Several complex variables

Vienna, February 2025

Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and let $f : \Omega \longrightarrow \mathbb{C}$ be a $\mathcal{C}^1$-function. We write $z_j = x_j + iy_j$ and consider for $P \in \Omega$ the differential

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(P)\, dx_j + \frac{\partial f}{\partial y_j}(P)\, dy_j \right).$$

We use the complex differentials

$$dz_j = dx_j + idy_j \quad , \quad d\overline{z}_j = dx_j - idy_j$$

and the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i\, \frac{\partial}{\partial y_j} \right) \quad , \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i\, \frac{\partial}{\partial y_j} \right)$$

and rewrite the differential df_P in the form

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}(P)\, dz_j + \frac{\partial f}{\partial \overline{z}_j}(P)\, d\overline{z}_j \right) = \partial f_P + \overline{\partial} f_P.$$

A general differential form is given by

$$\omega = \sum_{|J|=p,|K|=q}{}' \, a_{J,K} \, dz_J \wedge d\overline{z}_K,$$

where $\sum'_{|J|=p,|K|=q}$ denotes the sum taken only over all increasing multiindices $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$ and

$$dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p} \quad , \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}.$$

We call ω a (p,q)-form and we write $\omega \in \mathcal{C}^k_{(p,q)}(\Omega)$ if ω is a (p,q)-form with coefficients belonging to $\mathcal{C}^k(\Omega)$.

The derivative $d\omega$ of ω is defined by

$$d\omega = \sum_{|J|=p,|K|=q}{}' \, da_{J,K} \wedge dz_J \wedge d\overline{z}_K = \sum_{|J|=p,|K|=q}{}' \, (\partial a_{J,K} + \overline{\partial} a_{J,K}) \wedge dz_J \wedge d\overline{z}_K,$$

and we set

$$\partial\omega = \sum_{|J|=p,|K|=q}{}' \, \partial a_{J,K} \wedge dz_J \wedge d\overline{z}_K \text{ and } \overline{\partial}\omega = \sum_{|J|=p,|K|=q}{}' \, \overline{\partial} a_{J,K} \wedge dz_J \wedge d\overline{z}_K.$$

We have $d = \partial + \overline{\partial}$ and since $d^2 = 0$ it follows that

$$0 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial})\omega = (\partial \circ \partial)\omega + (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)\omega + (\overline{\partial} \circ \overline{\partial})\omega,$$

which implies $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$, by comparing the types of the differential forms involved.

Let $\Omega \subseteq \mathbb{C}^n$ be open. A function $f : \Omega \longrightarrow \mathbb{C}$ is called holomorphic on Ω if $f \in \mathcal{C}^1(\Omega)$ and f satisfies the system of partial differential equations

$$\frac{\partial f}{\partial \overline{z}_j}(z) = 0 \quad \text{for } 1 \leq j \leq n \text{ and } z \in \Omega, \tag{1}$$

equivalently, if f satisfies $\overline{\partial} f = 0$.
Let $P = P(a, r) = \{\zeta \in \mathbb{C} : |\zeta - a_j| < r_j\}$ be a polydisc in $\mathbb{C}^n$. Suppose that $f \in \mathcal{C}^1(\overline{P})$ and that f is holomorphic on P, i.e. for each $z \in P$ and $1 \leq j \leq n$, the function

$$\zeta \mapsto f(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n)$$

is holomorphic on $P(a, r)$. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\gamma_1} \cdots \int_{\gamma_n} \frac{f(\zeta)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} \, d\zeta_1 \dots d\zeta_n, \tag{2}$$

for $z \in P$, where $\gamma_j(t) = a_j + r_j e^{it}$, for $t \in [0, 2\pi]$ and $j = 1, \dots, n$.

