Alternating Sign Arrays

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

Littlewood-type identities

$$\sum_{\lambda} s_{\lambda}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i x_j}$$

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Outline

I. An old ASM story of a missing bijection: ASMs and TSSCPPs

II. Robbins (symmetric) polynomials

III. A new ASM story of missing bijections: ASTs, ASMs and TSSCPPs

IV. What do ASMs have to do with the Robbins polynomials ?

V. A Littlewood identity related to ASMs: The case $t = u = v = 1$.

VI. A Cauchy identity and another Littlewood identity: The case $t = 0$.

I. Alternating sign matrices (ASMs)

and

Totally symmetric self-complementary plane partitions (TSSCPPs)

Alternating Sign Matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

ASMs generalize permutation matrices !

The number of $n \times n$ ASMs

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

- Conjectured by **Mills, Robbins** and **Rumsey** in the 1980s.
- Zeilberger gave the first proof (of a generalization including an additional parameter) in 1996.
- Kuperberg gave another proof (of the special case) using methods from **statistical physics** such as the **Yang-Baxter equation.**

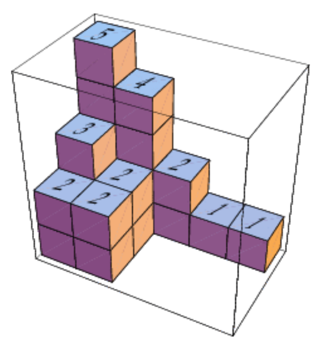
Plane partitions

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$



$a = 5, b = 3, c = 5$

Totally symmetric self-complementary plane partitions

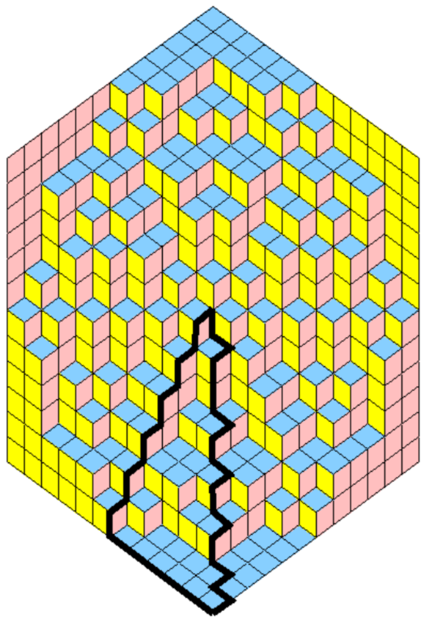


Figure by Di Francesco / Zinn-Justin

- **Totally symmetric:**
$(i,j,k) \in PP \Rightarrow \sigma(i,j,k) \in PP \ \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Theorem (Andrews 1994). The number of TSSCPPs in a $2n \times 2n \times 2n$ box is (also) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

No bijective proof known so far!

II. Robbins (symmetric) polynomials

Definition via a bialternant formula

We define the (Dave) Robbins polynomials as

$$R_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;t,u,v,w) = \frac{\mathbf{ASym}_{x_1,\ldots,x_n}\left[\prod_{1\le i\le j\le n}(tx_j+ux_ix_j+v+wx_i)\prod_{i=1}^{n}x_i^{k_i-1}\right]}{\prod_{1\le i<j\le n}(x_j-x_i)}$$

with

$$\mathbf{ASym}_{x_1,\ldots,x_n}F(x_1,\ldots,x_n)=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}\sigma\,F(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

Schur polynomials are a special case!

Two of several possibilities:

$$R_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n;0,0,0,1) = \frac{\mathbf{ASym}_{x_1,\ldots,x_n}\left[\prod_{i=1}^{n} x_i^{\lambda_i+n-i}\right]}{\prod_{1\le i<j\le n}(x_j-x_i)} = \frac{\det_{1\le i,j\le n}\left[x_i^{\lambda_j+n-j}\right]}{\prod_{1\le i<j\le n}(x_j-x_i)} = s_\lambda(x_1,\ldots,x_n)$$

$$R_{(\lambda_n+1,\ldots,\lambda_1+1)}(x_1,\ldots,x_n;1,0,0,0) = \frac{\mathbf{ASym}_{x_1,\ldots,x_n}\left[\prod_{i=1}^{n} x_i^{\lambda_{n-i}+i}\right]}{\prod_{1\le i<j\le n}(x_j-x_i)} = \frac{\det_{1\le i,j\le n}\left[x_i^{\lambda_j+n-j}\right]}{\prod_{1\le i<j\le n}(x_j-x_i)} = s_\lambda(x_1,\ldots,x_n)$$

