COUNTING INTEGER POINTS IN POLYTOPES ASSOCIATED WITH DIRECTED GRAPHS

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Abstract. We are interested in enumerating the integer points in certain polytopes that are naturally associated with directed graphs. These polytopes generalize Stanley's order polytopes and also (P,ω)-partitions. A classical result states that the number of integer points in any given rational polytope can be expressed by a formula that is piecewise a quasipolynomial in certain parameters of the polytope, and, remarkably, the domains of validity of the involved quasipolynomials overlap. In the case of our special polytopes, the quasipolynomials are shown to be polynomials. We investigate the domains of validity of these polynomials and demonstrate how the overlaps can be used to explore the zero set of the polynomials. We have a closer look at the counting of Gelfand-Tsetlin patterns, which can be phrased as the counting of integer points in a polytope associated with a particular directed graph. We conjecture that the zeros that can be deduced by studying the overlaps essentially determine the enumeration formula in this case.

Keywords: integer points in polytopes; order polytopes; Ehrhart theory; Gelfand-Tsetlin patterns
MSC: 05A

1. Directed graph polytope

In this article we address the problem of counting the integer points in the following (half-open) polyhedrons.

Definition 1 (Directed graph polyhedron)**.** *Let $D=(V,E,v_0)$ be a finite directed graph with vertex set V, edge set E and a root vertex $v_0 \in V$. Fix two disjoint subsets E_P and E_N of E and let $x: E \cup \{v_0\} \to \mathbb{Z}$ be a function. The directed graph polyhedron of D with respect to E_P, E_N, x is the set of functions $z: V \to \mathbb{R}$ with the following properties:*

(1) $z(v_0)=x(v_0)$
(2) $\forall\, e=(v,w) \in E_P$: $z(w)-z(v) \le x(e)$
(3) $\forall\, e=(v,w) \in E_N$: $z(w)-z(v) > x(e)$
(4) $\forall\, e=(v,w) \in E \setminus (E_P \cup E_N)$: $z(w)-z(v) = x(e)$

We denote this halfopen polyhedron by $\mathcal{P}_{D,E_P,E_N}(x)$ and the number of its integer points by $\mathrm{N}_{D,E_P,E_N}(x)$.

The author acknowledges support by the Austrian Science Foundation FWF, START grant Y463.

One motivation to study $\mathcal{P}_{D,E_P,E_N}(x)$ stems from the fact that it is a multivariate generalization of Stanley's *order polytope* [6] and that $\mathrm{N}_{D,E_P,E_N}(x)$ is a generalization of the *order polynomial*. Recall that the order polytope is the set of all order preserving maps from a given poset into the interval $[0,1]$. Concerning the order polynomial, consider the number of order preserving maps from the poset into the interval $\{1,2,\ldots,t\}$ which is easily seen to be polynomial function in t (the so-called order polynomial). If we perform the following four steps, we obtain the order polytope from $\mathcal{P}_{D,E_P,E_N}(x)$.

(1) To construct D, direct the edges in the Hasse diagram of the poset from the larger to the smaller element.
(2) Add a vertex v_0 and, for each maximal element and for each minimal element of the poset, we introduce an edge that connects it to v_0. For the maximal elements, v_0 is the tail, otherwise it is the head.
(3) We set $E_P = E$ and $E_N = \emptyset$.
(4) We set $x(v_0) = 0$ and $x(e) = 0$, except for the edges e that connect v_0 to a maximal element, where we have $x(e) = 1$.

Similarly, the order polynomial can be obtained from $\mathrm{N}_{D,E_P,E_N}(x)$. It suffices to modify x: we set $x(v_0) = 0$ and $x(e) = 0$ if v_0 is no endpoint of e, $x(e) = t$ if v_0 is the tail of e and $x(e) = -1$ if v_0 is the head of e. Moreover, since in our definition it is possible to have strict and non-strict inequalities, it also includes (P,ω)-partitions. (A different multivariate generalization of order polytopes – so-called marked order polytopes – were recently introduced in [1].)

The following proposition characterizes the triples (D, E_P, E_N) that satisfy $\mathrm{N}_{D,E_P,E_N}(x) < \infty$ for all x. For a set of edges F, let $\overline{F}$ denote the set of edges we obtain by reversing the direction of the edges. We define a directed graph $D(E_P, E_N)$ as follows.

$$D(E_P,E_N) = (V, E_P \cup \overline{E_N} \cup (E \setminus (E_P \cup E_N)) \cup \overline{(E \setminus (E_P \cup E_N))})$$

Proposition 1. *We have* $\mathrm{N}_{D,E_P,E_N}(x) < \infty$ *for all* x *if and only if* $D(E_P,E_N)$ *is strongly connected. Otherwise it holds* $\mathrm{N}_{D,E_P,E_N}(x) \in \{0,\infty\}$ *for all* x.

Proof. To see that $x(v)$ is bounded from above if $D(E_P,E_N)$ is strongly connected, consider the directed path Q from v_0 to v. Add up all inequalities, respectively equalities, corresponding to edges of $Q \cap (E \setminus E_N)$ and add the negatives of all inequalities corresponding to edges of $Q \cap E_N$. Similarly, to see that $x(v)$ is bounded from below, consider the directed path from v to v_0, add up all inequalities, respectively equalities, corresponding to edges of $Q \cap (E \setminus E_P)$ and add the negatives of all inequalities corresponding to edges of $Q \cap E_P$.

Suppose $D(E_P,E_N)$ is not strongly connected. Assume that there is a vertex v such that there is no directed path from v_0 to v. (The proof is similar if there is no directed path from v to v_0.) Let $W \subseteq V$ be the set of vertices w such that $D(E_P,E_N)$ contains a directed path from w to v. Choose $z : V \to \mathbb{Z}$ arbitrarily and let $x : E \cup \{v_0\} \to \mathbb{Z}$ such that $z \in \mathcal{P}_{D,E_P,E_N}(x)$. For any $c \in \mathbb{Z}$, define $z_c : V \to \mathbb{Z}$ as follows.

$$z_c(w) = \begin{cases} z(w) + c & w \in W, \\ z(w) & w \in V \setminus W. \end{cases}$$

Then $z_c \in \mathcal{P}_{D,E_P,E_N}(x)$ for all $c \geq 0$ as there is no edge of $E \setminus E_N$, resp. $E \setminus E_P$, in the cut $(W, V \setminus W)$ that has the head, resp. tail, in W. □

In the following, we consider only triples (D, E_P, E_N) such that $D(E_P, E_N)$ is strongly connected.

The purpose of this paper is to demonstrate how some general facts on the integer point enumeration in rational polytopes can be used to draw conclusions on the counting functions $\mathrm{N}_{D,E_P,E_N}(x)$. The paper is organized as follows: In Section 2, we provide the general facts that we apply in the following. In Section 3, we translate these general facts into our special setting of directed graph polytopes. In Section 4, we consider a specific directed graph that is related to Gelfand-Tsetlin patterns and show how the results from Section 2 can be used to determine many zeros of the (well-known) formula for the number of Gelfand-Tsetlin patterns with fixed bottom row. More concretely, we show that the number of Gelfand-Tsetlin patterns with bottom row $k_1, k_2, \dots, k_n$ is expressible by a polynomial in the k_i's of total degree no greater than $\binom{n}{2}$ that vanishes on

$$\bigcup_{1 \le i < j \le n} \{(k_1, \dots, k_n)\mathbb{Z}^n | \forall (p, q) \text{ with } i \le p < q \le j : |k_q + q - k_p - p| < j - i\}.$$

We conjecture that these zeros determine the following formula for the number of Gelfand-Tsetlin patterns with bottom row $k_1, \dots, k_n$ up to a constant (see Conjecture 1).

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$$

It is easy to see that the constant is 1, since it is obvious from the definition that the number of Gelfand-Tsetlin patterns with bottom row $(0, \dots, 0)$ is 1.

2. Useful facts on the integer point enumeration in rational polytopes

Given an $n \times d$ integer matrix A, two disjoint subsets $I_P, I_N \subseteq [n] := \{1, 2, \dots, n\}$ and an integer vector $x \in \mathbb{Z}^n$, we denote by $\mathrm{N}_{A,I_P,I_N}(x)$ the number of integer points $z \in \mathbb{Z}^d$ that satisfy the following conditions.

$$\begin{aligned} (A \cdot z)_i &\le x_i, & \forall i \in I_P \\ (A \cdot z)_i &> x_i, & \forall i \in I_N \\ (A \cdot z)_i &= x_i, & \forall i \in [n] \setminus (I_P \cup I_N) \end{aligned} \tag{2.1}$$

If there exists an x_0 with $\mathrm{N}_{A,I_P,I_N}(x_0) = \infty$, then $\mathrm{N}_{A,I_P,I_N}(x) \in \{0, \infty\}$ for all x; in the following we assume that $\mathrm{N}_{A,I_P,I_N}(x)$ is bounded. Letting $\langle A \rangle := \{A \cdot z | z \in \mathbb{R}^d\}$, it is obvious that the function $x \to \mathrm{N}_{A,I_P,I_N}(x)$ vanishes outside the cone

$$\mathcal{C}_{A,I_P,I_N} := \langle A \rangle + \sum_{i \in I_P} \mathbb{R}_{\ge 0}\, \mathbf{e}_i + \sum_{i \in I_N} \mathbb{R}_{\le 0}\, \mathbf{e}_i,$$

where $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ denotes the standard basis of $\mathbb{R}^n$. The boundedness of $\mathrm{N}_{A,I_P,I_N}(x)$ implies the linear independence of the columns of A and thus we have $n \ge d$. In the following we let $H_0 := \langle A \rangle + \sum_{i \in I_P \cup I_N} \mathbb{R}\, \mathbf{e}_i$ denote the minimal subspace containing the cone $\mathcal{C}_{A,I_P,I_N}$.

Definition 2 (Walls, hyperplanes, chambers and cells of (A, I_P, I_N))**.** *Let $J \subseteq I_P \cup I_N$ be maximal such that the vectors $(\mathbf{e}_i)_{i \in J}$ together with $\langle A \rangle$ span a hyperplane in H_0. Then the* *cone*

$$\mathrm{W}_{A,J} := \langle A \rangle + \sum_{j \in J \cap I_P} \mathbb{R}_{\geq 0}\, \mathbf{e}_j + \sum_{j \in J \cap I_N} \mathbb{R}_{\leq 0}\, \mathbf{e}_j$$

is said to be a wall *of* (A, I_P, I_N)*;* $\mathrm{H}_{A,J}$ *denotes the hyperplane containing* $\mathrm{W}_{A,J}$ *and is said to be a* hyperplane *of* (A, I_P, I_N)*. The connected components of the complement of the union of all walls in* H_0 *are said to be the* chambers *of* (A, I_P, I_N)*, while the connected components of the complement of the union of all hyperplanes in* H_0 *are said to be the* cells*.*

Definition 3. *A quasipolynomial is an expression of the form*

$$\sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} c_{i_1,i_2,\ldots,i_n}(x_1,\ldots,x_n)\, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where the coefficients $c_{i_1,i_2,\ldots,i_n} : \mathbb{Z}^n \to \mathbb{R}$ *are functions that are periodic in each component and almost all of them vanish.*

According to a result of McMullen [5], the counting function $\mathrm{N}_{A,I_P,I_N}(x)$ can be represented by a quasipolynomial if we restrict x to an open cone C such that, for all $x \in C$, the underlying polytope has the same combinatorial structure. The following theorem is a reformulation of this fact. As a matter of fact, the domains of validity of the quasipolynomials can be enlarged slightly and this causes overlappings of the domains of validity. To describe the extended cones C^{ext}, suppose $\mathbf{h}_1, \ldots, \mathbf{h}_m$ are normal vectors of the hyperplanes of (A, I_P, I_N) such that

$$C = \bigcap_{j=1}^{m} \{\mathbf{x} \in \mathbb{R}^n | \langle \mathbf{h}_j, \mathbf{x} \rangle > 0\}.$$

Then we define

$$C^{\mathrm{ext}} = \bigcap_{j=1}^{m} \{\mathbf{x} \in \mathbb{R}^n | \langle \mathbf{h}_j, \mathbf{x} \rangle > -\sum_{i \in I_P \cup I_N} \max(0, \mathbf{h}_j)_i)\}.$$

The fact that the domains of validity can be extended follows for instance from an analogous result on vector partition functions which probably first appeared in the work of Dahmen and Micchelli [2].

