

STABLE GABOR PHASE RETRIEVAL AND SPECTRAL CLUSTERING

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ABSTRACT. We consider the problem of reconstructing a signal f from its spectrogram, i.e., the magnitudes $|V_\varphi f|$ of its Gabor transform

$$V_\varphi f(x, y) := \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi i y t} dt, \quad x, y \in \mathbb{R}.$$

Such problems occur in a wide range of applications, from optical imaging of nanoscale structures to audio processing and classification.

While it is well-known that the solution of the above Gabor phase retrieval problem is unique up to natural identifications, the stability of the reconstruction has remained wide open. The present paper discovers a deep and surprising connection between phase retrieval, spectral clustering and spectral geometry. We show that the stability of the Gabor phase reconstruction is bounded by the reciprocal of the *Cheeger constant* of the flat metric on \mathbb{R}^2 , conformally multiplied with $|V_\varphi f|$. The Cheeger constant, in turn, plays a prominent role in the field of spectral clustering, and it precisely quantifies the ‘disconnectedness’ of the measurements $V_\varphi f$.

It has long been known that a disconnected support of the measurements results in an instability – our result for the first time provides a converse in the sense that there are no other sources of instabilities.

Due to the fundamental importance of Gabor phase retrieval in coherent diffraction imaging, we also provide a new understanding of the stability properties of these imaging techniques: Contrary to most classical problems in imaging science whose regularization requires the promotion of smoothness or sparsity, the correct regularization of the phase retrieval problem promotes the ‘connectedness’ of the measurements in terms of bounding the Cheeger constant from below. Our work thus, for the first time, opens the door to the development of efficient regularization strategies.

1. INTRODUCTION

1.1. Motivation. A signal is typically modeled as an element $f \in \mathcal{B}$ with \mathcal{B} an ∞ -dimensional Banach space. *Phase retrieval* refers to the reconstruction of a signal from phaseless linear measurements

$$(1.1) \quad (|\varphi_\omega(f)|)_{\omega \in \Omega},$$

where $\Phi = (\varphi_\omega)_{\omega \in \Omega} \subset \mathcal{B}'$, the dual of \mathcal{B} . Since for any $\alpha \in \mathbb{R}$ the signal $e^{i\alpha} f$ will yield the same phaseless linear measurements as f , a signal can only be reconstructed *up to global phase* e.g., up to the identification $f \sim e^{i\alpha} f$, where $\alpha \in \mathbb{R}$. If any $f \in \mathcal{B}$ can be uniquely reconstructed from its phaseless measurements (1.1), up to global phase, we say that Φ *does phase retrieval*.

Phase retrieval problems of the aforementioned type occur in a remarkably wide number of physical problems (often owing to the fact that the phase of a high-frequency wave cannot be measured), probably most prominently in coherent diffraction imaging [26, 29, 42, 41, 37] where Φ is either a Fourier- or a Gabor dictionary. Other applications include quantum mechanics [39], audio processing [10, 11] or Radar [35].

Given a concrete phase retrieval problem defined by a measurement system Φ it is notoriously difficult to study whether Φ does phase retrieval and there are only a few concrete instances where this is known. In the ∞ -dimensional setting, examples of such instances include phase retrieval from Poisson wavelet measurements [47], from Gabor measurements [1] or from masked Fourier

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measurements [48], while it is known that the reconstruction of a compactly supported function from its Fourier magnitude is in general not uniquely possible [32].

From a computational standpoint, solving a given phase retrieval problem is even more challenging: Assuming that Φ does phase retrieval, an algorithmic reconstruction of a signal f would require additionally that the reconstruction be stable in the sense that

$$(1.2) \quad d_{\mathcal{B}}(f, g) \leq c(f) \left| \|\Phi(f)\| - \|\Phi(g)\| \right|_{\mathcal{D}} \quad \text{for all } g \in \mathcal{B}$$

holds true, where we have put

$$(1.3) \quad d_{\mathcal{B}}(f, g) := \inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha} g\|_{\mathcal{B}},$$

$\Phi(f) := \begin{cases} \Omega & \rightarrow & \mathbb{C} \\ \omega & \mapsto & \varphi_{\omega}(f) \end{cases}$, and $\|\cdot\|_{\mathcal{D}}$ a suitable norm on the measurement space of functions $\Omega \rightarrow \mathbb{C}$.

1.2. Phase Retrieval is Severely Ill-Posed. Despite its formidable relevance, the study of stability properties of phase retrieval problems has seen little progress until in recent work [1, 13] a striking instability phenomenon has been identified by showing that $\sup_{f \in \mathcal{B}} c(f) = \infty$, whenever $\dim \mathcal{B} = \infty$ and some natural conditions on \mathcal{B} and \mathcal{D} are satisfied. Even worse, the stability of finite-dimensional approximations to such problems in general degenerates exponentially in a power of the dimension [3, 13]. This means that

every ∞ -dimensional (and therefore every practically relevant) phase retrieval problem, as well as any fine-grained finite-dimensional approximation thereof, is unstable – Phase retrieval is severely ill-posed.

