A method for proving polynomial enumeration formulas

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Proving polynomial enumeration formulas by explaining their zeros using an appropriate combinatorial extension to negative integer parameters.

A simple example:

$F(r,k)$ = Number of $(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r$ with $k \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$.

• Combinatorial interpretation for negative k:
$k < 0$: $F(r,k) = (-1)^r[\#(\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $k < \lambda_1 < \lambda_2 < \ldots < \lambda_r < 0]$.

• $F(r,k)$ is a polynomial in k, $\deg_k F(r,k) \leq r$
$\Leftrightarrow \Delta^{r+1} F(r,k) = 0$

Induction with respect to r: $F(1,k) = k+1$

$$\Delta F(r,k) = F(r,k+1) - F(r,k)$$
$= [\#(\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $k+1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ and $\lambda_1 = k+1] = F(r-1, k+1)$

Induction hypothesis: $\Delta^r F(r-1, k+1) = 0$
$\Rightarrow \Delta^{r+1} F(r,k) = 0$

• Integer zeros:
$F(r,-1) = F(r,-2) = \ldots = F(r,-r) = 0$
$F(r,0) = 1$

The method

Problem: Enumeration of objects depending on an integer parameter k. Suspect that the enumeration formula is a polynomial in k which factorizes totally over $\mathbb{Z}$.

1. Extension of the combinatorial interpretation. Typically the admissible domain of k is a set S of non-negative integers. Find new objects indexed by an arbitrary integer k, which are in bijection with the original objects for $k \in S$.

2. The extending objects are enumerated by a polynomial. Mostly done by a recursion. Compute the degree of the polynomial.

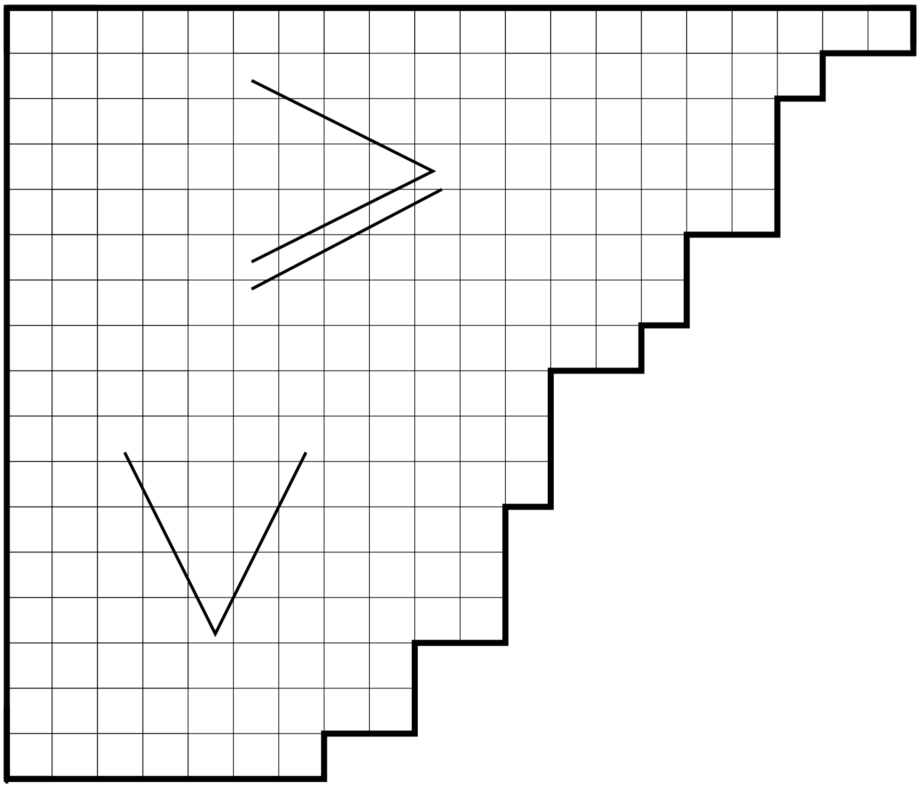
3. Explore the integer zeros of the polynomial. Plus one non-zero evaluation.