Theorem 1. *For each chamber* C *of* (A, I_P, I_N)*, the function* $x \to \mathrm{N}_{A,I_P,I_N}(x)$ *is a quasipolynomial function on* $C^{\mathrm{ext}} \cap \mathbb{Z}^n$ *of degree no greater than* $d + |I_P| + |I_N| - \dim \mathcal{C}_{A,I_P,I_N}$*.*

The quasipolynomials are in fact polynomials if the matrix A *is unimodular in the following sense: for any choice of* $n-d$ *rows deleted from* A*, the determinant of the remaining matrix is in* $\{0, 1, -1\}$*.*

If $I_P \cup I_N = [n]$, then the unimodularity of A is equivalent to the fact that the underlying polytope is integral for all x. Also note that the unimodularity of A implies that, for each hyperplane of (A, I_P, I_N), there exists a normal vector in $\{0, 1, -1\}^n$. (This property is equivalent to the unimodularity if $I_P \cup I_N = [n]$.)

The following corollary is now an easy consequence of the overlappings of the domains of validity. We say that two different chambers (resp. cells) are *adjacent* via the hyperplane H of (A, I_P, I_N) if they lie on the same side of all hyperplanes of (A, I_P, I_N) except for H.

Corollary 1. *Suppose A is unimodular. Let R_1, R_2 be two chambers of (A, I_P, I_N) that are adjacent via the hyperplane H of (A, I_P, I_N) and let $\mathbf{h}$ be a normal vector of H in $\{0, 1, -1\}^n$. Set*

$$a_1 := \#\{i \in I_P \cup I_N : \mathbf{h}_i = 1\}$$
$$a_2 := \#\{i \in I_P \cup I_N : \mathbf{h}_i = -1\}$$

and let $P_{R_1}(x), P_{R_2}(x)$ denote the enumeration polynomials that represent $\mathrm{N}_{A,I_P,I_N}(x)$ on R_1, R_2, respectively. Then the difference $P_{R_1}(x) - P_{R_2}(x)$ vanishes for all $x \in H_0$ with $\langle \mathbf{h}, x \rangle \in \{-a_1 + 1, -a_1 + 2, \ldots, a_2 - 1\}$.

Since the decomposition into cells refines the decomposition into chambers, Theorem 1 and Corollary 1 are also true if we replace chambers by cells. In the following we will work with cells.

An important point of this paper is to show how Corollary 1 can be used to find zeros of polynomial enumeration formulas of counting problems that can be phrased as the counting of integer points in certain integer polytopes. For instance, if a cell $Z \subseteq \mathcal{C}_{A,I_P,I_N}$ is adjacent to a cell outside of $\mathcal{C}_{A,I_P,I_N}$, then the corollary implies that the enumeration polynomial on Z has a number of linear factors of the form $\mathbf{h}_1 x_1 + \ldots + \mathbf{h}_n x_n + c$ (unless there is a unique $i \in I_P \cup I_N$ with $\mathbf{h}_i \neq 0$), where $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is a normal vector of the separating hyperplane and c ranges in some integer interval. I believe that this is the reason for many linear factors that appear in polynomial enumeration formulas. If the cell Z is located deeper inside the cone $\mathcal{C}_{A,I_P,I_N}$, then, by constructing a chain $Z = Z_1, \ldots, Z_{m+1}$ of cells such that Z_r and Z_{r+1} are adjacent and Z_{m+1} is outside of $\mathcal{C}_{A,I_P,I_N}$, the corollary can still be used to show that the enumeration polynomial on Z vanishes on certain affine subspaces of co-dimension no greater than m. For a non-trivial special case, this is illustrated in Lemma 1 and Proposition 7.

In order to be able to apply this, one needs to have some understanding of the hyperplanes of (A, I_P, I_N) and the decomposition into cells. In the following section, we show that, in case of the directed graph polytope, hyperplanes can be identified with particular circuits of the directed graph D and the decomposition into cells can be studied with the help of the vector space of flows on D. In our subsequent application to a polytope related to Gelfand-Tsetlin patterns, we demonstrate typical arguments that are useful in this setting.

Speaking of linear factors of polynomial enumeration formulas, we draw the reader's attention to a beautiful conjecture by Fonseca and Nadeau [3] on linear factors of an enumeration formula related to fully packed loop configurations. It would be interesting to explore whether our approach provides (unified) proofs of phenomena of this type frequently occurring, for instance, in connection with the enumeration of plane partitions, tilings, alternating sign matrices and related objects.

3. Hyperplanes and cells of (D, E_P, E_N)

Let $\mathcal{A}_D$ denote the following slight variant of the transpose of the incidence matrix of D: the rows are indexed by $E \cup \{v_0\}$ and the columns by V; the column of $v \in V$ is the following function $c_v : E \cup \{v_0\} \to \mathbb{R}$: if e is an edge then

$$c_v(e) = \begin{cases} 1 & \text{if } v \text{ is the head of edge } e \\ -1 & \text{if } v \text{ is the tail of edge } e \\ 0 & \text{otherwise} \end{cases}$$

and $c_v(v_0) = \delta_{v_0,v}$. Note that

$$\begin{aligned} \mathcal{P}_{D,E_P,E_N}(x) = \{z \in \mathbb{R}^V | z(v_0) = x(v_0), \\ (\mathcal{A}_D \cdot z)_e \leq x(e) \ \forall e \in E_P, \\ (\mathcal{A}_D \cdot z)_e > x(e) \ \forall e \in E_N, \\ (\mathcal{A}_D \cdot z)_e = x(e) \ \forall e \in E \setminus (E_P \cup E_N)\} \end{aligned}$$

and $\mathrm{N}_{\mathcal{A}_D,E_P,E_N}(x) = \mathrm{N}_{D,E_P,E_N}(x)$. We let $\mathcal{C}_{\mathcal{A}_D,E_P,E_N} =: \mathcal{C}_{D,E_P,E_N}$.

Next we aim at determining the hyperplanes of $(\mathcal{A}_D, E_P, E_N)$. We interpret the functions from $E \cup \{v_0\}$ to $\mathbb{R}$ as the real vector space $\mathbb{R}^{E\cup\{v_0\}}$ with the standard inner product

$$\langle f, g \rangle = \sum_{e \in E} f(e)g(e) + f(v_0)g(v_0).$$

As usual, an element of $f \in \mathbb{R}^E$ is said to be a *flow* of D if

$$\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$$

for all $v \in V$. A function $f \in \mathbb{R}^{E\cup\{v_0\}}$ is orthogonal to each column of $\mathcal{A}_D$ if and only if it satisfies the flow condition and $f(v_0) = 0$: This is because satisfying the flow condition for $v \in V \setminus \{v_0\}$ is equivalent with being orthogonal to the column corresponding to v. However, the following calculation shows that if f satisfies the flow condition for $v \in V \setminus \{v_0\}$, then it satisfies this condition also at v_0.

$$\begin{aligned} 0 &= \sum_{u\in V, v\in V:(u,v)\in E} f(u,v) - \sum_{v\in V, w\in V:(v,w)\in E} f(v,w) \\ &= \sum_{v\in V} \left(\sum_{u:(u,v)\in E} f(u,v) - \sum_{w:(v,w)\in E} f(v,w) \right) = \sum_{u:(u,v_0)\in E} f(u,v_0) - \sum_{w:(v_0,w)\in E} f(v_0,w) \end{aligned}$$

Thus, under the assumption that f satisfies the flow condition for all $v \in V \setminus \{v_0\}$, f is orthogonal to the column corresponding to v_0 if and only if $f(v_0) = 0$. Hence, the orthogonal complement of the space spanned by the columns of $\mathcal{A}_D$ can be identified with the flows of D.

The support $\{e \in E | f(e) \neq 0\} =: \mathrm{Supp}(f)$ of a flow f induces a subgraph P_f of D. Non-zero flows with minimal support are said to be *simple*; they are characterized by the fact that P_f are circuits (not necessarily directed and no repetitions of vertices allowed). Such flows are determined by P_f and a non-zero real constant determining the flow on P_f. Simple flows generate the vector space of all flows. To be more precise, we have the following.

Proposition 2. *For each non-zero flow $f : E \to \mathbb{R}$, there exist simple flows $(f_i)_{1\leq i\leq m}$ with $P_{f_i} \subseteq P_f$ and $f = \sum_{i=1}^{m} f_i$.*

Proof. Induction on the number of edges of P_f. There is nothing to prove if f is simple. Otherwise choose a simple flow f_1 with $P_{f_1} \subseteq P_f$ and $f_1(e) = f(e)$ for at least one edge in P_{f_1}. Apply the induction hypothesis to $f - f_1$. □

For $e \in E$, let $\mathbf{1}_e \in \mathbb{R}^{E \cup \{v_0\}}$ with $\mathbf{1}_e(f) = \delta_{e,f}$. We say that a hyperplane in

$$\langle \mathcal{A}_D \rangle + \sum_{e \in E_P \cup E_N} \mathbb{R} \cdot \mathbf{1}_e =: \langle \mathcal{A}_D \rangle_{E_P \cup E_N}$$

is a hyperplane of (D, E_P, E_N), if it is generated by $\langle \mathcal{A}_D \rangle$ and some $\mathbf{1}_e$, $e \in E_P \cup E_N$. Note that these are precisely the hyperplanes of $(\mathcal{A}_D, E_P, E_N)$. For each hyperplane F of $\langle \mathcal{A}_D \rangle_{E_P \cup E_N}$, there exists a normal vector f with

$$F = \{x \in \langle \mathcal{A}_D \rangle_{E_P \cup E_N} | \langle f, x \rangle = 0\}.$$

If $\langle \mathcal{A}_D \rangle \subseteq F$, then this implies that f is a flow and $P_f \cap (E_P \cup E_N) \neq \emptyset$.

Proposition 3. *The hyperplanes of (D, E_P, E_N) are the orthogonal complements of simple flows f of D such that $P_f \cap (E_P \cup E_N)$ is non-empty but minimal, i.e. there exists no simple flow g of D with $\emptyset \neq P_g \cap (E_P \cup E_N) \subsetneq P_f \cap (E_P \cup E_N)$.*

A hyperplane of (D, E_P, E_N) induces a facet of $\mathcal{C}_{D,E_P,E_N}$ if, for a normal vector f of the hyperplane, the graph P_f is a directed circuit if we reorient all edges of $P_f \cap E_N$ and possibly some edges of $P_f \cap (E \setminus (E_P \cup E_N))$.

