1, y_1)$.

LEMMA 2.25. *There is no directed $\overline{A \cup B}$-arc joining a vertex in an $A\overline{B}$-arc to a vertex in a $B\overline{A}$-arc.*

Proof: By symmetry it suffices to prove that no $\overline{A \cup B}$-arc is directed from a vertex in $A(x_1, y_1)$ to a vertex in $B(x_2, y_1)$. Suppose such an arc P exists, joining a vertex $v \in VA(x_1, y_1)$ to a vertex $y_1' \in VB(x_2, y_1)$ (see Figure 2.18). Let

$$A' = A(u_1, v) \cup P \cup B(y_1', v_1) \cup \{e_2\} \cup A(u_2, v_2) \cup \{e_1\}.$$

This is an f-alternating circuit containing e_1 and e_2. Observe that $A(v, y_1)$ is the only $A\overline{A'}$-arc. It follows that A and A' have the same clockwise parity. Henceforth A' will play the rôle previously assumed by A (see Figure 2.18, second picture, where A' is drawn as a circle). The circuit B will play the same rôle as before, but we define $x_1' = x_1$, $x_2' = x_2$ and $y_2' = y_2$. Note that $A(v, y_1)$ is an $\overline{A' \cup B}$-arc. Therefore, by Lemma 2.22, for some $i, j \in \{1, 2\}$ there exists a directed minimal path X from y_i' to x_j' including $A(v, y_1)$. Let P_1 be the $A'X$-arc with terminus v and P_2 be the $A'X$-arc with origin y_1. Then $P_1 <_X P_2$ and $P_1 <_{A'(u_1, v_1)} P_2$ in contradiction to the minimality of X. □

Now we show that there exists no directed $\overline{A \cup B}$-arc with origin in $A(y_1, v_1) \cup A(y_2, v_2)$ and terminus in $A(u_1, x_1) \cup A(u_2, x_2)$.

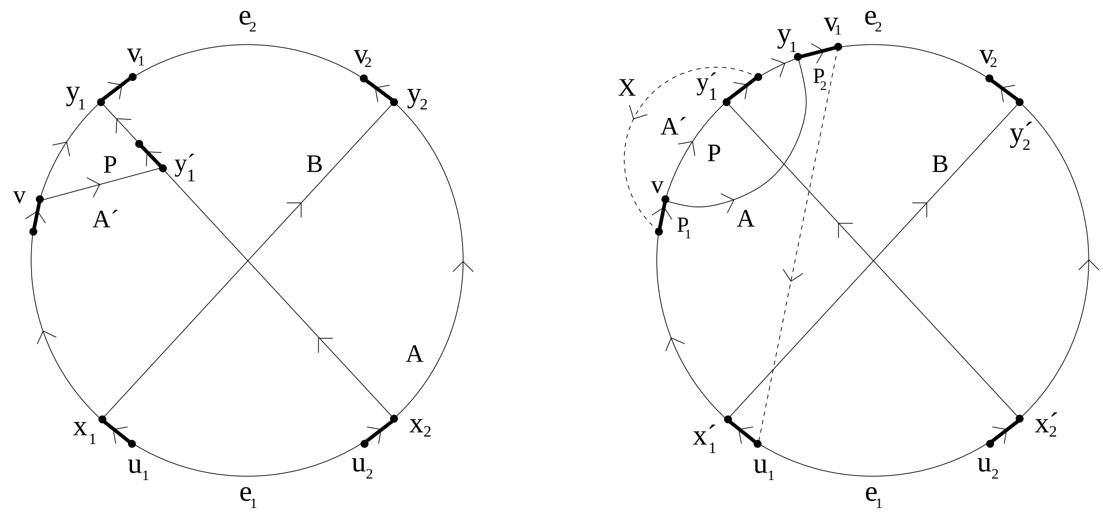


FIGURE 2.18. A directed arc joining a vertex in $A(x_1, y_1)$ to a vertex in $B(x_2, y_1)$.

LEMMA 2.26. *For each $i, j \in \{1, 2\}$ there is no directed $\overline{A \cup B}$-arc from a vertex in $A(y_i, v_i)$ to a vertex in $A(u_j, x_j)$.*

Proof: It suffices to prove the lemma for $i = j = 1$. Suppose such a directed arc P exists. Let P be directed from vertex v to vertex u. Define

$$C = A(u, v) \cup P.$$

Let $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= P \cup A(v, v_1) \cup \{e_2\} \cup A(u_2, v_2) \cup \{e_1\} \cup A(u_1, u) \end{aligned}$$

and

$$\begin{aligned} B' &= B + C \\ &= P \cup B(v, v_1) \cup \{e_2\} \cup B(x_1, v_2) \cup A(x_1, y_1) \cup B(u_2, y_1) \cup \{e_1\} \cup B(u_1, u). \end{aligned}$$

Thus A' and B' are f'-alternating circuits of opposite clockwise parity containing e_1 and e_2. However $A(x_2, y_2)$ is the only $A'\overline{B'}$-arc. This result contradicts the assumption that for every choice of A and B there are at least two AB-arcs. □

In the final lemma we rule out a certain pair of directed $\overline{A \cup B}$-arcs.

LEMMA 2.27. *Let P be a directed $A\overline{B}$-arc or a directed $B\overline{A}$-arc and let Q be a directed AB-arc in H having neither end in common with an end of P. Then there is no pair X, Y of $\overline{A \cup B}$-arcs such that X is directed from a vertex $u \in VP$ to a vertex $v \in VQ$, Y is directed from a vertex $w \in VQ$ to a vertex $x \in VP$, $x <_P u$ and $v <_Q w$ (see Figure 2.19).*

Proof: By the symmetry between A and B we may assume that $P = A(x_1, y_1)$. Therefore $Q = A(u_2, x_2)$ or $Q = A(y_2, v_2)$. By symmetry we may assume the latter case obtains.

Suppose X and Y exist. Let

$$C = A(v, w) \cup Y \cup A(x, u) \cup X.$$

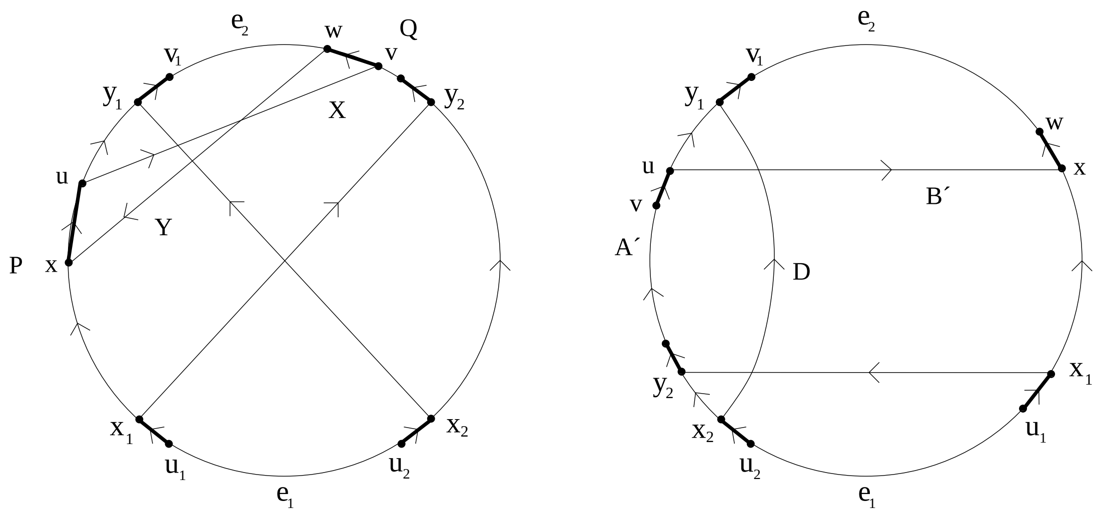


FIGURE 2.19. The situation in Lemma 2.27 before and after reorienting C.

Define $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= A(w, v_2) \cup \{e_2\} \cup A(u, v_1) \cup X \cup A(u_2, v) \cup \{e_1\} \cup A(u_1, x) \cup Y \end{aligned}$$

and

$$\begin{aligned} B' &= B + C \\ &= B(w, v_2) \cup \{e_2\} \cup B(u_2, v_1) \cup \{e_1\} \cup B(u_1, v) \cup X \cup A(x, u) \cup Y. \end{aligned}$$

Then A' and B' are f'-alternating circuits containing e_1 and e_2 and having opposite clockwise parity. Reorient C and define

$$D = B'(u_2, v_1) \cup \{e_2\} \cup A'(u_1, v_2) \cup \{e_1\}$$

(see Figure 2.19). Then D is f'-alternating and contains e_1 and e_2. If D is of opposite clockwise parity to A' then we have a contradiction because there is a unique $A'\overline{D}$-arc $A'(x_2, y_1)$; otherwise B' and D have opposite clockwise parity and there is another contradiction since $B'(x_1, x)$ is the only
$B'\overline{D}$-arc. □

4.3. Production of a list of cases to consider

We now introduce a notation to describe a minimal directed path X from y_i to x_j for $i, j \in \{1, 2\}$. Traversed from y_i to x_j, X meets a succession of directed $(A \cup B)X$-arcs in H. The *trace* of X is the sequence obtained from X by recording:
0 for each $A(x_j, y_i)X$-arc,
0′ for each $A(x_{3-j}, y_{3-i})X$-arc,
1 for each $B(x_{3-j}, y_i)X$-arc,
1′ for each $B(x_j, y_{3-i})X$-arc,
2 for each $(A \cap B)(y_{3-i}, v_{3-i})X$-arc,
2′ for each $(A \cap B)(u_{3-j}, x_{3-j})X$-arc.

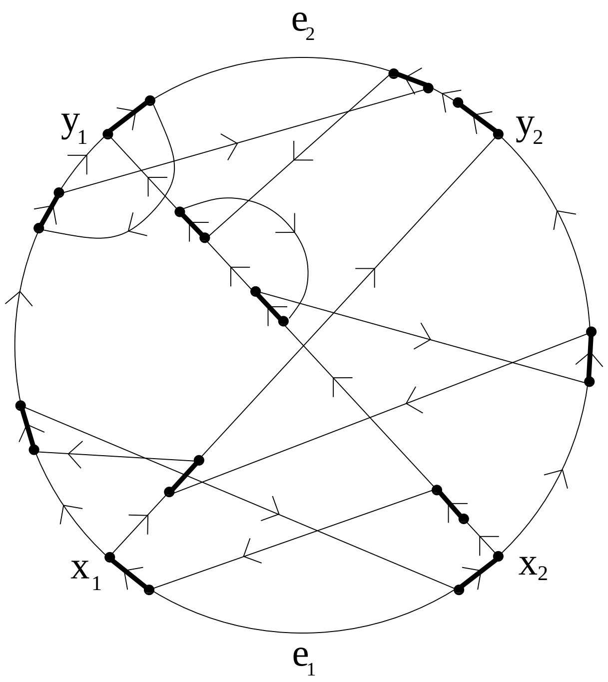


FIGURE 2.20. A directed minimal path with trace $02110'1'02'1$.

