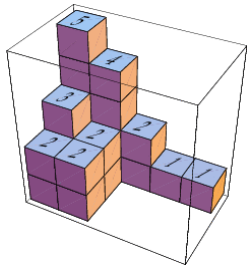
Alternating Sign Matrices and Plane Partitions

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Ilse Fischer

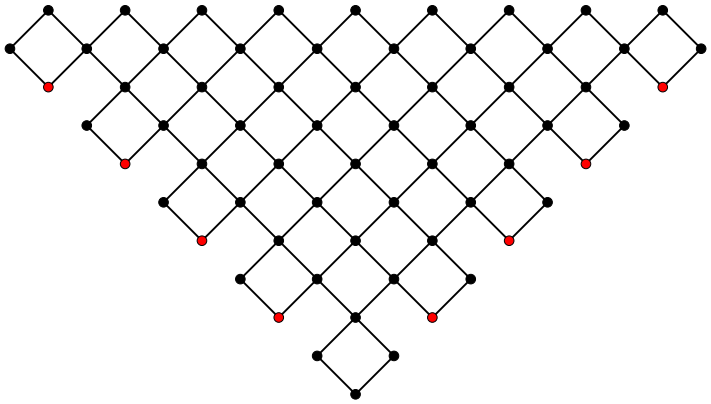
Universität Wien

FWF Österreichischer Wissenschaftsfonds

Enumeration of perfect matchings: some combinatorial tools

- Spider move (= urban renewal, square move)
- Kuo condensation
- Kuo condensation with free boundary
- Ciucu's factorization theorem

Exercise 5. Let T_n denote the graph indicated below where n is the number of "rows" ($n = 5$ in the example) and let W denote the set of red vertices.

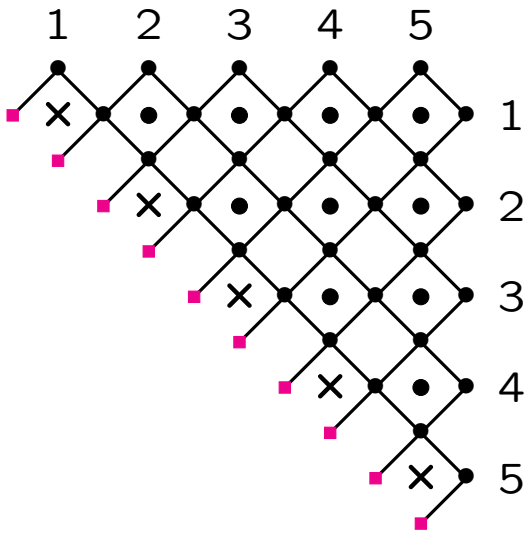


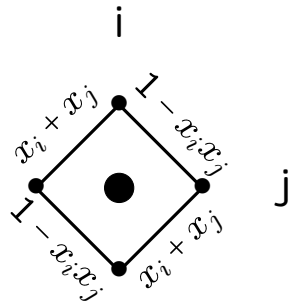
Prove

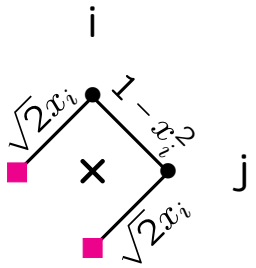
$$\sum_{U \subseteq W, |U| = n-1} (\#\text{ of perfect matchings of } T_n - U) = 2^{\binom{n+1}{2}}$$

by combinatorial means.

Exercise 7. Consider the graph below with the edge weights as indicated on the right.







Prove by combinatorial means that the matching generating function where only the vertices in magenta may be avoided is

$$\prod_{1 \le i \le j \le n} (1 + x_i x_j).$$

Exercise 14.

For any partition $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and integers $k_1, k_2 \geq \lambda_0$ show by combinatorial means that

$$\prod_{i=1}^{n} \left(x_i^{1/2} + x_i^{-1/2}\right) \left(s_{(k_1+\lambda_1,\ldots,k_1+\lambda_n,k_1-\lambda_{n-1},\ldots,k_1-\lambda_0)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) \right.$$
$$\left. + s_{(k_2+\lambda_1,\ldots,k_2+\lambda_{n-1},k_2-\lambda_n,\ldots,k_2-\lambda_0)}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})\right)$$
$$= (1+\delta_{\lambda_n,0})\, so^{\mathrm{odd}}_{(\lambda_0+1/2,\ldots,\lambda_{n-1}+1/2)}(x_1,\ldots,x_n)\, o^{\mathrm{even}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n).$$

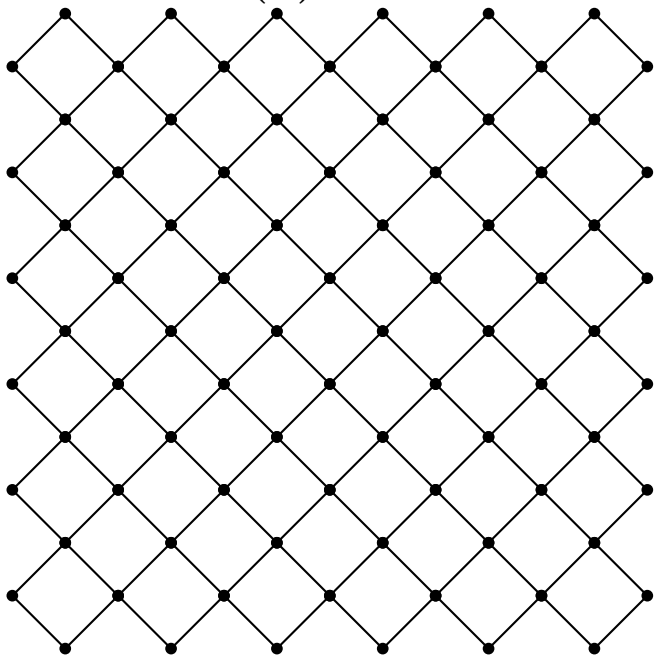
Then prove this identity also for half-partitions $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and half-integers $k_1, k_2 \geq \lambda_0$.

The spider move (=urban renewal, square move)

ASM

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

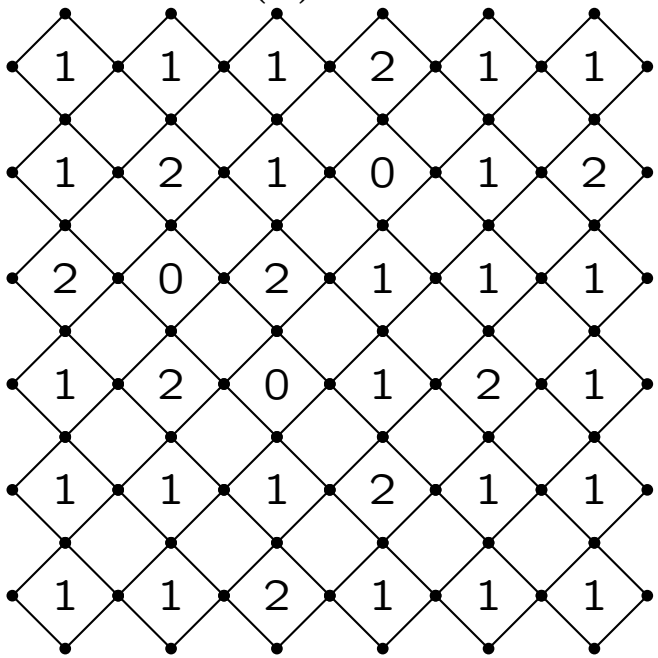
$AD(6)$



ASM

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

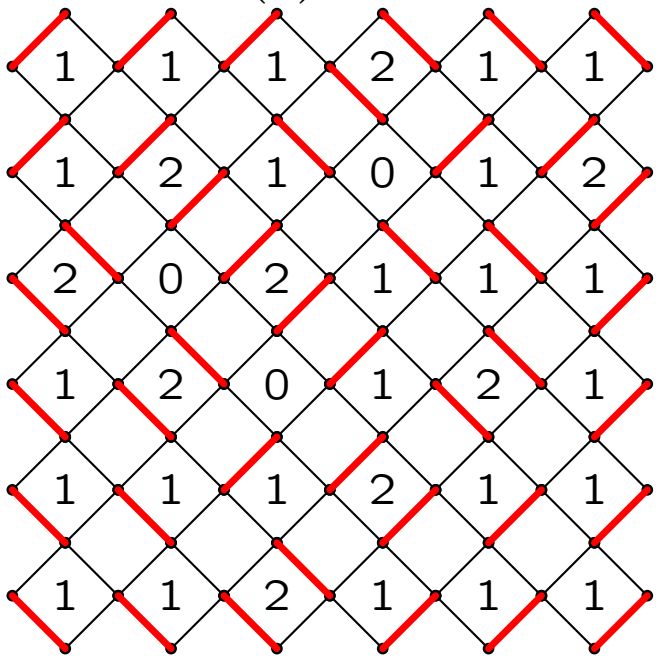
$AD(6)$



ASM

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$AD(6)$



Definition. An **alternating sign matrix** (ASM) is a square matrix with entries in $\{0, \pm 1\}$ such that, in each row and column, non-zero entries alternate and have a sum of 1.

Definition. A **matching** of a graph is a subset of the edges such that each vertex is contained in at most one selected edge. A **perfect matching** is a matching where each vertex is contained in precisely one selected edge.

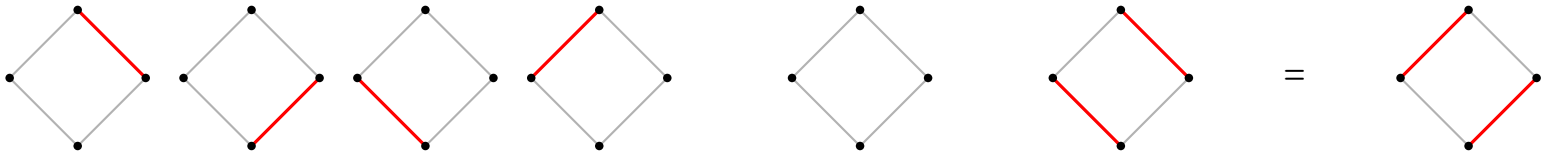
Instead of **perfect matching** people also say **dimer (covering)** or 1-**factor.**

Exercise 1. Every $n \times n$ ASM corresponds to $2^{\#\text{of 1's in the ASM}}$ perfect matchings of $AD(n)$, and, vice versa, every perfect matching of $AD(n)$ comes from an ASM in this way.

Hint. Use the six-vertex model point of view on ASMs.

Exercise 1A. Learn about the correspondence between ASMs and six-vertex configurations with domain wall boundary conditions.

The following configurations correspond to the six-vertex configurations.



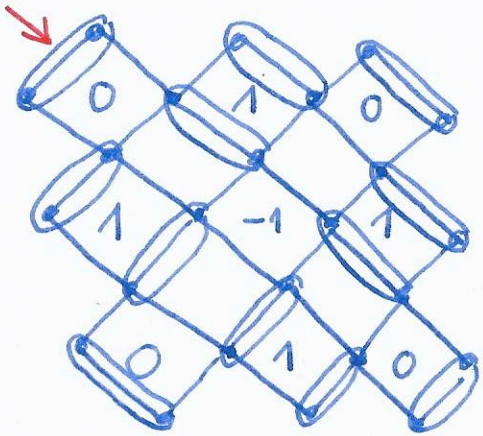
Therefore,

$$\sum_{A \text{ is an } n\times n \text{ ASM}} 2^{\#\text{of 1's in the ASM}}$$

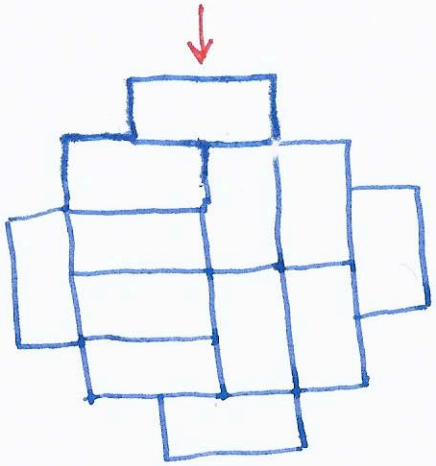
is the number of perfect matchings of $AD(n)$ (2-**enumeration of ASMs**).