Proof. Suppose F is a hyperplane of (D, E_P, E_N) and f is a normal vector of F. This implies automatically that f is a flow and $P_f \cap (E_P \cup E_N) \neq \emptyset$. Note that any other flow g is the normal vector of the same hyperplane if and only if $P_f \cap (E_P \cup E_N) = P_g \cap (E_P \cup E_N)$.

We use Proposition 2 to write f as a sum of simple flows $(f_i)_{1 \leq i \leq m}$ such that $P_{f_i} \subseteq P_f$ for all i. We may exclude the f_i's with $P_{f_i} \cap (E_P \cup E_N) = \emptyset$ from the sum and still obtain a normal vector of the same hyperplane. After this modification,

$$P_{f_i} \cap (E_P \cup E_N) = P_f \cap (E_P \cup E_N)$$

for all $i \in \{1, 2, \ldots, m\}$, because otherwise f_i is the normal vector of a hyperplane which lies strictly between F and $\langle \mathcal{A}_D \rangle_{E_P \cup E_N}$ and this is impossible for reasons of dimension. Thus each f_i is a normal vector of F that is also a simple flow. If there existed a simple flow g with $\emptyset \neq P_g \cap (E_P \cup E_N) \subsetneq P_f \cap (E_P \cup E_N)$ then again the orthogonal complement of g in $\langle \mathcal{A}_D \rangle_{E_P \cup E_N}$ would have to lie strictly between F and $\langle \mathcal{A}_D \rangle_{E_P \cup E_N}$.

Conversely, let f be a simple flow such that $P_f \cap (E_P \cup E_N)$ is non-empty but minimal and F_0 be a hyperplane of (D, E_P, E_N) that contains all $\mathbf{1}_e$, $e \in E_P \cup E_N$, with $\langle f, \mathbf{1}_e \rangle = f(e) = 0$. Let f_0 be a simple flow such that F_0 is the orthogonal complement of f_0. Hence $\emptyset \neq P_{f_0} \cap (E_P \cup E_N) \subseteq P_f \cap (E_P \cup E_N)$ and thus $P_{f_0} \cap (E_P \cup E_N) = P_f \cap (E_P \cup E_N)$. This implies that the orthogonal complements of f and f_0 in $\langle \mathcal{A}_D \rangle_{E_P \cup E_N}$ coincide.

For the second assertion, observe that the normal vectors $\mathbf{h}$ that constitute a hyperplane of $\mathcal{C}_{A,I_P,I_N}$ are characterized by the fact that two non-vanishing coordinates $\mathbf{h}_i$ and $\mathbf{h}_j$ have the same sign if and only if $i, j \in I_P$ or $i, j \in I_N$. Moreover note that in a simple flow f, the value of f on two different edges $e_1, e_2 \in P_f$ have the same sign if and only if they point in the same direction when running through the circuit in a fixed direction. □

The fact that each simple flow can be multiplied by a real constant such that its values on the edges are in $\{0, 1, -1\}$ implies that $\mathcal{A}_D$ is unimodular in the sense of Theorem 1. This establishes the following.

Corollary 2. *The quasipolynomials representing* $\mathrm{N}_{D,E_P,E_N}(x)$ *are in fact polynomials.*

Motivated by Proposition 3, we call a circuit c *admissible* if $c \cap (E_P \cup E_N)$ is non-empty but minimal. An *oriented circuit* is an admissible circuit together with one of the two possible orientations. Such an oriented circuit is uniquely determined by the flow $c : E \to \{0, 1, -1\}$, where $c(e) = 1$ if e is an edge of the circuit and its direction coincides with the chosen orientation, $c(e) = -1$ if e is an edge of the circuit and it is directed opposed to the chosen orientation, and $c(e) = 0$ otherwise; we identify the oriented circuits with these flows.

Since this is useful in the following, we assume that $\mathcal{C}_{D,E_P,E_N}$ is fulldimensional from now on.

A cell $Z \subseteq \mathbb{R}^{E \cup \{v_0\}}$ of (D, E_P, E_N)[1] is determined by a (unique) set of oriented circuits $c_1, c_2, \ldots, c_m$: the circuits constitute the facets of Z and Z lies on the "positive" side of each c_i, i.e.

- $Z = \{x \in \mathbb{R}^{E \cup \{v_0\}} | \langle c_i, x \rangle > 0, i = 1, 2, \ldots, m\}$, and
- any proper subset of $\{c_1, c_2, \ldots, c_m\}$ induces a larger subset of $\mathbb{R}^{E \cup \{v_0\}}$.

We address it in the following as the *set of oriented circuits associated with* Z, denoted by $C(Z)$.

Given a set of oriented circuits $c_1, \ldots, c_m$, we say that another oriented circuit c is *positively representable* by $c_1, \ldots, c_m$, if there exist non-negative real numbers $r_1, \ldots, r_m$ with $c = \sum_{i=1}^{m} r_i c_i$; if there exist non-positive numbers with this property, we say that c is *negatively representable.* If c is either positively or negatively representable then we say that c is *representable.* In the next proposition, we characterize the sets of oriented circuits that appear as $C(Z)$. For a set C of oriented circuits, the following subset of $\mathbb{R}^{E \cup \{v_0\}}$ is said to be the *open cone* of C.

$$\mathrm{Cone}^\circ(C) = \{x \in \mathbb{R}^{E \cup \{v_0\}} | \langle c, x \rangle > 0 \, \forall c \in C\}$$

Proposition 4. (1) *Let $C_1 \subseteq C_2$ be two sets of oriented circuits. The open cone of C_1 is non-empty iff no element $c \in C_1$ is negatively representable by elements in $C_1 \setminus \{c\}$. If this open cone is non-empty then the elements of C_1 constitute the facets of the open cone iff no element $c \in C_1$ is positively representable by elements in $C_1 \setminus \{c\}$. The open cone of C_1 is equal to the open cone of C_2 iff each element of C_2 is positively representable by elements in C_1.*

(2) *Assume $\dim \mathcal{C}_{D,E_P,E_N} = |E| + 1$ and let C be a set of oriented circuits. There exists a cell Z with $C(Z) = C$ if and only if every oriented circuit not in C is representable by elements in C, while each element $c \in C$ is not representable by elements in $C \setminus \{c\}$.*

Proof. This follows easily from the following consequence of Farkas' Lemma:

Let $h_1, \ldots, h_m, h \in \mathbb{R}^n$ such that the open cone of $h_1, h_2, \ldots, h_m$ is not empty and suppose that for all $x \in \mathbb{R}^n$ with $\langle h_i, x \rangle > 0$ for $i = 1, 2, \ldots, m$ we have $\langle h, x \rangle > 0$. Then there exist non-negative real numbers $r_1, r_2, \ldots, r_m$ with $h = r_1 h_1 + r_2 h_2 + \ldots + r_m h_m$. □

Remark 1. *Most of the time we will apply the following slight generalization of Proposition 4 (2) (the proof is the same): let Ω be a set of unoriented admissible circuits and define cells with*

[1] That is a cell of $(\mathcal{A}_D, E_P, E_N)$ in the sense of Definition 2.

respect to Ω as the connected components of the union of all hyperplanes induced by circuits in Ω. We call a set of oriented circuits an oriented subset *of Ω if the non-oriented versions of the oriented circuits lie in Ω.*

Now again every cell Z with respect to Ω is the open cone of a (unique) minimal oriented subset of Ω; we denote it by $C_\Omega(Z)$. For a subset $C_0 \subseteq \Omega$, there exists a cell Z with respect to Ω such that $C_0 = C_\Omega(Z)$ if and only if

- *every circuit in $\Omega \setminus C_0$ equipped with an orientation is representable by elements in C_0,*
- *every circuit $c \in C_0$ is not representable by elements in $C_0 \setminus \{c\}$.*

An oriented circuit c is said to be a *boundary circuit* if $c(E_P) \subseteq \{0, 1\}$ and $c(E_N) \subseteq \{0, -1\}$; they constitute the facets of $\mathcal{C}_{D,E_P,E_N}$ and they are oriented in such a way that $\mathcal{C}_{D,E_P,E_N}$ lies on their positive sides. According to Proposition 4, a cell Z lies in $\mathcal{C}_{D,E_P,E_N}$ if and only if every boundary circuit is positively representable by elements of $C_\Omega(Z)$. Using $C_\Omega(Z)$ it is also possible to identify adjacent cells.

Proposition 5. *Let Z_1, Z_2 be two cells with respect to a set Ω of admissible circuits. They are adjacent if and only if the following two conditions are fulfilled:*

- *There exists an oriented circuit $c \in C_\Omega(Z_1)$ with $-c \in C_\Omega(Z_2)$.*
- *Every circuit in Ω equipped with an orientation and different from $\pm c$ is positively representable by elements in $C_\Omega(Z_1)$ if and only if it is positively representable by elements in $C_\Omega(Z_2)$.*

4. A directed graph polytope associated with Gelfand-Tsetlin patterns

4.1. **A Gelfand-Tsetlin directed graph polytope.** In this section, we shall apply the theory to a special family of directed graphs, see Figure 1: for each positive integer n, we define a directed graph D_n as follows:

(1) The vertex set consists of all pairs (i, j) with $1 \le j \le i \le n$ and the root vertex v_0.
(2) For each i, j with $1 \le j \le i \le n-1$, there is an edge directed from $(i+1, j+1)$ to (i, j) and an edge directed from (i, j) to $(i+1, j)$.
(3) For each i, $1 \le i \le n$, there is an edge directed from v_0 to (n, i).

The set E_P consists of all edges from (2), while $E_N = \emptyset$. It is not hard to see that $\dim \mathcal{C}_{D,E_P,E_N} = |E| + 1$. In the remainder of this section, we fix a weakly increasing sequence of integers $(k_1, k_2, \ldots, k_n)$ and define $\widehat{x} : E \cup \{v_0\} \to \mathbb{R}$ as follows:

- $\widehat{x}(e) = 0$ for $e \in E_P$
- $\widehat{x}(e) = k_i$ for the edge directed from v_0 to (n, i)
- $\widehat{x}(v_0) = 0$

Then the integer points in $\mathcal{P}_{D_n,E_P,E_N}(\widehat{x})$ correspond to the Gelfand-Tsetlin patterns with bottom row $(k_1, k_2, \ldots, k_n)$: A *Gelfand-Tsetlin pattern* of order n is a triangular array of integers $(a_{i,j})_{1 \le j \le i \le n}$, often arranged as follows

$$\begin{array}{ccccccccc}
 & & & & a_{1,1} & & & & \\
 & & & a_{2,1} & & a_{2,2} & & & \\
 & & a_{3,1} & & a_{3,2} & & a_{3,3} & & \\
 & \iddots & & & & & & \ddots & \\
a_{n,1} & & \cdots & & \cdots & & \cdots & & a_{n,n}
\end{array},$$