In view of this negative result, any phase retrieval problem needs to be regularized and any regularization strategy for a given phase retrieval problem would require a deeper understanding of the behaviour of the local Lipschitz constant $c(f)$. This is a challenging problem requiring genuinely new methods: in [3] we show that all conventional regularization methods based on the promotion of smoothness or sparsity are unsuitable for the regularization of phase retrieval problems.

1.3. What are the Sources for Instability? We briefly summarize the current understanding of the situation.

A well-known source of instability (e.g., a very large constant $c(f)$), coined ‘multicomponent-type instability’ in [1] arises whenever the measurements $\Phi(f)$ are separated in the sense that $f = u + v$ with $\Phi(u)$ and $\Phi(v)$ concentrated in disjoint subsets of Ω . Intuitively, in this case the function $g = u - v$ will produce measurements $|\Phi(g)|$ very close to the original measurements $|\Phi(f)|$, while the distance $d_{\mathcal{B}}(f, g)$ is not small at all, resulting in an instability (see also Figure 1 for an illustration). If \mathcal{B} is a finite-dimensional Hilbert space over \mathbb{R} (i.e., the real-valued case where only a sign and not the full phase needs to be determined) the correctness of this intuition has been proved in [8] and generalized in [2] to the setting of ∞ -dimensional real or complex Banach spaces:

If the measurements $\Phi(f)$ are concentrated on a union of at least two disjoint domains, phase retrieval becomes unstable and correspondingly, the constant $c(f)$ becomes large.

If \mathcal{B} is a Banach space over \mathbb{R} it is not very difficult to show that the ‘multicomponent-type instability’ as just described is the only source of instability. More precisely, one can characterize $c(f)$, via the so-called σ -strong complement property (SCP) which indeeds provides a measure for the disconnectedness of the measurements, see [8, 2]. While these results provide a complete characterization of the stability of phase retrieval problems over \mathbb{R} , we hasten to add that the verification of the σ -strong complement property is computationally intractable which severely limits their applicability.

The (much more interesting) complex case is considerably more challenging and almost nothing is known. In this case the validity of the σ -SCP does not imply stability of the corresponding phase retrieval problem (it does not even imply uniqueness of the solution) [8].

Nevertheless, the results in the real-valued case suggest the following informal conjecture.

Conjecture 1.1. *Phase retrieval is unstable if and only if the measurements are concentrated on at least two distinct domains. In other words: if $c(f)$ is large, then it is possible to partition the parameter set Ω into two disjoint domains $\Omega_1, \Omega_2 \subset \Omega$ such that the measurements $\Phi(f) : \Omega \rightarrow \mathbb{C}$ are ‘clustered’ on Ω_1 , resp. Ω_2 .*

While this conjecture seems to be folklore in the phase retrieval community (for example in [46, page 1273] it is explicitly stated that ‘all instabilities [...] that we were able to observe in practice were of the form we described [...]’, meaning that they arise from measurements with disconnected components. Furthermore, [46] provides partial theoretical support for Conjecture 1.1 for phase retrieval problems based on wavelet measurements) and ensuring connectedness of the essential support of the measurements is a common empirical regularization strategy [46, 25, 8, 34], we are not aware of any mathematical result which resolves Conjecture 1.1 for any concrete phase retrieval problem.

1.4. Phase Retrieval and Spectral Clustering. Looking at Conjecture 1.1, clustering problems in data analysis come to mind. We may, as a matter of fact, look into this field to formalize what it could possibly mean that ‘data is clustered on two disjoint sets’. Let us suppose that $\Omega = \mathbb{R}^d$. We could interpret the measurements $|\Phi(f)| : \Omega \rightarrow \mathbb{R}_+$ as a density measure $d\mu = |\Phi(f)|dx$ (we shall also write μ^{d-1} for the induced surface measure) of data points and attempt to find two (or more) ‘clusters’ (i.e., subsets of Ω) on which this measure is concentrated. In data analysis, the standard notion which describes the degree to which it is possible to divide data points into clusters is the Cheeger constant which may be defined as

$$(1.4) \quad \inf_{C \subset \Omega} \frac{\mu^{d-1}(\partial C)}{\min(\mu(C), \mu(\Omega \setminus C))} = \inf_{C \subset \Omega, \mu(C) \leq \frac{1}{2}\mu(\Omega)} \frac{\mu^{d-1}(\partial C)}{\mu(C)},$$

see for example [36, 17, 43]. Looking at the above definition it becomes clear that the Cheeger constant indeed gives a measure of disconnectedness: if the constant above is small, there exists a partition of Ω into a set C and $\Omega \setminus C$ such that the volume of both C and $\Omega \setminus C$ is large, while the volume of the ‘interface’ ∂C is small.