Figure 2.20 shows an example of a directed minimal path from y_1 to x_1 with trace $02110'1'02'1$. By Lemma 2.21(a) the graph $G[A \cup B \cup X]$ is determined up to isomorphism by the trace of X. In particular, there are a unique $A(y_i, v_i)X$-arc and a unique $A(u_j, x_j)X$-arc.

One aim of this subsection is to produce a finite list of possible traces of a directed minimal path. In the course of doing so the lemmas from Subsection 3.3 will be very helpful. Unfortunately $G = G[A \cup B \cup S \cup T]$ is not determined up to homeomorphism by the traces of S and T. Therefore another aim of this subsection is to introduce some additional parameters of the graph, such that the traces of S and T plus these parameters determine G. These efforts result in Lemma 2.32 and Lemma 2.38, which give a finite list of graphs under which all minimal non-Pfaffian near bipartite graphs for this case can be found. This list will be studied in detail in Subsection 3.5.

The main purpose of the following lemma is to show that we can assume that the trace of S neither starts with 1 nor ends with $1'$. It is similar to Lemma 2.16.

LEMMA 2.28. *Let W be a string over $\{0, 0', 1, 1', 2, 2'\}$.*
(a) It is possible to choose A, B, f, x_1, x_2, y_1, y_2 and a directed minimal path X from y_1 to x_1 in G such that the trace of X is $0W$ if and only if it is possible to choose A', B', f', x_1', x_2', y_1', y_2' and a directed minimal path X' from y_1' to x_1' in G such that the trace of X' is $1W$.
Moreover it is possible to choose the circuits and paths so that

$$A \cup B \cup X = A' \cup B' \cup X'.$$

(b) It is possible to choose A, B, f, x_1, x_2, y_1, y_2 and a directed minimal path X from y_1 to x_1 in G such that the trace of X is $W0$ if and only if it is possible to choose A', B', f', x_1', x_2', y_1', y_2' and a directed minimal path X' from y_1' to x_1' in G such that the trace of X' is $W1'$.

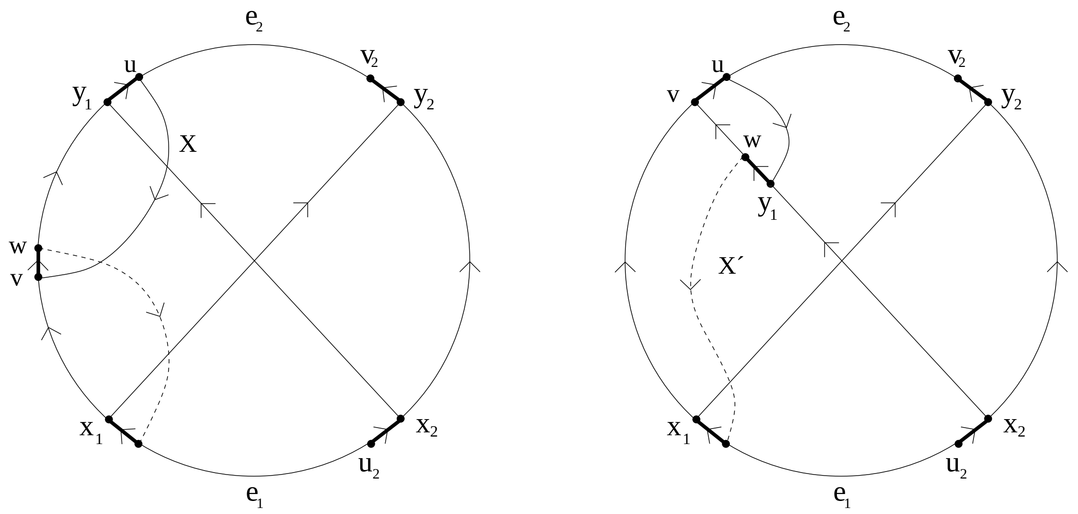


FIGURE 2.21. The situation in Lemma 2.28 before and after reorienting C.

Moreover it is possible to choose the circuits and paths so that

$$A \cup B \cup X = A' \cup B' \cup X'.$$

Proof: By symmetry, it suffices to prove (a). Suppose the trace of X is $0W$. There is an $A(x_1, y_1)X$-arc that corresponds to the first 0 in the trace of X. Let v be its origin, and let

$$C = X(y_1, v) \cup A(v, y_1).$$

Let u be the terminus of the $A(y_1, v_1)X$-arc and w the terminus of the $A(x_1, y_1)X$-arc with origin v (see Figure 2.21). Define $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= C(u, v) \cup A(u_1, v) \cup \{e_1\} \cup A(u_2, v_2) \cup \{e_2\} \cup A(u, v_1), \end{aligned}$$

$$\begin{aligned} B' &= B + C \\ &= C(u, y_1) \cup B(u_2, y_1) \cup \{e_1\} \cup B(u_1, v_2) \cup \{e_2\} \cup B(u, v_1) \end{aligned}$$

and

$$\begin{aligned} X' &= X + A(v, y_1) \\ &= C(w, v) \cup X(w, x_1). \end{aligned}$$

Then A' and B' are f'-alternating circuits containing e_1 and e_2 and having opposite clockwise parity. Moreover there are exactly two $A'B'$-arcs, one containing e_1 and the other containing e_2, and the vertices of degree 3 in $G[A' \cup B']$ are x_1, x_2, v, y_2. After reorientation of C, X' is a directed path from v to x_1. The trace of X' is $1W$, as $B'(y_1, w)$ is a $B'(x_2, v)X'$-arc which replaces the $A(x_1, y_1)X$-arc $A(v, w)$ (see Figure 2.21). Moreover X' is minimal: X' satisfies the condition in Lemma 2.21(b) since X does and

$$X'(v, w) \cap (A' \cup B') = C(v, u) \cup C(y_1, w).$$

For the converse in (a) note that $f = f' + C$, $A = A' + C$, $B = B' + C$ and $X = X' + A(v, y_1)$. $\square$

The following lemma shows that we can restrict our studies to graphs G where the trace of S is either 0 or $0'$.

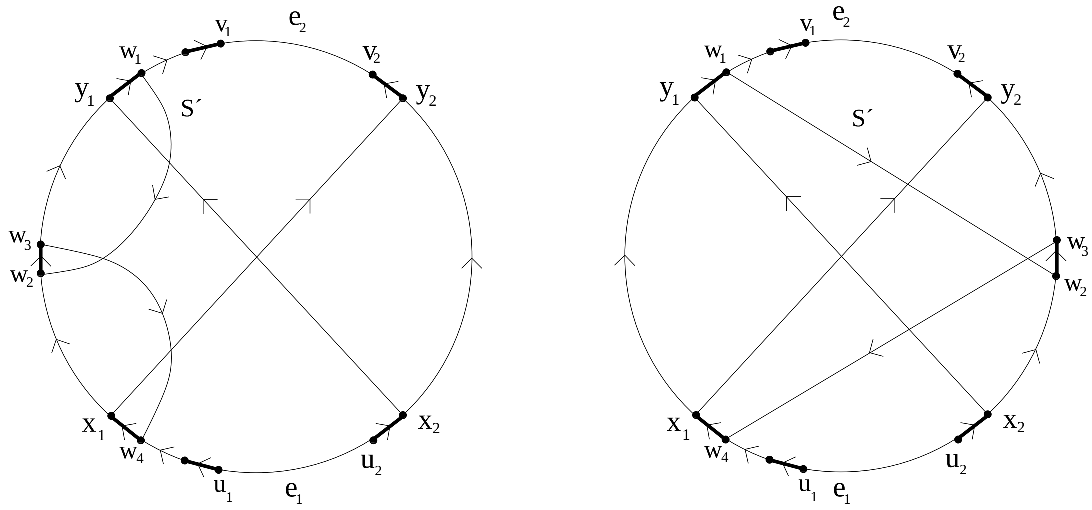


FIGURE 2.22. A directed minimal path S' with trace 0 and with trace $0'$.

LEMMA 2.29. *For some choice of* A, B, x_1, x_2, y_1, y_2, f *there is a directed minimal path* S' *from* y_1 *to* x_1 *with trace* 0 *or* $0'$ *(see Figure 2.22).*

Proof: We choose A, B, x_1, x_2, y_1, y_2, f and a directed path S' from y_1 to x_1 so that $A \cup B \cup S'$ is minimal. Thus S' is minimal.

Suppose the trace of S' contains 2. There is an $(A \cap B)(y_2, v_2)S'$-arc; let v be its terminus. Included in the set $A(y_2, v) \cup S'(v, x_1)$ is a directed path from y_2 to x_1, in contradiction to the minimality of $A \cup B \cup S'$ since $S'(y_1, v) - (A \cup B) \neq \emptyset$. Therefore the trace of S' does not contain 2, and similarly does not contain $2'$.

The trace of S' contains none of 00, 11, $0'0'$, $1'1'$ by Lemma 2.23, none of $00'$, $0'0$, $11'$, $1'1$ by Lemma 2.24, none of 01, $01'$, $0'1$, $0'1'$, 10, $10'$, $1'0$, $1'0'$ by Lemma 2.25, and is non-empty by Lemma 2.26. We infer that the trace of S is one of 0, $0'$, 1, $1'$. By Lemma 2.28 the case that the trace of S' is 1 or $1'$ can be reduced to the case that the trace of S' is 0. □

Because of this lemma we henceforth assume that the trace of S is 0 or $0'$. Given this choice for the trace of S, we now turn our attention to the trace of T. Note that this is the point where we break up the symmetry between S and T.

In the following we produce a finite list of possible traces of T and therefore a finite list of graphs we will consider in the following subsection. In order to do so we distinguish the two cases $S \cap T = \emptyset$ and $S \cap T \neq \emptyset$. First we assume that $S \cap T = \emptyset$.