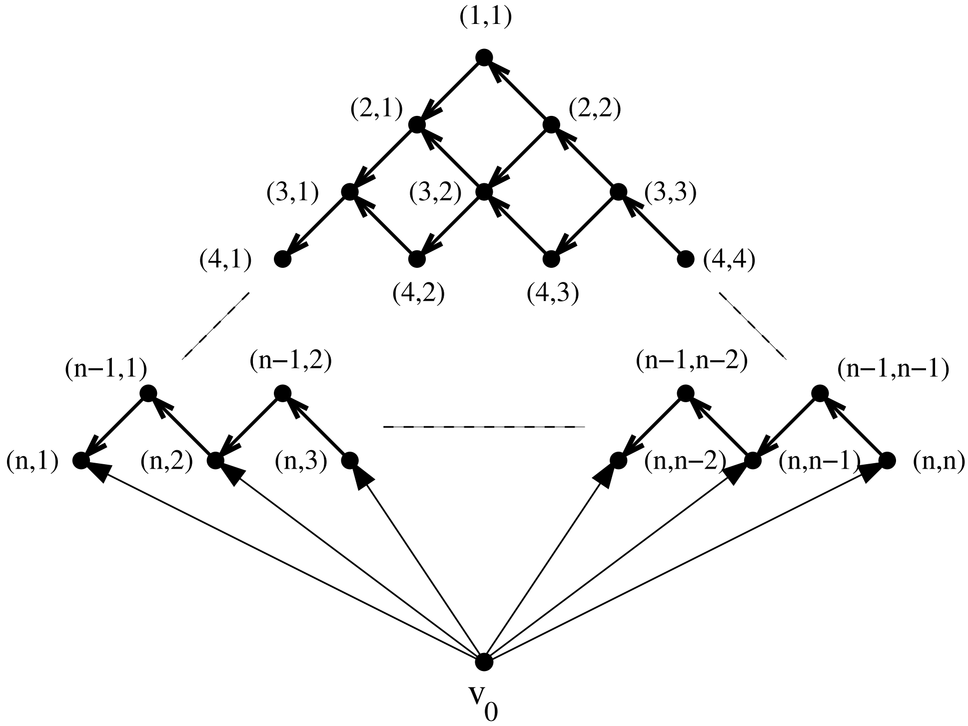

FIGURE 1.

with weak increase along North-East and South-East diagonals, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$. It is well-known that the number of the Gelfand-Tsetlin patterns with $a_{n,i} = k_i$, $i = 1, 2, \ldots, n$, is

$$\prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}, \tag{4.1}$$

if $k_1 \leq k_2 \leq \ldots \leq k_n$ (otherwise this number is obviously zero).

In [4] it was shown that Gelfand-Tsetlin patterns form bases of representations of $\mathrm{SL}(n)$; the formula then follows from the Weyl dimension formula. On the other hand, (4.1) can also be deduced from Stanley's hook-content formula [8]. In this section, we demonstrate how to use Corollary 1 to deduce a number of zeros of the formula. We conjecture that these zeros essentially determine the enumeration formula.

4.2. **Zeros.** We say that an admissible circuit of (D_n, E_P, E_N) is of *type 1* if it contains v_0; otherwise it is of *type* 2. It will turn out that type 2 circuits are of no relevance for us as $\widehat{x}$ is contained in every hyperplane induced by such a circuit. Admissible circuits of type 1 contain precisely two vertices in $\{(n,1), (n,2), \ldots, (n,n)\} =: B_n$ (this follows from the minimality condition in the definition of admissible circuits); these circuits are therefore characterized by paths in $D_n - v_0$, the two endpoints of which are in B_n and that contain no other vertex of B_n. These paths are addressed as *B-paths.* We say that a type 1 circuit is *elementary* if its B-path connects two "consecutive" vertices $(n,i), (n,i+1)$, $i = 1, 2, \ldots, n-1$, of B_n.

For each cell Z of (D_n, E_P, E_N), let $P_Z(x)$ denote the polynomial representing $\mathrm{N}_{D_n,E_P,E_N}(x)$ on Z. In the following proposition we find cells Z for which $P_Z(\widehat{z})$ is the number of Gelfand-Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$.

Proposition 6. (1) *The open cone of all counterclockwise oriented type 1 circuits is non-empty and a subset of* $\mathcal{C}_{D_n,E_P,E_N}$. *The elementary type 1 circuits constitute the facets of the open cone.*

(2) *Suppose* Z *is a cell of* (D_n, E_P, E_N) *that lies in the open cone of all counterclockwise oriented type 1 circuits. Then the evaluation* $P_Z(\widehat{z})$ *is the number of Gelfand-Tsetlin patterns with bottom row* $(k_1, \ldots, k_n)$.

Proof. re (1): We apply Proposition 4. The open cone of all counterclockwise oriented *elementary* type 1 circuits is non-empty because no counterclockwise oriented elementary type 1 circuit is negatively representable by the other counterclockwise oriented elementary type 1 circuits. Now it is clear that the open cone of *all* counterclockwise oriented type 1 circuits is non-empty since each counterclockwise oriented type 1 circuit can be written as a sum of counterclockwise oriented elementary type 1 circuits.

The open cone is contained in $\mathcal{C}_{D_n,E_P,E_N}$ because the boundary circuits are a subset of the counterclockwise oriented type 1 circuits.

Each elementary type 1 circuit is a facet since no counterclockwise oriented elementary type 1 circuit is positively representable by the other counterclockwise oriented elementary type 1 circuits.

re (2): First observe that the closure of a cell that lies in the open cone of all counterclockwise oriented type 1 circuits contains $\widehat{x}$: for any type 1 circuit c, we have $\langle c, \widehat{x} \rangle = \pm(k_j - k_i)$, where $(n, i), (n, j)$, $i < j$, are the two vertices of B_n contained in c and we have to choose the plus sign if and only if the orientation of c is counterclockwise. Moreover, $\hat{x}$ is contained in the hyperplane of each type 2 circuit.

Now Theorem 1 implies that, for each cell Z contained in $\mathcal{C}_{D_n,E_P,E_N}$, the domain of validity of the polynomial $P_Z(x)$ representing $\mathrm{N}_{D_n,E_P,E_N}(x)$ on Z includes the closure of Z: For each oriented circuit c that does not constitute a facet of $\mathcal{C}_{D_n,E_P,E_N}$, it holds $\{-1, 1\} \subseteq c(E_P)$, and, for each oriented boundary circuit c, it holds $1 \in c(E_P)$. These two facts imply that Z^{ext} contains the closure of Z. □

For any cell Z as described in the second part of the proposition, $P_Z(\widehat{x})$ is the same polynomial in the variables $(k_1, k_2, \ldots, k_n) =: \mathbf{k}$. It is denoted by $P(k_1, \ldots, k_n) = P(\mathbf{k})$. We now use Corollary 1 to deduce certain zeros of $P(\mathbf{k})$. The following lemma is fundamental.

Lemma 1. *Let* $1 \le i < j \le n$. *There exists a sequence of cells* $Z_1, Z_2, \ldots, Z_m$ *of* (D_n, E_P, E_N) *in* $\mathcal{C}_{D_n,E_P,E_N}$ *such that*

(1) Z_1 *is on the positive side of the hyperplanes of counterclockwise oriented circuits of type 1,*

(2) Z_m *has a facet that is induced by an oriented boundary circuit that contains the vertices* (n, i) *and* (n, j), *and,*

(3) *for each* r, $1 \le r < m$, Z_r *is adjacent to* Z_{r+1}, *either via a hyperplane of type 2 or via a hyperplane of type 1 such that, in the latter case, the associated admissible circuit* c *contains (see also Figure 2)*
  - *two vertices* $(n, p), (n, q)$ *with* $i \le p < q \le j$,
  - *a vertex* (s_1, t_1) *with* $t_1 \le i$, *and*
  - *a vertex* (s_2, t_2) *with* $s_2 - t_2 \le n - j$.

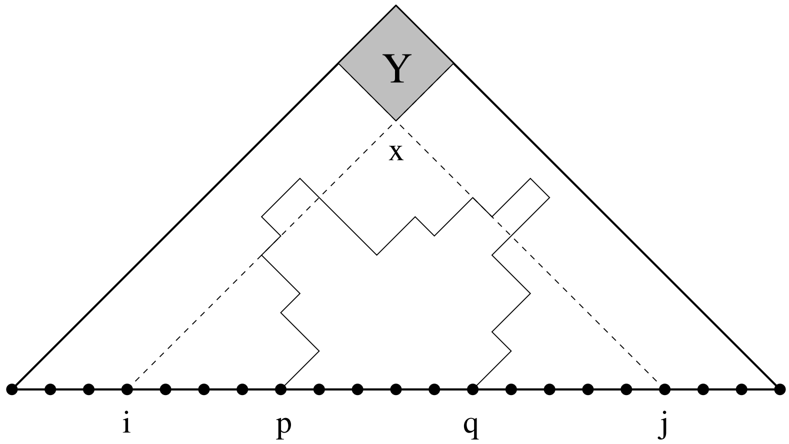

FIGURE 2. B-path of an (i,j)-feasible circuit

In the following, a type 1 circuit is said to be (i,j)-*feasible* if it has the three properties stated in (3).

In the proof of this lemma, a certain set C_0 of oriented type 1 circuits plays an important role. In order to define it, we need to consider the subset

$$Y = \{(s,t) | t \le i, s-t \le n-j\}$$

of the vertices of D_n as well as the special vertex $x = (n-j+i, i)$ of this set, see also Figure 2. Moreover, let e_1 denote the edge with endpoints x and $(n-j+i+1, i)$ and let e_2 denote the edge with endpoints x and $(n-j+i+1, i+1)$.

We define C_0 as follows: it consists of the oriented circuits whose B-paths fall into one of the following four categories.

(1) B-paths directed from (n,p) to $(n,p-1)$, for a $p = i+2, i+3, \ldots, j-1$, that contain no vertex from Y.
(2) B-paths directed from $(n,i+1)$ to (n,j) that contain x but no other vertex from Y.
(3) B-paths directed from (n,i) to $(n,j-1)$ that contain x but no other vertex from Y.
(4) B-paths directed from (n,j) to (n,i) that contain x but no other vertex from Y.

Note that B-paths as described in (4) traverse both e_1 and e_2 in accordance with their orientation, while elements as described in (2) and (3) traverse them in opposite direction. B-paths from (1) traverse neither e_1 nor e_2.

The proof of Lemma 1 is in turn based on the following lemma.

Lemma 2. *The open cone of C_0 is non-empty and each element of C_0 induces a facet.*

Proof. By Proposition 4, we need to show that no $c \in C_0$ is representable by elements in $C_0 \setminus \{c\}$. We divide this into showing that

(A) no $c \in C_0$ is *positively* representable by elements in $C_0 \setminus \{c\}$, and
(B) no $c \in C_0$ is *negatively* representable by elements in $C_0 \setminus \{c\}$.

re (A): Assume the contrary and suppose $c \in C_0$ is such a circuit. Suppose

$$c = \sum_k \alpha_k w_k + \sum_k \beta_k x_k + \sum_k \gamma_k y_k + \sum_k \delta_k z_k \tag{4.2}$$

where w_k is from (1), x_k is from (2), y_k is from (3) and z_k is from (4) and $\alpha_k, \beta_k, \gamma_k, \delta_k > 0$. We have to distinguish between the different possibilities for c.

Case c is from (1): Let a be the outgoing edge of (n, j) and b be the incoming edge of (n, j) not incident with v_0 (if it exists). Each x_k and each z_k contains precisely one of these edges, while the w_k's, the y_k's and also c contains none of these edges. Thus it follows that

$$\sum_{k:a\in x_k} \beta_k = \sum_{k:a\in z_k} \delta_k \quad \text{and} \quad \sum_{k:b\in x_k} \beta_k = \sum_{k:b\in z_k} \delta_k,$$

since a (resp. b) are traversed in opposite directions in x_k and z_k, and, after adding the two equations, $\sum_k \beta_k = \sum_k \delta_k$. Analogously, we obtain $\sum_k \gamma_k = \sum_k \delta_k$ by considering the edges incident with (n, i) but not with v_0. Moreover, on the right-hand side of (4.2), the value on edge e_1 is

$$\sum_k \delta_k - \sum_k \beta_k - \sum_k \gamma_k = -\sum_k \gamma_k.$$

Since this value is obviously 0 on the left-hand side, we can conclude $\sum_k \gamma_k = 0$ and thus $\gamma_k = \delta_k = \beta_k = 0$ for all k. This implies that the positive representation contains only circuits from (1), but this is impossible.