1.5. This Paper. The present paper establishes a surprising connection between the mathematical analysis of clustering problems and phase retrieval: we show that for a Gabor dictionary

$$\Phi(f) = \left(V_\varphi f(x, y) := \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi i t y} dt \right)_{(x,y) \in \mathbb{R}^2}$$

the Cheeger constant also characterizes the stability of the corresponding phase retrieval problem:

Given $f \in \mathcal{B}$ where \mathcal{B} denotes a certain modulation space and $\|\cdot\|_{\mathcal{D}}$ a natural norm on the measurement space of functions on $\Omega = \mathbb{R}^2$ our main result, Theorem 2.9, shows that the stability constant $c(f)$ can be bounded from above (up to a fixed constant, independent of f) by $h(f)^{-1}$, where

$$h(f) = \inf_{\substack{C \subset \mathbb{R}^2 \text{ open: } \partial C \text{ is smooth} \\ \text{and } \int_C |V_\varphi f| \leq \frac{1}{2} \int_{\mathbb{R}^2} |V_\varphi f|}} \frac{\|V_\varphi f\|_{L^1(\partial C)}}{\|V_\varphi f\|_{L^1(C)}}$$

denotes what we call the *Cheeger constant* of f . Note that the above definition is completely in line with (1.4) by setting $d\mu = |V_\varphi f(x, y)| dx dy$. The motivation for the term Cheeger constant stems from the fact that $h(f)$ is actually equal to the well-known Cheeger constant from Riemannian geometry [15] if we endow \mathbb{R}^2 with the Riemannian metric¹ induced by the metric tensor

$$\left(|V_\varphi f(x, y)| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)_{(x,y) \in \mathbb{R}^2}.$$

Such a metric is sometimes also called a *conformal multiplication of the flat metric* by $|V_\varphi f|$.

¹accepting the slight inaccuracy that $|V_\varphi f|$ may have zeros, so one does not in general get a Riemannian metric

We would like to stress that our result can be regarded as a formalization and as a proof of Conjecture 1.1: The fact that $h(f)$ is small precisely describes the fact that the measurement space $\Omega = \mathbb{R}^2$ can be partitioned into two sets C and $\mathbb{R}^2 \setminus C$ such that both $\|V_\varphi f\|_{L^1(C)}$ and $\|V_\varphi f\|_{L^1(\mathbb{R}^2 \setminus C)}$ are large, but on their separating boundary ∂C , the measurements are small. The quantity $h(f)$ is therefore a mathematical measure for the disconnectedness of the measurements. Indeed, as already mentioned, the Cheeger constant forms a crucial quantity in spectral clustering algorithms [44] and is, in the field of data science, a well-established quantity describing the degree of disconnectedness of data. Our results show that such a disconnectedness is the only possible source of instability of phase retrieval from Gabor measurements and we find it quite remarkable that the notion of Cheeger constant which is standard in clustering problems occurs as a natural characterization of the stability of phase retrieval.

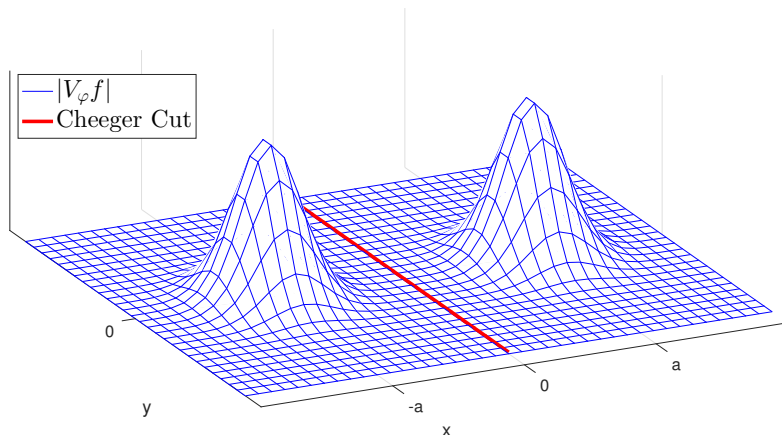


FIGURE 1. Standard examples of instabilities are constructed by adding functions whose measurements are essentially supported on sets that are far apart from each other. For the Gabor phase retrieval problem, such instabilities can be constructed as $f(\cdot) = \varphi(\cdot + a) + \varphi(\cdot - a)$, where $\varphi(\cdot) = e^{-\pi \cdot^2}$ denotes the Gaussian and $a > 0$ is a large real number. Since $V_\varphi f(x, y) = V_\varphi \varphi(x + a, y) + V_\varphi \varphi(x - a, y)$ holds true, Lemma A.5 yields that $|V_\varphi f| \approx |V_\varphi g|$, where $g(\cdot) = \varphi(\cdot + a) - \varphi(\cdot - a)$. Cutting the time-frequency plane along the line $x = 0$ results in two sets of equal measure w.r.t. $|V_\varphi f|(x, y) dx dy$. On the separating line (called a ‘Cheeger cut’) the weight is small, therefore also the Cheeger constant will be very small. Our main result shows that all instabilities ‘look like the above picture’.