Domino tilings of the Aztec diamond

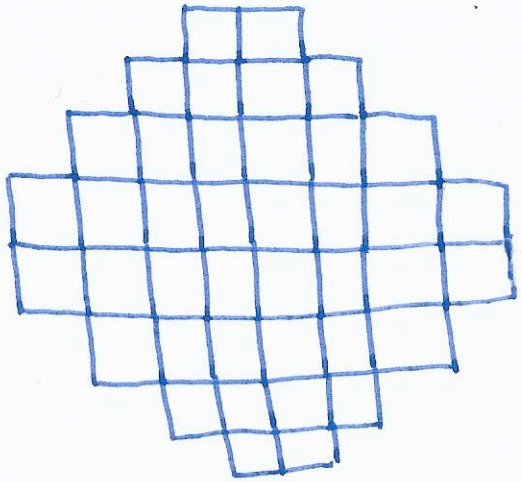
AD = Aztec diamond graph



45°
+
"dualizing"



* domino tiling of the Aztec diamond
* matching edges correspond to dominos
* AD is the dual of the Aztec diamond (ignoring the outer region)



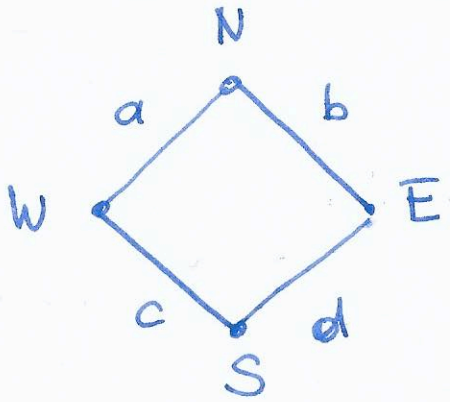
Aztec diamond of order 4

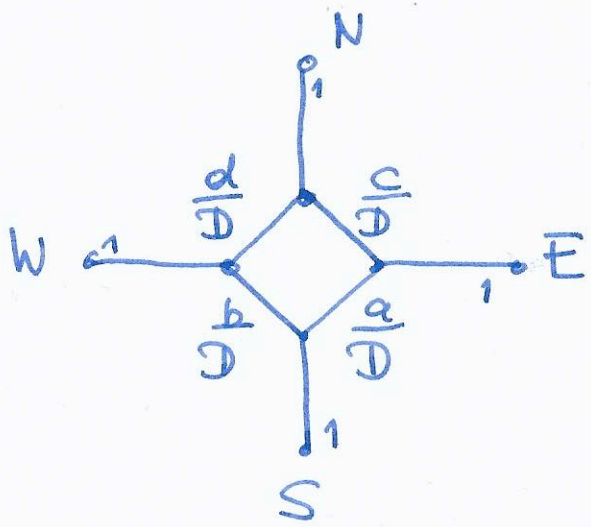
$$\#\, n\times n \text{ ASMs} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Zeilberger, 84 pages, 1996
Kuperberg, six-vertex model

# of perfect matchings of $AD(n) = 2^{\binom{n+1}{2}}$ Elkies, Kuperberg, Larsen, Propp, 1992 (*)

Lemma (Spider move):





$D = ad+bc$

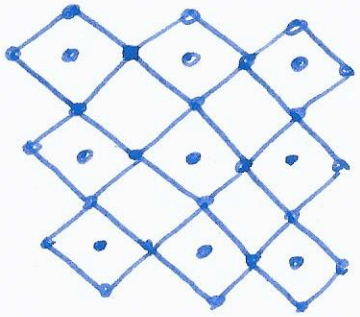
Then: $M(G) = (ad+bc)\, M(G')$

$M(G) :=$ edge generating function of the perfect matchings of G

Exercise 2 : Prove the lemma

Apply the lemma to prove (*):

$n = 3$ $\quad AD(3)$



$a = b = c = d = 1$

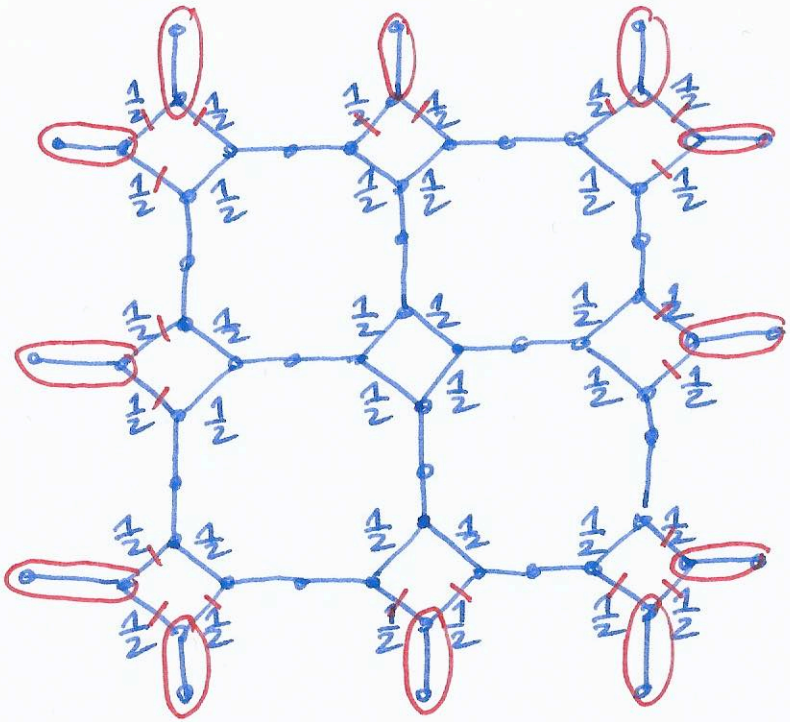
$D = 2$

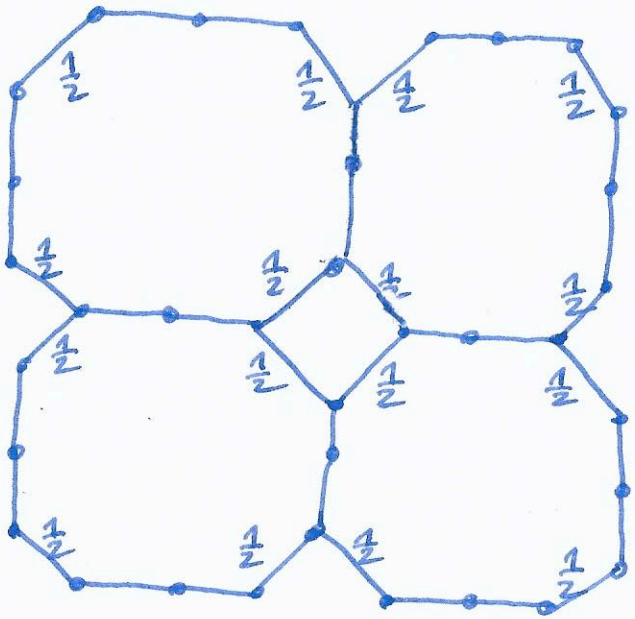
$\longrightarrow$

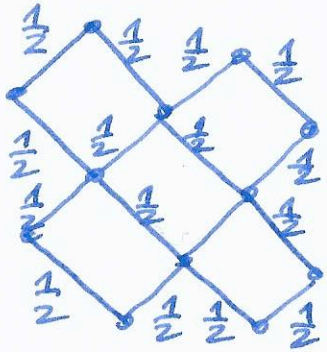
Apply the lemma to

$$\underbrace{(ad+bc)^{n^2}}_{2}$$

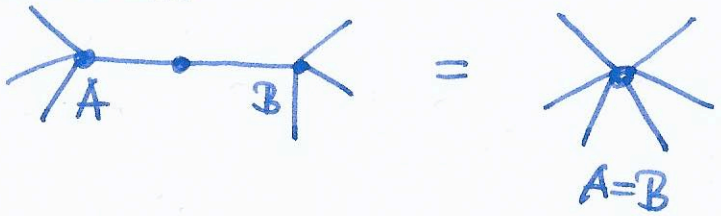
force edge



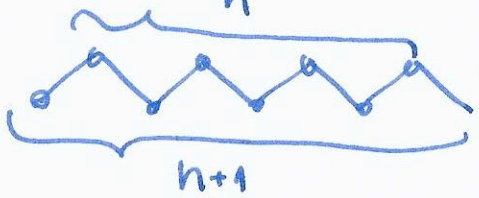




Lemma:



$AD(n)$ has $n \cdot (2n+1) + n = n \cdot (2n+2)$ vertices.



$\Rightarrow$ every perfect matching has $n \cdot (n+1)$ edges.

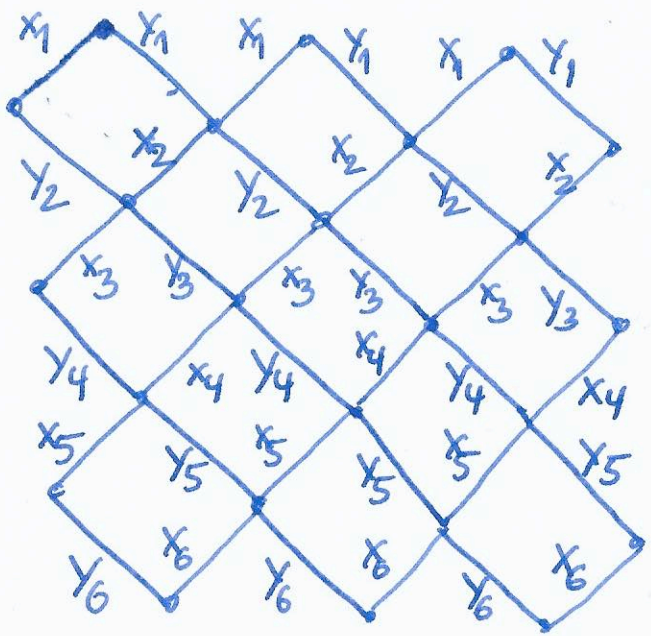
Therefore:

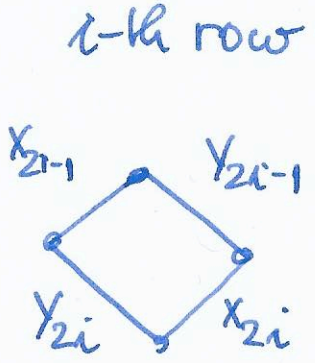
# of perfect matchings of $AD(n)$

$$= \underbrace{2^{n^2} \cdot \left(\frac{1}{2}\right)^{n\cdot(n-1)}}_{2^n} \cdot AD(n-1) \; ; \quad AD(1)=2, \text{ induction.}$$

