Diagonally and antidiagonally symmetric alternating sign matrices of odd order

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ASM = Alternating sign matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

Theorem (Zeilberger, 1995).

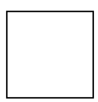
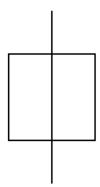
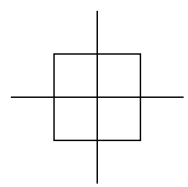
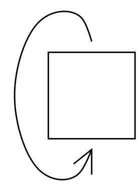
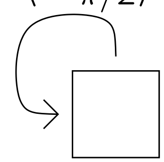
$$\# \text{ of } n \times n \text{ ASMs} = \frac{1!4!7! \cdots (3n-2)!}{n!(n+1)! \cdots (2n-1)!} = \prod_{i=1}^{n-1} \frac{\binom{3i+1}{i}}{\binom{2i}{i}}$$

Symmetry classes of ASMs

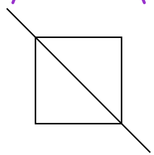
Stanley, Robbins, 1980s: Symmetry group of the square

$$\mathcal{D}_4 = \{\mathcal{I}, \underbrace{\mathcal{V}, \mathcal{H}, \mathcal{D}, \mathcal{A}}_{\text{reflections}}, \underbrace{\mathcal{R}_{\pi/2}, \mathcal{R}_{\pi}, \mathcal{R}_{-\pi/2}}_{\text{rotations}}\}$$

10 subgroups, 8 conjugacy classes

$\{\mathcal{I}\}$	$\langle\mathcal{V}\rangle \sim \langle\mathcal{H}\rangle$	$\langle\mathcal{V}, \mathcal{H}\rangle$	$\langle\mathcal{R}_{\pi}\rangle$	$\langle\mathcal{R}_{\pi/2}\rangle$
				
Zeilberger 1995	Kuperberg 2002	Okada 2004	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005

$\langle \mathcal{D} \rangle \sim \langle \mathcal{A} \rangle$

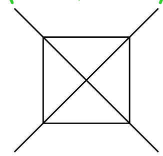


n	#
1	1
2	2
3	5
4	2^4
5	67
6	$2^4 \cdot 23$
7	$2 \cdot 5 \cdot 263$
8	$2^3 \cdot 11 \cdot 277$

$\mathcal{D}_4$

n	#
1	1
2	0
3	1
4	0
5	1
6	0
7	2
8	0
9	2^2
10	0
11	13
12	0
13	$2 \cdot 23$
14	0
15	$2^3 \cdot 31$
16	0
17	$2^2 \cdot 379$

$\langle \mathcal{D}, \mathcal{A} \rangle$



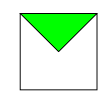
	n	#
$\rightarrow$	1	1
	2	2
$\rightarrow$	3	3
	4	2^3
$\rightarrow$	5	$3 \cdot 5$
	6	$2^2 \cdot 13$
$\rightarrow$	7	$2 \cdot 3^2 \cdot 7$
	8	$2^3 \cdot 71$
$\rightarrow$	9	$2 \cdot 3^4 \cdot 11$
	10	$2^2 \cdot 2609$
$\rightarrow$	11	$3^3 \cdot 11^2 \cdot 13$
	12	$2^3 \cdot 31 \cdot 1303$
$\rightarrow$	13	$2 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 17$

DASASMs

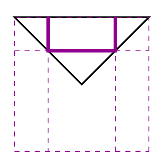
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Conjecture (Robbins 1980s). The number of $(2n+1) \times (2n+1)$ DASASMs is

$$\prod_{i=1}^{n} \frac{\binom{3i}{i}}{\binom{2i-1}{i}}.$$

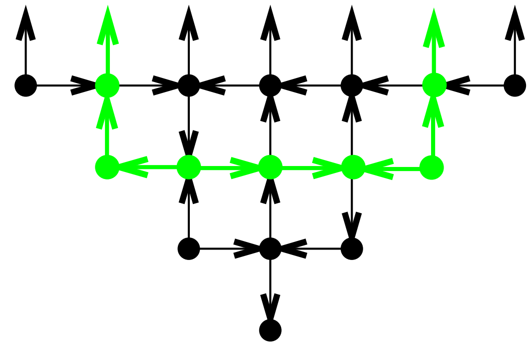
- It suffices to know the entries in the fundamental triangle: 