The following lemma is analogeously to Lemma 2.28: We show that we can assume that the trace of T neither starts with 1 nor ends with $1'$.

LEMMA 2.30. *Let* W *be a string over* $\{0, 0', 1, 1', 2, 2'\}$, *and let* $S \cap T = \emptyset$.
(a) Suppose that the trace of T *is* $1W$. *Then there exist* A', B', f', x_1', x_2', y_1', y_2', *a directed minimal path* S' *from* y_1' *to* x_1' *with trace* 0 *or* $0'$ *and a directed minimal path* T' *from* y_2' *to* x_2' *such that the trace of* T' *is* $0W$.
(b) Suppose that the trace of T *is* $W1'$. *Then there exist* A', B', f', x_1', x_2', y_1', y_2', *a directed minimal path* S' *from* y_1' *to* x_1' *with trace* 0 *or* $0'$ *and a directed minimal path* T' *from* y_2' *to* x_2' *such that the trace of* T' *is* $W0$.

Proof: That T' exists follows by a proof similar to that of the corresponding assertion in Lemma 2.28. The reorientation of the corresponding f-alternating circuit C does not affect S, since $S \cap C = \emptyset$. Therefore we may take $S' = S$. □

Thus we can assume that the first symbol in the trace of T is not 1 and that the last symbol in the trace of T is not $1'$. In the following we will refer to this property of T as (A).

The following lemma is to show that we can assume that the trace of T contains none of $20'$, $21'$, $12'$, $0'2'$. This is done by showing that if the trace of T contains one of these strings, we can find other circuits A', B', which are alternating with respect to another 1-factor f', and other directed minimal paths S' and T' such that $S' \cap T' \neq \emptyset$.

LEMMA 2.31. *Suppose $S \cap T = \emptyset$ and the trace of T contains one of $20'$, $21'$, $12'$, $0'2'$. Then there exist A', B', f', a directed minimal path S' from y_1 to x_1 with trace 0 or $0'$ and a directed minimal path T' from y_2 to x_2 such that $S' \cap T' \neq \emptyset$.*

Proof: It suffices to consider the case where the trace of T contains $20'$, for the other cases are similar. In this case there is a $T\overline{A \cup B}$-arc with origin $u \in VA(y_1, v_1)$ and terminus $v \in VA(x_1, y_1)$. Let u' be the origin of the $A(y_1, v_1)T$-arc with terminus u, and v' the terminus of the $A(x_1, y_1)T$-arc with origin v. Let w be the terminus of the unique $A(y_1, v_1)S$-arc (see Figure 2.23).

Since $S \cap T = \emptyset$ we have $w <_{A(y_1,v_1)} u'$. If S has trace 0, then let x be the terminus of the unique $A(x_1, y_1)S$-arc and x' its origin. We have $x <_{A(x_1,y_1)} v$: otherwise if we define

$$S^* = A(y_1, u) \cup T(u, v) \cup A(v, x) \cup S(x, x_1)$$

then $G = G[A \cup B \cup S^* \cup T]$ by Theorem 2.5, in contradiction to the fact that

$$S(w, x') \cap (A \cup B \cup S^* \cup T) = \emptyset.$$

In any case, define

$$C = T(u, v) \cup A(v, u).$$

This is an f-alternating circuit such that $C \cap S = A(y_1, w)$. Reorient C and define $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= C(v, u) \cup A(u_1, v) \cup \{e_1\} \cup A(u_2, v_2) \cup \{e_2\} \cup A(u, v_1), \end{aligned}$$

$$\begin{aligned} B' &= B + C \\ &= C(y_1, u) \cup B(u_2, y_1) \cup \{e_1\} \cup B(u_1, v_2) \cup \{e_2\} \cup B(u, v_1), \end{aligned}$$

$$\begin{aligned} S' &= S + C(v, y_1) \\ &= C(v, w) \cup S(w, x_1) \end{aligned}$$

and

$$\begin{aligned} T' &= T + C \\ &= T(y_2, u') \cup C(u', v') \cup T(v', x_2). \end{aligned}$$

Then A' and B' are f'-alternating circuits containing e_1 and e_2 and having opposite clockwise parity. Moreover there are exactly two $A'B'$-arcs and the vertices of degree 3 in $G[A' \cup B']$ are x_1, x_2, v and y_2, S' is a directed path from v to x_1 and T is a directed path from y_2 to x_2 (see Figure 2.23). Moreover S and S' have equal trace and $S' \cap T' = C(u', w) \neq \emptyset$. Finally S' and T' are minimal: both satisfy the condition in Lemma 2.21(b) since $S'(v, w) \cap (A' \cup B') = C(v, u)$ and $T'(u', v') \cap (A' \cup B') = C(y_1, v')$. □

Thus we assume that the trace of T contains none of $20'$, $21'$, $12'$, $0'2'$, if $S \cap T = \emptyset$. In the following we will refer to this property of T as (B).

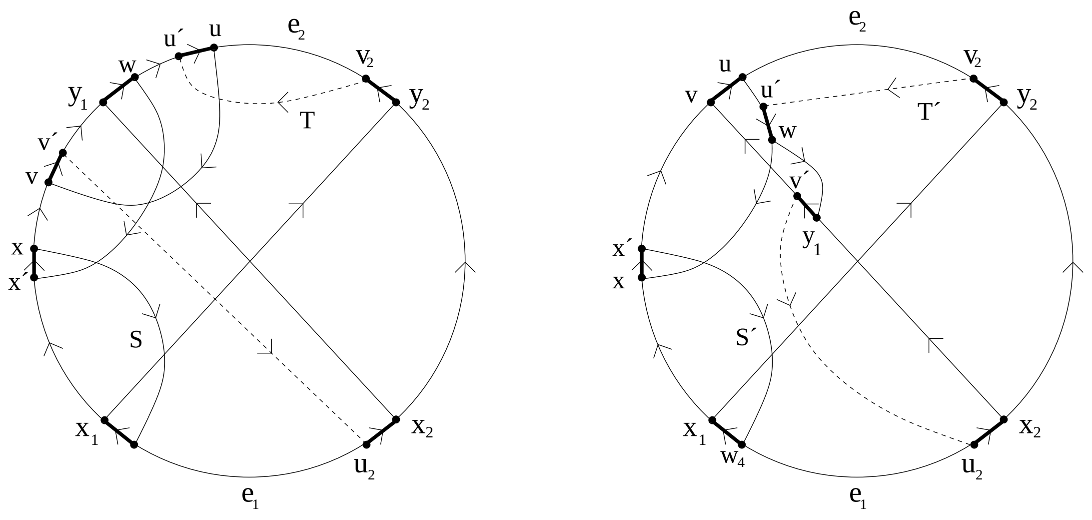


FIGURE 2.23. The situation in Lemma 2.31 before and after reorientating C.

The following lemma gives a complete list of graphs to be considered if $S \cap T = \emptyset$. We use $*$ to denote an arbitrary string of symbols, and Λ to denote the empty string.

LEMMA 2.32. *Suppose $S \cap T = \emptyset$ and that T has properties (A) and (B). Then the trace of T is one of* 0, $0'$, 20, 21, $02'$, $1'2'$.

Proof: First we see that the symbols in the trace of T alternate between the sets $\{0, 0', 1, 1'\}$ and $\{2, 2'\}$, for the trace of T contains none of 00, 11, 22, $0'0'$, $1'1'$, $2'2'$ by Lemma 2.23, none of $00'$, $0'0$, $11'$, $1'1$ by Lemma 2.24, none of 01, $01'$, $0'1$, $0'1'$, 10, $10'$, $1'0$, $1'0'$ by Lemma 2.25, and does not contain $22'$ by Lemma 2.26 or $2'2$ by Lemma 2.21(a).

Next we show that the trace of T does not contain both 2 and $2'$. Suppose that the trace of T contains $2' * 2$. Thus $x_1, y_1 \in VT$ by Lemma 2.21. This is a contradiction to $S \cap T = \emptyset$. Suppose that the trace of T contains $2 * 2'$. Choose 2 and $2'$ in the trace of T, with the chosen $2'$ appearing later than the chosen 2. Let u be the terminus of the $A(y_1, v_1)T$-arc that corresponds to the chosen 2 and let v be the origin of the $A(u_1, x_1)T$-arc that corresponds to the chosen $2'$. There is a directed path from y_1 to x_1 included in

$$A(y_1, u) \cup T(u, v) \cup A(v, x_1).$$

By Theorem 2.5 we have $G = G[A \cup B \cup T]$, in contradiction to

$$(S - (A \cup B)) \cap (A \cup B \cup T) = \emptyset.$$

We distinguish the following cases:

1. The trace of T contains neither 2 nor $2'$.
2. The trace of T contains 2 and consequently does not contain $2'$.
3. The trace of T contains $2'$ and consequently does not contain 2.

Case 1: By Lemma 2.26 the trace is not empty and therefore the trace of T is one of 0, $0'$, 1, $1'$ in this case. By (A) the case that the trace is 1 or $1'$ is not possible.

Case 2: The symbols in the trace of T alternate between the sets $\{2\}$ and $\{0, 0', 1, 1'\}$. By (B) and Lemma 2.26 every 2 in the trace must be immediately followed by 0 or 1. Therefore the trace contains at most one 2 by Lemma 2.27 and consequently exactly one 2. In fact the trace of T is $x2y$, where $x \in \{\Lambda, 0, 0', 1, 1'\}$ and $y \in \{0, 1\}$.