Robbins polynomials also generalize the **Hall-Littlewood polynomials** in the following sense:

$$P_\lambda(x_1,\ldots,x_n;t) = R_\lambda(x_1,\ldots,x_n;-t,0,0,1)\,(1-t)^{-n+|\lambda|}\prod_{i\ge 1}\prod_{j=1}^{\#i \text{ in } \lambda}(1-t^j)$$

• Schur polynomials are the "generating functions" of semistandard Young tableaux (SSYT) of fixed shape !

• **Is there a combinatorial model underlying the Robbins polynomials ?**

Gelfand-Tsetlin patterns

A **Gelfand-Tsetlin pattern** is a triangular array of integers of the form

$$\begin{array}{ccccccccc} & & & & a_{1,1} & & & & \\ & & & a_{2,1} & & a_{2,2} & & & \\ & & \dots & & \dots & & \dots & & \\ & a_{n,1} & & \dots & & \dots & & a_{n,n} & \end{array}$$

with weak increase in $\nearrow$- and $\searrow$-direction. The weight of a Gelfand-Tsetlin pattern is $\prod_{i=1}^{n} x_i^{\sum \text{row}_i - \sum \text{row}_{i-1}}$ and $s_\lambda(x_1, \dots, x_n)$ is the sum of weights of all Gelfand-Tsetlin patterns with bottom row $(0, \dots, 0, \lambda_l, \dots, \lambda_1)$.

Example:

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 3 & & 5 & & & \\ & & 2 & & 4 & & 6 & & \\ & 1 & & 3 & & 5 & & 7 & \\ 1 & & 1 & & 4 & & 7 & & 8 \end{array} \quad \Leftrightarrow$$

1	1	1	2	2	3	4	5
2	2	2	3	4	5	5	
3	3	4	5				
4							
5							

Bijection between GT patterns and SSYTs: The i-th row of the GT pattern is the shape (in reverse order) of the entries $\leq i$ in the SSYT.

Example $\lambda = (4,2,2)$

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 2 & & 2 & & 4 \end{array}$$

$$\mathbf{x}_1^2\mathbf{x}_2^2\mathbf{x}_3^4 \qquad \mathbf{x}_1^2\mathbf{x}_2^3\mathbf{x}_3^3 \qquad \mathbf{x}_1^3\mathbf{x}_2^2\mathbf{x}_3^3$$

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 4 & \\ 2 & & 2 & & 4 \end{array}$$

$$\mathbf{x}_1^2\mathbf{x}_2^4\mathbf{x}_3^2 \qquad \mathbf{x}_1^3\mathbf{x}_2^3\mathbf{x}_3^2 \qquad \mathbf{x}_1^4\mathbf{x}_2^2\mathbf{x}_3^2$$

$$s_{(4,2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 x_3^2 (x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2)$$

Case $t = 0$: Arrowed monotone triangles

An arrowed monotone triangle is a Gelfand-Tsetlin where each entry is decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\nearrow\}$ such that for the **little triangles**

$$\begin{array}{ccc} & \mathbf{y} & \\ \mathbf{x} & & \mathbf{z} \end{array}$$

in the pattern we have the following:

- $\text{decor}(x) = \nwarrow, \text{decor}(z) = \nearrow : y \in [x, z]$
- $\text{decor}(x) = \nwarrow, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\nearrow\} : y \in [x, z-1]$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) = \nearrow : y \in [x+1, z]$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\nearrow\} : y \in [x+1, z-1]$

Summary: In principle, we have again weak increase in $\nearrow$- and $\searrow$-direction (as in Gelfand-Tsetlin patterns), with the additional requirement that an arrow indicates strict increase.