□

Exercise 3: Stanley's weighting scheme



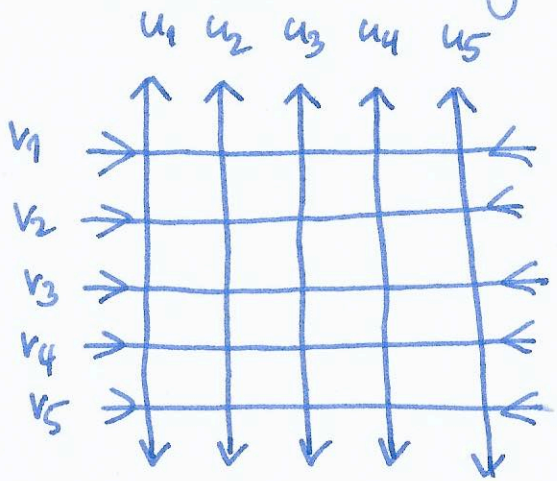


edge weights

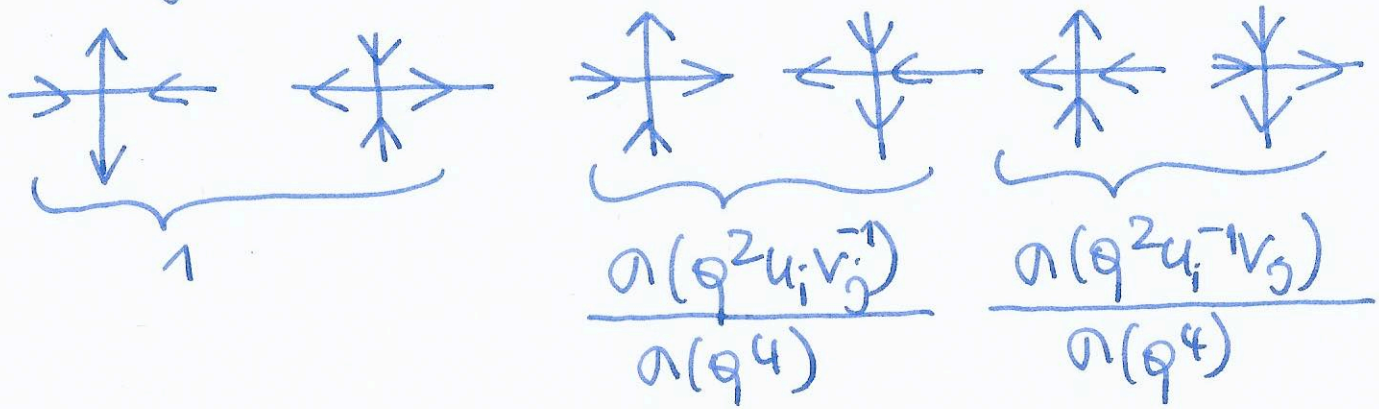
Show that the matching generating function is equal to

$$\prod_{\substack{1\le i<j\le n \\ i \text{ odd} \\ j \text{ even}}} (x_i x_j + y_i y_j)$$

Exercise 4: Goal: another generalization

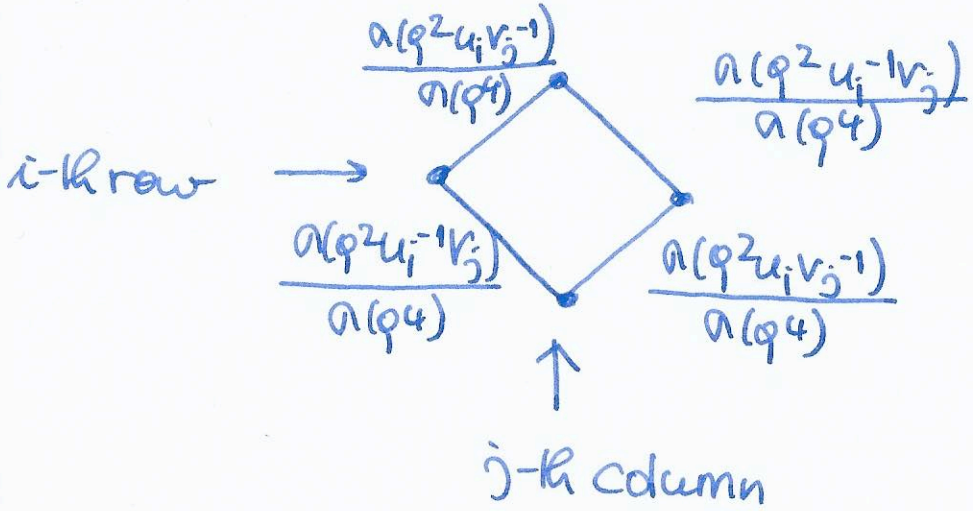


Weights: $\Omega(x) = x - x^{-1}$



Setting $q = e^{\frac{i\pi}{8}}$, we obtain a generalization of the 2-enumeration of ASM. Prove it combinatorially!

Hint: Exercise 1 (if done using the hint there).



Crucial:

$$\left[\frac{a(q^2u_iv_j^{-1})}{a(q^4)}\right]^2+\left[\frac{a(q^2u_i^{-1}v_j)}{a(q^4)}\right]^2=1$$

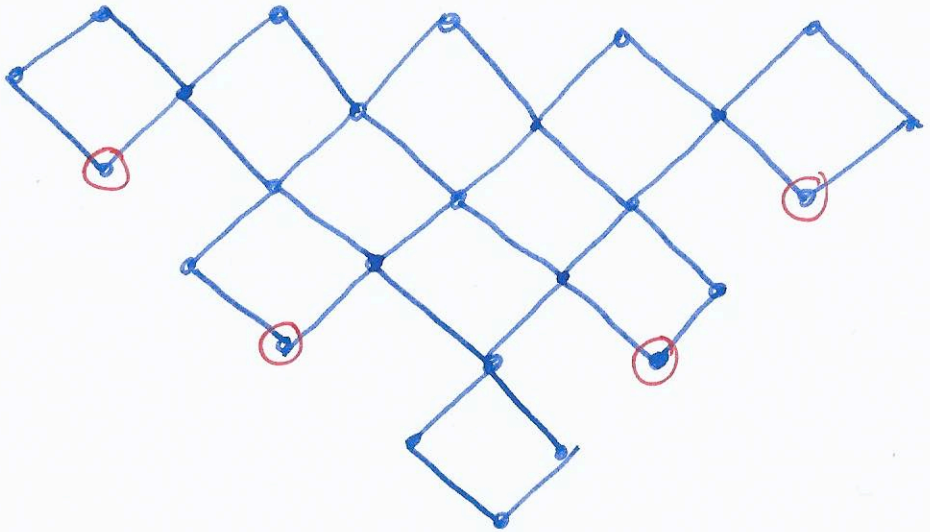
for $q=e^{\frac{i\pi}{8}}$

The partition function is

$$\frac{\prod_{i,j=1}^{n} a(q^2u_iv_j^{-1})\,a(q^2u_i^{-1}v_j)}{a(q^4)^{n(n-1)}\prod_{1\le i<j\le n} a(u_iu_j^{-1})\,a(v_i^{-1}v_j)}\det_{1\le i,j\le n}\left(\frac{1}{a(q^2u_iv_j^{-1})\,a(q^2u_i^{-1}v_j)}\right)$$

This factorizes for $q=e^{\frac{i\pi}{8}}$. Why? [Cauchy determinant]

Exercise 5 (open problem)



$n=3$

T_3

W = red vertices

bipartite graph: # of vertices in Class 1: $1+3+5+\dots+(2n-1)+(2n-1) = n^2+2n-1$

# of vertices in Case 2: $2+4+6+\dots+2n = n\cdot(n+1)$

What is the number of perfect matchings? 0

There are $n-1$ more vertices in Class 1 than in Class 2.

$\longrightarrow$ Delete $n-1$ vertices in Class 1

THEOREM (Ayyer–Behrend–T. 2020)

$$\sum_{\substack{U \subseteq W \\ |U|=n-1}} (\#\text{ of perfect matchings of } \overline{T}_n - U) = 2^{\binom{n+1}{2}}$$

$$= !\,(\#\text{ of perfect matchings of } AD(n))$$

Give a combinatorial proof (using the spider move or by giving a direct bijection with perfect matchings of $AD(n)$)!

Why is this interesting – especially if we construct a bijection with the perfect matchings of $AD(n)$?

* perfect matchings of $AD(n) \widehat{=}$ 2-enumeration of ASMs

* perfect matchings of T_n with free boundary at the bottom $\widehat{=}$ 2-enumeration of ASTs (alternating sign triangles)

THEOREM (Ayyer, Behrend, F, 2020).

$$\#\, n \times n \text{ ASMs} = \#\text{ ASTs with } n \text{ rows}$$

Remark: ASMs and ASTs have the same partition function upon specializing $v_i = q_i^{-1}$ on the ASM-side. This implies in particular that the 2-enumerations have to agree too.

Exercise 5a: (open but most likely very difficult)

Find a bijection between nxn ASMs and ASTs with n rows.

A solution of Exercise 5 could help.

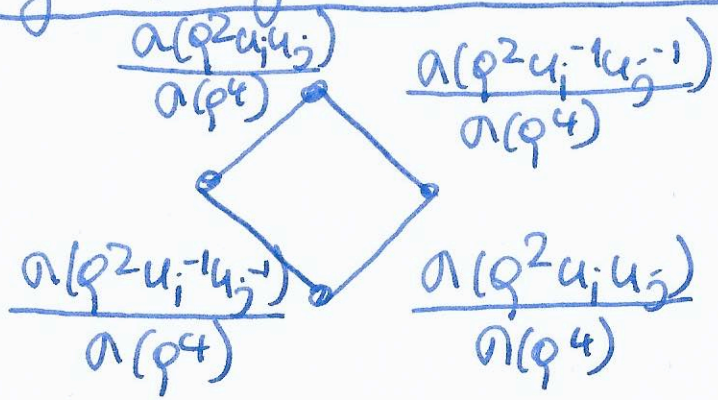
Reference: Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order

Ayyer, Behrend, Fischer, Adv. Math. 2020

Exercise 6: Find a multiparameter generalization by specializing the partition function of ASTS and prove it by combinatorial means.

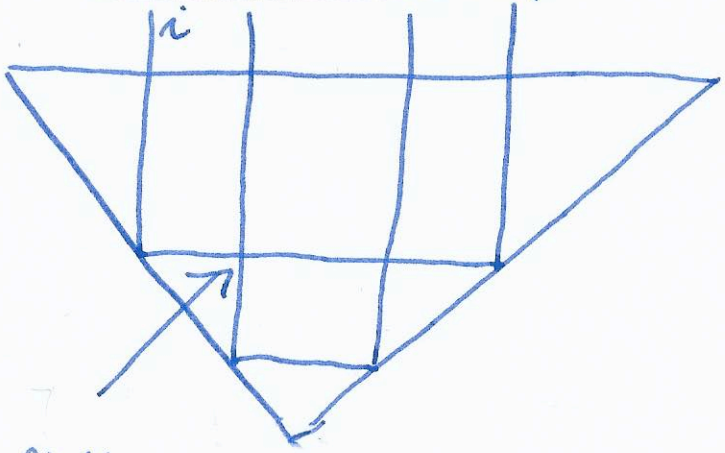
Hint: Check the paper above; set $q = e^{\frac{i\pi}{5}}$

Edge weights in the bulk



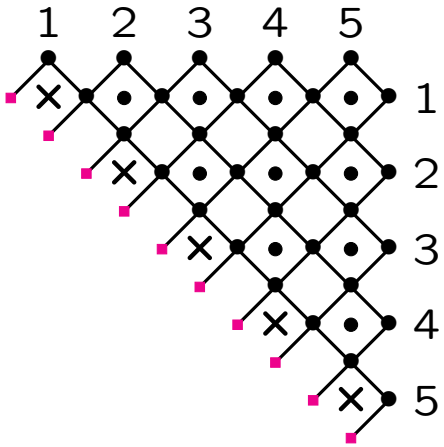
$\Lambda(x) = x - x^{-1}$

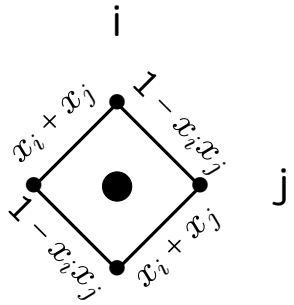
Parameters

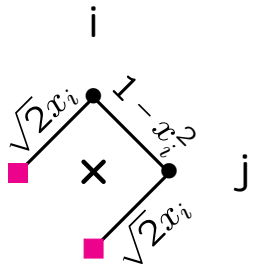


To be done:
* Work out weights on the left and right boundary.
* Partition function factorizes for $q = e^{\frac{i\pi}{5}}$. Why?