- Conversely, a triangular array is the fundamental triangle of a DASASM if 1s and -1s alternate and add up to 1 along paths of the following type:



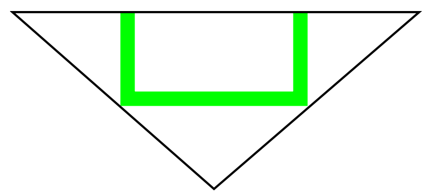
DASASM-triangle $\rightarrow$ 6-vertex configuration

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\ & & & 1 & & -1 & & 0 & & 1 & & 0 & \\ & & & & 0 & & 1 & & -1 & & & & \\ & & & & & & -1 & & & & & & \end{array}$$



Transformation:

1) Top edges are oriented upward.

2) Work through all paths of type  :

- Along straight lines, change orientation iff ± 1.
- As for turns, change orientation iff 0.

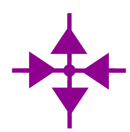
Triangular 6-vertex configuration

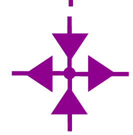
We obtain orientations of triangular graphs s.t.

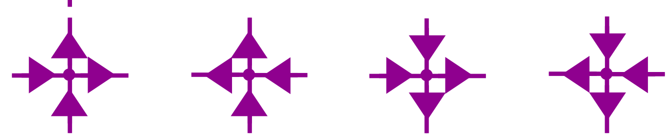
- all degree 4 vertices are "balanced",
- all top edges are oriented upward.

1-1-correspondence with DASASMs

Dictionary:

$1 \leftrightarrow$ 

$-1 \leftrightarrow$ 

$0 \leftrightarrow$ 

Weighted enumeration

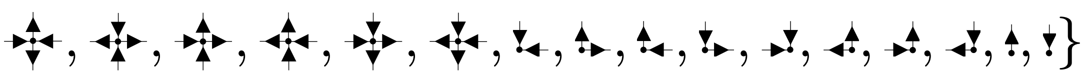
- Assign to each vertex v a weight $\mathsf{W}(v)$.

- Weight $\mathsf{W}(C)$ of a configuration C: $\mathsf{W}(C) = \prod_{v \in C} \mathsf{W}(v)$

- Generating function=partition function:

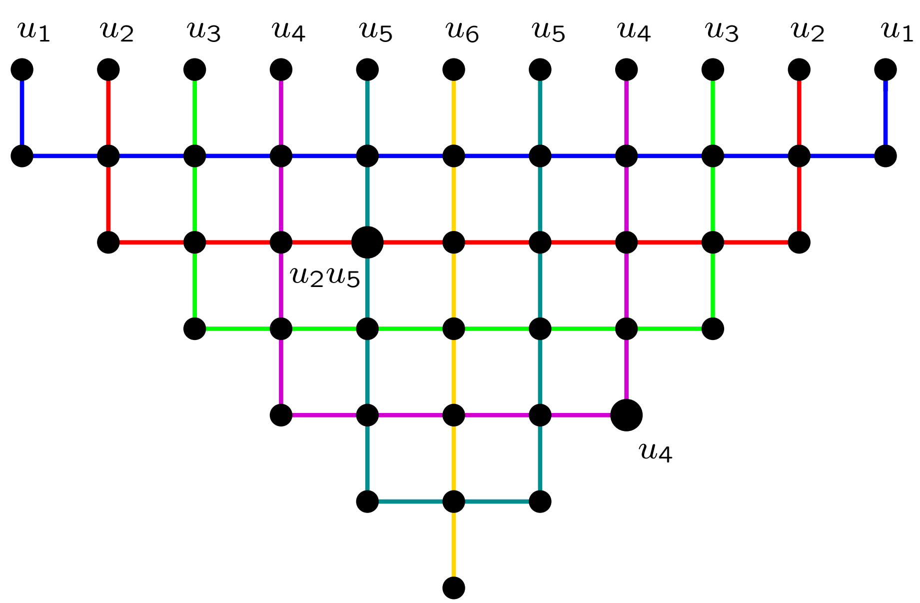
$$Z_n = \sum_{C \text{ 6-vertex configuration}} \mathsf{W}(C)$$