Example:

$$\begin{array}{ccccccccc}
 & & & & {}^{\nwarrow}3^{\nearrow} & & & & \\
 & & & {}^{\nwarrow}3 & & {}^{\nwarrow}5 & & & \\
 & & 2^{\nearrow} & & {}^{\nwarrow}4 & & {}^{\nwarrow}6 & & \\
 & 1^{\nearrow} & & 3^{\nearrow} & & 4^{\nearrow} & & {}^{\nwarrow}7^{\nearrow} & \\
{}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} & & 3^{\nearrow} & & {}^{\nwarrow}7 & & {}^{\nwarrow}8
\end{array}$$

Case $t = 0$: Arrowed monotone triangles

Weight:

$$u^{\# \nearrow} v^{\# \nwarrow} w^{\# \nwarrow\!\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum \mathrm{row}_i - \sum \mathrm{row}_{i-1} + \# \nearrow \text{ in row } i - \# \nwarrow \text{ in row } i}$$

Theorem (F. and Schreier-Aigner, 2023) Suppose n is a positive integer and $k_1 < k_2 < \ldots < k_n$ is a finite sequence of integers, then the Robbins polynomial at $t = 0$, that is

$$R_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n; 0, u, v, w),$$

is the generating function of arrowed monotone triangles with bottom row $k_1, \ldots, k_n$.

$t = 0$: Arrowed monotone triangles with arbitrary bottom row

Signed interval:

$$\underline{[a,b]} = \begin{cases} [a,b] & a \le b \\ [b+1, a-1] & b < a-1 \\ \varnothing & b = a-1 \end{cases}$$

In the second case, the interval has sign -1.

To obtain the extension to arbitrary bottom rows, replace all intervals in the definition of arrowed monotone triangles with strictly increasing bottom row by signed intervals!

Weight:

$$(-1)^{\#\text{ of intervals with sign }-1} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum \text{row}_i - \sum \text{row}_{i-1} + \#\nearrow \text{ in row } i - \#\nwarrow \text{ in row } i}$$

Example:

$$
\begin{array}{ccccccccc}
 & & & & {}^{\nwarrow}3^{\nearrow} & & & & \\
 & & & 4^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}3 & & & \\
 & & {}^{\nwarrow}4 & & 4^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}2 & & \\
 & 4^{\nearrow} & \leftrightsquigarrow & 3^{\nearrow} & & 4^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}2^{\nearrow} & \\
5^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}2^{\nearrow} & & 3^{\nearrow} & & {}^{\nwarrow}6 & \leftrightsquigarrow & {}^{\nwarrow}2
\end{array}
$$

Negative intervals are indicated using "$\leftrightsquigarrow$".

Theorem (F. and Schreier-Aigner, 2023) Suppose n is a positive integer and $k_1, k_2, \ldots, k_n$ is any finite sequence of integers, then the Robbins polynomial at $t = 0$, that is

$$R_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n; 0, u, v, w),$$

is the generating function of arrowed monotone triangles with bottom row $k_1, \ldots, k_n$.

Arrowed Gelfand-Tsetlin patterns for general t

An arrowed monotone triangle is a Gelfand-Tsetlin where each entry is decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\!\nearrow, \emptyset\}$ such that for the **little triangles**

$$\begin{array}{ccc} & \mathbf{y} & \\ \mathbf{x} & & \mathbf{z} \end{array}$$

in the pattern we have the following:

- $\text{decor}(x) \in \{\nwarrow, \emptyset\}, \text{decor}(z) \in \{\nearrow, \emptyset\} \Rightarrow y \in \underline{[x, z]}$
- $\text{decor}(x) \in \{\nwarrow, \emptyset\}, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\} \Rightarrow y \in \underline{[x, z-1]}$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) \in \{\nearrow, \emptyset\} \Rightarrow y \in \underline{[x+1, z]}$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\} \Rightarrow y \in \underline{[x+1, z-1]}$

Example:

$$\begin{array}{ccccccccc}
 & & & & {}^{\nwarrow}3^{\nearrow} & & & & \\
 & & & 4^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}3 & & & \\
 & & {}^{\nwarrow}4 & & 4^{\nearrow} & \leftrightsquigarrow & {}^{\nwarrow}2 & & \\
 & 4^{\nearrow} & \leftrightsquigarrow & 3^{\nearrow} & & 4 & \leftrightsquigarrow & {}^{\nwarrow}2^{\nearrow} & \\
5 & \leftrightsquigarrow & 2 & & 3^{\nearrow} & & {}^{\nwarrow}6 & \leftrightsquigarrow & {}^{\nwarrow}2
\end{array}$$

Arrowed Gelfand-Tsetlin patterns for general t

Weight:

$$(-1)^{\#\text{ of intervals with sign }-1} t^{\#\varnothing} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum \text{row}_i - \sum \text{row}_{i-1} + \#\nearrow \text{ in row } i - \#\nwarrow \text{ in row } i}$$

Theorem (F. and Schreier-Aigner, 2023) Suppose n is a positive integer and $k_1, k_2, \ldots, k_n$ is a finite sequence of integers, then the Robbins polynomial

$$R_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n; t, u, v, w)$$

is the generating function of arrowed Gelfand-Tsetlin patterns with bottom row $k_1, \ldots, k_n$.