Case c is from (2): In a similar way as before, we can conclude

$$\sum_k \delta_k - \sum_k \beta_k = -1$$

and

$$\sum_k \delta_k - \sum_k \gamma_k = 0.$$

The value on the edge e_1 is

$$\sum_k \delta_k - \sum_k \beta_k - \sum_k \gamma_k = -1 - \sum_k \gamma_k.$$

Since this values is -1 on the left-hand side, we can conclude $\gamma_k = \delta_k = 0$ for all k. Thus, there are no circuits from (3) or (4) in the positive representation. Also the edge of x_k incident with (n, j) must be the same as in c for all k, and $\sum_k \beta_k = 1$. This implies that also the edges of x_k incident with $(n, i + 1)$ must be the same as in c for all k. Therefore and by Proposition 2,

$$c - \sum_k \beta_k x_k = \sum_k \beta_k (c - x_k)$$

is a non-trivial linear combination of type 2 circuits since $x_k \neq c$ for all k. However, such a linear combination is obviously not positively representable by circuits from (1).

Case c is from (3): Similar to the previous case.

Case c is from (4): In this case, we have

$$\sum_k \delta_k - \sum_k \beta_k = \sum_k \delta_k - \sum_k \gamma_k = 1$$

and this implies in particular $\sum_k \beta_k = \sum_k \gamma_k$. On the right-hand side of (4.2), the value on edge e_1 is

$$\sum_k \delta_k - \sum_k \beta_k - \sum_k \gamma_k = 1 - \sum_k \gamma_k.$$

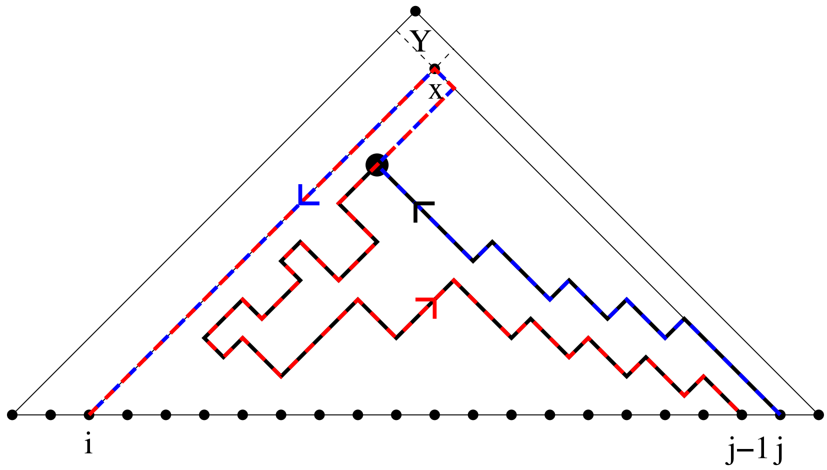

FIGURE 3. Explanation of (B) in Lemma 1

Since this value is obviously 1 on the left-hand side, $\beta_k = \gamma_k = 0$ and this implies that there are no circuits from (2) or (3) in the positive representation in (4.2). Again the edges of all z_k incident with (n, j) and (n, i) must be the same as in c and $\sum_k \delta_k = 1$. By Proposition 2,

$$c - \sum_k \delta_k z_k = \sum_k \delta_k (c - z_k)$$

is a non-trivial linear combination of type 2 circuits. This leads to the same contradiction as in the case when c is from (2).

re (B): Assume the contrary and suppose $c \in C_0$ is such an element. Suppose

$$c = \sum_k \alpha_k w_k + \sum_k \beta_k x_k + \sum_k \gamma_k y_k + \sum_k \delta_k z_k,$$

where w_k is from (1), x_k is from (2), y_k is from (3) and z_k is from (4) and $\alpha_k, \beta_k, \gamma_k, \delta_k < 0$.

Case c is from (1): As in the analogous case in (A), we can conclude $\beta_k = \gamma_k = \delta_k = 0$ and this is a contradiction.

Case c is from (2): As in the analogous case in (A), we can conclude $\gamma_k = \delta_k = 0$, thus circuits of type (3) or (4) do not appear in the representation and this is a contradiction.

Case c is from (3): Similar to the previous case.

Case c is from (4): Also here we can conclude that circuits from (2) or (3) do not appear in the linear combination and this is impossible. □

Proof of Lemma 1. We may choose Z_1 to be any cell of (D_n, E_P, E_N) that is in the open cone of all counterclockwise oriented type 1 circuits. We need to show that it is possible to assign orientations to the type 1 circuits such that clockwise orientation is only permissible if the circuit is (i, j)-feasible and the following conditions are fulfilled:

- The open cone of the set of type 1 circuits equipped with these orientations is non-empty.
- This open cone has a facet as described in (2) in the statement of the lemma.

(At the end of the proof it is explained how this gives automatically the required sequence.)

First we show that every counterclockwise oriented type 1 circuit associated with a B-path that connects two vertices (n, p) and (n, q) with $i \le p < q \le j$ and contains no vertex from

Y is positively representable by elements from C_0, except for possibly some circuits that are (i,j)-feasible.

(A) If $i < p < q < j$, this is clear because only B-paths from (1) are needed.
(B) If $p = j-1$ and $q = j$, then we need B-paths from (3) and (4), see also Figure 3. We can restrict to B-paths that do not contain a vertex (s,t) with $t = i$ since the others are obviously (i,j)-feasible
(C) If $p = i$ and $q = i+1$, then the B-paths from (2) and (4) are used in the positive representation. Again we may restrict to B-paths not containing a vertex (s,t) with $s - t = n - j$.
(D) For the general case $i \le p < q \le j$, we can combine B-paths from (A), (B) and (C).

Now let D be the counterclockwise oriented type 1 circuits that are induced by B-paths with starting point (n,s) and endpoint (n,t) such that either $s > j$ or $t < i$. Let $\mathcal{C}$ denote the open cone of $C_0 \cup D$. Here are some important facts on $\mathcal{C}$:

- $\mathcal{C}$ is non-empty: This is because there is no element $d \in C_0 \cup D$ that is negatively representable by elements from $(C_0 \cup D) \setminus \{d\}$.
- Let $c_0 \in C_0$ be the boundary circuit induced by the B-path directed from (n,j) to (n,i) containing x. It induces a facet of $\mathcal{C}$ because it is not positively representable by elements from $(C_0 \cup D) \setminus \{c_0\}$.
- Each counterclockwise oriented type 1 circuit is positively representable by elements from $C_0 \cup D$, except for some circuits the B-path of which is (i,j)-feasible (note that a B-path connecting two vertices of $\{(n,i),(n,i+1),\ldots,(n,j)\}$ and containing a vertex from Y is (i,j)-feasible). We denote the set of these *unoriented* exceptional admissible circuits by E.

Now equip each circuit in E with an orientation and denote the resulting set by $\overrightarrow{E}$ such that the open cone of $C_0 \cup D \cup \overrightarrow{E}$ is non-empty and c_0 induces a facet of this open cone. Let Z be a cell contained in the open cone such that c_0 also constitutes a facet of Z. There must now be a chain $Z_1, Z_2, \ldots, Z_t = Z$ of cells as described in the statement, where we possibly need to relax the condition that the cells are inside of $\mathcal{C}_{D,E_P,E_N}$.

If one of the hyperplanes that we need to overcome, say when going from Z_m to Z_{m+1}, is a boundary hyperplane then the corresponding B-path connects (n,i) and (n,j). We choose m minimal with this property (where $t = m$ if we do not have to overcome a boundary hyperplane) and have finally constructed our sequence. □

The following proposition is a direct consequence of the previous lemma.

Proposition 7. (1) *The total degree of $P(\mathbf{k})$ is no greater than $\binom{n}{2}$.*
(2) *For each pair (i,j) with $1 \le i < j \le n$, the polynomial $P(\mathbf{k})$ vanishes for all integer n-tuples $(k_1,k_2,\ldots,k_n)$ with the following property: for all pairs (p,q) with $i \le p < q \le j$, we have $i - j + p - q + 1 \le k_q - k_p \le j - i + p - q - 1$.*

Proof. As for the degree, we use the bound $d + |I_P| + |I_N| - \dim \mathcal{C}_{A,I_P,I_N}$ given in Theorem 1: d is the number of vertices, which is $\binom{n+1}{2} + 1$ in our case, $|I_P| = |E_P|$ is the number of edges, except for those incident with v_0, $|E_N| = 0$ and $\mathcal{C}_{D,E_P,E_N}$ is fulldimensional, i.e. the dimension is the number of edges plus 1. Thus, the upper bound is $\binom{n+1}{2} + 1 - n - 1 = \binom{n}{2}$.

Choose $Z_1, Z_2, \ldots, Z_m$ as in Lemma 1. By Proposition 6, $P_{Z_1}(\widehat{z}) \equiv P(\mathbf{k})$. Moreover $P_{Z_m}(\widehat{z})$ vanishes if $\mathbf{k}$ fulfills the requirement in the lemma: Let c be the oriented boundary circuit containing (n,i) and (n,j) that constitutes a facet of c. The cell Z_m is adjacent to a cell outside of $\mathcal{C}_{D,E_P,E_N}$ via this hyperplane. By Corollary 1, $P_{Z_m}(x)$ vanishes if $-2j+2i+1 \le \langle c,x\rangle \le -1$ for all integer valued functions x, because c contains $2j-2i$ edges with value 1 not incident with v_0. Now $\widehat{x}$ has this property by assumption, since $\langle c,\widehat{x}\rangle = k_j - k_i$.

For $1 \le r < m$, let c_r be the admissible circuit of the hyperplane between Z_r and Z_{r+1}. By Corollary 1, it suffices to show that $\langle c_r,\widehat{x}\rangle \in \{-a_1+1,\ldots,a_2-1\}$, where a_1 and a_2 are defined as in the corollary, since then $P_{Z_{r-1}}(\widehat{x}) = P_{Z_r}(\widehat{x})$.

If c_r is of type 2, then this is obvious because $\langle c_r,\widehat{x}\rangle = 0$ and $a_1, a_2 > 0$.

If c_r is of type 1, then the circuit is (i,j)-feasible and let p,q be as in the definition. We denote by P the associated B-path.

We show $a_1 \ge j-i+q-p$ and $a_2 \ge j-i+p-q$, where c_r is counterclockwise oriented so that $\langle c_r,\widehat{x}\rangle = k_q - k_p$. By assumption, there exists a vertex $(n-j+x,x)$ and a vertex (y,i) in P; it can be assumed that $(n-j+x,x)$ appears before (y,i) when traversing P from (n,q) to (n,p).

If $x<q$, then there have already been $q-x$ forward edges (i.e. $c_r(e)=1$) before $(n-j+x,x)$ in NW-direction when traversing P from (n,q) to (n,p). If $n-p > y-i$, then there are at least $n-p-y+i$ forward edges in SW direction after (y,i) when traversing P from (n,q) to (n,p). There are at least $x-i+y-i-n+j$ forward edges from $(n-j+x,x)$ to (y,i). If we add all these quantities, we obtain the lower bound for a_1.