- Specialize parameters $\rightarrow$ # of $(2n+1) \times (2n+1)$ DASASMs

- The vertex weights $\mathsf{W}(c, u)$ depend on the orientations c of surrounding edges, i.e. $c \in \{$  $\}$, and the label u of the vertex.

Label of a vertex

Each colored path is assigned a parameter u_i as follows.

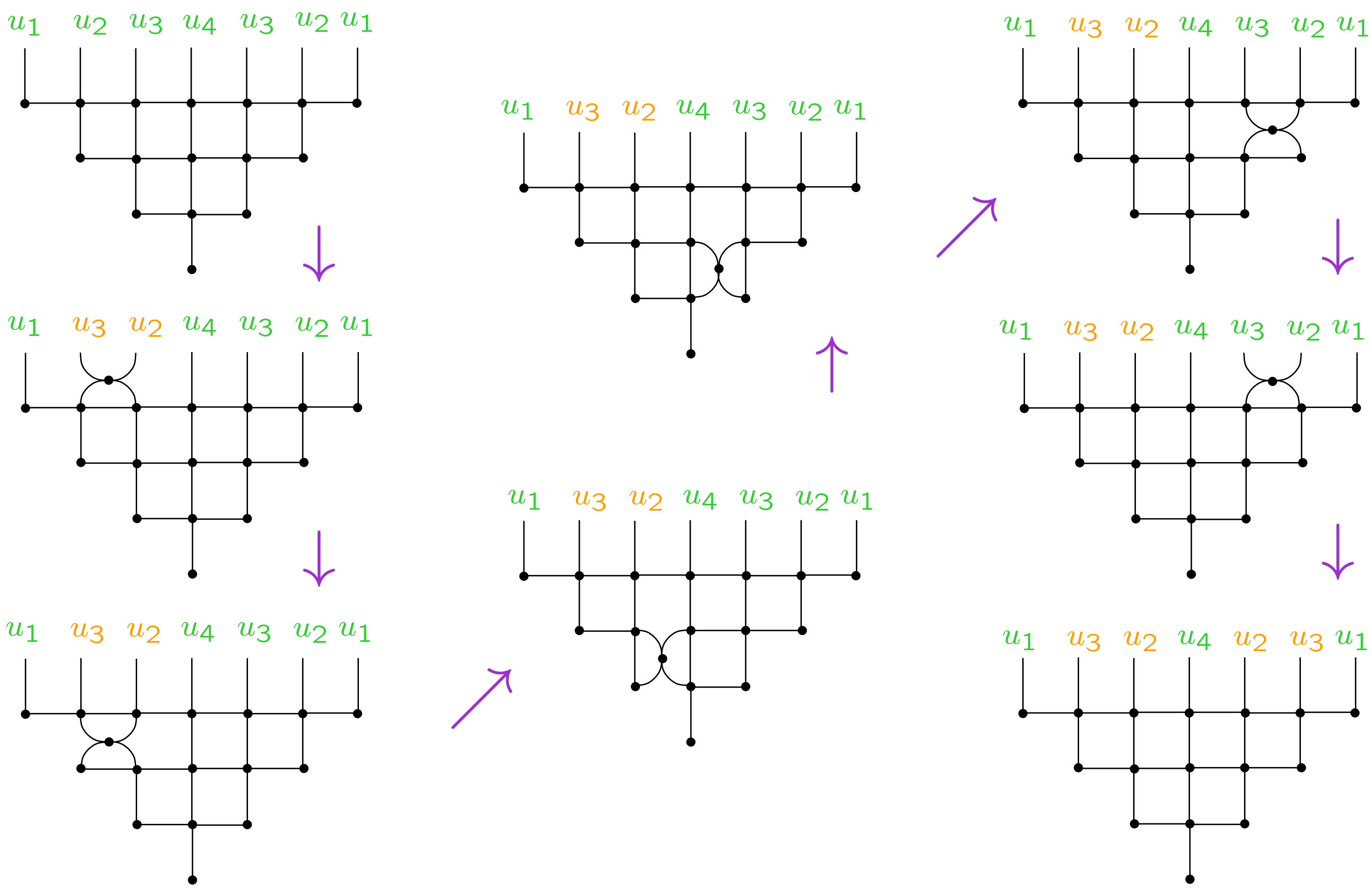


• A degree 4 vertex is contained in two colored paths u_i and u_j $\Rightarrow$ label u_iu_j

• All boundary vertices have a unique path u_i $\Rightarrow$ label u_i

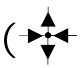
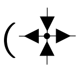
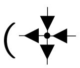
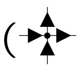
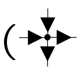
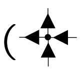
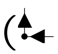
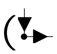