Exercise 7. Consider the graph below with the edge weights as indicated on the right.







Prove by combinatorial means that the matching generating function where only the vertices in magenta may be avoided is

$$\prod_{1 \le i \le j \le n} (1 + x_i x_j).$$

Reference: A Littlewood-type identity for Robbins polynomials, F. & Höngesberg, Alg. Combin. 2026.

Kuo condensation

Kuo's graphical condensation

For an edge-weighted graph G, $\mathrm{M}(G)$ denotes the generating function of perfect matchings.

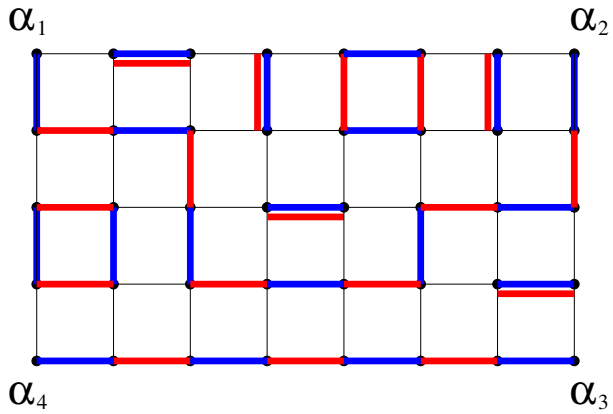
Theorem. Let G be a planar edge-weighted graph with four vertices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that appear in this cyclic order on a face of G. Then

$$\mathrm{M}(G)\,\mathrm{M}(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) + M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\})$$
$$= \mathrm{M}(G - \{\alpha_1, \alpha_2\})\,\mathrm{M}(G - \{\alpha_3, \alpha_4\}) + M(G - \{\alpha_1, \alpha_4\})M(G - \{\alpha_2, \alpha_3\}).$$

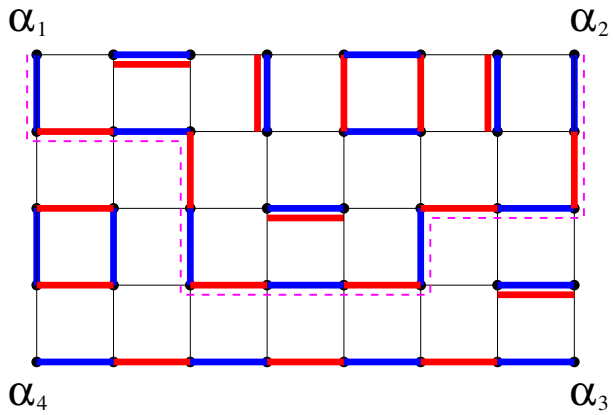
G bipartite: $V = V_1 \sqcup V_2$
$\alpha_1, \alpha_3 \in V_1, \alpha_2, \alpha_4 \in V_2, |V_1| = |V_2|$: second term vanishes
$\alpha_1, \alpha_2 \in V_1, \alpha_3, \alpha_4 \in V_2, |V_1| = |V_2|$: third term vanishes
$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V_1, |V_1| = |V_2| + 2$: first term vanishes

Idea of the proof

• $\mathrm{M}(G)\,\mathrm{M}(G-\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})$: Superimpose a perfect matching of G (blue) and a perfect matching of $G-\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ (red).

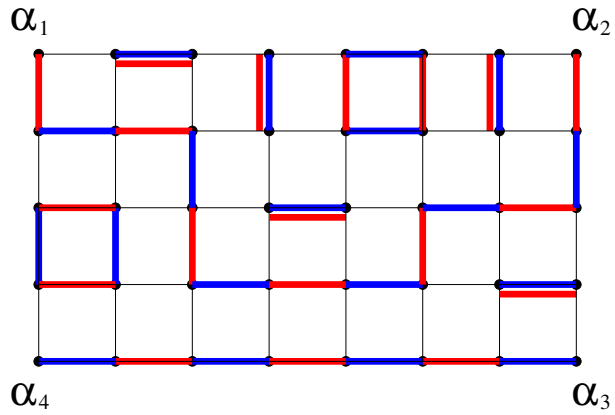


- There is a blue-red-alternating path from α_1 to α_i for an $i \in \{2,3,4\}$.



- Two blue-red-alternating paths cannot cross and thus $i \neq 3$.

• Switch the edges in the path of α_1 and obtain a pair of matchings of $G - \{\alpha_1, \alpha_2\}$ and $G - \{\alpha_3, \alpha_4\}$ or of $G - \{\alpha_1, \alpha_4\}$ and $G - \{\alpha_2, \alpha_3\}$. (This corresponds to the terms on the right-hand side of the identity.)

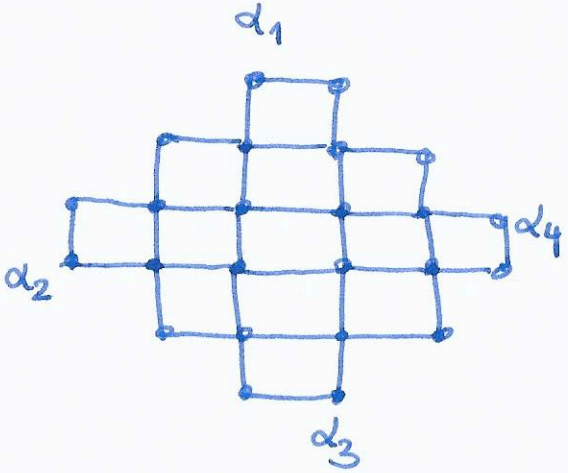


• When thinking about the reverse mapping, one observes that this is not a bijection and that the term $M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\})$ is also necessary.

Exercise 8. Use Kuo condensation to reprove that the number of perfect matchings of $AD(n)$ is $2^{\binom{n+1}{2}}$.

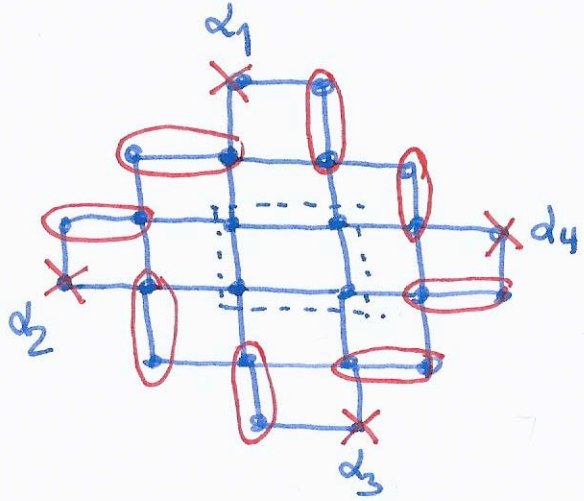
Solution to Exercise 8 :

Choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as follows.



AD(3)

$G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$



AD(n) is bipartite and balanced. α_1, α_3 are in the same vertex class and α_2, α_4 are in the other vertex class.

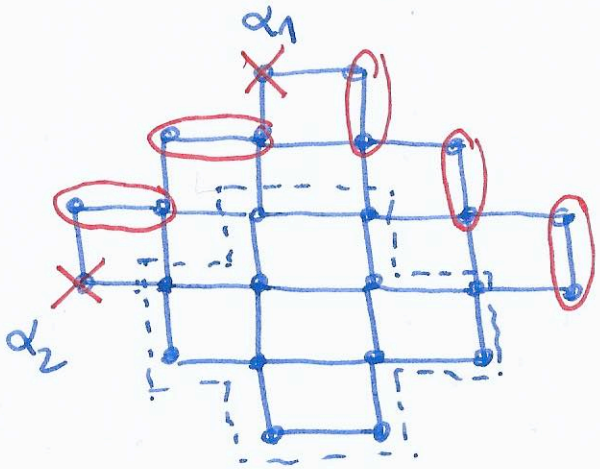
$$\Rightarrow M(G - \{\alpha_1, \alpha_3\}) = M(G - \{\alpha_2, \alpha_4\}) = \emptyset$$

We use

$$\underbrace{M(G)}_{\Pi(AD(n))} \cdot \underbrace{\Pi(G-\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})}_{\Pi(AD(n-2))} = \Pi(G-\{\alpha_1,\alpha_2\})\,\Pi(G-\{\alpha_3,\alpha_4\}) + \Pi(G-\{\alpha_1,\alpha_4\})\,\Pi(G-\{\alpha_2,\alpha_3\})$$

to give a recursion for $\Pi(AD(n))$.

What is $G-\{\alpha_1,\alpha_2\}$?

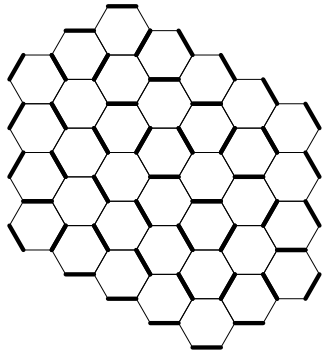


$G-\{\alpha_1,\alpha_2\} \cong$
$G-\{\alpha_3,\alpha_4\} \cong$
$G-\{\alpha_1,\alpha_4\} \cong$
$G-\{\alpha_2,\alpha_3\}$

$$\Rightarrow \Pi(AD(n)) \cdot \Pi(AD(n-2)) = 2 \cdot \Pi(AD(n-1))$$

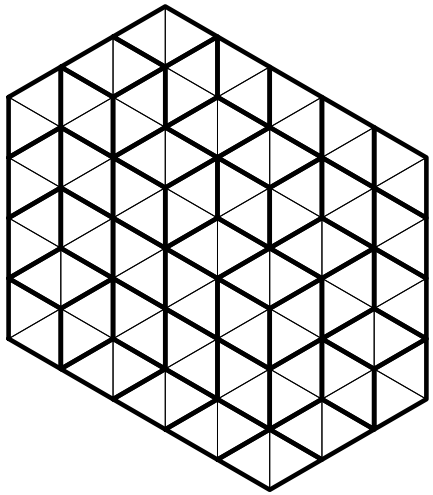
Exercise 9. Use Kuo condensation to prove that the number of perfect matchings of the honeycomb graph of hexagonal shape with side lengths a, b, c, a, b, c is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$



$a = 3, b = 5, c = 4$

What does this have to do with tilings and plane partitions? And what is a plane partition?



Exercise 10. Is Kuo condensation helpful for Exercise 5?

Exercise 11. Is Kuo condensation helpful for Exercise 7?

Problem with the unsolved exercises: **free boundary**

Kuo condensation with free boundary

Mihai Ciucu has established a free boundary version of Kuo condensation.

Reference: Correlation of a macroscopic dent in a wedge with mixed boundary conditions. **TAMS** 2020.

Setting:

- G edge-weighted planar graph

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ vertices that appear in that cyclic order on the boundary of a face F of G.

- S subset of vertices on the boundary of F not containing $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

• Let $\mathrm{M}_f(G)$ be the sum of weights of all the (not necessarily perfect) matchings of G in which all the vertices that are not in S are matched (but those in S are free to be matched or not).

We say that S is α_1, α_3-separated if there are not mutually disjoint paths P_1, P_2, P_3 in G so that

- P_1 connects α_1 to α_3
- P_2 connects α_2 to some vertex in S
- P_3 connects α_4 to some other vertex in S.