1.6. Implications. Aside from providing the first ever stability bounds for any realistic ∞ -dimensional phase retrieval problem, our result has a number of important implications:

- Given measurements $V_\varphi f$, estimating the Cheeger constant $h(f)$ is a computationally tractable procedure [5, 38]. In this way one can decide from the measurements how noise-stable the reconstruction is expected to be.
- Our results (in particular Corollary 2.10 below) for the first time open the door to the construction of regularization methods for the notoriously ill-posed phase retrieval problem from Gabor measurements. Any useful regularizer will have to promote the connectedness of the measurements in terms of keeping the value $h(f)$ above a certain threshold. To put it more pointedly:

Contrary to most classical problems in imaging science whose regularization requires the promotion of smoothness or sparsity, the correct regularization of the

phase retrieval problem promotes the ‘connectedness’ of the measurements in terms of the Cheeger constant!

We will explore algorithmic implications in future work.

- Often one has a priori knowledge on the data f to be measured in the sense that f belongs to a compact subset $\mathcal{C} \subset \mathcal{B}$ (such as, for example, piecewise smooth nonnegative functions). By studying the quantity $\inf_{f \in \mathcal{C}} h(f)$ we can for the first time decide what type of a priori knowledge is useful for the phase retrieval problem. We also expect our stability results to lead to insights on how to design masks ω such that the Gabor phase retrieval problem of the masked signal ωf becomes stable.
- In [1] it has been observed that for various applications, such as audio processing, the multi-component-type instability is actually harmless because the assignment of different bulk phases to different connected components of the measurements is not recognizable by the human ear. Our results show that in fact no other instabilities occur which, for these applications, makes phase retrieval a stable problem! In particular, using our insights we expect to be able to make the concept of ‘multicomponent instability’ of [1] rigorous. Furthermore in Section 2.2.3 we outline how to algorithmically find multicomponent decompositions for unstable Gabor measurements using well-established spectral clustering algorithms which are precisely based on minimizing the Cheeger constant associated with the data [44].
- The quantity $h(f)^{-1}$ has another interpretation: it provides a bound for the *Poincaré constant* on the weighted $L^1(\mathbb{R}^2, \mu)$ space with measure $d\mu = |V_\varphi f| dx dy$. In fact, our results show that the stability of Gabor phase retrieval is controlled by the Poincaré constant. There exists a huge body of research providing bounds on such weighted Poincaré constants in terms of properties of $|V_\varphi f|$. By our results, every such result directly implies a stability result for phase retrieval from Gabor measurements.
- In Section 2.2.1 we outline an intimate connection between Gabor phase retrieval and the solution of the backward heat equation. Our results therefore also have implications on the latter problem which we will study in detail in future work.

Our proof techniques are not restricted to the case of Gabor measurements but crucially assume that, up to multiplication with a smooth function, the measurements $\Phi(f)$ constitute a holomorphic function which is for example also satisfied if the measurements arise from a wavelet transform with a Poisson wavelet [47]. In terms of practical applications, the case of Gabor measurements is already of great relevance: Such measurements arise for instance in Ptychography, a subfield of diffraction imaging where an extended object is scanned through a highly coherent X-ray beam, producing measurements which can be modeled as Gabor measurements [30, 41, 42]. Another application area is in audio processing where phase retrieval from Gabor measurements arises in the so-called ‘phase coherence problem’ for phase vocoders [27, 7, 40].

2. SUMMARY OF OUR MAIN RESULT

2.1. Main Results of This Paper. This section summarizes our main results. We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwartz test functions and with $\mathcal{S}'(\mathbb{R})$ its dual, the space of tempered distributions [45]. The short-time Fourier transform (STFT) is then defined as follows.

Definition 2.1. *Let $g \in \mathcal{S}(\mathbb{R})$. Then the short-time Fourier transform (STFT) (with window function g) of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ is defined as*

$$V_g f(x, y) := \left(f, \overline{g(\cdot - x)} e^{-2\pi i y \cdot} \right)_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} . \quad 2$$

If $g(t) = \varphi(t) := e^{-\pi t^2}$ we call the arising STFT the Gabor transform.

The functional analytic properties of the STFT are best studied within the framework of modulation spaces as defined below.

²If f is a regular tempered distribution, i.e., abusing notation, $(f, g)_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} := \int_{\mathbb{R}} f(t) g(t) dt$ for $g \in \mathcal{S}(\mathbb{R})$, we would get the usual formula $V_g f(x, y) = \int_{\mathbb{R}} f(t) \overline{g(t - x)} e^{-2\pi i y t} dt$

Definition 2.2. Given $1 \leq p \leq \infty$, the Modulation space $M^{p,p}(\mathbb{R})$ is defined as

$$M^{p,p}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : V_g f \in L^p(\mathbb{R}^2)\},$$

with induced norm

$$\|f\|_{M^{p,p}(\mathbb{R})} := \|V_g f\|_{L^p(\mathbb{R}^2)}.$$

Its definition is independent of $g \in \mathcal{S}(\mathbb{R})$, see [31].

Our goal will be to restore a signal f in a modulation space $M^{p,p}(\mathbb{R})$ from its phaseless Gabor measurements $|V_\varphi f| : \mathbb{R}^2 \rightarrow \mathbb{R}_+$, up to a global phase.