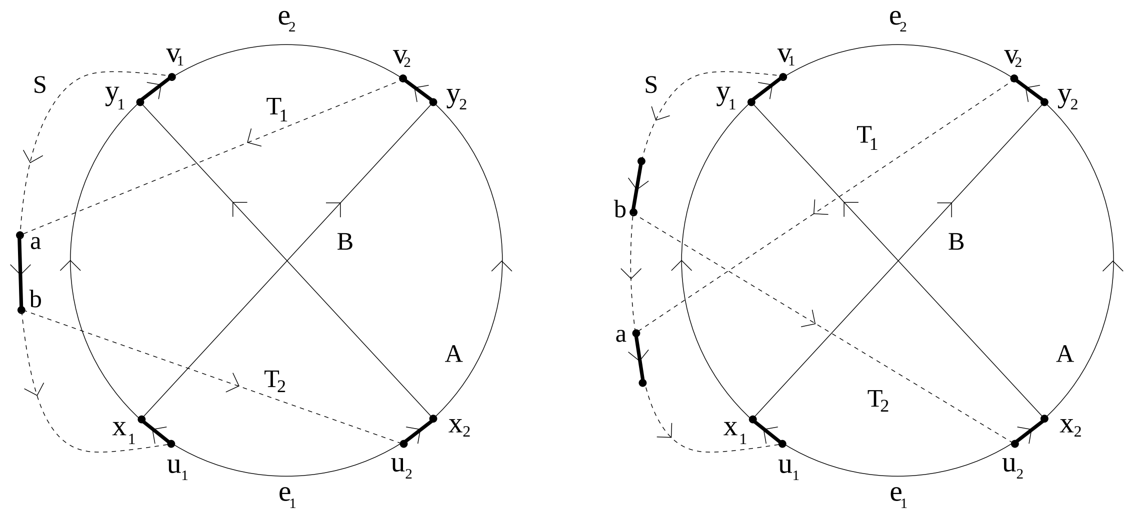


FIGURE 2.24. There are two possible cases: Either $a <_S b$ or $b <_S a$.

We show that $x = \Lambda$. We have $x \notin \{0', 1'\}$, for otherwise $y_1 \in VS \cap VT$ by Lemma 2.21(a). Suppose $x = 0$. If $y = 0$, we have a contradiction by Lemma 2.27. If $y = 1$, we have a contradiction by Lemma 2.27 also, since in this case there exist A', B', f', x_1', x_2', y_1', y_2' in G and a directed minimal path T' from y_2' to x_2' with trace 121 by Lemma 2.28. Similarly we obtain a contradiction if we suppose that $x = 1$. Therefore the trace of T is either 20 or 21, if the trace contains 2.

Case 3: Similarly the trace of T is either $02'$ or $1'2'$, if it contains $2'$. □

REMARK 2.3. *The case where the trace of T is either $02'$ or $1'2'$ can be reduced to the case where the trace of T is either 20 or 21. In order to see this suppose that the trace of T is either $02'$ or $1'2'$ and switch to the reference orientation with respect to (N, M, f). Then S is a directed path from $y_1' = x_1$ to $x_1' = y_1$ with trace 0 or $0'$ and T is a directed path from $y_2' = x_2$ to $x_2' = y_2$ with trace 20 or 21.*

Now we consider the case where $S \cap T \neq \emptyset$. First we prove the following consequence of Theorem 2.5.

COROLLARY 2.1. *Let S' be a directed path from y_1 to x_1, T_1 a directed path from y_2 to a vertex in S' and T_2 a directed path from a vertex in S' to x_2. Then*

$$G = G[A \cup B \cup S' \cup T_1 \cup T_2].$$

Proof: By Theorem 2.5 we have to show that there exists a directed path from y_2 to x_2 in $G[A \cup B \cup S' \cup T_1 \cup T_2]$. Let a be the terminus of T_1 and b the origin of T_2. Then such a path is included in

$$T_1 \cup S(a, x_1) \cup A(x_1, y_1) \cup S(y_1, b) \cup T_2.$$

□

Since $S \cap T \neq \emptyset$, there exists a first vertex a in T that is also in S, and a last vertex b in T that is also in S (see Figure 2.24). Let $T_1 = T(y_2, a)$ and $T_2 = T(b, x_2)$. By Corollary 2.1,

$$G = G[A \cup B \cup S \cup T_1 \cup T_2].$$

We define the trace of T_1 and T_2 in a manner analogous to the definition of the trace of a directed minimal path from y_2 to x_2. Note that T_1 and T_2 satisfy the condition in

Lemma 2.21(b), since they are directed subpaths of the directed minimal path T. Again we have a lemma that is analogeous to Lemma 2.28 and Lemma 2.30.

LEMMA 2.33. *Let W be a string over $\{0, 0', 1, 1', 2, 2'\}$, and let $S \cap T \neq \emptyset$.*
(a) Suppose that the trace of T_1 is $1W$. Then there exist A', B', f', x_1', x_2', y_1', y_2', a directed minimal path S' from y_1' to x_1' with trace 0 or $0'$ and a directed minimal path T' from y_2' to x_2' such that the trace of T_1' is $0W$.
(b) Suppose that the trace of T_2 is $W1'$. Then there exist A', B', f', x_1', x_2', y_1', y_2', a directed minimal path S' from y_1' to x_1' with trace 0 or $0'$ and a directed minimal path T' from y_2' to x_2' such that the trace of T_2' is $W0$.

Proof: Similar to the proof of Lemma 2.30. □

Therefore we can assume that the first symbol in the trace of T_1 is not 1 and that the last symbol in the trace of T_2 is not $1'$. In the following we will refer to this property of T_1 and T_2 as (A').

In the directed path S there are exactly 6 vertices of degree 3 in $G[A \cup B \cup S]$, the first being y_1 and the last being x_1. We label the other such vertices w_1, w_2, w_3 and w_4 in the order they occur when we traverse S from y_1 to x_1 (see Figure 2.22). Note that $a \notin \{w_1, w_3, x_1\}$ and $b \notin \{y_1, w_2, w_4\}$, since the vertices of G have indegree 1 or outdegree 1. Before we are able to present a list of possible traces for T_1 and T_2 we need the following lemma.

LEMMA 2.34. *The vertices a and b are not both in $VS(y_1, w_1)$ and not both in $VS(w_4, x_1)$.*

Proof: By symmetry it suffices to show that $\{a, b\} \not\subseteq VS(y_1, w_1)$. Suppose the contrary.

First we assume that the trace of S is 0. We define

$$C = S(w_1, w_2) \cup A(w_2, w_1).$$

This is an f-alternating circuit. Furthermore we define $f' = f + C$,

$$\begin{aligned} A' &= A + C \\ &= C(w_1, w_2) \cup A(u_1, w_2) \cup \{e_1\} \cup A(u_2, v_2) \cup \{e_2\} \cup A(w_1, v_1), \end{aligned}$$

$$\begin{aligned} B' &= B + C \\ &= C(w_1, y_1) \cup B(u_2, y_1) \cup \{e_1\} \cup B(u_1, v_2) \cup \{e_2\} \cup B(w_1, v_1). \end{aligned}$$

By Lemma 2.4, A' and B' are f'-alternating circuits containing e_1 and e_2, of opposite clockwise parity, such that there are exactly two $A'B'$-arcs. The vertices of degree 3 in $G[A' \cup B']$ are x_1, x_2, w_2, y_2.

First we assume $a <_S b$. In this case $a = y_1$ and $b = w_1$, for otherwise we would have vertices of degree 2. We define the paths

$$X' = T_1 \cup C(w_3, a) \cup S(w_3, x_1)$$

and

$$Y' = C(b, w_2) \cup T_2.$$

These paths would become directed from y_2 to x_1 and from w_2 to x_2, respectively, if C were reoriented. Therefore

$$G = G[A' \cup B' \cup X' \cup Y']$$

by Theorem 2.5, in contradiction to

$$C(a, b) \cap (A' \cup B' \cup X' \cup Y') = \emptyset.$$

Now we assume that $b <_S a$. We reorient C and define the directed path

$$\begin{aligned} T' &= T + C \\ &= T_1 \cup C(a,b) \cup T_2 \end{aligned}$$

and a directed path S' from w_2 to x_1 included in

$$C(w_2,b) \cup T_2 \cup B(x_2,y_1) \cup C(y_1,w_3) \cup S(w_3,x_1).$$

By Theorem 2.5

$$G = G[A' \cup B' \cup S' \cup T'],$$

in contradiction to

$$C(b,y_1) \cap (A' \cup B' \cup S' \cup T') = \emptyset.$$

Next we consider the case that the trace of S is $0'$. We define

$$D = S(w_1,w_4) \cup A(w_4,w_1).$$

This is an f-alternating circuit. Furthermore we define $f'' = f + D$,

$$\begin{aligned} A'' &= A + D \\ &= D(w_1,w_2) \cup A(u_2,w_2) \cup \{e_1\} \cup A(u_1,w_4) \cup \\ &\quad\ D(w_3,w_4) \cup A(w_3,v_2) \cup \{e_2\} \cup A(w_1,v_1), \end{aligned}$$

$$\begin{aligned} B'' &= B + D \\ &= D(w_1,w_4) \cup B(u_1,w_4) \cup \{e_1\} \cup B(u_2,y_1) \cup \\ &\quad\ D(x_1,y_1) \cup B(x_1,v_2) \cup \{e_2\} \cup B(w_1,v_1). \end{aligned}$$

By Lemma 2.4, A'' and B'' are f''-alternating circuits including e_1 and e_2, of opposite clockwise parity, such that there are exactly two $A''B''$-arcs. The vertices of degree 3 in $G[A'' \cup B'']$ are w_3, x_2, w_2, y_2.

First we assume $a <_S b$. Again we have $a = y_1$ and $b = w_1$, for otherwise G would have vertices of degree 2. We define the paths

$$X'' = D(b,w_2) \cup T_2$$

and

$$Y'' = T_1 \cup D(w_3,a).$$

These paths would become directed from w_2 to x_2 and from y_2 to w_3, respectively, if D were reoriented. Therefore

$$G = G[A'' \cup B'' \cup X'' \cup Y'']$$

by Theorem 2.5, in contradiction to

$$D(a,b) \cap (A'' \cup B'' \cup X'' \cup Y'') = \emptyset.$$

Now we assume that $b <_S a$. We reorient D and define the directed path

$$\begin{aligned} T'' &= T + D \\ &= T_1 \cup D(a,b) \cup T_2 \end{aligned}$$

and a directed path S'' from w_2 to w_3 included in

$$D(w_2,b) \cup T_2 \cup B(x_2,y_1) \cup D(y_1,w_3).$$

By Theorem 2.5

$$G = G[A'' \cup B'' \cup S'' \cup T''],$$

in contradiction to

$$D(b,y_1) \cap (A'' \cup B'' \cup S'' \cup T'') = \emptyset.$$

□

LEMMA 2.35. *The traces of T_1 and T_2 are either empty or* 0.

Proof: By symmetry it suffices to show that the trace of T_1 is either empty or 0.

First we show that the edge in T_1 incident on a is not in $A \cup B$. This edge exists, for $T_1 \neq \emptyset$ since the trace of S is 0 or $0'$. Let a^* be its origin and suppose that $(a^*, a) \in A \cup B$. Note that $(a^*, a) \notin f$ because S is f-alternating. Thus the edge of f incident on a^* is in T_1 and in $A \cup B$. Now we have the contradiction that a^* is a vertex of degree 2, since $a^* \notin VS \cup VT_2$.