Theorem (Ciucu 2020). If S is α_1, α_3-separated, then we have

$$\begin{aligned}
&\mathrm{M}_f(G) \cdot \mathrm{M}_f(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) + M_f(G - \{\alpha_2, \alpha_4\}) \cdot \mathrm{M}_f(G - \{\alpha_1, \alpha_3\}) \\
&\quad + M_f(G - \{\alpha_2\}) \cdot \mathrm{M}_f(G - \{\alpha_1, \alpha_3, \alpha_4\}) + M_f(G - \{\alpha_4\}) \cdot \mathrm{M}_f(G - \{\alpha_1, \alpha_2, \alpha_3\}) \\
&= M_f(G - \{\alpha_1, \alpha_4\}) \cdot \mathrm{M}_f(G - \{\alpha_2, \alpha_3\}) + M_f(G - \{\alpha_1, \alpha_2\}) \cdot \mathrm{M}_f(G - \{\alpha_3, \alpha_4\}) \\
&\quad + M_f(G - \{\alpha_1\}) \cdot \mathrm{M}_f(G - \{\alpha_2, \alpha_3, \alpha_4\}) + M_f(G - \{\alpha_1, \alpha_2, \alpha_4\}) \cdot \mathrm{M}_f(G - \{\alpha_3\}).
\end{aligned}$$

Exercise 12. Is this of any use for Exercise 5 or 7? If not, is there a different free-boundary-condensation tool?

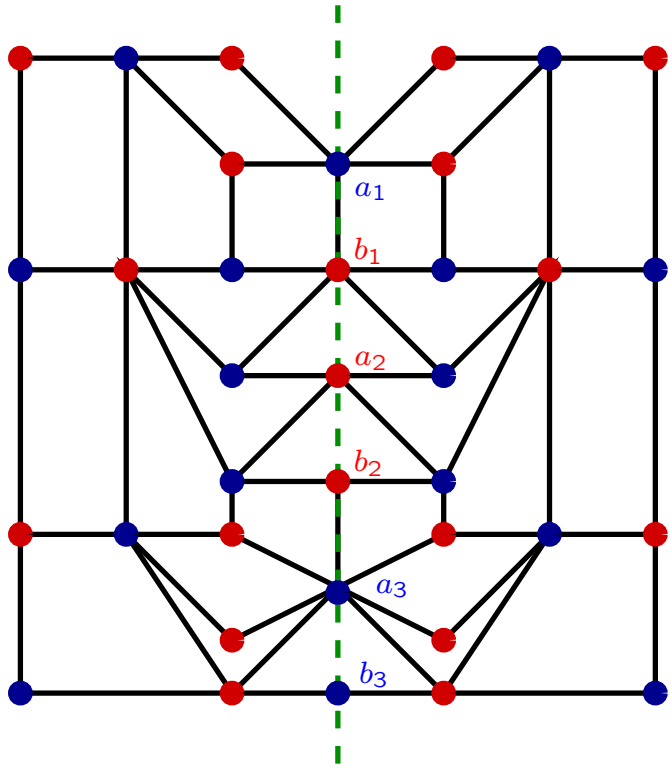
Ciucu's factorization theorem

Ciucu's factorization theorem I

Suppose G is an edge-weighted graph and we want to compute the (perfect) matching generating function $\mathrm{M}(G)$.

Assume G has the following properties:

- Planar, bipartite and connected.
- G symmetric w.r.t. a vertical symmetry axis ℓ, including the edge-weights!
- Removing ℓ disconnects G.
- W.l.o.g. there are an even number $2\,\mathrm{N}(G)$ of vertices on the symmetry axis. (If not, G has no perfect matching.)

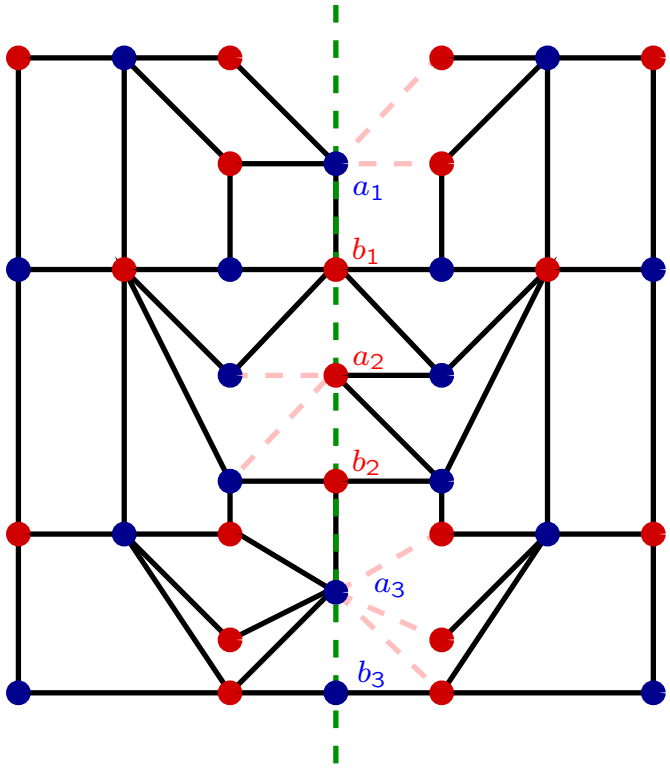


Let $a_1, b_1, a_2, b_2, \ldots, a_{\mathrm{N}(G)}, b_{\mathrm{N}(G)}$ be the vertices on the symmetry axis from top to bottom.

Ciucu's factorization theorem II

For a vertex v on ℓ, we define cutting operations: *"Cutting left (resp. right) of v" means that we delete all incident edges left (resp. right) of ℓ.*

A reduced subgraph of G is a graph that is obtained from G by cutting left or right of each vertex a_i.



Lemma (Ciucu 1997). All $2^{\mathrm{N}(G)}$ reduced subgraphs of G have the same matching generating function.

Proof

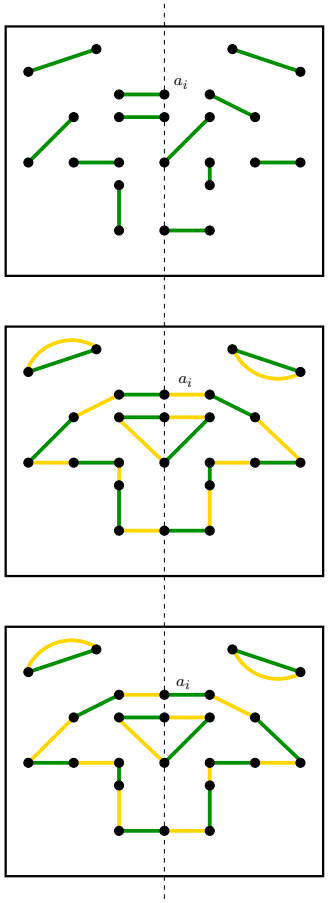
• Let G_1 and G_2 be two reduced subgraphs that differ only around a single vertex a_i.

• Let μ be a matching of G_1, and μ' be the matching of G obtained from μ by reflecting across ℓ. Then $\nu := \mu \cup \mu'$ ("multiset union") is a disjoint union of even cycles (with every other edge in μ and μ', respectively) that cover all vertices.

• Let C be the cycle containing a_i. Since ν is symmetric, C is also symmetric with respect to ℓ, and C contains precisely one other vertex of ℓ besides a_i.

• This vertex is one of $b_1, b_2, \ldots, b_{\mathrm{N}(G)}$. Otherwise, the set encircled by C has an odd number of elements on ℓ. Since this set is symmetric about ℓ, it follows that it has an odd number of elements, contradicting that ν is a disjoint union of even cycles.

• Define μ'' to be the matching of G obtained from μ by replacing $\mu \cap C$ by $\mu' \cap C$. Then μ'' is a matching of G_2 and the correspondence $\mu \to \mu''$ is a weight-preserving involution between the matchings of G_1 and those of G_2.

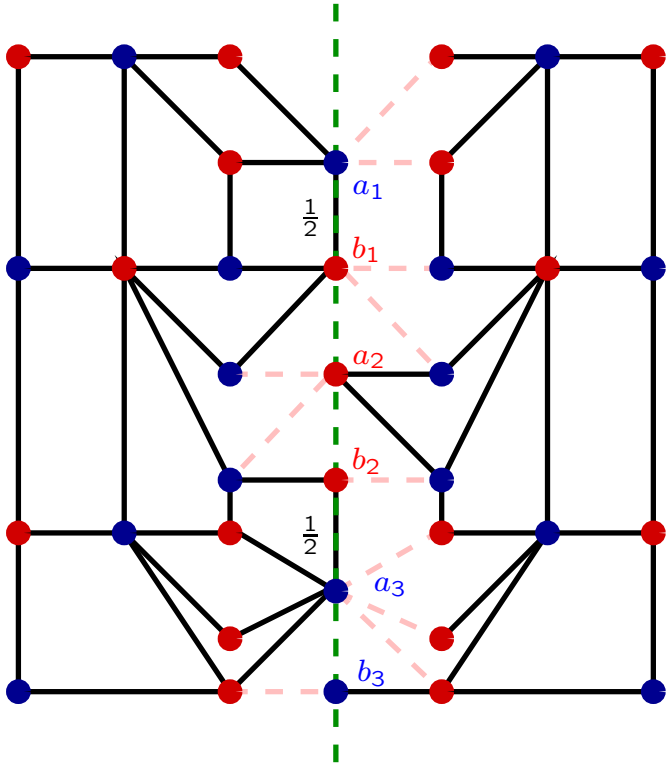


Ciucu's factorization theorem II

Now we define two subgraphs of G as follows:

- **We perform the cutting operation right of all a_i's and b_i's, and left of a_i's and b_i's.**

- Reduce the weights of the edges on ℓ by half and leave all other weights unchanged. We obtain two disconnected graphs, and denote by G^+ the left graph and by G^- the right graph.



Theorem (Ciucu 1997). We have the following factorization for the perfect matching generating function $\mathrm{M}(G)$.

$$\mathrm{M}(G) = 2^{\mathrm{N}(G)}\,\mathrm{M}(G^+)\,\mathrm{M}(G^-)$$

Proof (Sketch).

Step 1. Reduction to the case when all vertices on ℓ are independent.

Step 2. It suffices to show that $\mathrm{M}(G^+)\,\mathrm{M}(G^-)$ is the matching generating function of some (hence any) of the $2^{\mathrm{N}(G)}$ reduced subgraphs. This is proved for the reduced subgraph H where the cutting was performed right of all a_i's, and left of all a_i's.

We need to show that in every matching of this reduced subgraph, all b_i's are matched to the left and all b_i's are matched to the right.

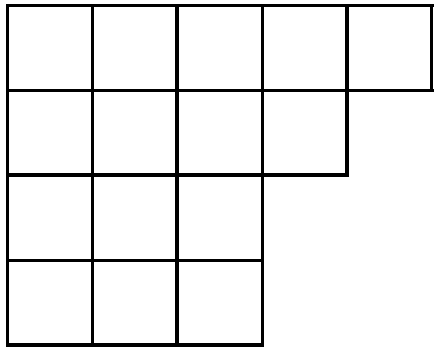
Exercise 13. Give a full proof of the theorem.

Two Schur polynomial factorizations

Partitions and Young diagrams

• A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a sequence of weakly decreasing non-negative integers, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

• The Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a left-justified arrangement of cells with λ_i cells in row i. The Young diagram of $(5, 4, 3, 3)$ is the following.



• Such diagrams are for instance useful to construct all representations of the symmetric group and also of other groups.