To compute the lower bound for a_2, observe that there are at least $j-q$ backward edges from (n,q) to $(n-j+x,x)$ in NE direction, while there are at least $p-i$ backward edges in SE-direction after (y,i). □

4.3. **Do we have enough zeros to determine $P(\mathbf{k})$?**

Now it is more convenient to work with the following transform of $P(\mathbf{k})$:

$$Q(x_1,\ldots,x_n) := P(x_1-1, x_2-2, \ldots, x_n-n)$$

For integers i,j,n with $1 \le i < j \le n$, let

$$\mathcal{Z}_{i,j,n} = \bigcup_{c\in\mathbb{Z}} \left(\mathbb{Z}^{i-1} \times [c+i+1, c+j]^{j-i+1} \times \mathbb{Z}^{n-j}\right)$$

where $[l,u] = \{l, l+1, \ldots, u\}$ and set

$$\mathcal{Z}_n = \bigcup_{1\le i<j\le n} \mathcal{Z}_{i,j,n}.$$

In Proposition 7(2), we have shown that every element of $\mathcal{Z}_n$ is a zero of $Q(x_1,\ldots,x_n)$. This becomes obvious after observing that

$$\mathcal{Z}_{i,j,n} = \{(x_1,\ldots,x_n) \in \mathbb{Z}^n | \forall (p,q) \text{ with } i \le p < q \le j : |x_q - x_p| < j-i\},$$

since the requirement just means that the difference between the maximal and the minimal element of $\{x_i, x_{i+1}, \ldots, x_j\}$ is less than $j-i$.

A natural question to ask is whether we have gathered enough information on $Q(x_1,\ldots,x_n)$ to determine it. We have checked that this is the case for $n \le 5$ (see the appendix) and so we conjecture the following.

Conjecture 1. *Let n be a positive integer. Each polynomial in $x_1, \ldots, x_n$ over $\mathbb{R}$ of total degree no greater than $\binom{n}{2}$ that vanishes for all elements of $\mathcal{Z}_n$ is of the form*

$$c \cdot \prod_{1 \le i < j \le n} (x_j - x_i)$$

for an appropriate constant $c \in \mathbb{R}$.

In our case we can easily determine the constant c, since there is only one Gelfand-Tsetlin pattern with bottom row $(0, 0, \ldots, 0)$ and thus

$$Q(1, 2, \ldots, n) = P(0, 0, \ldots, 0) = 1.$$

Phrased differently, we conjecture that the non-zero polynomials of minimal degree in the vanishing ideal $\mathcal{I}(\mathcal{Z}_n)$ are of the form

$$c \cdot \prod_{1 \le i < j \le n} (x_j - x_i)$$

for an appropriate $c \in \mathbb{R}$. Is it possible to compute this vanishing ideal? Observe that

$$\mathcal{I}(\mathcal{Z}_n) = \bigcap_{\substack{1 \le i < j \le n \\ c \in \mathbb{Z}}} \mathcal{I}(\mathbb{Z}^{i-1} \times [c+i+1, c+j]^{j-i+1} \times \mathbb{Z}^{n-j})$$

and

$$\mathcal{I}(\mathbb{Z}^{i-1} \times [c+i+1, c+j]^{j-i+1} \times \mathbb{Z}^{n-j}) = ((x_i - j - c)_{j-i}, (x_{i+1} - j - c)_{j-i}, \ldots, (x_j - j - c)_{j-i}).$$

From the combinatorial interpretation it is also obvious that $P(\mathbf{k})$ has another property: it is invariant under shifting each variable by the same amount, i.e.

$$P(k_1, \ldots, k_n) = P(k_1 + c, \ldots, k_n + c)$$

for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and all $c \in \mathbb{Z}$. The transform $Q(x_1, \ldots, x_n)$ then has the same property. We address this property as **1-shift-invariance**, where **1** stands for the vector $(1, 1, \ldots, 1)$.

Hence for our purposes it is enough to prove the conjecture for **1**-shift-invariant polynomials. However, in the following proposition we argue how to deduce Conjecture 1 once we know it for **1**-shift-invariant polynomials. This is useful when testing the conjecture for small values of n.

In the following, it is sometimes necessary to perform an affine transformation of variables from, say, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. This means that $\mathbf{x}$ and $\mathbf{y}$ are related by an invertible $n \times n$ matrix A and a translation by a vector $\mathbf{b} \in \mathbb{R}^n$, i.e. $\mathbf{x} = A \cdot \mathbf{y} + \mathbf{b}$. The total agree of a polynomial in $\mathbf{x}$ agrees with the total degree in $\mathbf{y}$.

We need to introduce a notation: suppose $R(x_1, \ldots, x_n)$ is a function in $x_1, \ldots, x_n$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. The difference operator $\Delta_{\mathbf{a}}$ with respect to $\mathbf{a}$ is defined as

$$\Delta_{\mathbf{a}} R(x_1, \ldots, x_n) = R(x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n) - R(x_1, x_2, \ldots, x_n).$$

The application of $\Delta_{\mathbf{a}}$ decreases the total degree by 1 at least.

Proposition 8. *Suppose each* **1***-shift-invariant polynomial in the variables* $x_1, \ldots, x_n$ *of degree no greater than* $\binom{n}{2}$ *that vanishes on* $\mathcal{Z}_n$ *is of the form*

$$c \cdot \prod_{1 \le i < j \le n} (x_j - x_i)$$

for an appropriate constant $c \in \mathbb{R}$. *Then Conjecture 1 is true.*

Proof. Suppose there exists a polynomial $R(x_1, \ldots, x_n)$ in $x_1, x_2, \ldots, x_n$ of degree no greater than $\binom{n}{2}$ that vanishes on $\mathcal{Z}_n$ and is not **1**-shift-invariant. We express it as a polynomial in the variables

$$y_1 = x_2 - x_1, y_2 = x_3 - x_2, \ldots, y_{n-1} = x_n - x_{n-1}, y_n = x_1 + x_2 + \ldots + x_n.$$

Since $y_1, \ldots, y_{n-1}$ are **1**-shift-invariant in the variables $x_1, \ldots, x_n$, the degree in y_n is at least 1. Let d denote this degree and $S(x_1, \ldots, x_n)$ be the coefficient of y_n^d. This coefficient is **1**-shift-invariant (w.r.t. $x_1, x_2, \ldots, x_n$) and of degree less than $\binom{n}{2}$. Moreover,

$$\Delta_{\mathbf{1}}^d R(x_1, \ldots, x_n) = n^d d! \, S(x_1, \ldots, x_n).$$

We can conclude that $S(x_1, \ldots, x_n)$ also vanishes on $\mathcal{Z}_n$ (since $\mathcal{Z}_n + \mathbf{1} = \mathcal{Z}_n$) and this is a contradiction to the assumption. □

This leads us to another version of Conjecture 1. We transform the variables as described in the proof of Proposition 8, and, since we can restrict to **1**-shift-invariant polynomials, the variable y_n will not appear. That is, we are now interested in the polynomial $Q(0, y_1, y_1 + y_2, \ldots, y_1 + \ldots + y_{n-1}) =: R(y_1, \ldots, y_{n-1})$. The zeros of $P(\mathbf{k})$ can be translated into zeros of $R(y_1, \ldots, y_{n-1})$.

For integers i, j, n with $1 \le i \le j \le n$, we define

$$\mathcal{Z}^*_{i,j,n} = \{(y_1, \ldots, y_n) \in \mathbb{Z}^n | \forall (p,q) \text{ with } i \le p \le q \le j : |y_p + y_{p+1} + \ldots + y_q| \le j - i\}$$

and

$$\mathcal{Z}^*_n = \bigcup_{1 \le i \le j \le n} \mathcal{Z}^*_{i,j,n}.$$

Then each element of $\mathcal{Z}^*_n$ is a zero of $R(y_1, \ldots, y_n)$. The following conjecture is equivalent to Conjecture 1. Since we have reduced the number of variables by 1, the n variable case in the following conjecture corresponds to the $n+1$ variable case in Conjecture 1.

Conjecture 2. *Let* n *be a positive integer. Each polynomial in* $y_1, \ldots, y_n$ *of total degree no greater than* $\binom{n+1}{2}$ *that vanishes for all elements of* $\mathcal{Z}^*_n$ *is of the form*

$$c \cdot \prod_{1 \le i \le j \le n} (y_i + y_{i+1} + \ldots + y_j)$$

for an appropriate constant $c \in \mathbb{R}$.

Appendix A. How to attack Conjectures 1 and 2?

We present some thoughts on how to attack the conjectures. The approach we present is then used to verify Conjecture 2 for $n \leq 4$.

Definition 4. *Let d be a non-negative integer and $\mathcal{Z} \subseteq \mathbb{R}^n$. We say that $\mathcal{Z}$ is* d**-defining** *if a polynomial in n variables of degree at most d is uniquely determined by its values on this set.*

More generally, if $\mathcal{G} \subseteq \mathbb{R}^n$ is a second set, then we say that $\mathcal{Z}$ is $(d, \mathcal{G})$**-defining** *if a polynomial function in n variables of degree at most d is on $\mathcal{G}$ uniquely determined by its values on $\mathcal{Z}$.*

Note that in this definition, we do not require that for each choice of values on $\mathcal{Z}$, there is a polynomial. In our application, $\mathcal{G}$ is usually an affine subspace of $\mathbb{R}^n$.

Conjecture 1 is equivalent to proving that $\mathcal{Z}_n \cup \{(1, 2, \ldots, n)\}$ is an $\binom{n}{2}$-defining set, while Conjecture 2 is equivalent to proving that $\mathcal{Z}_n^* \cup \{(1, 1, \ldots, 1)\}$ is an $\binom{n+1}{2}$-defining set.

Definition 5. *Let $\mathcal{Z} \subseteq \mathbb{R}^n$. Suppose $\mathbf{a} \in \mathbb{R}^n$ and H is an affine hyperplane of $\mathbb{R}^n$ with $\mathbb{R}^n = \mathbb{R}\,\mathbf{a} + H$. The pair $(\mathbf{a}, H)$ is said to be $\mathcal{Z}$-compatible if the following is fulfilled.*

1. *For each $\mathbf{z} \in \mathcal{Z}$, there exist $\mathbf{h} \in H \cap \mathcal{Z}, y \in \mathbb{Z}$ with $\mathbf{z} = \mathbf{h} + y\,\mathbf{a}$.*
2. *Suppose $\mathbf{h} \in H \cap \mathcal{Z}, y \in \mathbb{Z}$ with $\mathbf{h} + y\,\mathbf{a} \in \mathcal{Z}$. Then $\mathbf{h} + t\,\mathbf{a} \in \mathcal{Z}$ for all $t \in \{0, 1, \ldots, y\}$, respectively $t \in \{y, y+1, \ldots, 0\}$.*

Example 1. *For non-negative integers $a_1, a_2, \ldots, a_n$, define*

$$\mathcal{D}_{a_1,\ldots,a_n} = \bigcup_{c \in \mathbb{Z}} [c, c + a_1] \times \cdots \times [c, c + a_n].$$

If $a_i \geq a_j$, then $(\mathbf{e}_i, x_i = x_j)$ is a $\mathcal{D}_{a_1,\ldots,a_n}$-compatible pair. Recall that

$$\mathcal{Z}_n = \bigcup_{1 \leq i < j \leq n} \left(\mathbb{Z}^{i-1} \times \mathcal{D}_{j-i-1,j-i-1,\ldots,j-i-1} \times \mathbb{Z}^{n-j} \right).$$

The following proposition constitutes a recursive procedure for proving or disproving that a certain set is d-defining.