It is well-known that for any suitable window function the resulting phase retrieval problem is uniquely solvable:

Theorem 2.3. Suppose that $g \in \mathcal{S}(\mathbb{R})$ is such that its ambiguity function

$$A(g)(x, y) := \int_{\mathbb{R}} g(t) \overline{g(t-x)} e^{-2\pi i t y} dt, \quad (x, y) \in \mathbb{R}^2$$

is nonzero everywhere. Then, for any $f, h \in \mathcal{S}'(\mathbb{R})$ with $|V_g f| = |V_g h|$ there exists $\alpha \in \mathbb{R}$ such that $f = e^{i\alpha} h$.

Proof. This is essentially folklore. For the convenience of the reader we provide a proof in Appendix A. \square

Since the Gabor window $\varphi(t) = e^{-\pi t^2}$ satisfies the assumptions of Theorem 2.3, we know that any f is uniquely, up to global phase, determined by its Gabor transform magnitudes $|V_\varphi f|$. For nice signals we even have an explicit reconstruction formula (see Theorem A.3 in the appendix and denoting \mathcal{F}_2 the Fourier transform in the second coordinate and S the transform defined by $SF(x, y) = F(y, x)$):

$$f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1} \left(SF |V_g f|^2 / Ag \right) (t, t).$$

We do not know however how to exploit this formula for the question of stability of our phase retrieval problem and our methods do not make use of it.

What makes the Gabor transform special is that it possesses a lot of additional structure as compared to an ordinary STFT. For instance, it turns out that the Gabor transform of a tempered distribution is, after simple modifications, a holomorphic function.

Theorem 2.4. Let $z := x + iy \in \mathbb{C}$. Define $\eta(z) := e^{\pi \left(\frac{|z|^2}{2} - ixy \right)}$. Then for every $f \in \mathcal{S}'(\mathbb{R})$ the function $x + iy \mapsto \eta(x, y) \cdot V_\varphi f(x, -y)$ is an entire function.

Proof. This is again well-known, at least for $f \in L^2(\mathbb{R})$, see for example [6] where it is also shown that φ is essentially the only window function with this property. For the convenience of the reader we present a proof in Appendix A. \square

We are interested in stability estimates of the form (1.2). To this end we need to put a norm $\|\cdot\|_{\mathcal{D}}$ on the measurement space $\mathcal{S}'(\mathbb{R}^2)$. A suitable family of norms on the measurement space turns out to be the following.

Definition 2.5. For a bivariate tempered distribution $F \in \mathcal{S}'(\mathbb{R}^2)$ and $D \subset \mathbb{R}^2$ we define the norms

$$\|F\|_{\mathcal{D}_{p,q}^{r,s}(D)} := \|F\|_{L^p(D)} + \|F\|_{L^q(D)} + \|\nabla^r F\|_{L^p(D)} + \|(|x| + |y|)^s F(x, y)\|_{L^q(D)}$$

where ∇^r denotes the r -th order total differential of a bivariate tempered distribution.

If $D = \mathbb{R}^2$ we simply write $\mathcal{D}_{p,q}^{r,s}$ instead of $\mathcal{D}_{p,q}^{r,s}(\mathbb{R}^2)$.

Remark 2.6. It may appear slightly irritating that the norms on measurement space include a polynomial weight. It turns out that without any polynomial weight (for example putting $p = q = 2$ and $r = s = 0$), the stability constant $c(f)$ will in general be infinite (as a nontrivial exercise the reader may verify this for the function $f(t) = \frac{1}{1+t^2}$). In a sense the norms $\mathcal{D}_{p,q}^{r,s}(D)$ possess some symmetry between the space domain and the Fourier domain in the sense that they promote both spatial as well as Fourier-domain localization.

The norms as just introduced measure the time-frequency concentration of F in terms of both smoothness and spatial localization. Note that the last term $\|(|x| + |y|)^s F(x, y)\|_{L^q(\mathbb{R}^2)}$ in its definition is not translation-invariant and therefore it will be convenient to apply the norm to, what we call, centered functions.

Definition 2.7. A function $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ is centered if $|F|$ possesses a maximum at the origin $(x, y) = (0, 0)$.

Our setup is now complete; with $\Phi = (\varphi(\cdot - x)e^{2\pi iy})_{(x,y) \in \mathbb{R}^2}$, $\mathcal{B} = M^{p,p}(\mathbb{R})$ a modulation space and \mathcal{D} the norm as defined above we are interested in estimating the constant $c(f)$ as defined in (1.2).

The main insight of this paper is that the constant $c(f)$ behaves like the reciprocal of, what we call, the p -Cheeger constant of f . It is defined as follows.

Definition 2.8. For $f \in \mathcal{S}'(\mathbb{R})$, $D \subset \mathbb{R}^2$ and $p \in [1, \infty)$ define its p -Cheeger constant

$$(2.1) \quad h_{p,D}(f) := \inf_{\{C \subset D \text{ open: } \partial C \cap D \text{ is smooth, and } \int_C |V_\varphi f|^p \leq \frac{1}{2} \int_D |V_\varphi f|^p\}} \frac{\|V_\varphi f\|_{L^p(\partial C)}^p}{\|V_\varphi f\|_{L^p(C)}^p}.$$

If $D = \mathbb{R}^2$ we simply write $h_p(f)$ instead of $h_{p,\mathbb{R}^2}(f)$.