Schur polynomials

• A semistandard tableau of shape λ is a filling of the Young diagram of λ such that rows increase weakly and columns increase strictly. The Schur polynomial of the partition λ is

$$s_\lambda(x_1,\ldots,x_n) = \sum_T x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \cdots x_n^{\#n's \text{ in } T},$$

where the sum is over all semistandard tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Example. For $n=3$ and $\lambda=(2,1)$ we have the following semistandard tableaux:

$$\begin{array}{ccccccccccccccccc} 1 & 1 & , & 1 & 2 & , & 1 & 3 & , & 1 & 1 & , & 1 & 2 & , & 1 & 3 & , & 2 & 2 & , & 2 & 3 & , \\ 2 & & & 2 & & & 2 & & & 3 & & & 3 & & & 3 & & & 3 & & & 3 & & \end{array}$$

and so

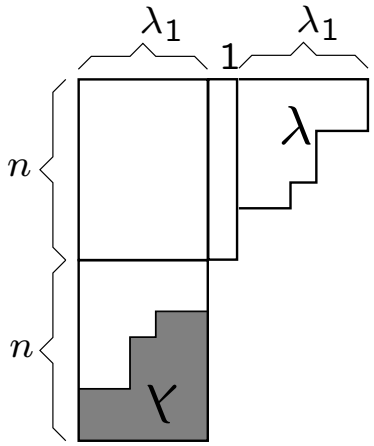
$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

• Schur polynomials are the irreducible characters of the polynomial representations of the general linear group and an important basis of the space of symmetric functions.

Determinantal formula. Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, then

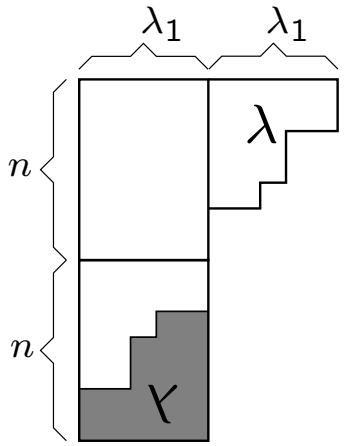
$$s_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\leq i,j\leq n}\left(x_i^{\lambda_j+n-j}\right)}{\det_{1\leq i,j\leq n}\left(x_i^{n-j}\right)}.$$

Two families of partitions



$(\lambda_1+1,\ldots,\lambda_n+1,-\lambda_n,\ldots,-\lambda_1)+\lambda_1$

$(11,9,2,0)$



$(\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)+\lambda_1$

$(10,8,2,0)$

for $(\lambda_1,\lambda_2)=(5,3)$

$$s_{(11,9,2,0)}(x_1,\bar{x}_1,x_2,\bar{x}_2) = \frac{1}{x_1^{11}x_2^{11}}$$

$$\times\left(x_1^{10}x_2^8+x_1^{10}x_2^6+x_1^{10}x_2^4+x_1^{10}x_2^2+x_1^9x_2^9+2x_1^9x_2^7+2x_1^9x_2^5+2x_1^9x_2^3+x_1^9x_2+x_1^8x_2^{10}+3x_1^8x_2^8+4x_1^8x_2^6+4x_1^8x_2^4+3x_1^8x_2^2\right.$$
$$+x_1^8+2x_1^7x_2^9+4x_1^7x_2^7+5x_1^7x_2^5+4x_1^7x_2^3+2x_1^7x_2+x_1^6x_2^{10}+4x_1^6x_2^8+6x_1^6x_2^6+6x_1^6x_2^4+4x_1^6x_2^2+x_1^6+2x_1^5x_2^9+5x_1^5x_2^7$$
$$+6x_1^5x_2^5+5x_1^5x_2^3+2x_1^5x_2+x_1^4x_2^{10}+4x_1^4x_2^8+6x_1^4x_2^6+6x_1^4x_2^4+4x_1^4x_2^2+x_1^4+2x_1^3x_2^9+4x_1^3x_2^7+5x_1^3x_2^5+4x_1^3x_2^3+2x_1^3x_2$$
$$\left.+x_1^2x_2^{10}+3x_1^2x_2^8+4x_1^2x_2^6+4x_1^2x_2^4+3x_1^2x_2^2+x_1^2+x_1x_2^9+2x_1x_2^7+2x_1x_2^5+2x_1x_2^3+x_1x_2+x_2^8+x_2^6+x_2^4+x_2^2\right)$$
$$\times\left(x_1^{12}x_2^{10}+x_1^{12}x_2^2+x_1^{11}x_2^{11}+x_1^{11}x_2^9+x_1^{11}x_2^3+x_1^{11}x_2+x_1^{10}x_2^{12}+x_1^{10}x_2^{10}+x_1^{10}x_2^8+x_1^{10}x_2^4+x_1^{10}x_2^2+x_1^{10}+x_1^9x_2^{11}+x_1^9x_2^9\right.$$
$$+x_1^9x_2^7+x_1^9x_2^5+x_1^9x_2^3+x_1^9x_2+x_1^8x_2^{10}+x_1^8x_2^8+2x_1^8x_2^6+x_1^8x_2^4+x_1^8x_2^2+x_1^7x_2^9+2x_1^7x_2^7+2x_1^7x_2^5+x_1^7x_2^3+2x_1^6x_2^8+2x_1^6x_2^6$$
$$+2x_1^6x_2^4+x_1^5x_2^9+2x_1^5x_2^7+2x_1^5x_2^5+x_1^5x_2^3+x_1^4x_2^{10}+x_1^4x_2^8+2x_1^4x_2^6+x_1^4x_2^4+x_1^4x_2^2+x_1^3x_2^{11}+x_1^3x_2^9+x_1^3x_2^7+x_1^3x_2^5+x_1^3x_2^3+x_1^3x_2$$
$$\left.+x_1^2x_2^{12}+x_1^2x_2^{10}+x_1^2x_2^8+x_1^2x_2^4+x_1^2x_2^2+x_1^2+x_1x_2^{11}+x_1x_2^9+x_1x_2^3+x_1x_2+x_2^{10}+x_2^2\right)$$

(Notation: $\bar{x} = x^{-1}$)

$$s_{(10,8,2,0)}(x_1,\bar{x}_1,x_2,\bar{x}_2) = \frac{1}{x_1^{10}x_2^{10}}$$
$$\times\left(x_2^8x_1^{10}-x_2^7x_1^{10}+x_2^6x_1^{10}-x_2^5x_1^{10}+x_2^4x_1^{10}-x_2^3x_1^{10}+x_2^2x_1^{10}+x_2^9x_1^9-2x_2^8x_1^9+3x_2^7x_1^9-3x_2^6x_1^9+3x_2^5x_1^9-3x_2^4x_1^9+3x_2^3x_1^9\right.$$
$$-2x_2^2x_1^9+x_2x_1^9+x_2^{10}x_1^8-2x_2^9x_1^8+4x_2^8x_1^8-5x_2^7x_1^8+6x_2^6x_1^8-6x_2^5x_1^8+6x_2^4x_1^8-5x_2^3x_1^8+4x_2^2x_1^8-2x_2x_1^8+x_1^8-x_2^{10}x_1^7$$
$$+3x_2^9x_1^7-5x_2^8x_1^7+7x_2^7x_1^7-8x_2^6x_1^7+9x_2^5x_1^7-8x_2^4x_1^7+7x_2^3x_1^7-5x_2^2x_1^7+3x_2x_1^7-x_1^7+x_2^{10}x_1^6-3x_2^9x_1^6+6x_2^8x_1^6-8x_2^7x_1^6$$
$$+10x_2^6x_1^6-10x_2^5x_1^6+10x_2^4x_1^6-8x_2^3x_1^6+6x_2^2x_1^6-3x_2x_1^6+x_1^6-x_2^{10}x_1^5+3x_2^9x_1^5-6x_2^8x_1^5+9x_2^7x_1^5-10x_2^6x_1^5+11x_2^5x_1^5$$
$$-10x_2^4x_1^5+9x_2^3x_1^5-6x_2^2x_1^5+3x_2x_1^5-x_1^5+x_2^{10}x_1^4-3x_2^9x_1^4+6x_2^8x_1^4-8x_2^7x_1^4+10x_2^6x_1^4-10x_2^5x_1^4+10x_2^4x_1^4-8x_2^3x_1^4$$
$$+6x_2^2x_1^4-3x_2x_1^4+x_1^4-x_2^{10}x_1^3+3x_2^9x_1^3-5x_2^8x_1^3+7x_2^7x_1^3-8x_2^6x_1^3+9x_2^5x_1^3-8x_2^4x_1^3+7x_2^3x_1^3-5x_2^2x_1^3+3x_2x_1^3-x_1^3$$
$$+x_2^{10}x_1^2-2x_2^9x_1^2+4x_2^8x_1^2-5x_2^7x_1^2+6x_2^6x_1^2-6x_2^5x_1^2+6x_2^4x_1^2-5x_2^3x_1^2+4x_2^2x_1^2-2x_2x_1^2+x_1^2+x_2^9x_1-2x_2^8x_1+3x_2^7x_1$$
$$-3x_2^6x_1+3x_2^5x_1-3x_2^4x_1+3x_2^3x_1-2x_2^2x_1+x_2x_1+x_2^8-x_2^7+x_2^6-x_2^5+x_2^4-x_2^3+x_2^2\Big)$$
$$\times\left(x_2^8x_1^{10}+x_2^7x_1^{10}+x_2^6x_1^{10}+x_2^5x_1^{10}+x_2^4x_1^{10}+x_2^3x_1^{10}+x_2^2x_1^{10}+x_2^9x_1^9+2x_2^8x_1^9+3x_2^7x_1^9+3x_2^6x_1^9+3x_2^5x_1^9+3x_2^4x_1^9+3x_2^3x_1^9\right.$$
$$+2x_2^2x_1^9+x_2x_1^9+x_2^{10}x_1^8+2x_2^9x_1^8+4x_2^8x_1^8+5x_2^7x_1^8+6x_2^6x_1^8+6x_2^5x_1^8+6x_2^4x_1^8+5x_2^3x_1^8+4x_2^2x_1^8+2x_2x_1^8+x_1^8+x_2^{10}x_1^7$$
$$+3x_2^9x_1^7+5x_2^8x_1^7+7x_2^7x_1^7+8x_2^6x_1^7+9x_2^5x_1^7+8x_2^4x_1^7+7x_2^3x_1^7+5x_2^2x_1^7+3x_2x_1^7+x_1^7+x_2^{10}x_1^6+3x_2^9x_1^6+6x_2^8x_1^6+8x_2^7x_1^6$$
$$+10x_2^6x_1^6+10x_2^5x_1^6+10x_2^4x_1^6+8x_2^3x_1^6+6x_2^2x_1^6+3x_2x_1^6+x_1^6+x_2^{10}x_1^5+3x_2^9x_1^5+6x_2^8x_1^5+9x_2^7x_1^5+10x_2^6x_1^5+11x_2^5x_1^5$$
$$+10x_2^4x_1^5+9x_2^3x_1^5+6x_2^2x_1^5+3x_2x_1^5+x_1^5+x_2^{10}x_1^4+3x_2^9x_1^4+6x_2^8x_1^4+8x_2^7x_1^4+10x_2^6x_1^4+10x_2^5x_1^4+10x_2^4x_1^4+8x_2^3x_1^4$$
$$+6x_2^2x_1^4+3x_2x_1^4+x_1^4+x_2^{10}x_1^3+3x_2^9x_1^3+5x_2^8x_1^3+7x_2^7x_1^3+8x_2^6x_1^3+9x_2^5x_1^3+8x_2^4x_1^3+7x_2^3x_1^3+5x_2^2x_1^3+3x_2x_1^3+x_1^3$$
$$+x_2^{10}x_1^2+2x_2^9x_1^2+4x_2^8x_1^2+5x_2^7x_1^2+6x_2^6x_1^2+6x_2^5x_1^2+6x_2^4x_1^2+5x_2^3x_1^2+4x_2^2x_1^2+2x_2x_1^2+x_1^2+x_2^9x_1+2x_2^8x_1+3x_2^7x_1$$
$$+3x_2^6x_1+3x_2^5x_1+3x_2^4x_1+3x_2^3x_1+2x_2^2x_1+x_2x_1+x_2^8+x_2^7+x_2^6+x_2^5+x_2^4+x_2^3+x_2^2\Big)$$