Let $f \in \mathcal{H}(P(a,r))$. Then the Taylor series of f at a converges to f uniformly on all compact subsets of $P(a,r)$, that is

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha, \tag{3}$$

for $z \in P(a,r)$.
In addition, we get the Cauchy estimates: for $f \in \mathcal{H}(P(a,r))$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$: let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$, furthermore set $r^\alpha = r_1^{\alpha_1} \dots r_n^{\alpha_n}$, then

$$|D^\alpha f(a)| = \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(a) \right| \leq \frac{\alpha!}{r^\alpha} \sup\{|f(z)| : z \in P(a,r)\}. \tag{4}$$

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let

$$g = \sum_{j=1}^{n} g_j \, d\bar{z}_j$$

be a $(0,1)$-form with coefficients $g_j \in \mathcal{C}^1(\Omega)$, for $j = 1, \ldots, n$. We want to find a function $f \in \mathcal{C}^1(\Omega)$ such that

$$\bar{\partial} f = g, \tag{5}$$

in other words

$$\frac{\partial f}{\partial \bar{z}_j} = g_j, \; j = 1, \ldots, n. \tag{6}$$

f is called a solution to the inhomogeneous CR equation $\bar{\partial} f = g$.

Since $\overline{\partial}^2 = 0$, a necessary condition for solvability of (5) is that the right hand side g satisfies $\overline{\partial} g = 0$. So, the $(0,2)$-form $\overline{\partial} g$ satisfies

$$\overline{\partial} g = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial g_j}{\partial \overline{z}_k} \, d\overline{z}_k \wedge d\overline{z}_j = 0,$$

which means that

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j}, \; j,k = 1, \dots, n.$$

Theorem Let $n \geq 2$ and let $g = \sum_{j=1}^{n} g_j \, d\overline{z}_j$ be a $(0,1)$-form with coefficients $g_j \in \mathcal{C}_0^k(\mathbb{C}^n), \; j = 1, \dots, n$, where $1 \leq k \leq \infty$ and suppose that $\overline{\partial} g = 0$. Then there exists $f \in \mathcal{C}_0^k(\mathbb{C}^n)$ such that $\overline{\partial} f = g$.

$\mathcal{C}_0^k(\mathbb{C}^n)$ is the space of $\mathcal{C}^k$-functions with compact support.

For $n = 1$ the above theorem is false:
Suppose that $\int_{\mathbb{C}} g(\zeta)\, d\lambda(\zeta) \neq 0$ and that there is a compactly supported solution f of the equation $\frac{\partial f}{\partial \overline{z}} = g$. Then there exists $R > 0$ such that $f(\zeta) = 0$ for $|\zeta| \geq R$. Applying Stokes' Theorem we obtain for $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$

$$\begin{aligned} 0 &= \int_\gamma f(\zeta)\, d\zeta \\ &= \int_{D_R(0)} \frac{\partial f}{\partial \overline{\zeta}}\, d\overline{\zeta} \wedge d\zeta \\ &= 2i \int_{D_R(0)} g(\zeta)\, d\lambda(\zeta) \\ &\neq 0, \end{aligned}$$

whenever $D_R(0)$ contains the support of g. That is a contradiction.

Distributions

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$ the space of $\mathcal{C}^\infty$-functions with compact support (test functions).
A sequence $(\phi_j)_j$ tends to 0 in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\text{supp}(\phi_j) \subset K$ for every j and

$$\frac{\partial^{|\alpha|}\phi_j}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \to 0$$

uniformly on K for each $\alpha = (\alpha_1, \dots, \alpha_n)$.
A distribution is a linear functional u on $\mathcal{D}(\Omega)$ such that for every compact subset $K \subset \Omega$ there exists $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and a constant $C > 0$ with

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in K} \left| \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|,$$

for each $\phi \in \mathcal{D}(\Omega)$ with support in K. We denote the space of distributions on Ω by $\mathcal{D}'(\Omega)$.
It is easily seen that $u \in \mathcal{D}'(\Omega)$ if and only if $u(\phi_j) \to 0$ for every sequence $(\phi_j)_j$ in $\mathcal{D}(\Omega)$ converging to 0 in $\mathcal{D}(\Omega)$.