We use this observation to show that the trace of T_1 contains none of $0'$, $1'$, $2'$.

Suppose that the trace of T_1 contains $0'$. Let a' be the origin of an $A(x_1, y_1)T_1$-arc. Clearly $a' \neq a$. Included in

$$T_1(y_2, a') \cup A(a', y_1)$$

is a directed path T_1' from y_2 to a vertex in S with

$$T_1' \cap (T_1(a', a) - (A \cup B)) = \emptyset.$$

By Corollary 2.1 we have

$$G = G[A \cup B \cup S \cup T_1' \cup T_2].$$

Since

$$(T_1(a', a) - (A \cup B)) \cap (A \cup B \cup S \cup T_1' \cup T_2) = \emptyset,$$

it follows that

$$T_1(a', a) - (A \cup B) = \emptyset.$$

Therefore we have the contradiction that the edge of T_1 incident on a is in $A \cup B$. The proof that the trace of T_1 contains neither $1'$ nor $2'$ is similar. Likewise the trace of T_2 contains none of $0'$, 1 and 2.

Next we show that the trace of T_1 does not contain 2. Assume the contrary and let v be the terminus of the last $A(y_1, v_1)T_1$-arc. Then there is a directed path S' from y_1 to x_1 included in

$$A(y_1, v) \cup T_1(v, a) \cup S(a, x_1)$$

with

$$S' \cap (S(y_1, a) - (A \cup B)) = \emptyset.$$

Therefore

$$S(y_1, a) \subseteq A \cup B \cup T,$$

since $G = G[A \cup B \cup S' \cup T]$ by Theorem 2.5. Consequently,

$$a \notin (VS(w_1, w_2) - \{w_1\}) \cup (VS(w_3, w_4) - \{w_3\}),$$

for otherwise the edge of $S(y_1, a)$ incident on a is in T (since it is not in $A \cup B$), and we have a contradiction to the choice of a. Thus

$$a \in VS(y_1, w_1) \cup (VS(w_2, w_3) - \{w_2\}) \cup (VS(w_4, x_1) - \{w_4\}).$$

Now we distinguish three cases according to which set of this union contains a.

Suppose that $a \in VS(y_1, w_1)$. We already know that the trace of T_1 is a string over $\{0, 1, 2\}$. The symbols alternate between 2 and members of the set $\{0, 1\}$ for the trace of T_1 contains none of 00, 11, 22 by Lemma 2.23 and neither 01 nor 10 by Lemma 2.25. Therefore 2 is either the last symbol or the penultimate symbol in the trace of T_1. If 2 is the last symbol in the trace of T_1 we have a contradiction by Lemma 2.23, and if 2 is the penultimate symbol in the trace of T_1 then the last symbol of the trace of T_1 is either 0 or 1 and we have a contradiction by Lemma 2.21(a) and Lemma 2.27.

Suppose that $a \in VS(w_4, x_1) - \{w_4\}$. Note that a cannot be adjacent to w_4 since both vertices have indegree more than 1. Thus $b \in VS(w_4, a) - \{w_4\}$, for otherwise there would be a vertex of degree 2 in G. Therefore $\{a, b\} \subseteq VS(w_4, x_1)$ in contradition to Lemma 2.34.

Therefore $a \in VS(w_2, w_3) - \{w_2\}$. Then $b \in VS(w_2, a)$, for otherwise there would be a vertex of degree 2 in G. First we consider the case that the trace of S is 0. We define

$$T' = T_1 \cup S(a, x_1) \cup A(x_1, b) \cup T_2.$$

This is a directed path from y_2 to x_2 with

$$T' \cap (S(y_1, a) - (A \cup B)) = \emptyset.$$

By Theorem 2.5 we have

$$G = G[A \cup B \cup S' \cup T'],$$

a contradiction, since

$$S(w_1, w_2) \cap (A \cup B \cup S' \cup T') = \emptyset.$$

Now we consider the case that the trace of S is $0'$. By Lemma 2.23 and Lemma 2.25 the path $T_1(v, a)$ is an $\overline{A \cup B}$-arc. We define the following directed minimal path S'' from y_1 to x_1:

$$S'' = A(y_1, v) \cup T_1(v, a) \cup S(a, x_1).$$

The trace of S'' is $0'$. Let u be the origin of the $A(y_1, v_1)T_1$-arc with terminus v. Note that

$$T = T_1 \cup S(a, x_1) \cup A(x_1, y_1) \cup S(y_1, b) \cup T_2.$$

Then u is the first vertex in T that is also in S'' and w_1 is the last vertex in T that is also in S'', since the trace of T_2 does not contain 2. If we replace S by S'', this is a contradiction by Lemma 2.34 and finally shows that the trace of T_1 does not contain 2.

Now we know that the trace of T_1 is a string over $\{0, 1\}$. By Lemma 2.23 the trace of T_1 contains neither 00 nor 11, and by Lemma 2.25 it contains neither 01 nor 10. Therefore the trace of T_1 is either empty, 0 or 1. Since the directed path T_1 has property (A') the case that the trace of T_1 is 1 is not possible. $\square$

REMARK 2.4. *An argument similar to the one that showed that the trace of T_1 does not contain $0'$ leads to the following observation: If the trace of S is $0'$, the symbol 0 in the trace of T_1 corresponds to an $A(w_3, y_2)T_1$-arc and the symbol 0 in the trace of T_2 corresponds to an $A(x_2, w_2)T_2$-arc.*

Now we have a finite list of possible traces for S, T_1 and T_2. Unfortunately a minimal non-Pfaffian graph G is not determined by the traces of S, T_1 and T_2. But it is determined by these traces plus some information about the position of a and b. In Lemma 2.34 we already learned something about possible positions of a and b. In the following we prove two lemmas of a similar flavour.

LEMMA 2.36. *Suppose $b <_S a$. Then $S(b, a) \subseteq S(w_2, w_3)$.*

Proof: The assertion can be deduced from Lemma 2.34 after we show that $S(b, a) \subseteq A \cup B$. Define

$$X = S(y_1, b) \cup T_2$$

and

$$Y = T_1 \cup S(a, x_1).$$

Then X is a directed path from y_1 to x_2 and Y is a directed path from y_2 to x_1. Since $S(b, a) \cap (X \cup Y) = \emptyset$ and

Proof: The assertion can be deduced from Lemma 2.34 after we show that $S(b, a) \subseteq A \cup B$.

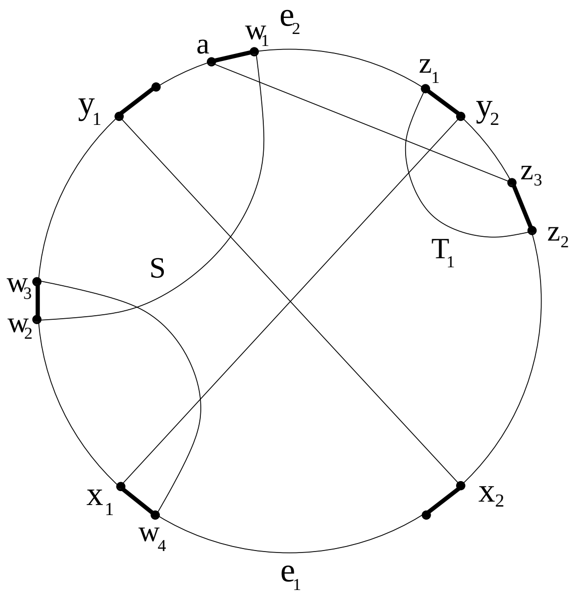


FIGURE 2.25. This graph contains an even subdivision of $K_{3,3}$.

Define

$$X = S(y_1, b) \cup T_2$$

and

$$Y = T_1 \cup S(a, x_1).$$

Then X is a directed path from y_1 to x_2 and Y is a directed path from y_2 to x_1. Since $S(b, a) \cap (X \cup Y) = \emptyset$ and

$$G = G[A \cup B \cup X \cup Y]$$

by Theorem 2.5, we have $S(b, a) \subseteq A \cup B$. □

LEMMA 2.37. *If the trace of T_1 is empty then $a \notin VS(w_3, x_1)$.*
If the trace of T_1 is 0 then $a \notin VS(y_1, w_3)$.
If the trace of T_2 is empty then $b \notin VS(y_1, w_2)$.
If the trace of T_2 is 0 then $b \notin VS(w_2, x_1)$.

Proof: By symmetry it suffices to show the assertions for T_1. The first assertion is an immediate consequence of Lemma 2.26.

Now suppose that the trace of T_1 is 0. If the trace of S is 0 then $a \notin VS(w_1, w_3)$ by Lemma 2.24; if the trace of S is $0'$ then $a \notin VS(w_1, w_3)$ by Lemma 2.23.

Suppose the trace of S is 0, and that $a \in VS(y_1, w_1)$. Let z_1, z_2, z_3 be the vertices of $VT_1 - \{y_2, a\}$ in the order in which they appear as T_1 is traced from y_2. Then

$$G[(A \cup B \cup S \cup T_1) - (A(y_2, z_1) \cup A(z_2, z_3) \cup B(x_2, y_1))]$$

is an even subdivision of $K_{3,3}$ (see Figure 2.25), in contradiction to the fact that G is minimal non-Pfaffian. If the trace of S is $0'$ then $a \notin VS(y_1, w_1)$ by Lemma 2.27 and Remark 2.4. □

The following lemma gives a complete list of graphs to be considered if $S \cap T \neq \emptyset$.

LEMMA 2.38. *Suppose $S \cap T \neq \emptyset$ and that the directed path T has property (A'). Then one of the following cases is true:*

1. *the traces of T_1 and T_2 are Λ and 0 respectively, $a \in VS(w_1, w_2) - \{w_2\}$, $b \in VS(w_1, w_2) - \{w_1\}$, $a <_S b$,*

2. *the traces of* T_1 *and* T_2 *are empty,* $a \in VS(y_1, w_1)$, $b \in VS(w_2, w_3)$,
3. *the traces of* T_1 *and* T_2 *are empty,* $a \in VS(y_1, w_1)$, $b \in VS(w_4, x_1)$,
4. *the traces of* T_1 *and* T_2 *are empty,* $a \in VS(w_2, w_3)$, $b \in VS(w_2, w_3)$, $a <_S b$,
5. *the traces of* T_1 *and* T_2 *are empty,* $a \in VS(w_2, w_3)$, $b \in VS(w_2, w_3)$, $b <_S a$,
6. *the traces of* T_1 *and* T_2 *are empty,* $a \in VS(w_2, w_3)$, $b \in VS(w_4, x_1)$,
7. *the traces of* T_1 *and* T_2 *are 0 and* Λ *respectively,* $a \in VS(w_3, w_4) - \{w_4\}$, $b \in VS(w_3, w_4) - \{w_3\}$ *and* $a <_S b$.