Characters of classical groups

- Symplectic characters ($Sp_{2n}(\mathbb{C})$):

$$sp_\lambda(x_1,\dots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1} - \overline{x}_i^{\lambda_j+n-j+1}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j+1} - \overline{x}_i^{n-j+1}\right)}.$$

- Even orthogonal characters ($O_{2n}(\mathbb{C})$):

$$o_\lambda^{\text{even}}(x_1,\dots,x_n) = (1+[\lambda_n \neq 0])\frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j} + \overline{x}_i^{\lambda_j+n-j}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j} + \overline{x}_i^{n-j}\right)}.$$

- Odd orthogonal characters ($SO_{2n+1}(\mathbb{C})$):

$$so_\lambda^{\text{odd}}(x_1,\dots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j+1/2} - \overline{x}_i^{\lambda_j+n-j+1/2}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j+1/2} - \overline{x}_i^{n-j+1/2}\right)}.$$

- A half-integer is an odd integer divided by 2, and a half-integer partition is a finite weakly decreasing sequence of positive half-integers. In the case of the even orthogonal group, the character formula has a representation theoretic meaning as characters of spin covering groups when λ is a half-integer partition.

- General linear characters ($GL_n(\mathbb{C})$):

$$s_\lambda(x_1,\dots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j}\right)}$$

The factorization theorem

Theorem (Ayyer, Behrend, 2019). Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition.

- For $\widehat{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) + \lambda_1$, we have

$$s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = sp_\lambda(x_1, \ldots, x_n)\, o^{\mathsf{even}}_{\lambda+1}(x_1, \ldots, x_n).$$

- For $\widehat{\lambda} = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1) + \lambda_1$, we have

$$s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \prod_{i=1}^{n} (x_i^{1/2} + \bar{x}_i^{1/2})^{-1} so^{\mathsf{odd}}_\lambda(x_1, \ldots, x_n)\, o^{\mathsf{even}}_{\lambda+\frac{1}{2}}(x_1, \ldots, x_n).$$

Goal of our work: Bijective proof

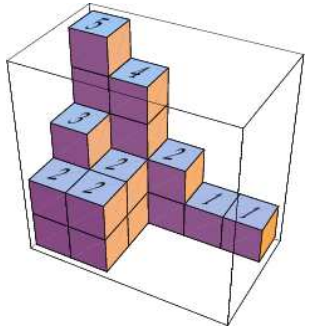
Why do we care? plane partitions!

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$



$a = 5, b = 3, c = 5$

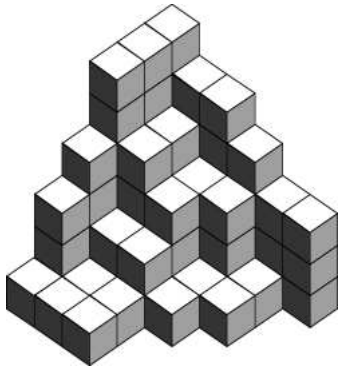
Cyclically symmetric plane partitions = CSPPs

An $n \times n \times n$ plane partition PP is cyclically symmetric if

$$(i,j,k) \in PP \Rightarrow (j,k,i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$

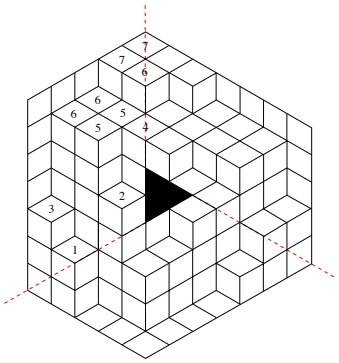


Other symmetry classes

TSPP(n) = # totally symmetric plane partitions in an $n \times n \times n$-box. (Lozenge tilings are invariant under the rotation by 120° as well as under the reflection in one type of symmetry axis.)

CSTCPP(n) = # cyclically symmetric transpose complementary plane partitions in an $n \times n \times n$-box. (Lozenge tilings are invariant under the rotation by 120° as well as under the reflection in the other type of symmetry axis.)

$\mathrm{CSPP}_2(n)$ = # cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central hole of size 2.



Similar definition for $\mathrm{CSTCPP}_2(n)$.

Factorizations that follow from the theorem of Ayyer and Behrend

$$\mathrm{CSPP}(2n) = \mathrm{TSPP}(2n) \cdot \mathrm{CSTCPP}(2n)$$
$$\mathrm{CSPP}(2n+1) = 2\,\mathrm{TSPP}(2n) \cdot \mathrm{CSTCPP}(2n+2)$$
$$\mathrm{CSPP}_2(2n+1) = R(2n+1) \cdot \mathrm{CSTCPP}_2(2n+1)$$
$$\mathrm{CSPP}_2(2n) = 2R(2n) \cdot \mathrm{CSTCPP}_2(2n+1)$$

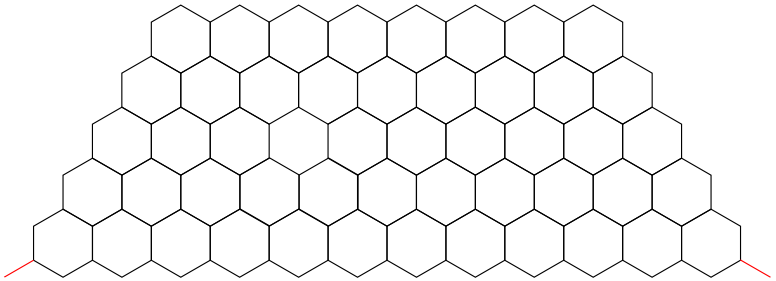
Open problem: Find bijective proofs!

$\mathrm{CSPP}(n), \mathrm{TSPP}(n) \ldots$ can be written in terms of specializations of characters (Schur polynomials, ...) when setting $x_1 = x_2 = \ldots = x_n = 1$, up to powers of 3.

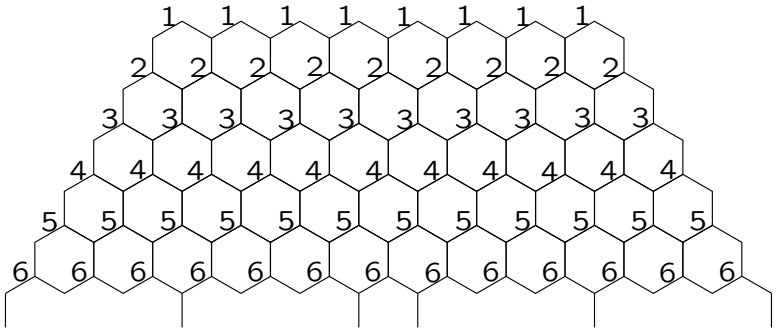
The characters have underlying combinatorial "models" (i.e., they are multivariate generating functions of certain objects), however, the bijections that explain the relations to these classes of plane partitions are still missing (?).

Graphical models of characters

Trapezoidal honeycomb graph



$T_{6,8}$



$T_{6,8}^{1,4,7,8,11,14}(x_1,\ldots,x_6)$

(Edge weight x_i is abbreviated by i.)

Theorem. For a partition $\lambda = (\lambda_1,\ldots,\lambda_n)$, we have

$$s_\lambda(x_1,\ldots,x_n) = \mathrm{M}(\mathrm{T}_{n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\ldots,\lambda_1+n}(x_1,\ldots,x_n)),$$

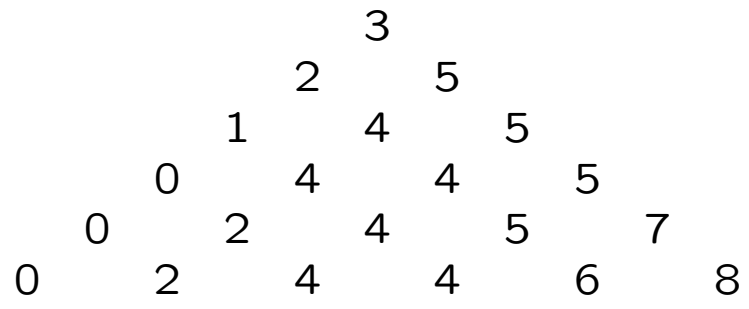
where $\mathrm{M}(G)$ denotes the perfect matching generating function of the edge weighted graph G.

From a tableau of shape λ to a matching of $\top_{n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\ldots,\lambda_1+n}$

Semistandard tableau of shape $(8,6,4,4,2,0)$:

1	1	1	2	2	5	5	6
2	2	3	3	5	6		
3	4	4	4				
5	5	6	6				
6	6						

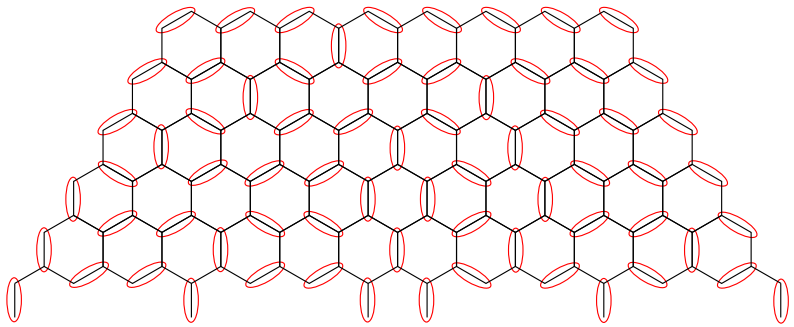
Corresponding Gelfand-Tsetlin pattern:



We add i to the i-th $\nearrow$-diagonal:

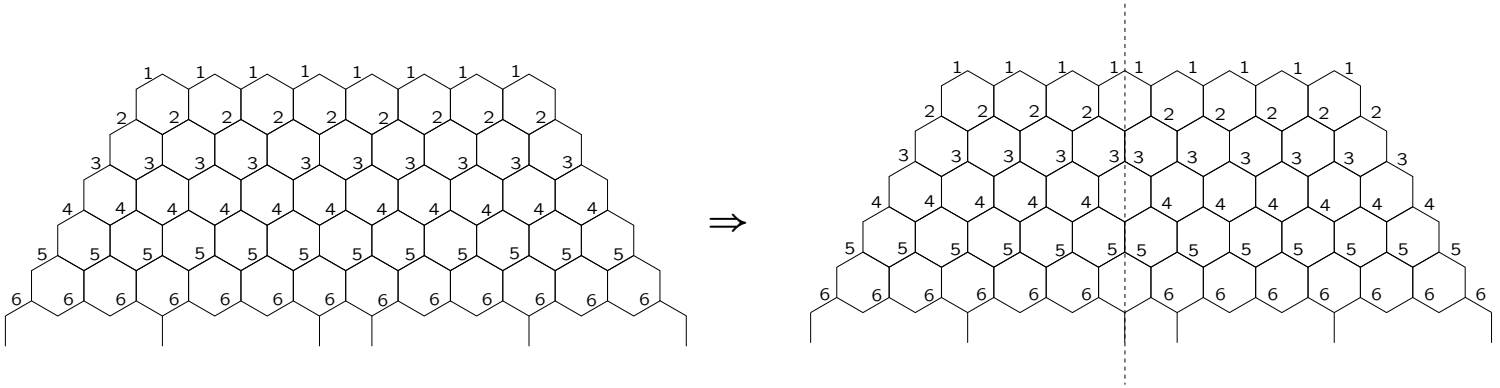
					4							
				3		7						
			2		6		8					
		1		6		7		9				
	1		4		7		9		12			
1		4		7		8		11		14		

Corresponding perfect matching of $T_{6,8}^{1,4,7,8,11,14}(x_1, \ldots, x_6)$:



Symmetrizing the weights using a "randomized" bijection

Trapezoidal honeycomb graph with different edge weights



$T^{1,4,7,8,11,14}_{6,8}(x_1,\ldots,x_6)$ $\qquad$ $\mathsf{ST}^{1,4,7,8,11,14}_{6,8,4}(x_1\ldots,x_6)$

Important: Vertical line contains "peak vertex" in top row!