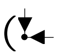
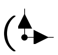
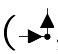
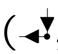
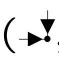
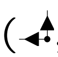
Generating function: $Z_n(u_1, \ldots, u_n; u_{n+1})$.

Symmetry in $u_1, u_2, \ldots, u_n$



Vertex weights

Notation: $x^{-1} = \bar{x}$ and $\sigma(x) = x - \bar{x}$

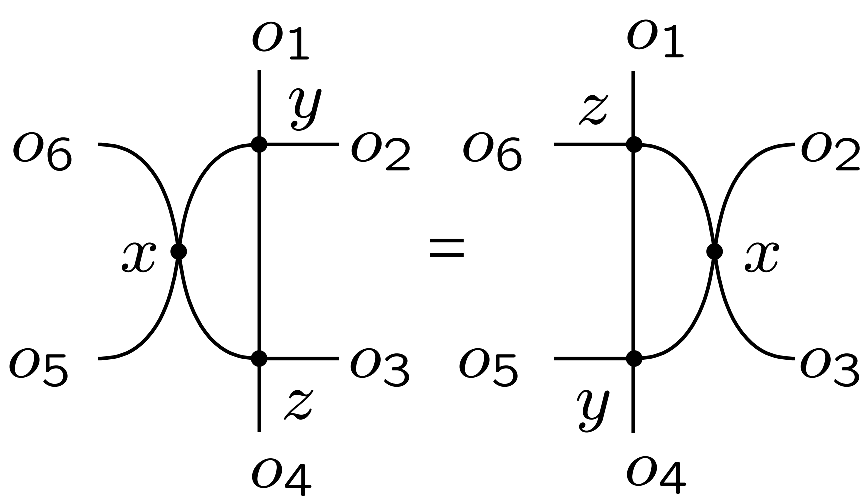
bulk	left	right
W(, u) $=$ W(, u) $= 1$ W(, u) $=$ W(, u) $= \frac{\sigma(q^2 u)}{\sigma(q^4)}$ W(, u) $=$ W(, u) $= \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$	W(, u) $=$ W(, u) $= 1$ W(, u) $=$ W(, u) $= \frac{\sigma(qu)}{\sigma(q)}$	W(, u) $=$ W(, u) $= 1$ W(, u) $=$ W(, u) $= \frac{\sigma(q\bar{u})}{\sigma(q)}$

Degree 1 vertices have weight 1.