Examples

Let $f \in L^1_{loc}(\Omega)$, where

$$L^1_{loc}(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}\, \text{measurable} : f\,|_K \in L^1(K)\, \forall K \subset \Omega,\ K\, \text{compact}\}.$$

The mapping $T_f(\phi) = \int_\Omega f(x)\phi(x)\,d\lambda(x)$, $\phi \in \mathcal{D}(\Omega)$, is a distribution.

Let $a \in \Omega$ and $\delta_a(\phi) := \phi(a)$, which is the point evaluation in a. The distribution δ_a is called Dirac delta distribution.

Let

$$D_k = \frac{\partial}{\partial x_k} \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index. The partial derivative of a distribution $u \in \mathcal{D}'(\Omega)$ is defined by

$$(D_k u)(\phi) := -u(D_k \phi), \quad \phi \in \mathcal{D}(\Omega);$$

higher order mixed derivatives are defined as

$$(D^\alpha u)(\phi) := (-1)^{|\alpha|}\, u(D^\alpha \phi), \quad \phi \in \mathcal{D}(\Omega).$$

This definition stems from integrating by parts:

$$\int_\Omega (D_k f)\phi \, d\lambda = -\int_\Omega f(D_k\phi)\, d\lambda,$$

where $f \in \mathcal{C}^1(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$.
Let $\Omega = \{z \in \mathbb{C}^n \ : \ r(z) < 0\}$, where r is a real-valued $\mathcal{C}^1$-function with

$$\nabla_z r := \left(\frac{\partial r}{\partial z_1}, \dots, \frac{\partial r}{\partial z_n}\right) \neq 0$$

on $b\Omega = \{z \ : \ r(z) = 0\}$. Then r is called a defining function for Ω.
Without loss of generality we can suppose that $|\nabla_z r| = |\nabla r| = 1$ on $b\Omega$.
For $u, v \in \mathcal{C}^\infty(\overline{\Omega})$ and

$$(u, v) = \int_\Omega u(z)\overline{v(z)}\, d\lambda(z)$$

we have

$$(u_{x_k}, v) = -(u, v_{x_k}) + \int_{b\Omega} u(z)\overline{v(z)}\, r_{x_k}(z)\, d\sigma(z),$$

where $d\sigma$ is the surface measure on $b\Omega$.

This follows from the Gauß–Green Theorem: for $\omega \subseteq \mathbb{R}^n$ we have

$$\int_\omega \nabla . F(x)\, d\lambda(x) = \int_{b\omega} (F(x), \nu(x))\, d\sigma(x),$$

where $\nu(x) = \nabla r(x)$ is the normal to $b\omega$ at x, and F is a $\mathcal{C}^1$ vector field on $\overline{\omega}$, and

$$\nabla . F(x) = \sum_{j=1}^n \frac{\partial F_j}{\partial x_j}.$$

For $k = 1$ and $F = (u\overline{v}, 0, \dots, 0)$ one gets

$$(u_{x_1}, v) = -(u, v_{x_1}) + \int_{b\Omega} u(z)\overline{v(z)}\, r_{x_1}(z)\, d\sigma(z),$$

similarly one obtains

$$\left(\frac{\partial u}{\partial z_k}, v\right) = -\left(u, \frac{\partial v}{\partial \overline{z}_k}\right) + \int_{b\Omega} u(z)\,\overline{v(z)}\,\frac{\partial r}{\partial z_k}(z)\, d\sigma(z). \tag{7}$$