Proof: First we deal with the case that

$$a \in (VS(w_1, w_2) - \{w_2\}) \cup (VS(w_3, w_4) - \{w_4\})$$

or

$$b \in (VS(w_1, w_2) - \{w_1\}) \cup (VS(w_3, w_4) - \{w_3\}).$$

Since there are no vertices of degree 2 in G, the vertex a is in $VS(w_1, w_2) - \{w_2\}$ if and only if $b \in VS(w_1, w_2) - \{w_1\}$. Furthermore $a <_S b$ in this case and by Lemma 2.37 the trace of T_1 is empty and the trace of T_2 is 0. This situation corresponds to the first case in the lemma.

Similarly $a \in VS(w_3, w_4) - \{w_4\}$ if and only if $b \in VS(w_3, w_4) - \{w_3\}$. In this case $a <_S b$, the trace of T_1 is 0 and the trace of T_2 is empty. This situation corresponds to the last case in the lemma.

Therefore we may now assume that

$$\{a, b\} \subset VS(y_1, w_1) \cup VS(w_2, w_3) \cup VS(w_4, x_1).$$

We show that the trace of T_1 is empty. Suppose the contrary, that the trace of T_1 is 0. By Lemma 2.37 and our assumption we have $a \in VS(w_4, x_1)$. By Lemma 2.36 we have $a <_S b$ and therefore $b \in VS(w_4, x_1)$. This is a contradiction to Lemma 2.34.

Similarly the trace of T_2 is empty.

By Lemma 2.37 and our assumption we have

$$a \in VS(y_1, w_1) \cup VS(w_2, w_3)$$

and

$$b \in VS(w_2, w_3) \cup VS(w_4, x_1).$$

From this result we deduce the following list of cases to be considered.

1. $a \in VS(y_1, w_1)$, $b \in VS(w_2, w_3)$
2. $a \in VS(y_1, w_1)$, $b \in VS(w_4, x_1)$
3. $a \in VS(w_2, w_3)$, $b \in VS(w_2, w_3)$, $a <_S b$
4. $a \in VS(w_2, w_3)$, $b \in VS(w_2, w_3)$, $b <_S a$
5. $a \in VS(w_2, w_3)$, $b \in VS(w_4, x_1)$.

□

REMARK 2.5. *If we change from the reference orientation with respect to* (M, N, f) *to the reference orientation with respect to* (N, M, f) *the first case in Lemma 2.38 changes to the last case. Therefore we do not have to consider the last case. The same is true for the second case and the sixth case in Lemma 2.38, and so we do not consider the sixth case.*

4.4. The minimal non-Pfaffian near bipartite graphs II

In this subsection we study the list of graphs we have established in the previous subsection in Lemma 2.32 and Lemma 2.38. In some cases we find that the graph or a contraction of it is isomorphic to either Γ_1 or Γ_2. In many other cases we obtain a contradiction based on requirements for the clockwise parities of various circuits. For that purpose we need the following corollary, which is an immediate consequence of Lemma 2.4.

COROLLARY 2.2. *The sum $A+B$ is a clockwise even circuit under our extended Pfaffian orientation.*

Proof: Note that

$$A+B = A(x_1,y_1) \cup B(x_2,y_1) \cup A(x_2,y_2) \cup B(x_1,y_2),$$

which is a circuit. Since A and B are of opposite clockwise parity, the result follows from Lemma 2.4. $\square$

We divide the argument into cases according to whether or not $S \cap T = \emptyset$.

Case 1: In the case where $S \cap T = \emptyset$ it follows from Lemma 2.32 and Remark 2.3 that the trace of S may be assumed to be either 0 or $0'$ and that of T may be assumed to be one of 0, $0'$, 20, 21. Let the vertices in $VT - \{y_2, x_2\}$ be $z_1, z_2, \ldots, z_n$ in the order in which they appear.

Subcase 1.1: Suppose the trace of T is 0.

Subcase 1.1.1: Suppose the trace of S is 0. Consider the circuits

$$\begin{aligned}
C_1 &= S(w_1,w_2) \cup A(w_2,w_1),\\
C_2 &= S(w_3,w_4) \cup A(w_4,w_3),\\
C_3 &= T(z_1,z_2) \cup A(z_2,z_1),\\
C_4 &= T(z_3,z_4) \cup A(z_4,z_3),\\
C_5 &= S \cup B(x_1,y_2) \cup T \cup B(x_2,y_1).
\end{aligned}$$

Their sum is $A+B$. However, under our reference orientation all of C_1, C_2, C_3, C_4, C_5, $A+B$ are clockwise even, but under our extended Pfaffian orientation of H only $A+B$ is clockwise even, by Corollary 2.2. This result contradicts Lemma 2.4.

Subcase 1.1.2: Suppose the trace of S is $0'$. By symmetry we may assume that $w_3 <_{A(x_2,y_2)} z_2$ (see Figure 2.27). Then G is isomorphic to Γ_1. The isomorphism ϕ from Γ_1 and G is given by $\phi(a) = x_2$, $\phi(b) = y_1$, $\phi(c) = w_1$, $\phi(d) = w_2$, $\phi(e) = w_3$, $\phi(f) = w_4$, $\phi(g) = x_1$, $\phi(h) = y_2$, $\phi(i) = z_1$, $\phi(j) = z_2$, $\phi(k) = z_3$, $\phi(l) = z_4$.

Subcase 1.2: Suppose the trace of T is $0'$.

Subcase 1.2.1: The case where the trace of S is 0 is symmetric to Subcase 1.1.2.

Subcase 1.2.2: Suppose the trace of S is $0'$. Consider the circuits

$$\begin{aligned}
C_1 &= S \cup A(x_1,y_1),\\
C_2 &= T \cup A(x_2,y_2),\\
C_3 &= S \cup B(x_1,y_2) \cup T \cup B(x_2,y_1).
\end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.

Subcase 1.3: Suppose the trace of T is 20.

Subcase 1.3.1: Suppose the trace of S is 0. This case is similar to Subcase 1.1.1 except that vertices z_2, z_3, z_4 in Subcase 1.1.1 are replaced by z_4, z_5, z_6 respectively.

Subcase 1.3.2: Suppose the trace of S is $0'$. Note that $z_1 = v_2$ and $z_3 = v_1$. Contract the circuit $T(z_1,z_3) \cup \{e_2\}$. The resulting graph is isomorphic to Γ_1 whether or not $w_3 <_{A(x_2,y_2)} z_4$.

Subcase 1.4: Suppose the trace of T is 21.

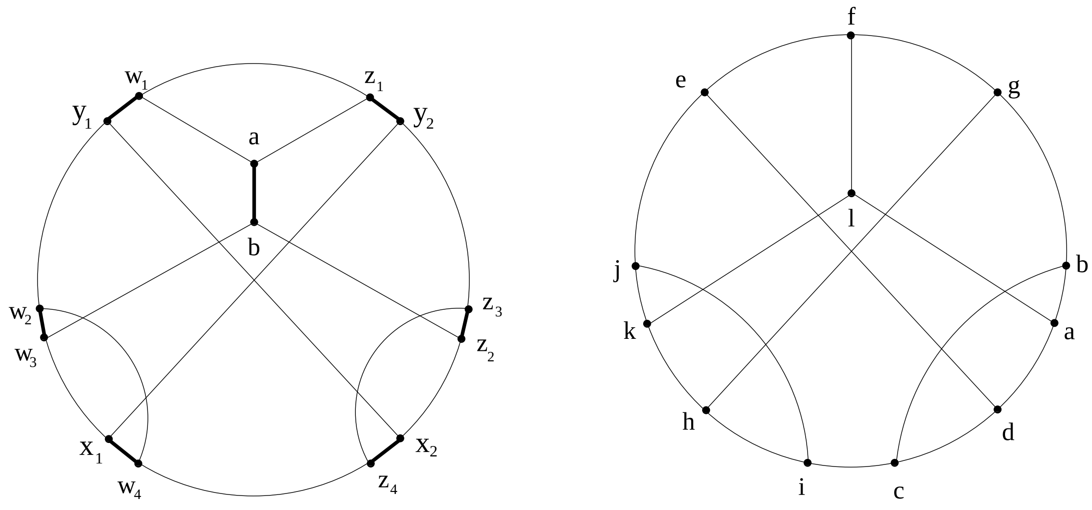


FIGURE 2.26. Subcase 2.1.1

Subcase 1.4.1: Suppose the trace of S is 0. Consider the circuits

$$\begin{aligned} C_1 &= S(w_1, w_2) \cup A(w_2, w_1), \\ C_2 &= S(w_3, w_4) \cup A(w_4, w_3), \\ C_3 &= T \cup A(x_2, y_2), \\ C_4 &= T(y_2, z_5) \cup B(z_5, y_2), \\ C_5 &= S \cup B(x_1, z_5) \cup T(z_5, x_2) \cup B(x_2, y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 1.4.2: Suppose the trace of S is $0'$. Contract the circuit $T(z_1, z_3) \cup \{e_2\}$. The resulting graph is isomorphic to Γ_1.