Strict plane partitions

$$\begin{matrix} 7 & 5 & 5 & 4 & 2 \\ 6 & 4 & 3 & 3 & 1 \\ 3 & 2 & 2 & 2 & \\ 2 & 1 & & & \end{matrix}$$

strict plane partition of shape $(5, 5, 4, 2)$

rows weakly decreasing, columns strictly decreasing



Why considering (strict) plane partitions?

• A natural generalization to two dimensions of ordinary partitions.

• Enumerations subject to a variety of different restrictions lead to "nice formulas". Proofs are usually much more complicated as in the one-dimensional case.

• Related to other combinatorial objects: rhombus tilings, tableaux, alternating sign matrices.

Bender-Knuth (ex-)Conjecture

Theorem (Andrews 1977, Gordon 1983, Macdonald 1979, Proctor 1984): The number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$ and at most c columns is

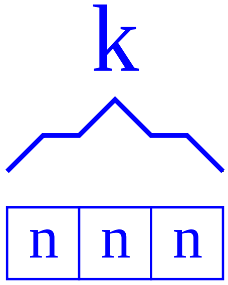
$$\prod_{i=1}^{n} \frac{(c+i)_i}{(i)_i}.$$

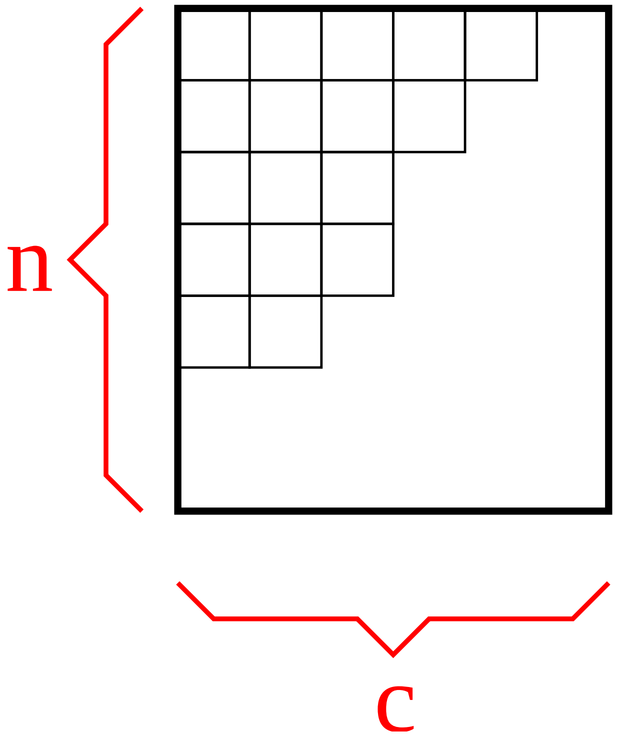
Rising shifted factorial:

$$(a)_n = \prod_{i=0}^{n-1} (a+i)$$

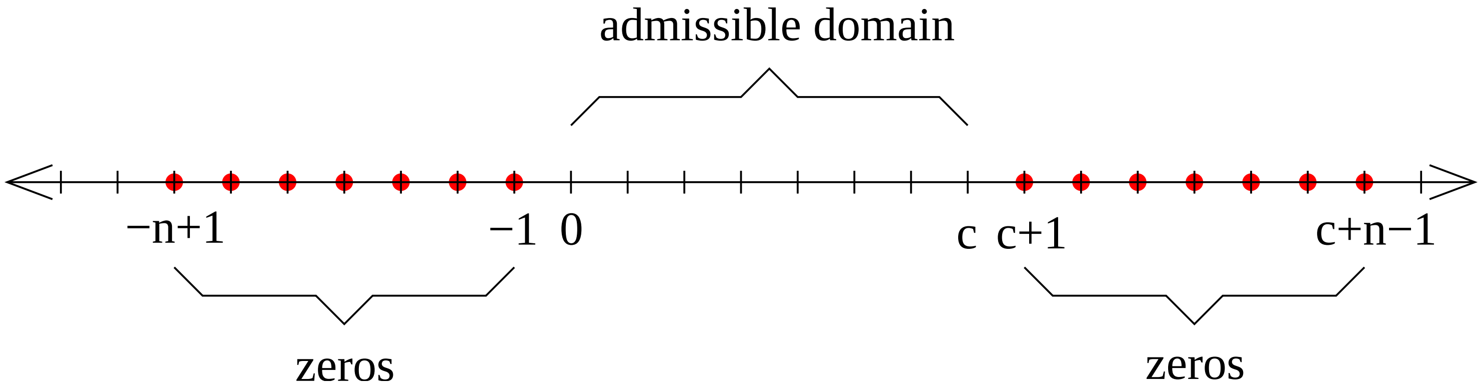
Theorem (F. 2002): The number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n is