Proposition 9. *Let $\mathcal{Z} \subseteq \mathbb{R}^n$ and G be an affine subspace of $\mathbb{R}^n$. Suppose $\mathbf{a} \in \mathbb{R}^n$ and H is an affine hyperplane with $\mathbb{R}^n = \mathbb{R}\,\mathbf{a} + H$. Moreover we assume $\mathbf{a} + G = G$ and that H does not contain G. If*

1. *the set $\mathcal{Z} \cap (\mathcal{Z} + \mathbf{a}) =: \mathcal{Z}_{\mathbf{a}}$ is $(d-1, G)$-defining, and*
2. *the set $\mathcal{Z}$ is $(d, G \cap H)$-defining,*

then $\mathcal{Z}$ is (d, G)-defining. The converse is also true if $(\mathbf{a}, H)$ is $\mathcal{Z}$-compatible and $G = \mathbb{R}^n$. Condition (2) may also be replaced by

(2') *$\mathcal{Z} \cap G \cap H$ is d-defining for polynomials on $G \cap H$.*[2]

[2] A polynomial in $\mathbf{x} = (x_1, \ldots, x_n)$ on an affine subspace F is a polynomial for which there exists a linear transformation $\mathbf{x} = A \cdot \mathbf{y} + \mathbf{b}$ of the variables and $k \in \{1, 2, \ldots, n\}$ such that

$$F = \mathbb{R}\mathbf{a}_1 + \ldots + \mathbb{R}\mathbf{a}_k + \mathbf{b},$$

where $\mathbf{a}_i$ denotes the i-th column of A and the polynomial is in fact independent of $y_{k+1}, \ldots, y_n$.

Example 2. *Note that*

$$(\mathcal{D}_{a_1,\ldots,a_n})_{-\mathbf{e}_i} = \begin{cases} \mathcal{D}_{a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_n} & \text{if } a_i > 0, \\ \mathcal{D}_{a_1-1,\ldots,a_{i-1}-1,a_i,a_{i+1}-1,\ldots,a_n-1} - \mathbf{e}_i & \text{if } a_i = 0. \end{cases}$$

Observe that (2') is stronger than (2). The proposition allows to reduce the problem to two subproblems of the same type, where we decreased the degree in one problem, while we decreased the number of variables in the other (if we use (2')). To determine whether a set is d-defining is trivial if $n = 1$ or $d = 0$.

We will use the following natural extension of summation:

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \ldots + f(b) & \text{if } a \leq b, \\ 0 & \text{if } b = a-1, \\ -f(b+1) - f(b+2) - \ldots - f(a-1) & \text{if } b+1 \leq a-1. \end{cases}.$$

This definition is convenient because if $p(i)$ is a polynomial in i, then $\sum_{i=0}^{j} p(i)$ is a polynomial in j with an increase of the degree by one.

Proof of Proposition 9. Let $V : \mathcal{Z} \to \mathbb{R}$. We need to show that all polynomials $U(z_1, \ldots, z_n)$ of degree no greater than d with $U(z_1, \ldots, z_n) = V(z_1, \ldots, z_n)$ on $\mathcal{Z}$ agree on G. In the following, suppose that $U(z_1, \ldots, z_n)$ is such a polynomial.

The polynomial

$$\delta_{\mathbf{a}} U(z_1, \ldots, z_n) := U(z_1, \ldots, z_n) - U(z_1 - a_1, z_2 - a_2, \ldots, z_n - a_n) =: U_{\mathbf{a}}(z_1, \ldots, z_n)$$

is then a polynomial of degree $d-1$ at most that has the values $\delta_{\mathbf{a}} V$ on $\mathcal{Z} \cap (\mathcal{Z} + \mathbf{a})$. By assumption, this determines $U_{\mathbf{a}}(z_1, \ldots, z_n)$ on G.

By assumption, $G = \mathbb{R}\,\mathbf{a} + (G \cap H)$, and thus two polynomials agree on G if they agree on $\mathbb{Z}_{\geq 0}\,\mathbf{a} + (G \cap H)$. Let

$$\mathbf{z} = y\,\mathbf{a} + \mathbf{h}$$

be an element in this set, i.e. $y \in \mathbb{Z}_{\geq 0}$ and $\mathbf{h} \in G \cap H$. Now it is clear that $U(\mathbf{z})$ can be expressed with the help of $U_{\mathbf{a}}$ on G and the restriction of U to $G \cap H$. Indeed, by telescoping,

$$U(\mathbf{z}) = U(\mathbf{h}) + \sum_{t=1}^{y} (U(\mathbf{h} + t\,\mathbf{a}) - U(\mathbf{h} + (t-1)\,\mathbf{a})) = U(\mathbf{h}) + \sum_{t=1}^{y} U_{\mathbf{a}}(\mathbf{h} + t\,\mathbf{a}).$$

For the converse, we assume first that $\mathcal{Z}_{\mathbf{a}}$ is not $(d-1)$-defining and let $V_{\mathbf{a}} : \mathcal{Z}_{\mathbf{a}} \to \mathbb{R}$ be such that there are two different polynomials $U_1(\mathbf{z}), U_2(\mathbf{z})$ of degree at most $d-1$ that agree on $\mathcal{Z}_{\mathbf{a}}$ with $V_{\mathbf{a}}$. We define $V : \mathcal{Z} \to \mathbb{R}$ as follows: For $\mathbf{z} \in \mathcal{Z}$, let $\mathbf{h} \in \mathcal{Z} \cap H$ and $y \in \mathbb{Z}$ with $\mathbf{z} = \mathbf{h} + y\,\mathbf{a}$. Then

$$V(\mathbf{z}) = \sum_{t=1}^{y} V_{\mathbf{a}}(\mathbf{h} + t\mathbf{a}).$$

Choose $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b} \in \mathbb{R}^n$ such that

$$H = \mathbb{R}\mathbf{a}_1 + \mathbb{R}\mathbf{a}_2 + \ldots + \mathbb{R}\mathbf{a}_{n-1} + \mathbf{b}.$$

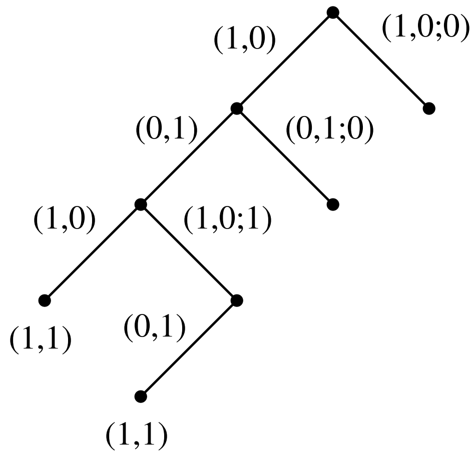

FIGURE 4. The case $n = 2$ of Conjecture 2.

We perform a change of variables from $\mathbf{x}$ to $\mathbf{y}$ where $\mathbf{x} = A \cdot \mathbf{y} + \mathbf{b}$ and A is the matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{a}$. Now it is clear that

$$\sum_{t=1}^{y_n} U_i(y_1\mathbf{a}_1 + \ldots + y_{n-1}\mathbf{a}_{n-1} + \mathbf{b} + t\,\mathbf{a}),$$

$i = 1, 2$, constitute two different polynomials of degree at most d that agree with V on $\mathcal{Z}$.

If, on the other hand, the set $\mathcal{Z} \cap H$ is not d-defining for polynomials on H, then let $V : \mathcal{Z} \cap H \to R$ be a function and two different polynomials U_1, U_2 on H of degree d at most that agree with V but do not agree on H. We choose $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$ as above and may then consider U_1, U_2 as polynomials in $y_1, \ldots, y_{n-1}$. We extend V to $\mathcal{Z}$ as follows: If $\mathbf{z} = \mathbf{h} + y\,\mathbf{a} \in \mathcal{Z}$ with $\mathbf{h} \in H$ and $y \in \mathbb{Z}$, then set $V(\mathbf{z}) = V(\mathbf{h})$. We may then consider U_1, U_2 also as polynomials on $\mathbb{R}^n$. They agree on V and this is a contradiction. □

We have applied this proposition to prove Conjecture 2 for $n \leq 3$ – see Figures 4 and 5 (the case $n = 1$ is trivial). We use the version with (2') and have $G = \mathbb{R}^n$. The binary trees have to be interpreted as follows.

(1) The root (=topmost vertex) is identified with the set $\mathcal{Z}_n$.
(2) The left branch of a vertex $\mathcal{Z}$ corresponds to subproblem (1) in the proposition. Its label is the vector $\mathbf{a}$ and the endpoint of the branch is identified with $\mathcal{Z}_{\mathbf{a}}$.
(3) The right branch of a vertex $\mathcal{Z}$ corresponds to subproblem (2'). Its label is the affine hyperplane H (we write $(h_1, \ldots, h_n; d)$ if $H = \{(x_1, \ldots, x_n) | h_1 x_1 + \ldots + h_n x_n = d\}$) and the endpoint of the branch is identified $\mathcal{Z} \cap H$.
(4) The procedure terminates at a left branch if it is the d-th left branch in the unique path from the root to the leaf (since we have arrived at a degree 0 case). The set of this leaf has to be non-empty and below the leaf we write a representative of this set. Note that it has to be contained in all hyperplanes that appear as labels of the unique path from the leaf to the root.
(5) The procedure terminates at a right branch if the set corresponding to the leaf is H (in which case the set is of course defining).