Lemma. Let $1 \le j \le k$. Then the matching generating function

$$\mathsf{M}(\mathsf{T}^{\mathrm{p}}_{2n,k}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n))$$

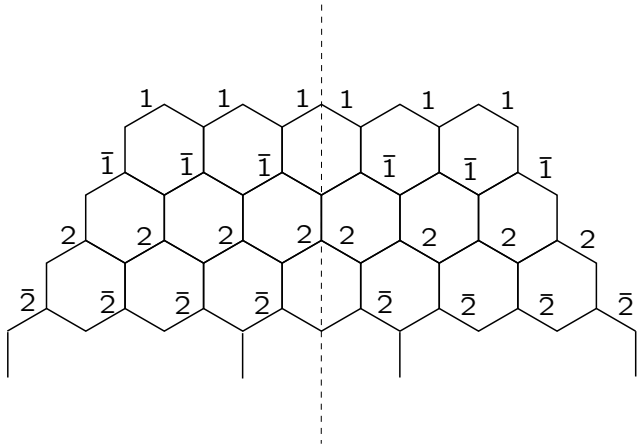
is equal to

$$\frac{1}{2^n}\sum_{\sigma\in\mathcal{I}_n}\mathsf{M}(\sigma\,\mathsf{ST}^{\mathrm{p}}_{2n,k,j}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n)),$$

where $\mathcal{I}_n$ denotes the permutations of the edge weights $\{x_1,\bar{x}_1,x_2,\bar{x}_2,\ldots,x_n,\bar{x}_n\}$ that are generated by the transpositions $(x_1,\bar{x}_1),(x_2,\bar{x}_2),\ldots,(x_n,\bar{x}_n)$.

First identity: $s_{\widehat{\lambda}}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = sp_\lambda(x_1, \ldots, x_n)\, o^{\text{even}}_{\lambda+1}(x_1, \ldots, x_n)$

For $n = 2$ and $\lambda = (2, 0)$ we obtain the graph $\mathrm{ST}^{1,4,6,9}_{4,5,3}(x_1, \bar{x}_1, x_2, \bar{x}_2)$:



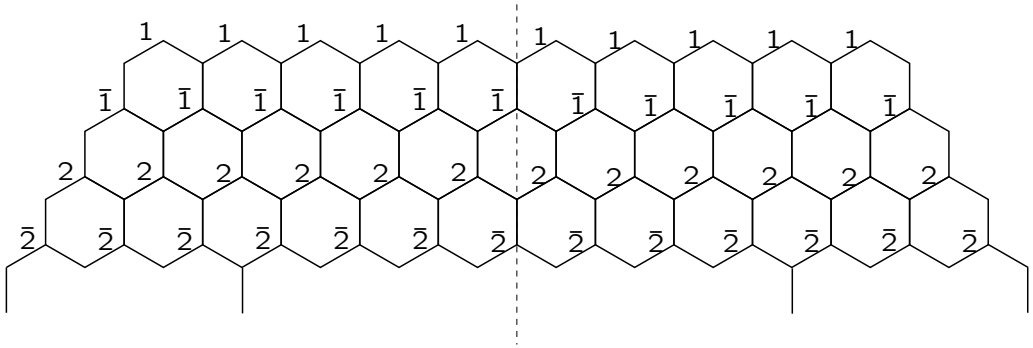
The weighted graph is symmetric with respect to the vertical symmetry axis!

(Almost) doubling the graph

Second identity:

$$s_{\widehat{\lambda}}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n) = \prod_{i=1}^{n}(x_i^{1/2}+\bar{x}_i^{1/2})^{-1} so_{\lambda}^{\text{odd}}(x_1,\ldots,x_n)\, o_{\lambda+\frac{1}{2}}^{\text{even}}(x_1,\ldots,x_n)$$

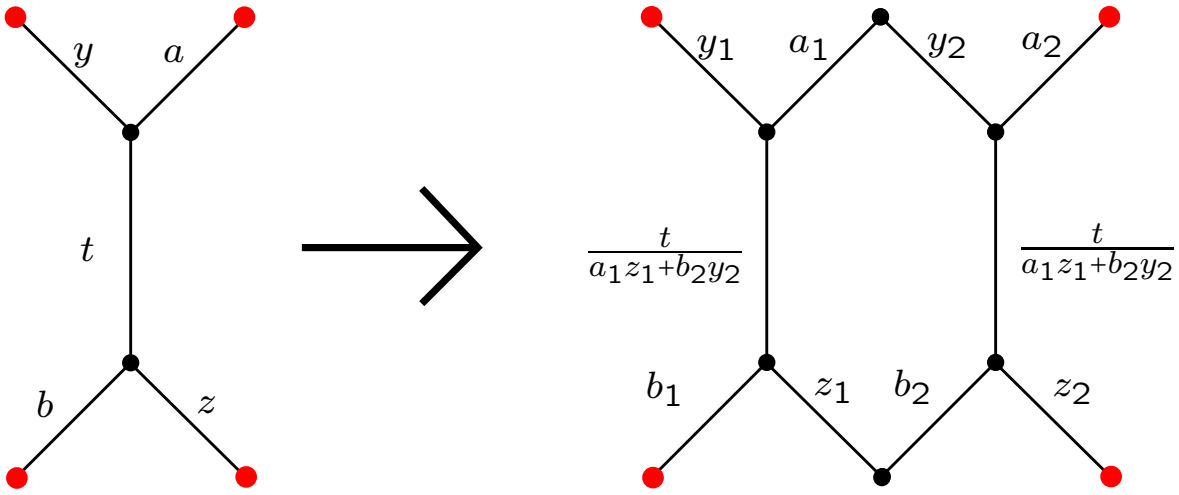
Example $\lambda = (5,3)$, $\widehat{\lambda} = (10,8,2,0)$**:**



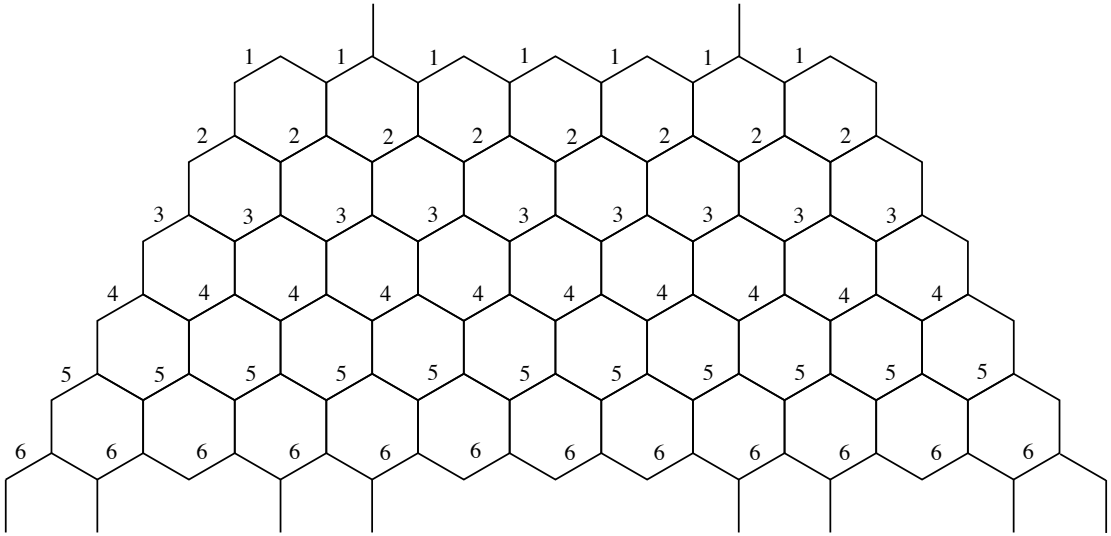
Wrong type of symmetry axis: In this case, the lemma cannot be applied to achieve symmetric edge weights!

A graph operation is the key

Lemma. Let $a_1 \cdot a_2 = a$, $b_1 \cdot b_2 = b$, $y_1 \cdot y_2 = y$ and $z_1 \cdot z_2 = z$. The following replacement rule in a weighted graph leaves the matching generating function invariant, where in the replacement the degree of the black vertices does not change and the red vertices are the connecting points.



The power of bijective proofs: we get the skew case for free!



Other identities in the Ayyer-Behrend paper

For any partition or half-integer partition $(\lambda_0, \ldots, \lambda_n)$,

$$\prod_{i=1}^{n}(x_i^{1/2}+\bar{x}_i^{1/2})\left(s_{(\lambda_1+\lambda_0,\lambda_2+\lambda_0,\ldots,\lambda_n+\lambda_0,-\lambda_{n-1}+\lambda_0,-\lambda_{n-2}+\lambda_0,\ldots,0)}(x_1,\bar{x}_1,x_2,\bar{x}_2,\ldots,x_n,\bar{x}_n)\right.$$
$$\left.+s_{(\lambda_1+\lambda_0,\lambda_2+\lambda_0,\ldots,\lambda_{n-1}+\lambda_0,-\lambda_n+\lambda_0,-\lambda_{n-1}+\lambda_0,\ldots,0)}(x_1,\bar{x}_1,x_2,\bar{x}_2,\ldots,x_n,\bar{x}_n)\right)$$
$$=(1+\delta_{\lambda_n,0})so^{\text{odd}}_{(\lambda_0+1/2,\ldots,\lambda_{n-1}+1/2)}(x_1,\ldots,x_n)o^{\text{even}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)$$

This is **Exercise 14.**

For any partition or half-integer partition $(\lambda_0, \ldots, \lambda_n)$,

$$2\prod_{i=1}^{n}(x_i^{1/2}+\bar{x}_i^{1/2})s_{(\lambda_1+\lambda_0,\lambda_2+\lambda_0,\ldots,\lambda_n+\lambda_0,-\lambda_n+\lambda_0,-\lambda_{n-1}+\lambda_0,\ldots,0)}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n,1)$$
$$= so^{\text{odd}}_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n)o^{\text{even}}_{(\lambda_0+1/2,\ldots,\lambda_n+1/2)}(x_1,\ldots,x_n,1)$$

and

$$2\prod_{i=1}^{n}(x_i^{1/2}+\bar{x}_i^{1/2})s_{(\lambda_0+\lambda_0,\lambda_1+\lambda_0,\ldots,\pm\lambda_n+\lambda_0,-\lambda_{n-1}+\lambda_0,\ldots,0)}(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n,1)$$
$$= (1+\delta_{\lambda_n,0})so^{\text{odd}}_{(\lambda_0+1/2,\ldots,\lambda_{n-1}+1/2)}(x_1,\ldots,x_n)o^{\text{even}}_{(\lambda_0,\ldots,\lambda_n)}(x_1,\ldots,x_n,1).$$

Reference: Bijective proofs of skew Schur polynomial factorizations, Ayyer & F., JCTA 2020.