As already mentioned in the introduction we borrowed here a term from spectral geometry. Indeed, our definition of $h_p(f)$ is equal to the usual Cheeger constant of the flat Riemannian manifold \mathbb{R}^2 , conformally multiplied with $|V_\varphi f(x, y)|^p$, see [15].

We are ready to give an appetizer to our results by stating the following theorem which confirms that disconnected measurements form the only source of instabilities for Gabor phase retrieval.

Theorem 2.9. Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered. Then there exists a constant $c > 0$ only depending on p, q and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that

$$d_{M^{p,p}(\mathbb{R})}(f, g) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}}.$$

Theorem 2.9 is proved in Section 5.3 as a special case of the more general Theorems 5.11 and 5.12.

The theorem above also establishes a noise-stability result for reconstruction of a signal from noisy spectrogram measurements

$$\text{noisy measurements} = |V_\varphi f| + \eta.$$

Corollary 2.10. Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered. Then there exists a constant $c > 0$ only depending on p, q and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ such that for any $\eta \in \mathcal{D}_{p,q}^{1,6}$ with $\|\eta\|_{\mathcal{D}_{p,q}^{1,6}} \leq \nu$ and any

$$h \in \operatorname{argmin}_{g \in M^{p,p}(\mathbb{R})} \| (|V_\varphi f| + \eta) - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}}.$$

it holds that

$$d_{M^{p,p}(\mathbb{R})}(f, h) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \nu.$$

Due to its simplicity, we present the proof here.

Proof. By Theorem 2.9, it holds that

$$\begin{aligned} d_{M^{p,p}(\mathbb{R})}(f, h) &\leq c \cdot (1 + h_p(f)^{-1}) \cdot \| |V_\varphi f| - |V_\varphi h| \|_{\mathcal{D}_{p,q}^{1,6}} \\ &\leq c \cdot (1 + h_p(f)^{-1}) \cdot \left(\| (|V_\varphi f| + \eta) - |V_\varphi h| \|_{\mathcal{D}_{p,q}^{1,6}} + \nu \right). \end{aligned}$$

To finish the argument we note that, due to the definition of h , it holds that

$$\| (|V_\varphi f| + \eta) - |V_\varphi h| \|_{\mathcal{D}_{p,q}^{1,6}} \leq \| (|V_\varphi f| + \eta) - |V_\varphi f| \|_{\mathcal{D}_{p,q}^{1,6}} \leq \nu.$$

□

Typically, one is mainly interested in the reconstruction of a specific time-frequency regime of f . To this end, we will also establish a local stability result of which we here offer a special case in the following theorem.

Theorem 2.11. *Let $p \in [1, 2)$, $q \in (2p/(2-p), \infty]$ and $R > 0$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered. Suppose further that f is ε -concentrated on a ball $B_R(0) \subset \mathbb{R}^2$ in the sense that*

$$\int_{\mathbb{R}^2 \setminus B_R(0)} |V_\varphi f(x, y)|^p dx dy \leq \varepsilon^p$$

Then there exists a constant $c > 0$ only depending on p, q and

$$\max \left\{ \frac{\|V_\varphi f\|_{L^p(B_R(0))}}{\|V_\varphi f\|_{L^\infty(B_R(0))}}, \frac{\|V_{\varphi'} f\|_{L^\infty(B_R(0))}}{\|V_\varphi f\|_{L^\infty(B_R(0))}} \right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ which is ε -concentrated in $B_R(0)$ it holds that

$$d_{M^{p,p}(\mathbb{R})}(f, g) \leq c \cdot \left((1 + h_{p, B_R(0)}(f))^{-1} \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,0}(B_R(0))} + \varepsilon \right).$$

Similar to Corollary 2.10, also a local noise-stability result can be deduced in an obvious way. We leave the details to the reader.

2.2. Putting our Results in Perspective. In this subsection we briefly relate our results to the stable solution of the backwards heat equation and our previous work [1].

2.2.1. Connections with the Backwards Heat Equation. We would like to draw the reader's attention to an intricate connection between phase retrieval and the solution of the backwards heat equation.

Consider the heat equation in the plane:

(2.2)

$$u_t(t, x, y) = \Delta u(t, x, y) = u_{xx}(t, x, y) + u_{yy}(t, x, y) \quad \text{and} \quad u(0, x, y) = f(x, y), \quad x, y \in \mathbb{R}, t > 0.$$

The backward heat equation problem, i.e. (stably) reconstructing the initial value f given $u(t, \dots)$ for fixed t is known to be severely ill-posed. Solving the heat equation in the frequency domain yields

$$\widehat{u}(t, \xi, \eta) = \widehat{f}(\xi, \eta) \cdot e^{-4\pi^2(\xi^2 + \eta^2)t}.$$

Therefore solving the backward heat equation problem amounts to deconvolving $u(t, \cdot, \cdot)$ with a Gaussian kernel.