If $u = 1$ and $q = e^{i\pi/6}$, all weights are 1!

Yang-Baxter equation

Theorem. If $xyz = q^2$ and $o_1, o_2, \ldots, o_6 \in \{\text{in}, \text{out}\}$, then

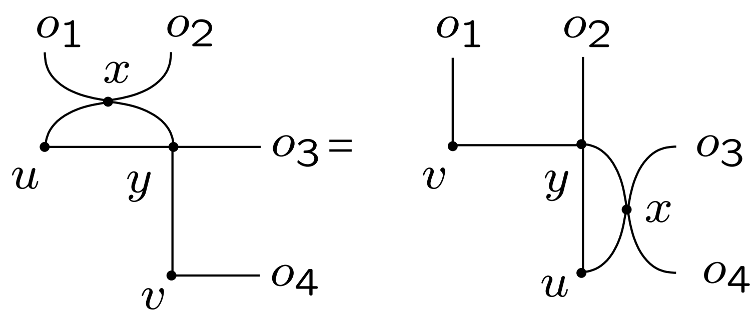


.

A diagram stands for the generating function of all orientations of the graph such that the external edges have the prescribed orientations $o_1, o_2, \ldots, o_6$, degree 4 vertices are balanced, and the vertex weights are as given in the table, where the letter close to a vertex indicates its label (rotate until the label is in the SW corner).

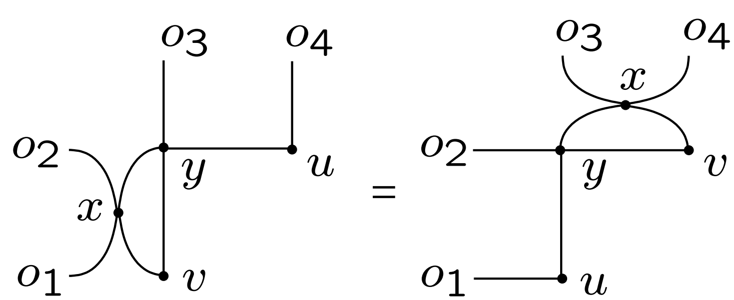
Reflection equations

Theorem. Suppose $o_1, o_2, o_3, o_4 \in \{\text{in}, \text{out}\}$. If $x = q^2\bar{u}v$ and $y = uv$, then



,

and if $x = q^2\bar{u}v$ and $y = \bar{u}\bar{v}$, then



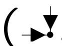
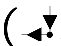
.

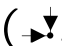
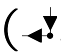
$\Rightarrow$ Symmetry of $Z_n(u_1, \ldots, u_n; u_{n+1})$ in $u_1, \ldots, u_n$.

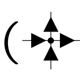
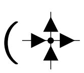
Another important property

Lemma.

$Z_n(u_1, \ldots, u_n; q^2\bar{u}_1)$

$= \frac{1}{2}\Big(($W( $, u_1) +$ W( $, u_1) +$ W( $, u_1) +$ W( $, u_1))\,Z_{n-1}(u_2, \ldots, u_n; u_1)$

$+(-1)^{n+1}($W( $, u_1) +$ W( $, u_1) -$ W( $, u_1) -$ W( $, u_1))\,Z_{n-1}(u_2, \ldots, u_n; -u_1)\Big)$

$\times$ W( $, u_1) \prod_{i=2}^{n}$ W( $, u_1 u_i)$ W( $, q^2\bar{u}_1 u_i).$

$Z_n(u_1,\ldots,u_n;u_{n+1})$ at $u_{n+1}=1$

Theorem (BFK 2015).