Case 2: If $S \cap T \neq \emptyset$ then by Remark 2.5 we see that only cases 1–5 in Lemma 2.38 need to be considered. Case i of the lemma is dealt with in Subcase 2.i below. Let the vertices in $(VT_1 \cup VT_2) - \{y_2, x_2, a, b\}$ be $z_1, z_2, \ldots, z_n$ in the order in which they appear in T.
Subcase 2.1: The traces of T_1 and T_2 are Λ and 0 respectively, $a \in VS(w_1, w_2) - \{w_2\}$, $b \in VS(w_1, w_2) - \{w_1\}$, $a <_S b$.
Subcase 2.1.1: Suppose the trace of S is 0 (see Figure 2.26). Note that $z_1 = v_2$ and $w_1 = v_1$. Contraction of the circuit $T(z_1, a) \cup S(w_1, a) \cup \{e_2\}$ yields a graph isomorphic to Γ_1 (see the second picture in Figure 2.26, where the contraction is fulfilled and the isomorphism is indicated).
Subcase 2.1.2: Suppose the trace of S is $0'$. By Remark 2.4 we have $z_3 <_{A(x_2,y_2)} w_2$. Define

$$S' = S(y_1, b) \cup T(b, z_3) \cup A(z_3, w_2) \cup S(w_2, x_1).$$

By Theorem 2.5 we find that $G = G[A \cup B \cup S' \cup T]$, in contradiction to the fact that

$$S(b, w_2) \cap (A \cup B \cup S' \cup T) = \emptyset.$$

Subcase 2.2: The traces of T_1 and T_2 are empty, $a \in VS(y_1, w_1)$, $b \in VS(w_2, w_3)$. Note that $a = y_1$ and $b = w_3$.

Subcase 2.2.1: Suppose the trace of S is 0. Consider the circuits

$$\begin{aligned} C_1 &= S(w_1,w_2)\cup A(w_2,w_1), \\ C_2 &= S(w_3,w_4)\cup A(w_4,w_3), \\ C_3 &= T_1\cup S(y_1,w_3)\cup T_2\cup A(x_2,y_2), \\ C_4 &= T_1\cup S\cup B(x_1,y_2), \\ C_5 &= S(y_1,w_3)\cup T_2\cup B(x_2,y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 2.2.2: Suppose the trace of S is $0'$. Consider the circuits

$$\begin{aligned} C_1 &= S\cup A(x_1,y_1), \\ C_2 &= T_1\cup S(y_1,w_3)\cup A(w_3,y_2), \\ C_3 &= T_2\cup A(x_2,w_3), \\ C_4 &= T_1\cup S\cup B(x_1,y_2), \\ C_5 &= S(y_1,w_3)\cup T_2\cup B(x_2,y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 2.3: The traces of T_1 and T_2 are empty, $a\in VS(y_1,w_1)$, $b\in VS(w_4,x_1)$. Note that $a=y_1$ and $b=x_1$.
Subcase 2.3.1: Suppose the trace of S is 0. Consider the circuits

$$\begin{aligned} C_1 &= S(w_1,w_2)\cup A(w_2,w_1), \\ C_2 &= S(w_3,w_4)\cup A(w_4,w_3), \\ C_3 &= T_1\cup S\cup T_2\cup A(x_2,y_2), \\ C_4 &= T_1\cup S\cup B(x_1,y_2), \\ C_5 &= S\cup T_2\cup B(x_2,y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 2.3.2: Suppose the trace of S is $0'$. Consider the circuits

$$\begin{aligned} C_1 &= S\cup A(x_1,y_1), \\ C_2 &= T_1\cup S(y_1,w_3)\cup A(w_3,y_2), \\ C_3 &= S(w_2,x_1)\cup T_2\cup A(x_2,w_2), \\ C_4 &= T_1\cup S\cup B(x_1,y_2), \\ C_5 &= S\cup T_2\cup B(x_2,y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 2.4: The traces of T_1 and T_2 are empty, $a\in VS(w_2,w_3)$, $b\in VS(w_2,w_3)$, $a<_S b$. Note that $a=w_2$ and $b=w_3$.
Subcase 2.4.1: Suppose the trace of S is 0. Consider the circuits

$$\begin{aligned} C_1 &= S(w_1,w_2)\cup A(w_2,w_1), \\ C_2 &= S(w_3,w_4)\cup A(w_4,w_3), \\ C_3 &= T_1\cup S(w_2,w_3)\cup T_2\cup A(x_2,y_2), \\ C_4 &= T_1\cup S(w_2,x_1)\cup B(x_1,y_2), \\ C_5 &= S(y_1,w_3)\cup T_2\cup B(x_2,y_1). \end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1.
Subcase 2.4.2: Suppose the trace of S is $0'$. This case is symmetric to Subcase 2.4.1.

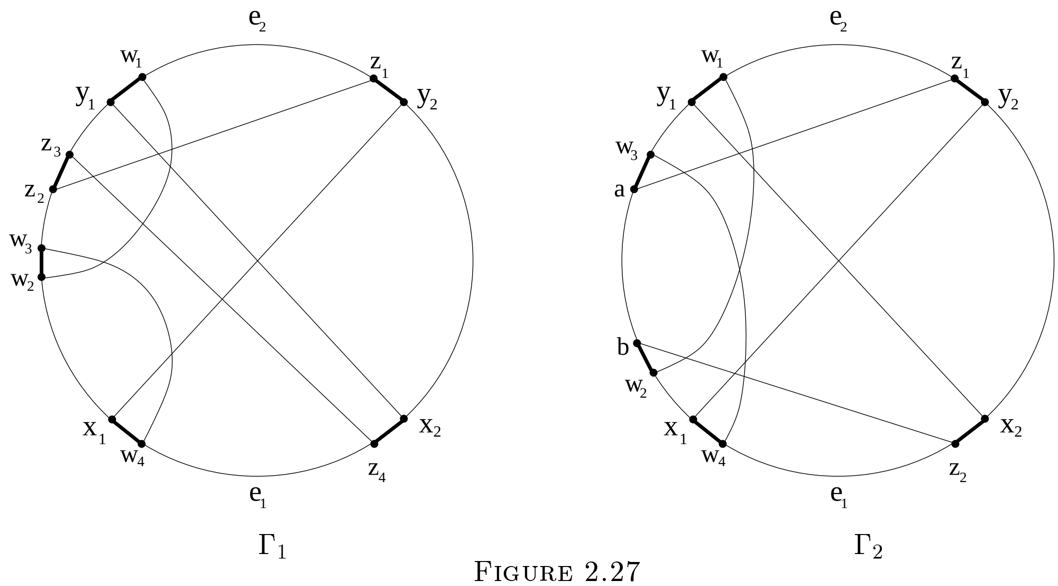


FIGURE 2.27

Subcase 2.5: The traces of T_1 and T_2 are empty, $a \in VS(w_2, w_3)$, $b \in VS(w_2, w_3)$, $b <_S a$.
Subcase 2.5.1: Suppose the trace of S is 0 (see Figure 2.27). Then G is isomorphic to Γ_2.
Subcase 2.5.2: Suppose the trace of S is $0'$. Consider the circuits

$$\begin{aligned}
C_1 &= S \cup A(x_1, y_1), \\
C_2 &= T_1 \cup A(a, y_2), \\
C_3 &= T_2 \cup A(x_2, b), \\
C_4 &= T_1 \cup S(a, x_1) \cup B(x_1, y_2), \\
C_5 &= S(y_1, b) \cup T_2 \cup B(x_2, y_1).
\end{aligned}$$

We obtain a contradiction by the method in Subcase 1.1.1 and conclude the proof of Theorem 2.2.

5. Non-reduction of Γ_1 and Γ_2 to $K_{3,3}$

We conclude this thesis by showing that neither Γ_1 nor Γ_2 is reducible to an even subdivision of $K_{3,3}$.

LEMMA 2.39. *Both Γ_1 and Γ_2 are minimal non-Pfaffian graphs.*

Proof: Let Γ_1 and Γ_2 be oriented as in Figure 2.28.

First we consider Γ_1, which we have already seen to be non-Pfaffian. Suppose therefore that Γ_1 is not minimal. Let x be an edge such that $\Gamma_1 - \{x\}$ is non-Pfaffian. The 1-factors of Γ_1 are

$$\begin{aligned}
f_1 &= \{(a,b),(c,d),(f,e),(h,g),(j,i),(l,k)\},\\
f_2 &= \{(l,a),(b,c),(d,e),(g,f),(i,h),(k,j)\},\\
f_3 &= \{(b,c),(a,d),(f,e),(h,g),(j,i),(l,k)\},\\
f_4 &= \{(a,b),(c,d),(j,e),(g,f),(i,h),(l,k)\},\\
f_5 &= \{(a,d),(b,c),(j,e),(g,f),(i,h),(l,k)\},\\
f_6 &= \{(l,a),(b,c),(d,e),(g,f),(k,h),(j,i)\},\\
f_7 &= \{(l,a),(b,g),(i,h),(k,j),(c,d),(f,e)\},\\
f_8 &= \{(l,a),(b,g),(k,h),(c,d),(f,e),(j,i)\},\\
f_9 &= \{(l,f),(d,e),(c,i),(k,j),(a,b),(h,g)\},\\
f_{10} &= \{(a,d),(b,g),(c,i),(j,e),(k,h),(l,f)\}.
\end{aligned}$$

Observe that the figure for the undirected graph Γ_1 is symmetric about the edge (l,f). Therefore we can assume that $x \notin f_1$. All 1-factors are associated with a plus sign except f_{10}. Thus $x \notin f_{10}$, for otherwise $\Gamma_1 - \{x\}$ is Pfaffian, and therefore $x \in f_2$.

Suppose that $x = (d,e)$ or $x = (k,j)$. We obtain a Pfaffian orientation of $\Gamma_1 - \{x\}$ if we change the orientation of (l,f), since every 1-factor of G that contains (l,f) also contains x except for f_{10}. If $x = (g,f)$ or $x = (i,h)$, we obtain a Pfaffian orientation of $\Gamma_1 - \{x\}$ by changing the orientation of (j,e). If $x = (l,a)$, we obtain a Pfaffian orientation of $\Gamma_1 - \{x\}$ by changing the orientation of (b,g). If $x = (b,c)$, we obtain a Pfaffian orientation of $\Gamma_1 - \{x\}$ by changing the orientation of (a,d). In all cases we have a contradiction to the fact that $\Gamma_1 - \{x\}$ was non-Pfaffian. Therefore Γ_1 is minimal non-Pfaffian.