Let

$$L^2_{(0,1)}(\Omega) := \left\{ u = \sum_{j=1}^{n} u_j \, d\overline{z}_j \ : \ u_j \in L^2(\Omega), \, j = 1, \dots, n \right\}$$

be the space of $(0,1)$-forms with coefficients in $L^2(\Omega)$. For $u, v \in L^2_{(0,1)}(\Omega)$ we define the inner product by

$$(u, v) = \sum_{j=1}^{n} (u_j, v_j).$$

In this way $L^2_{(0,1)}(\Omega)$ becomes a Hilbert space. $(0,1)$-forms with compactly supported $\mathcal{C}^\infty$ coefficients are dense in $L^2_{(0,1)}(\Omega)$.
Let $f \in \mathcal{C}_0^\infty(\Omega)$ and set

$$\overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \, d\overline{z}_j,$$

then

$$\overline{\partial} : \mathcal{C}_0^\infty(\Omega) \longrightarrow L^2_{(0,1)}(\Omega).$$

$\overline{\partial}$, with $\mathcal{C}_0^\infty(\Omega)$ as domain, is a densely defined unbounded operator on $L^2(\Omega)$.

We have to extend the domain to get a densely defined unbounded operator $\overline{\partial}$ with closed graph: the domain $\mathrm{dom}(\overline{\partial})$ of $\overline{\partial}$ consists of all functions $f \in L^2(\Omega)$ such that $\overline{\partial} f$, in the sense of distributions, belongs to $L^2_{(0,1)}(\Omega)$, i.e. $\overline{\partial} f = g = \sum_{j=1}^n g_j \, d\overline{z}_j$, and for each $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_\Omega f \left(\frac{\partial \phi}{\partial z_j} \right)^{-} d\lambda = - \int_\Omega g_j \, \overline{\phi} \, d\lambda, \quad j = 1, \dots, n. \tag{8}$$

For $f_k \in \mathrm{dom}(\overline{\partial})$ and $f_k \to f$ in $L^2(\Omega)$ and $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$, we have to show that $f \in \mathrm{dom}(\overline{\partial})$ and $\overline{\partial} f = g$.

By Cauchy-Schwarz we have

$$|\int_\Omega (f - f_k)\left(\frac{\partial \phi}{\partial z_j}\right)^- \, d\lambda| \leq \|f - f_k\|_2 \, \|\left(\frac{\partial \phi}{\partial z_j}\right)^- \|_2, \tag{9}$$

which implies that

$$\int_\Omega f\left(\frac{\partial \phi}{\partial z_j}\right)^- \, d\lambda = \lim_{k\to\infty} \int_\Omega f_k \left(\frac{\partial \phi}{\partial z_j}\right)^- \, d\lambda = \lim_{k\to\infty} (-1) \int_\Omega \frac{\partial f_k}{\partial \overline{z}_j} \overline{\phi} \, d\lambda$$

$$= (-1) \int_\Omega g_j \overline{\phi} \, d\lambda,$$

which gives $f \in \mathrm{dom}(\overline{\partial})$ and from (9) we have $\overline{\partial} f_k \to \overline{\partial} f$ in $L^2_{(0,1)}(\Omega)$, so we have finally $\overline{\partial} f = g$.

Unbounded operators

Let H_1, H_2 be Hilbert spaces and $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined linear operator, i.e. $\mathrm{dom}(T)$ is a dense linear subspace of H_1. Let $\mathrm{dom}(T^*)$ be the space of all $y \in H_2$ such that $x \mapsto (Tx, y)_2$ defines a continuous linear functional on $\mathrm{dom}(T)$. Since $\mathrm{dom}(T)$ is dense in H_1 there exists a uniquely determined element $T^*y \in H_1$ such that $(Tx, y)_2 = (x, T^*y)_1$. The map $y \mapsto T^*y$ is linear and $T^* : \mathrm{dom}(T^*) \longrightarrow H_1$ is the adjoint operator to T.
T is called a closed operator, if the graph

$$\mathcal{G}(T) = \{(f, Tf) \in H_1 \times H_2 \, : \, f \in \mathrm{dom}(T)\}$$

is a closed subspace of $H_1 \times H_2$.
The inner product in $H_1 \times H_2$ is

$$((x, y), (u, v)) = (x, u)_1 + (y, v)_2.$$

If $\tilde{V}$ is a linear subspace of H_1 which contains $\mathrm{dom}(T)$ and $\tilde{T}x = Tx$ for all $x \in \mathrm{dom}(T)$ then we say that $\tilde{T}$ is an extension of T.