In Appendix A we show that

$$\mathcal{F}|V_g f|^2(\eta, \xi) = \mathcal{A}f(\xi, \eta) \cdot \mathcal{A}g(\xi, \eta),$$

(where \mathcal{F} denotes the two-dimensional Fourier transform) as well as the fact that $\mathcal{A}g$ is a 2-dimensional Gaussian for the Gaussian window $g = e^{-\pi \cdot^2}$. Thus, reconstructing the ambiguity function of f from the absolute values of its Gabor transform amounts to solving the backward heat equation problem. Consequently, the Gabor phase retrieval problem and the backward heat equation problem, as well as their stabilization, are closely related. We consider the investigation of the consequences of our results for the stabilization of the backwards heat equation an interesting problem for future work.

2.2.2. Comparison with the Results of [1]. Our result is very much inspired by stability results in recent work [1] by Rima Alaifari, Ingrid Daubechies, Rachel Yin and one of the authors and in fact grew out of this work.

In order to put our current results in perspective and to exemplify the improvement of our present results as compared to those in [1] we give a short comparison between the main stability results of [1] and the present paper.

In [1] it is shown that, for certain measurement scenarios (including Gabor and Poisson wavelet measurements), stable phase reconstruction is locally possible on subsets $\Omega' \subset \Omega$ on which the variation of the measurements, namely

$$\frac{\sup_{\omega \in \Omega'} |\varphi_\omega(f)|}{\inf_{\omega \in \Omega'} |\varphi_\omega(f)|}$$

is bounded. However, in an ∞ -dimensional problem, the quantity $\sup_{\omega \in \Omega'} |\varphi_\omega(f)| / \inf_{\omega \in \Omega'} |\varphi_\omega(f)|$ will not be bounded and therefore the results of [1] do not provide bounds for $c(f)$.

For concreteness we compare the sharpness of our result to the results of [1] at hand of a very simple example, namely a Gaussian signal $f = e^{-\pi t^2}$. A simple calculation (see Lemma A.5) reveals that

$$|V_\varphi f(x, y)| = r e^{-\pi/2(x^2+y^2)}$$

for some positive number r . Clearly, f is ε -concentrated on $B_R(0)$ with $\varepsilon \lesssim e^{-\pi/2R^2}$.

The results of [1] rely on the assumption that the measurements $V_\varphi f$ are of little variation on the domain of interest, which for our particular example is $B_R(0)$. The main parameter governing the stability in the results of [1] would be

$$\frac{\sup_{(x,y) \in B_R(0)} |V_\varphi f(x, y)|^2}{\inf_{(x,y) \in B_R(0)} |V_\varphi f(x, y)|^2} = e^{\pi R^2}$$

and the best stability bound that can be achieved using the results of [1] is thus of the form

$$(2.3) \quad \inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha} g\|_{M^{2.2}(\mathbb{R})} \leq c \cdot \left(e^{\pi R^2} \cdot \| |V_\varphi f| - |V_\varphi g| \|_{W^{2,1}(B_R(0))} + e^{-\pi/2R^2} \right),$$

where g is an arbitrary function which is also ε -concentrated on $B_R(0)$.

We see that the stability bound obtainable from the results of [1] grows exponentially in R^2 which still suggests that the problem to reconstruct f from its spectrogram is severely ill-posed.

It turns out that this is not the case. In the Appendix (Theorem B.12) we see that $h_{p, B_R(0)}(f) \gtrsim 1$ with the implicit constant independent of R (in fact, this is well-known and follows from the Gaussian isoperimetric inequality and geometric arguments).

We can thus directly apply Theorem 2.11 and get the following.

Theorem 2.12. *Let $f(t) = e^{-\pi t^2}$. Let $p \in [1, 2)$, $q \in (2p/(2-p), \infty]$ and $\varepsilon > 0$. Then there exists a constant $c > 0$ only depending on p, q and ε such that for any $R > 1$ and $g \in M^{p,p}(\mathbb{R})$ which is ε -concentrated in $B_R(0)$ it holds that*

$$\inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha} g\|_{M^{p,p}(\mathbb{R})} \leq c \cdot \left(\| |V_\varphi f| - |V_\varphi g| \|_{W^{p,1}(B_R(0))} + R^6 \cdot \| |V_\varphi f| - |V_\varphi g| \|_{L^q(B_R(0))} + e^{-\pi/2R^2} \right).$$

We remark that a more careful analysis (which exploits the specific form of f) would yield an estimate of the form

$$(2.4) \quad \inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha} g\|_{M^{p,p}(\mathbb{R})} \leq c \cdot \left(R \cdot \| |V_\varphi f| - |V_\varphi g| \|_{W^{p,1}(B_R(0))} + e^{-\pi/2R^2} \right),$$

valid for every $p \in [1, \infty]$.

Comparing our result (2.4) with the bound (2.3) from [1] we see that our bound is much tighter. In particular,

our bound turns a superexponential growth of the stability constant into a low-order polynomial growth!