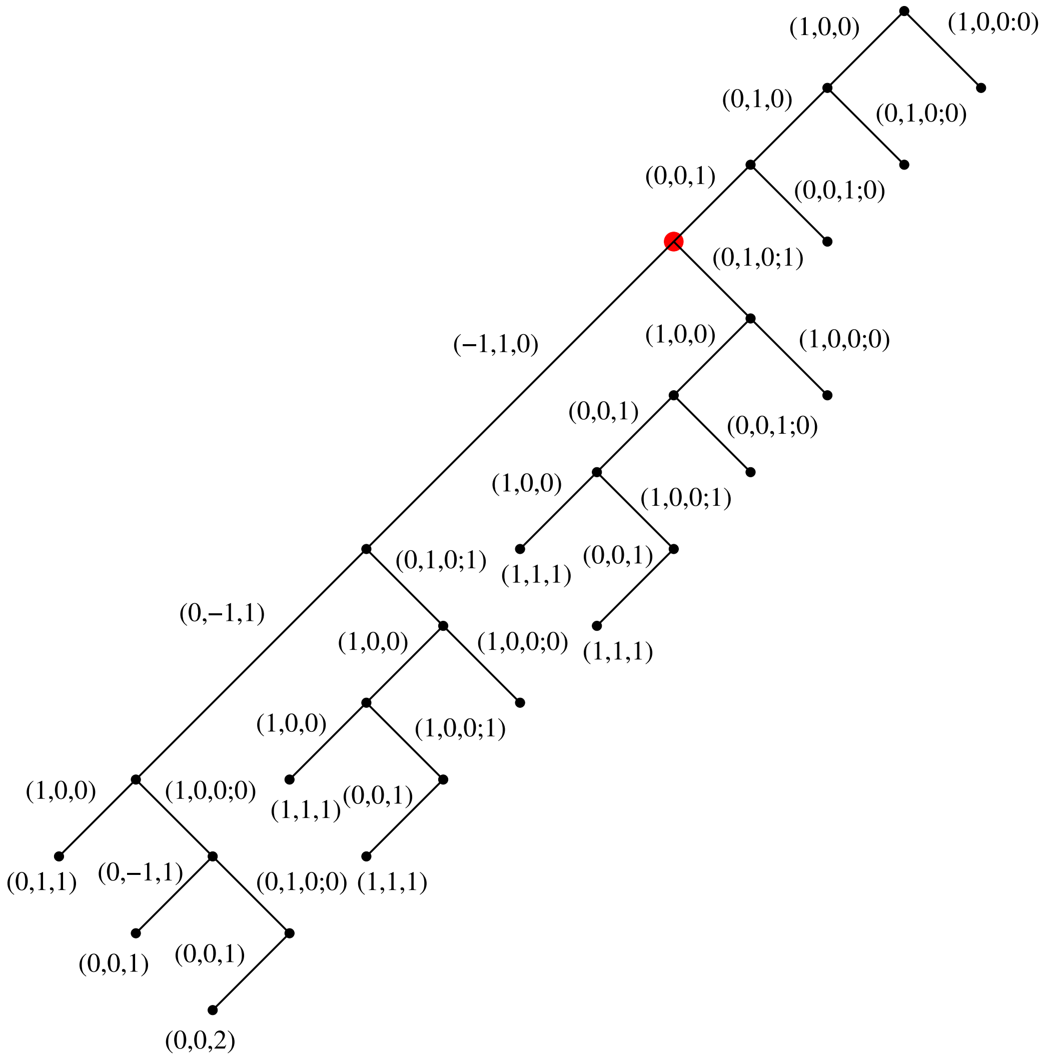

FIGURE 5. The case $n = 3$ of Conjecture 2.

(6) Right branches that are the $(n-1)$-st right branch in the path from the root to the branch need not to have another right branch. This is clear if the procedure terminates at this point. But also if there is a left branch, then each element of the set in the left branch could serve as the "affine hyperplane" of the right branch since we are in the no-variable case then.

(7) For each label $\mathbf{a}$ of a left branch, consider all normal vectors of affine hyperplanes that appear as labels of the path from the left branch to the root. Then $\mathbf{a}$ has to be orthogonal to these normal vectors.

The red vertex in Figure 5 indicates that the pair $(\mathbf{a}, H)$ is not compatible with the set corresponding to the vertex since property (2) in Definition 5 is not fulfilled. We were unable to find compatible pairs in all cases.

Since it is unclear to us how to generalize the "proofs" in Figures 4 and 5, we present a slightly different type for the case $n = 3$ in Figure 6. Here we use condition (2) of Proposition 9 and,

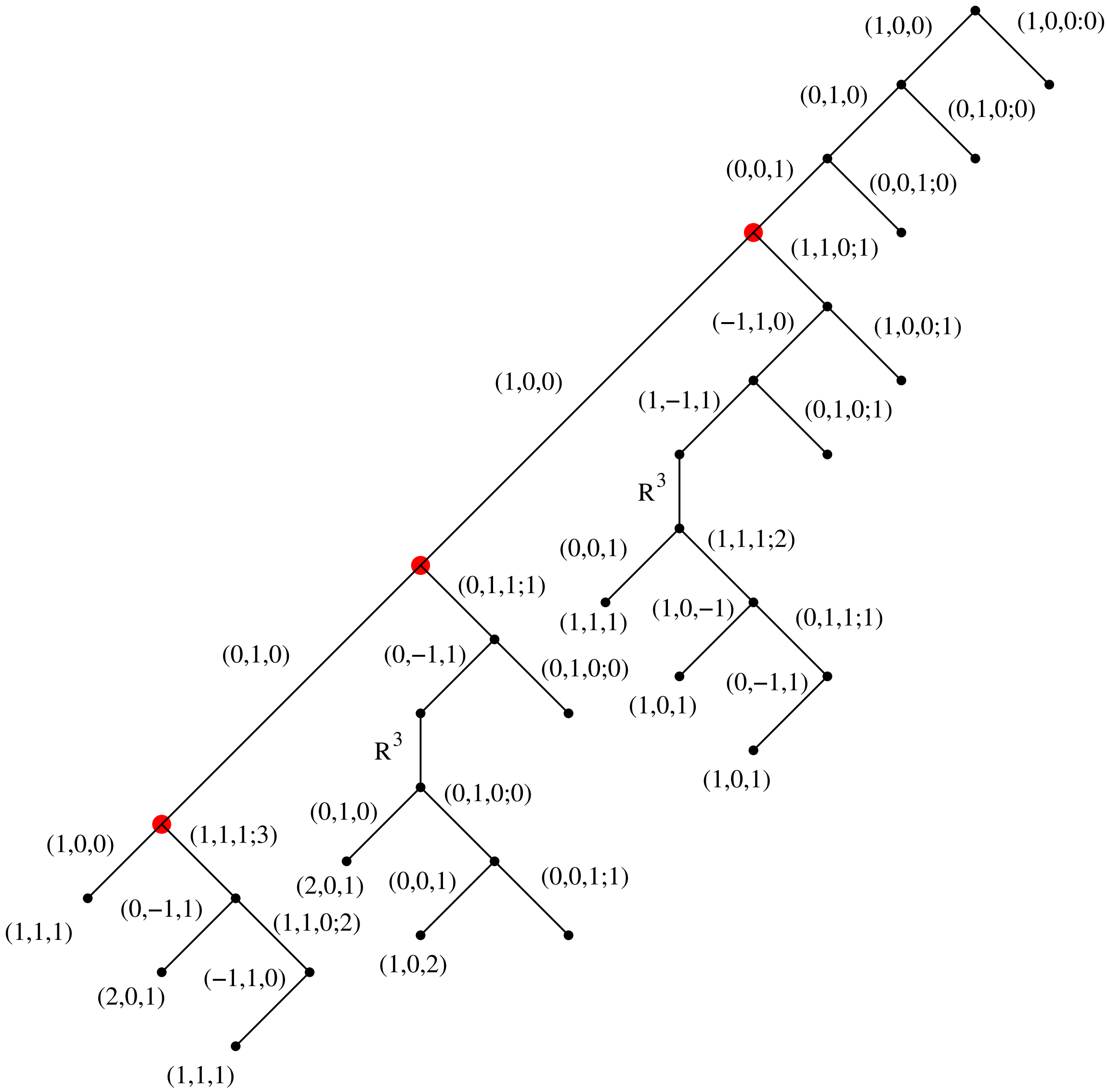

Figure 6. Another proof of the case $n = 3$ of Conjecture 2.

in general, we do not have $G = \mathbb{R}^n$. Here the pairs that are not compatible (again indicated by red vertices) are in a sense closer to being compatible.

Vertices now correspond to a pair $(\mathcal{Z}, G)$ where $\mathcal{Z} \subseteq \mathbb{R}^n$ and G is an affine subspace of $\mathbb{R}^n$; the root is identified with the pair $(\mathcal{Z}_n, \mathbb{R}^n)$. For left branches labelled with $\mathbf{a}$, vertex $(\mathcal{Z}, G)$ is transformed into $(\mathcal{Z}_{\mathbf{a}}, G)$, while for right branches labelled with H, vertex $(\mathcal{Z}, G)$ is transformed into $(\mathcal{Z}, G \cap H)$. However, a vertex v may now also have just one successor which is connected via a vertical edge. The label of the edge is an affine subspace F that contains G (a set $\mathcal{Z}$ is surely (d, G)-defining if it is (d, F)-defining) and leads to a vertex identified with $(\mathcal{Z}, F)$. Consequently, we have increased the number of variables again so that we need again more right branches (depending on the dimension of F) to terminate the procedure.

Finally, in Figure 7 we sketch the proof of the case $n = 4$. The red circles around leaves indicated that we checked these cases directly.

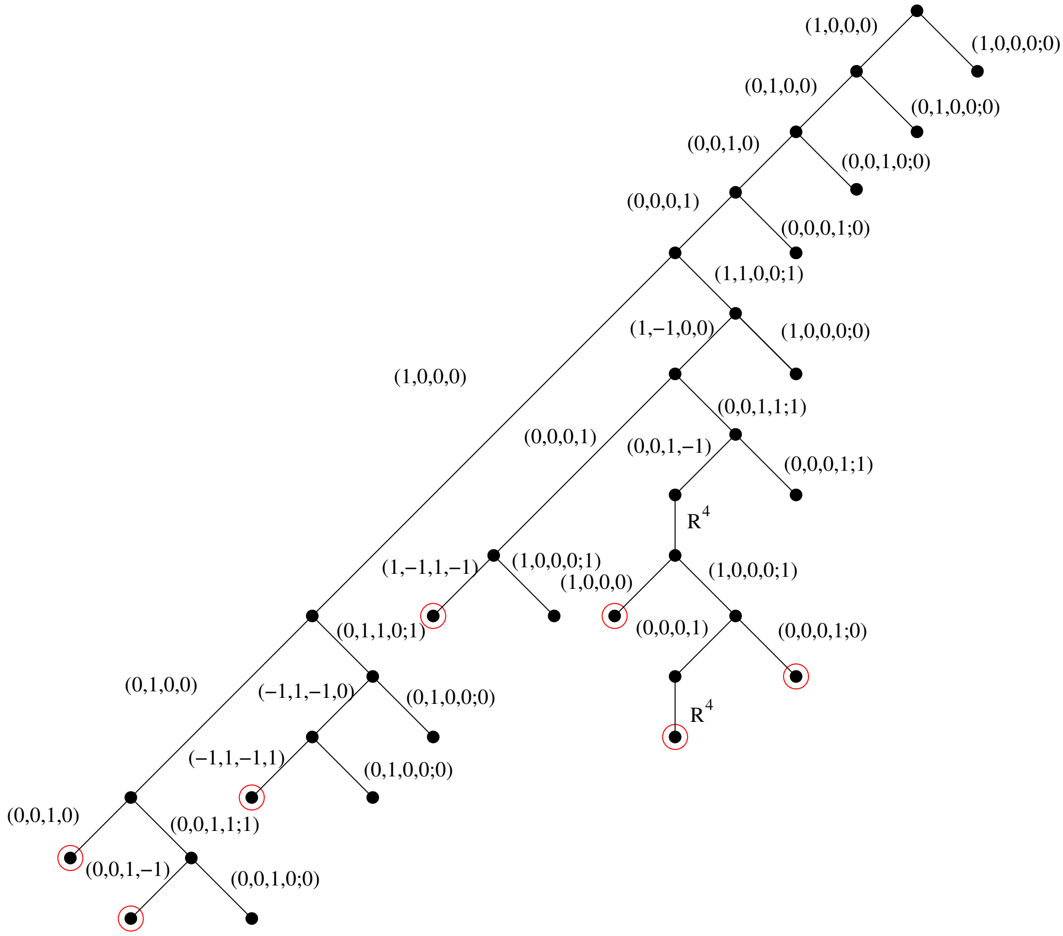

FIGURE 7. Sketch of proof of the case $n = 4$ of Conjecture 2.

Acknowledgement. I thank Philipp Puaschunder for an enlightening discussion concerning Conjecture 1.

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