Some basic results:

Let $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined linear operator and define $V : H_1 \times H_2 \longrightarrow H_2 \times H_1$ by $V((x,y)) = (y, -x)$. Then

$$\mathcal{G}(T^*) = [V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp);$$

in particular T^* is always closed.

Let $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $\mathrm{dom}(T^*)$ is dense in H_2 and $T^{**} = T$.

Let $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined linear operator. Then $\ker T^* = (\mathrm{im}\, T)^\perp$, which means that $\ker T^*$ is closed.

Let $T : \mathrm{dom}(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $\ker T$ is a closed linear subspace of H_1.

For our applications to the $\overline{\partial}$-equation it will be important to know whether the differential operators involved have closed range or are even surjective.

Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator. $T(H_1)$ is closed if and only if $T|_{(\ker T)^\perp}$ is bounded from below, i.e.

$$\|Tf\| \geq C\|f\| \ , \ \forall f \in (\ker T)^\perp.$$

Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator. $\operatorname{im} T$ is closed in H_2 if and only if $T|_{\operatorname{dom}(T)\cap(\ker T)^\perp}$ is bounded from below, i.e.

$$\|Tf\| \geq C\|f\| \ \ , \ \forall f \in \operatorname{dom}(T) \cap (\ker T)^\perp.$$

Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator. $\operatorname{im} T$ is closed if and only if $\operatorname{im} T^*$ is closed.

Let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator and G a closed subspace of H_2 with $G \supseteq \operatorname{im} T$. Suppose that $T^*|_{\operatorname{dom}(T^*)\cap G}$ is bounded from below, i.e. $\|f\| \leq C\|T^*f\|$ for all $f \in \operatorname{dom}(T^*) \cap G$, where $C > 0$ is a constant. Then $G = \operatorname{im} T$.

In the following we introduce the fundamental concept of an unbounded self-adjoint operator, which will be crucial for both spectral theory and its applications to complex analysis.

Let $T : \text{dom}(T) \longrightarrow H$ be a densely defined linear operator. T is symmetric if $(Tx, y) = (x, Ty)$ for all $x, y \in \text{dom}(T)$. We say that T is self-adjoint if T is symmetric and $\text{dom}(T) = \text{dom}(T^*)$. This is equivalent to requiring that $T = T^*$ and implies that T is closed.

Let T be a densely defined, symmetric operator.
(i) If $\text{dom}(T) = H$, then T is self-adjoint and T is bounded.
(ii) If T is self-adjoint and injective, then $\text{im}(T)$ is dense in H, and T^{-1} is self-adjoint.
(iii) If $\text{im}(T)$ is dense in H, then T is injective.
(iv) If $\text{im}(T) = H$, then T is self-adjoint, and T^{-1} is bounded.

Theorem

Let T be a densely defined closed operator, $dom(T) \subseteq H_1$ and $T : dom(T) \longrightarrow H_2$. Then $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are everywhere defined and bounded, $\|B\| \leq 1, \ \ \|C\| \leq 1$; in addition B is self-adjoint and positive, i.e. $(Bu, u) > 0$, for all $u \in H_1$.

Now we consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0 \,, \tag{10}$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J\, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \, \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J, \tag{11}$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices $J = (j_1, \dots, j_q)$.
The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.

We consider the $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\Omega), \tag{12}$$

for $1 \leq q \leq n-1$.

Theorem

The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$, defined on the domain $dom(\Box) = \{u \in L^2_{(0,q)}(\Omega) : u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*), \overline{\partial}u \in dom(\overline{\partial}^*), \overline{\partial}^* u \in dom(\overline{\partial})\}$ acts as an unbounded, densely defined, closed and self-adjoint operator on $L^2_{(0,q)}(\Omega)$, for $1 \leq q \leq n$, which means that $\Box = \Box^*$ and $dom(\Box) = dom(\Box^*)$.