$$\frac{(k+1)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$





parts in {1,2,...,n}


admissible domain
−n+1
−1
0
c
c+1
c+n−1
zeros
zeros

Step 1: Extension of the combinatorial interpretation

What is a strict plane partition with parts in $\{1,2,3,4,5\}$, at most 8 columns and -15 parts equal to 5 ?

From strict plane partitions to Gelfand patterns:

7	5	5	4	2
6	4	3	3	1
3	2	2	2	
2	1			

strict plane partition

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 0 & & 1 & & 3 & & & & \\
 & & & 0 & & 0 & & 2 & & 4 & & & \\
 & & 0 & & 0 & & 1 & & 4 & & 4 & & \\
 & 0 & & 0 & & 1 & & 4 & & 4 & & 5 & \\
0 & & 0 & & 0 & & 2 & & 4 & & 5 & & 5
\end{array}$$

Gelfand pattern

The ith row of the Gelfand pattern is the shape of entries which are greater than $n - i$ in the strict plane partition.

(n, c)-Gelfand pattern

$n = 5$:

$$
\begin{array}{ccccccccccccc}
 & & & 0 & & a_{5,5} & & c & & & & & \\
 & & 0 & & a_{4,4} & & a_{4,5} & & c & & & & \\
 & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & c & & & \\
0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & c & & \\
 & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} & & &
\end{array}
$$

Rule: $0 \leq a_{n,n} \leq c$ and $a_{i-1,j} \in [a_{i,j}, a_{i,j+1}] := \{a_{i,j}, a_{i,j} + 1, \ldots, a_{i,j+1}\}$

(n, c)-Gelfand patterns $\Leftrightarrow$ Strict plane partitions with parts in $\{1, 2, \ldots, n\}$ and at most c columns

$a_{n,n} = \#$ of parts equal to n in the strict plane partition $=$ "polynomial parameter"

Example: $n = 5, c = 8$

$$
\begin{array}{ccccccccccc}
 & & & 0 & & -15 & & 8 & & & \\
 & & 0 & & -13 & & -3 & & 8 & & \\
 & 0 & & -4 & & -10 & & 0 & & 8 & \\
0 & & -2 & & -5 & & -8 & & 2 & & 8 \\
 & -1 & & -4 & & -6 & & -1 & & 5 &
\end{array}
$$

Definition: A generalized (n,c)-Gelfand pattern is an array $(a_{i,j})_{1 \le i \le n, i-1 \le j \le n+1}$ with

- $a_{i,i-1} = 0$ and $a_{i,n+1} = c$,
- $a_{i-1,j} \in [a_{i,j}, a_{i,j+1}]$ with

$$
[x, y] := \begin{cases} \{x, x+1, \ldots, y\} & x \le y \\ \{y+1, y+2, \ldots, x-1\} & x > y \end{cases}.
$$

$$
F(n, c; k) := \sum_{(a_{i,j}) \text{ is } (n,c)\text{-pattern},\ a_{n,n} = k} (-1)^{\text{inv}}
$$

If $0 \le k \le c$ then $F(n, c; k)$ is the number of (n, c)-Gelfand patterns with $a_{n,n} = k$.