Why was it necessary to introduce signed sets for the general case?

Even it the rows are weakly increasing we can have the following situation:

$$\begin{array}{ccc} & y & \\ x^{\nearrow} & & {}^{\nwarrow}x \end{array}$$

Then, by definition,

$$y \in \underline{[x+1, x-1]} = [x-1+1, x+1-1] = [x, x] = \{x\}$$

and this configuration contributes -1 to the weight.

III. A new ASM story of missing bijections:

Alternating sign triangles

Alternating sign triangles = ASTs

An AST of order n is a triangular array of 1's, −1's and 0's with n centered rows

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & & \\ & & & \bullet & & & \end{array}$$

such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

ASTs of order 3

$$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}\;\Bigg|\;\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}\;\Bigg|\;\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}\;\Bigg|\;\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$$

$$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}\;\Bigg|\;\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}\;\Bigg|\;\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$$

Theorem (Ayyer, Behrend, and F., 2020). The number of ASTs with n rows is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

Number of -1's in ASMs and ASTs

Let A be an ASM or an AST. Then we define

$$\mu(A) = \# \text{ of } -1\text{'s in } A.$$

We have $|\{A \in \mathrm{ASM}(n) \mid \mu(A) = 0\}| = n! = |\{A \in \mathrm{AST}(n) \mid \mu(A) = 0\}|$.

Why?

ASTs with no -1: Unique 1 in each row and at most one occurrence of 1 in each column.

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & 0 & 1 & & \\ & & & 1 & & & \end{array}$$

Enumeration of ASTs with no -1: Work from bottom to top to choose the column of the unique 1 in each row. There are $(2i-1)-(i-1)=i$ columns to choose in row i from the bottom, so in total we obtain $1 \cdot 2 \cdot 3 \cdot \dots \cdot n = n!$.

Generalization of our theorem: Let m, n be non-negative integers. Then

$$|\{A \in \mathrm{ASM}(n) \mid \mu(A) = m\}| = |\{A \in \mathrm{AST}(n) \mid \mu(A) = m\}|.$$

Inversion numbers

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation and A be the permutation matrix of π, that is π_i is the column of the unique 1 in row i. Then

$$\text{inv}(A) = \sum_{1 \le i' < i \le n, 1 \le j' \le j \le n} a_{i'j} a_{ij'}$$

is the number of inversions in π.

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & (i',j) & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & (i,j') & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

We use this to define the inversion number of ASMs.

Let $A = (a_{i,j})_{1 \le i \le n, i \le j \le 2n-i}$ be an AST. We define

$$\text{inv}(A) = \sum_{i' < i, j' \le j} a_{i'j} a_{ij'}.$$

Generalization of the generalization of our theorem: Let m, n, i be non-negative integers. Then

$$|\{A \in \text{ASM}(n) \mid \mu(A) = m, \text{inv}(A) = i\}| = |\{A \in \text{AST}(n) \mid \mu(A) = m, \text{inv}(A) = i\}|.$$

The case $n = 3$

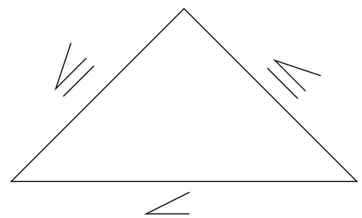
ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
AST	$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}$

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
AST	$\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$

IV. What do ASMs have to do with the Robbins polynomials ?

Monotone triangles

Triangular arrays of integers with monotonicity requirements:



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

- Monotone triangles with bottom row $1, 2, \ldots, n \Leftrightarrow n \times n$ ASMs

- **Monotone triangles** are defined as Gelfand-Tsetlin patterns with **strictly increasing rows.**

Recall: Arrowed monotone triangles

An arrowed monotone triangle is a Gelfand-Tsetlin where each entry is decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\nearrow\}$ such that for the **little triangles**

$$\begin{array}{ccc} & \mathbf{y} & \\ \mathbf{x} & & \mathbf{z} \end{array}$$

in the pattern we have the following:

- $\text{decor}(x) = \nwarrow, \text{decor}(z) = \nearrow \Rightarrow y \in [x, z]$
- $\text{decor}(x) = \nwarrow, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\nearrow\} \Rightarrow y \in [x, z-1]$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) = \nearrow \Rightarrow y \in [x+1, z]$
- $\text{decor}(x) \in \{\nwarrow\!\!\!\!\nearrow, \nearrow\}, \text{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\nearrow\} \Rightarrow y \in [x+1, z-1]$

Weight:

$$u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum \text{row}_i - \sum \text{row}_{i-1} + \#\nearrow \text{ in row } i - \#\nwarrow \text{ in row } i}$$

Homework: If the bottom row of an arrowed monotone triangle is strictly increasing then all rows of the underlying Gelfand-Tsetlin pattern are strictly increasing !

Arrowed monotone triangles $\rightarrow$ monotone triangles

Claim: When setting $u = v = 1, w = -1$ and $(x_1, \ldots, x_n) = (1, \ldots, 1)$ in the generating function of arrowed monotone triangles, we obtain the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$.

Why?

• Fix a **monotone triangle** and consider all arrowed monotone triangles that can be obtained by decorating the entries of that monotone triangle.

• Namely, an entry in the monotone triangle that is **equal to its $\nwarrow$-neighbor can only be decorated by $\nearrow$**, while an entry that is **equal to its $\nearrow$-neighbor can only be decorated by $\nwarrow$.**

• Let l **be the number of entries of the first type** and r **be the number of entries of the second type.** All other entries can be decorated by any element in $\{\nwarrow, \nearrow, \nwarrow\!\!\!\nearrow\}$ and we let f be their number.

• Setting $(x_1, \ldots, x_n) = (1, \ldots, 1)$ in the generating function, we see that the contribution of the fixed monotone triangle is

$$u^l v^r (u + v + w)^f$$

and this reduces to 1 when setting $u = v = 1$ and $w = -1$.

V. A Littlewood-type identity related to ASMs:

The case $t = u = v = 1$.

The classical (unbounded) Littlewood identity

$$\sum_{\lambda} s_{\lambda}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j}$$

Proof: RSK and exploiting its symmetry.

The key ingredient in the work of myself and of Hans Höngesberg to show that there is a same number of ASTs as there is of ASMs is the following identity.

$$\sum_{0\le k_1<k_2<\ldots<k_n} R_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,1,w) = \prod_{i=1}^{n} \frac{x_i^{-1}+(1+w)+x_i}{1-x_i} \prod_{1\le i<j\le n} \frac{1+x_i+x_j+w x_i x_j}{1-x_i x_j}$$

Combinatorial interpretation of the RHS of the Littlewood-type identity: decorated two-line arrays (straightforward).

Moritz Gangl, Hans Höngesberg, Florian Schreier-Aigner and myself have worked out an RSK-like bijective proof of this result for $w=1$ (forthcoming).

Bounded classical Littlewood identity

Bounded? $\sum_{0\le k_1<k_2<\ldots<k_n} \to \sum_{0\le k_1<k_2<\ldots<k_n\le m}$

$$\sum_{\lambda\subseteq(m^n)} s_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{j-1} - x_i^{m+2n-j}\right)}{\prod_{i=1}^n(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}$$

Macdonald in his book.

Bounded Littlewood identity related to ASMs

Theorem (F., 2024+).

$$\sum_{0 \le k_1 < k_2 < \ldots < k_n \le m} R_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,1,w)$$
$$= \frac{\det_{1 \le i,j \le n}\left(x_i^{j-1}(1+x_i)^{j-1}(1+wx_i)^{n-j} - x_i^{m+2n-j}(1+x_i^{-1})^{j-1}(1+wx_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-x_i)\prod_{1 \le i < j \le n}(1-x_i x_j)(x_j - x_i)}$$

• The proof (of a more general Q-analog) has more than 7 pages, but it is elementary.

• Moritz Gangl has a nice application of this relating a (very refined) AST-counting to a (very refined) TSSCPP-counting. The significance stems from the fact that there are some analogous old conjectures of Mills, Robbins and Rumsey and Krattenthaler on the relation between an analogous (very refined) ASM-counting and an analogous (very refined) TSSCPP-counting that are still open!

$$\mathbf{1, 4, 60, 3328, 678912 \ldots}$$

Setting all $x_i = 1, w = -1$ and $m = n - 1$, we obtain

$$1, 4, 60, 3328, 678912, \ldots = 2^{n(n-1)/2} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}.$$

• This is a consequence of our Theorem 1 below.