Now suppose that the non-Pfaffian graph Γ_2 is not minimal, and let x be an edge such that $\Gamma_2 - \{x\}$ is non-Pfaffian. The 1-factors of Γ_2 are

$$\begin{aligned}
f_1 &= \{(a,b),(c,d),(e,f),(g,h),(i,j),(k,l)\},\\
f_2 &= \{(c,b),(e,d),(g,f),(i,h),(k,j),(a,l)\},\\
f_3 &= \{(a,b),(c,j),(e,d),(g,f),(i,h),(k,l)\},\\
f_4 &= \{(a,b),(c,d),(e,l),(g,f),(i,h),(k,j)\},\\
f_5 &= \{(c,b),(d,h),(k,g),(a,l),(e,f),(i,j)\},\\
f_6 &= \{(b,f),(e,d),(c,j),(a,i),(k,l),(g,h)\},\\
f_7 &= \{(b,f),(a,i),(k,j),(e,l),(c,d),(g,h)\},\\
f_8 &= \{(e,d),(b,f),(a,l),(k,g),(i,h),(c,j)\},\\
f_9 &= \{(g,f),(d,h),(c,b),(a,i),(k,j),(e,l)\},\\
f_{10} &= \{(b,f),(e,l),(k,g),(d,h),(c,j),(a,i)\}.
\end{aligned}$$

Observe that there is an automorphism of Γ_2 that interchanges the f_1-alternating circuits f_1+f_2 and f_1+f_{10}. Therefore we can assume that $x \in f_1+f_{10}$. The figure for the undirected graph Γ_2 is symmetric about the line through the midpoints of the edges (a,b) and (g,h). Therefore we can assume that

$$x \in (f_1+f_{10}) - \{(b,f),(k,g),(c,j)\} = \\ \{(a,b),(c,d),(e,f),(g,h),(i,j),(k,l),(e,l),(d,h),(a,i)\}.$$

If $x \in f_1$ then the given orientation is a Pfaffian orientation of $\Gamma_2 - \{x\}$, since the sign of f_1 is the opposite of that of the other 1-factors. Therefore $x \in \{(e,l),(d,h),(a,i)\}$. If $x = (e,l)$, we obtain a Pfaffian orientation of $\Gamma_2 - \{x\}$ by changing the orientation of (c,d), since every

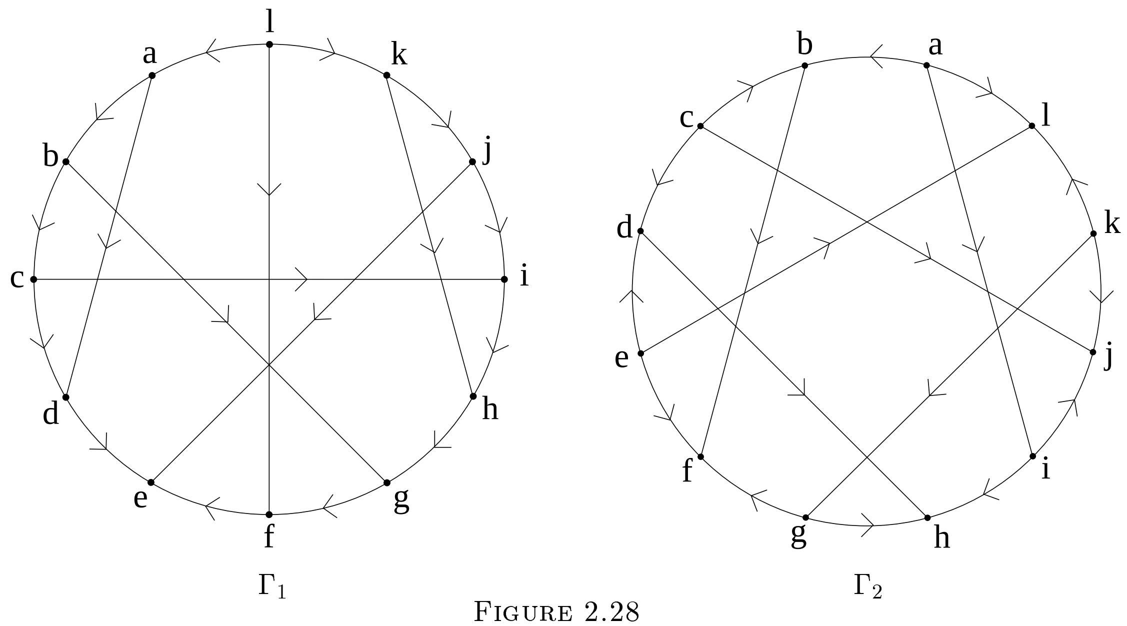


FIGURE 2.28

1-factor that contains (c, d) also contains x except for f_1. If $x = (d, h)$, we obtain a Pfaffian orientation of $\Gamma_2 - \{x\}$ by changing the orientation of (e, f). If $x = (a, i)$, we obtain a Pfaffian orientation of $\Gamma_2 - \{x\}$ by changing the orientation of (g, h). In all case we have a contradiction to the fact that $\Gamma_2 - \{x\}$ was Pfaffian. Therefore Γ_2 is minimal non-Pfaffian.

COROLLARY 2.3. *Neither Γ_1 nor Γ_2 contains an even subdivision of $K_{3,3}$.*

Proof: If Γ_1 or Γ_2 contained an even subdivision of $K_{3,3}$ then Γ_1 or Γ_2 itself would be an even subdivision of $K_{3,3}$, since Γ_1 and Γ_2 are minimal non-Pfaffian and every even subdivision of $K_{3,3}$ is non-Pfaffian. But Γ_1 and Γ_2 both have 12 vertices of degree 3 whereas an even subdivision of $K_{3,3}$ has only 6. □

In order to see that Γ_1 and Γ_2 are not reducible to an even subdivision of $K_{3,3}$, we need the following two lemmas.

LEMMA 2.40. *Let G be a minimal non-Pfaffian graph and C a circuit of odd length in G. Suppose that the graph G^C obtained by contracting VC to a vertex v is also non-Pfaffian. Let $VC = \{v_1, v_2, \ldots, v_n\}$. Then*

$$\deg v = \sum_{i=1}^{n} \deg v_i - 2n.$$

Moreover if w is a vertex in $VG - VC$ then w is also a vertex of G^C, and

$$\deg_G w = \deg_{G^C} w.$$

Proof: Let $N(u)$ denote the set of vertices in G that are adjacent to the vertex u in G. Moreover we assume that v_i is adjacent to v_{i+1} in C, for all $i < n$. We define $v_0 = v_n$ and $v_{n+1} = v_1$.

First we observe the following. Let $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. We claim that

$$N(v_i) \cap N(v_j) \subseteq \{v_{i-1}, v_{i+1}\}.$$

Indeed, suppose $u \in N(v_i) \cap N(v_j)$. Then there are edges e_i and e_j joining u to v_i and v_j respectively. If $u \notin VC$, then the graph obtained from $G - \{e_i\}$ by contracting VC is G^C.

Since G^C is supposed to be non-Pfaffian, it follows from Lemma 2.17 that $G - \{e_i\}$ is non-Pfaffian too. This is a contradiction to the fact that G was minimal non-Pfaffian. Therefore $u \in VC$. If $e_i \notin C$ or $e_j \notin C$ we can conclude in a similar way that either $G - \{e_i\}$ or $G - \{e_j\}$ is non-Pfaffian, and therefore have a contradiction again. Thus $\{e_i, e_j\} \subseteq C$, and the claim is proved. Thus C has no chords, and any vertex not in C has at most one neighbour in C.

We have

$$N(v) = \bigcup_{i=1}^{n} N(v_i) - VC = \bigcup_{i=1}^{n} (N(v_i) - \{v_{i-1}, v_{i+1}\}),$$

and if we apply the observation this gives

$$\deg v = |N(v)| = \sum_{i=1}^{n} |N(v_i) - \{v_{i-1}, v_{i+1}\}| = \sum_{i=1}^{n} (|N(v_i)| - 2) = \sum_{i=1}^{n} \deg v_i - 2n.$$

Finally the degree of a vertex w that is not in C does not change upon contraction of C, since w is adjacent to at most one vertex in C. $\square$

LEMMA 2.41. *Let G be a minimal non-Pfaffian graph that is cubic and does not contain a circuit of length* 3. *Then G is not reducible to an even subdivision of $K_{3,3}$.*

Proof: Suppose the contrary, that is that there exists a sequence $C_0, C_1, \ldots, C_{p-1}$ of circuits and a sequence $G_0, G_1, \ldots, G_p$ of graphs, such that $G_0 = G$, G_p is an even subdivision of $K_{3,3}$ and, for all $i < p$, G_{i+1} is obtained from G_i by contracting VC_i. Note that for all i the graph G_i is minimal non-Pfaffian by Lemma 2.17, since G_p is non-Pfaffian.

First we show that G_1 contains a vertex of degree at least 5. Since G_0 does not contain a circuit of length 3, the circuit C_0 must have length at least 5. Let $VC_0 = \{v_1, v_2, \ldots, v_n\}$ and let v be the corresponding vertex in G_1. Then, by Lemma 2.40,

$$\deg v = \sum_{i=1}^{n} \deg v_i - 2n = 3n - 2n = n \geq 5.$$

Again by Lemma 2.40 all the other vertices in G_1 have degree 3, since G_0 is cubic.

Now we show by induction that all the vertices in G_i, where $i \geq 1$, are of degree at least 3 and that there exists a vertex in G_i with degree at least 5. Therefore let us assume that the induction hypothesis is true for G_i and show it for G_{i+1}. First we show that every vertex in G_{i+1} is of degree at least 3. For all vertices in G_{i+1} except the one that corresponds to C_i this is an immediate consequence of Lemma 2.40 and the induction hypothesis. Let w be the vertex in G_{i+1} that corresponds to C_i and let $VC_i = \{w_1, w_2, \ldots, w_m\}$. Then, by Lemma 2.40 and the induction hypothesis,

$$\deg w = \sum_{i=1}^{m} \deg w_i - 2m \geq 3m - 2m = m \geq 3.$$

Now let u be the vertex of degree at least 5 in G_i. If $u \notin VC_i$ then $u \in VG_{i+1}$ and $\deg_{G_{i+1}} u = \deg_{G_i} u$, by Lemma 2.40. Therefore suppose that $u \in VC_i$, and without loss of generality assume that $u = w_1$. Then, by Lemma 2.40 and the induction hypothesis,

$$\deg w = \deg u + \sum_{i=2}^{m} \deg w_i - 2m \geq \deg u + 3(m-1) - 2m = \deg u + m - 3 \geq \deg u.$$

Therefore G_p contains a vertex of degree at least 5. This is a contradiction, since G_p was an even subdivision of $K_{3,3}$. $\square$

COROLLARY 2.4. *Neither Γ_1 nor Γ_2 is reducible to an even subdivison of $K_{3,3}$.*

Proof: This is an immediate consequence of Lemma 2.41 since both Γ_1 and Γ_2 are minimal non-Pfaffian, cubic and do not contain a circuit of length 3. $\square$

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