Step 2: $F(n, c; k)$ is a polynomial in k

$n = 3$:

$$\begin{array}{ccccccccc} & 0 & & k & & c & \\ 0 & & a_{2,2} & & a_{2,3} & & c \\ & a_{1,1} & & a_{1,2} & & a_{1,3} & \end{array}$$

$F(3, c; k) =$

$$\sum_{a_{2,2}=0}^{k} \sum_{a_{2,3}=k}^{c} \sum_{a_{1,1}=0}^{a_{2,2}} \sum_{a_{1,2}=a_{2,2}}^{a_{2,3}} \sum_{a_{1,3}=a_{2,3}}^{c} 1$$

In fact: $F(n, c; k)$ is a polynomial in k of degree $2n - 2$.

Step 3: Exploring the zeros of $F(n, c; k)$

$F(n, c; k) = 0$ for $k = -1, -2, \ldots, -n + 1$ and $k = c + 1, c + 2, \ldots, c + n - 1$.

Reason:

If $k < 0$ then $0 > a_{1,1} > a_{2,2} > \ldots > a_{n,n} = k$ and therefore there exists no (n, c)-Gelfand pattern with $k \in \{-1, -2, \ldots, -n + 1\}$.

$$
\begin{array}{ccccccccccc}
 & & & 0 & > & k = a_{5,5} & & c & & & \\
 & & 0 & & a_{4,4} & & a_{4,5} & & c & & \\
 & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & c & \\
0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & c \\
 & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} &
\end{array}
$$

Therefore ...

$$F(n, c; k) = (k+1)_{n-1}(1+c-k)_{n-1} \cdot X_n$$

Computation of X_n:

$$\begin{array}{ccccccccccc}
 & & & 0 & & k=c & & c & & & \\
 & & 0 & & a_{4,4} & & c & & c & & \\
 & 0 & & a_{3,3} & & a_{3,4} & & c & & c & \\
0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & c & & c \\
 & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & c &
\end{array}$$

$$\Rightarrow F(n, c; c) = \sum_{a_{4,4}=0}^{c} F(n-1, c; a_{4,4})$$

Generation function version

Norm of a plane partition = sum of its parts

Theorem (F. 2002): The generating function of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n is

$$\frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \times \prod_{i=1}^{n} \frac{[c+i+1;q]_{i-1}}{[i;q]_i},$$

where $[n;q] = 1 + q + q^2 + \ldots + q^{n-1}$ and $[a;q]_n = \prod_{i=0}^{n-1} [a+i;q]$.

The method can be adapted to q-polynomials (= polynomials in $[X;q]$).

Properties of q-polynomials:

- If $p(X)$ is a q-polynomial of degree n then $\sum_{x=1}^{y} p(x)q^x$ is a q-polynomial in $[y;q]$ of degree $n+1$.

- If $p(X)$ is a q-polynomial and a is an integer zero of $p(X)$ then there exists a q-polynomial $r(X)$ with

$$p(X) = ([X;q] - [a;q])r(X).$$

Refined alternating sign matrix theorem

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Square matrix of 0s, -1s, 1s for which

- row-sums and column-sums are equal to 1
- non-zero entries alternate in each row and column.

$$\text{Number of } n \times n \text{ ASM} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

Number of $n \times n$ ASM where the unique 1 in the first row is in the k-th column $=$

$$\frac{(k)_{n-1}(n+1-k)_{n-1}}{(n-1)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} =: A_{n,k}.$$

ASM:

$$A_{n,k} = \frac{(k)_{n-1}(n+1-k)_{n-1}}{(n-1)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

Strict plane partitions:

$$F(n, n-1; k) = \frac{(k+1)_{n-1}(n-k)_{n-1}}{(n-1)!} \prod_{i=1}^{n-1} \frac{(n+i)_{i-1}}{(i)_i}$$

$$\frac{\#(n, n-1)\text{-patterns with } a_{n,n} = k-1}{A_{n,k}} =$$

$\#(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions.

$n \times n$ ASM are in bijection with Gelfand patterns with strictly increasing rows called monotone triangles.