We demonstrate the method for the $\overline{\partial}$-Neumann problem first in its finite dimensional analog: let E, F, G denote finite dimensional vector spaces over $\mathbb{C}$ with inner product. We consider an exact sequence of linear maps

$$E \overset{S}{\longrightarrow} F \overset{T}{\longrightarrow} G,$$

which means that $\mathrm{im} S = \ker T$, hence $TS = 0$.
Given $f \in \mathrm{im} S = \ker T$, we want to solve $Su = f$ with $u \perp \ker S$, then u will be called the canonical solution.
For this purpose we investigate

$$E \underset{S^*}{\overset{S}{\rightleftarrows}} F \underset{T^*}{\overset{T}{\rightleftarrows}} G$$

and observe that $\ker T = (\mathrm{im}\, T^*)^{\perp}$ and $\ker T^* = (\mathrm{im}\, T)^{\perp}$. We claim that the operator $SS^* + T^*T : F \longrightarrow F$ is bijective. Let $N = (SS^* + T^*T)^{-1}$. Then

$$u = S^*Nf$$

is the canonical solution to $Su = f$.

We return to the $\Box$-operator on $(0, q)$-forms and suppose now that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. It will be shown that

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \geq c\, \|u\|^2, \tag{13}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, $c > 0$, (basic estimate).

Theorem

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then $\overline{\partial}$ and $\overline{\partial}^*$ have closed range.

Theorem

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then $\Box : dom(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$ *is bijective and has a bounded inverse*

$$N : L^2_{(0,q)}(\Omega) \longrightarrow dom(\Box).$$

N is called $\overline{\partial}$-Neumann[1] operator. In addition

$$\|Nu\| \leq \frac{1}{c}\,\|u\|. \tag{14}$$

We consider the embedding

$$j : \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \longrightarrow L^2_{(0,q)}(\Omega),$$

where $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ is endowed with the graph-norm

$$u \mapsto (\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}.$$

The graph-norm stems from the inner product

$$Q(u,v) = (u,v)_Q = (\Box u, v) = (\overline{\partial} u, \overline{\partial} v) + (\overline{\partial}^* u, \overline{\partial}^* v).$$

The basic estimates (13) imply that j is a bounded operator with operator norm

$$\|j\| \leq \frac{1}{\sqrt{c}} \text{ and } N = j \circ j^*.$$

Finally we give a few examples of the so-called Kohn calculus:
The operators

$$\overline{\partial}N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q+1)}(\Omega) \text{ and } \overline{\partial}^* N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega)$$

are both bounded.

Let N_q denote the $\overline{\partial}$-Neumann operator on $L^2_{(0,q)}(\Omega)$. Then

$$N_{q+1}\overline{\partial} = \overline{\partial}N_q, \tag{15}$$

on $\text{dom}(\overline{\partial})$ and

$$N_{q-1}\overline{\partial}^* = \overline{\partial}^* N_q, \tag{16}$$

on $\text{dom}(\overline{\partial}^*)$.

The main results are the following

Theorem

Let $\alpha \in L^2_{(0,q)}(\Omega)$*, with* $\overline{\partial}\alpha = 0$*. Then* $u_0 = \overline{\partial}^* N_q \alpha$ *is the canonical solution of* $\overline{\partial} u = \alpha$*, this means* $\overline{\partial} u_0 = \alpha$ *and* $u_0 \perp \ker \overline{\partial}$*, and*

$$\|\overline{\partial}^* N_q \alpha\| \leq c^{-1/2} \, \|\alpha\|. \tag{17}$$

Theorem

Let $P_q : L^2_{(0,q)}(\Omega) \longrightarrow \ker\overline{\partial}$ *denote the orthogonal projection, which is the Bergman projection for* $q = 0$*. Then*

$$P_q = I - \overline{\partial}^* N_{q+1} \overline{\partial}, \tag{18}$$

on $dom(\overline{\partial})$*.*