$$Z_n(u_1,\ldots,u_n;1)$$
$$=\frac{\sigma(q^2)^n}{\sigma(q)^{2n}\sigma(q^4)^{n^2}}\prod_{i=1}^{n}\sigma(qu_i)\sigma(q\bar{u}_i)\sigma(q^2u_i)\sigma(q^2\bar{u}_i)$$
$$\times\prod_{1\le i<j\le n}\left(\frac{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)}\right)^2\mathrm{det}_{1\le i,j\le n}\left(\frac{q^2+\bar{q}^2+u_i^2+\bar{u}_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}\right).$$

Yet another problem: $(u_1,\ldots,u_n)=(1,\ldots,1)\Rightarrow\frac{0}{0}$

Schur function expression for $Z_n(u_1, \ldots, u_n; 1)$ at $q = e^{i\pi/6}$ à la Soichi Okada

Theorem (BFK 2015).

$$Z_n(u_1, \ldots, u_n; 1)\big|_{q=e^{i\pi/6}} = 3^{-\binom{n}{2}} \times \mathsf{s}_{(n,n-1,n-1,n-2,n-2,\ldots,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2, 1)$$

Now we may use the formula

$$\mathsf{s}_\lambda(1, \ldots, 1) = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

to conclude the proof of the DASASM (ex-)conjecture. □

Stroganov's refined DASASM conjecture

Observation: The central entry of an odd order DASASM is ± 1.

Conjecture (Stroganov, 2008).

$$\frac{\text{\# of DASASMs } (a_{i,j})_{1 \le i,j \le 2n+1} \text{ with } a_{n+1,n+1} = -1}{\text{\# of DASASMs } (a_{i,j})_{1 \le i,j \le 2n+1} \text{ with } a_{n+1,n+1} = 1} = \frac{n}{n+1}$$

Combinatorial proof?

Refined generating functions

$Z_n^{\pm}(u_1, \ldots, u_n; u_{n+1}) =$ generating function where we restrict to DASASMs that have a ± 1 in the center.

Lemma.

$$Z_n^{\pm}(u_1, \ldots, u_n; u_{n+1}) = \frac{1}{2}\Big(Z_n(u_1, \ldots, u_n; u_{n+1}) \\ \pm (-1)^n Z_n(u_1, \ldots, u_n; -u_{n+1})\Big)$$

$Z_n(u_1,\dots,u_n;u_{n+1})$ at arbitrary u_{n+1}

Theorem (BFK 2015).

$$Z_n(u_1,\dots,u_n;u_{n+1}) = \frac{\sigma(q^2)^n}{\sigma(q)^{2n}\sigma(q^4)^{n^2}} \prod_{i=1}^{n} \frac{\sigma(u_i)\sigma(qu_i)\sigma(q\bar{u}_i)\sigma(q^2u_iu_{n+1})\sigma(q^2\bar{u}_i\bar{u}_{n+1})}{\sigma(u_i\bar{u}_{n+1})}$$

$$\times \prod_{1\le i<j\le n} \left(\frac{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)}\right)^2 \left(\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+\bar{q}^2+u_i^2+\bar{u}_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\ \frac{u_{n+1}-1}{u_j^2-1}, & i=n+1\end{cases}\right)\right.$$

$$\left. + \det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+\bar{q}^2+\bar{u}_i^2+u_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\ \frac{\bar{u}_{n+1}-1}{\bar{u}_j^2-1}, & i=n+1\end{cases}\right)\right)$$

Schur function expression for $Z_n(u_1, \ldots, u_n; u_{n+1})$ at $q = e^{i\pi/6}$

Theorem (BFK 2015).

$$
\begin{aligned}
&Z_n(u_1, \ldots, u_n; u_{n+1})\big|_{q=e^{i\pi/6}} \\
&\quad = 3^{-\binom{n}{2}} \Big(\frac{u_{n+1}^n}{u_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2) \\
&\qquad\qquad + \frac{\bar{u}_{n+1}^n}{\bar{u}_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2, u_{n+1}^2) \Big).
\end{aligned}
$$

This implies Stroganov's conjecture. $\square$