• In fact Theorem 1 involves the **additional parameter** m, that is we do not need to specify $m = n - 1$.

• Note that $m = n - 1$ just means that we consider arrowed Gelfand-Tsetlin patterns with bottom row $(0, 1, \ldots, n - 1)$.

• These numbers also appear in recent work of Di Francesco as the conjectural number of certain **twenty vertex model configurations** and **domino tilings.** The conjecture was subsequently proven by Koutschan.

Explicit product formulas in case $x_i = 1$ and $w = -1$

Theorem 1 (F. and Schreier-Aigner, 2024). For $(x_1, \ldots, x_n) = (1, \ldots, 1)$ and $w = -1$, we have that

$$\frac{\det_{1 \le i,j \le n}\left(x_i^{j-1}(1+x_i)^{j-1}(1+wx_i)^{n-j} - x_i^{m+2n-j}(1+x_i^{-1})^{j-1}(1+wx_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-x_i) \prod_{1 \le i<j \le n}(1-x_i x_j)(x_j - x_i)}$$

is

$$2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i},$$

where $(a)_n = a(a+1)\ldots(a+n-1)$.

We have signless versions for the objects (decorated Gelfand-Tsetlin patterns) in the theorem.

Open problem: Find a bijection between Di Francesco's twenty vertex configurations and our objects for $m = n-1$.

VI. A Cauchy identity and another Littlewood related to ASMs: The case $t = 0$.

Generalizations of the classical cases

Theorem (F. and Höngesberg, 2024+). For each positive integer n, there is a symmetric polynomial $t(x_1,\dots,x_n)$ with

$$\sum_{0\le k_1<k_2<\ldots<k_n} R_{(k_1,\dots,k_n)}(x_1,\dots,x_n;0,u,v,w)=\prod_{i=1}^{n}\frac{1}{1-x_i}\frac{t(x_1,\dots,x_n)}{\prod_{1\le i<j\le n}(1-x_ix_j)}.$$

The polynomials $t(x_1,\dots,x_n)$ have several nice properties.

Theorem (F. and Gangl, 2024+). For each positive integer n, there is a symmetric polynomial $d(x_1,\dots,x_n;y_1,\dots,y_n)$ with

$$\sum_{0\le k_1<k_2<\ldots<k_n} R_{(k_1,\dots,k_n)}(x_1,\dots,x_n;0,u,v,w)R_{(k_1,\dots,k_n)}(y_1,\dots,y_n;0,v,u,w)$$
$$=\frac{d(x_1,\dots,x_n;y_1,\dots,y_n)}{\prod_{1\le i,j\le n}(1-x_iy_j)}.$$

The polynomials $d(x_1,\dots,x_n;y_1,\dots,y_n)$ have several nice properties.

Thank you!

Refined ASM-Theorem

Observation: There is a unique 1 in the top row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in the top row and column r is

$$\binom{n+r-2}{n-1} \frac{(2n-r-1)!}{(n-r)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} = A_{n,r}.$$

Find a statistic on ASTs that has the same distribution as the column of the 1 in the top row of an ASM!

Equivalent statistic on ASTs

In an AST, the columns add up to 0 or 1. We say that a column is a 1-column if they add up to 1.

Let T be an AST with n rows. Define

$$\rho(T) = (\#\text{1-columns in the left half of } T \text{ that have a 0 at the bottom}) \\ + (\#\text{1-columns in the right half of } T \text{ that have a 1 at the bottom}) + 1.$$

This statistic was introduced by Behrend.

Theorem (F., 2019). The number of ASTs T with n rows and $\rho(T) = r$ is equal to $A_{n,r}$.

The case $n = 3$

ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
Top 1	1	1	2	2
ρ	1	3	2	2
AST	$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}$

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
Top 1	3	3	2
ρ	1	3	2
AST	$\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$

Remark. The statistics inv can also be replaced by inv of the rotated or reflected ASM and/or the reflected AST. Rotation by 90° and reflection replaces inv by $3 - \text{inv}$. However, also after such a replacement it is **not possible to have a bijection that preserves all three statistics.**