2.2.3. *An Algorithm for finding a meaningful partition of the data and estimating the corresponding multi-component stability constant.* Again we want to take up an idea from [1], where the concept of multi-component phase retrieval was introduced: The multi-component paradigm amounts to the following identification of measurements $F = V_\varphi f$, $G = V_\varphi g$:

$$F = \sum_{j=1}^k F_j \sim G = \sum_{j=1}^k e^{i\alpha_j} F_j$$

for any $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ where the components F_1, \dots, F_k are essentially supported on mutually disjoint domains D_1, \dots, D_k . This means we consider F and G to be close to each other whenever the quantity

$$\inf_{\alpha_1, \dots, \alpha_k} \sum_{j=1}^k \|F - e^{i\alpha_j} G\|_{L^p(D_j)}$$

is small. Thus we no longer demand that there is a global phase factor but allow different phase factors which are constant on the distinct subdomains D_i . Since the human ear cannot recognize an identification $F \sim G$ whenever the measurements F_j are distant from each other, this notion of distance is sensible for the purpose of applications in audio.

Assume we are given a signal f such that the Cheeger constant $h_p(f)$ is small meaning that we will expect the phase retrieval problem to be very unstable. A natural question to ask is whether it is possible to partition the time-frequency plane in subdomains D_1, \dots, D_k such that Gabor phase retrieval is stable in the multi-component sense, i.e.,

$$(2.5) \quad \inf_{\alpha_1, \dots, \alpha_k \in \mathbb{R}} \sum_{j=1}^k \|V_\varphi f - e^{i\alpha_j} V_\varphi g\|_{L^p(D_j)} \leq B \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}},$$

for moderately large $B > 0$ and all g .

Obviously the finer the partition the smaller B will become. However in view on the motivation from audio applications we will not want to choose a very fine partition, because then the corresponding multi-component distance will not be naturally meaningful.

The challenge therefore is to find – given a signal f – a partition D_1, \dots, D_k such that

- (i) B is small and
- (ii) the measurements $V_\varphi f \cdot \chi_{D_j}$ and $V_\varphi f \cdot \chi_{D_l}$ are distant for all $j \neq l$

simultaneously hold.

Corollary 5.14 tells us that B can essentially be bounded by the quantity

$$(2.6) \quad \min_{j=1, \dots, k} (1 + h_{p, D_j}(f)^{-1}) \cdot (1 + \frac{\kappa_j^p}{\delta_j^2}),$$

where

$$\delta_j = \min\{\sup\{r > 0 : B_r(z) \subset D, \inf_{\zeta \in B_r(z)} |V_\varphi f(\zeta)| \geq \frac{1}{2} \|V_\varphi f\|_{L^\infty(D_j)}\}, 1\}$$

and

$$\kappa_j = \frac{\|V_\varphi f\|_{L^p(D_j)}}{\|V_\varphi f\|_{L^\infty(D_j)}}.$$

Since in practice one only has finitely many samples of $|V_\varphi f|$ at hand, we consider a discrete version of this partitioning problem. Spectral Clustering methods from Graph theory provide algorithms that aim at finding partitions minimizing a discrete Cheeger ratio [12].

We now suggest an iterative approach. Once the domain D is partitioned into two components C and $D \setminus C$ (see Figure 3) we can again measure the disconnectedness of these two sets by estimating their respective Cheeger constants (see Figure 4). If this estimate lies above a given threshold we leave the set untouched in view of (ii). Otherwise we partition again. After carrying out this iterative procedure a few times, we expect to arrive at a partition C_1, \dots, C_l of D such that each C_j is well connected (in terms of the Cheeger constant being large) and simultaneously for any $k \neq j$ the set $C_k \cup C_j$ is very disconnected (in terms of the Cheeger constant being small). We hence find a partition such that $h_{p, C_j}(f)$ is moderately large for all j .

However to use Theorem 5.14 we also need δ_j not to be too small and κ_j not to be too large, which can be verified a posteriori.

In Appendix C we describe the algorithm we used for the experiment illustrated in Figures 2 to 5 in detail.

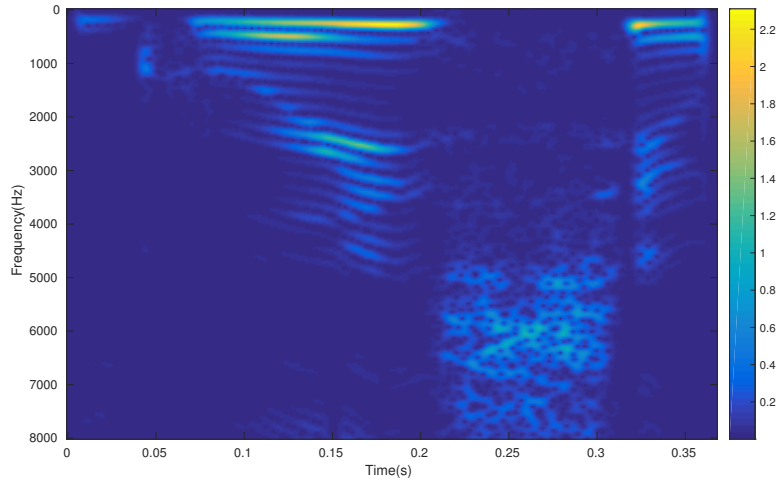


FIGURE 2. Magnitudes of the discrete Gabor transform of the signal "greasy" from the LTFAT toolbox.

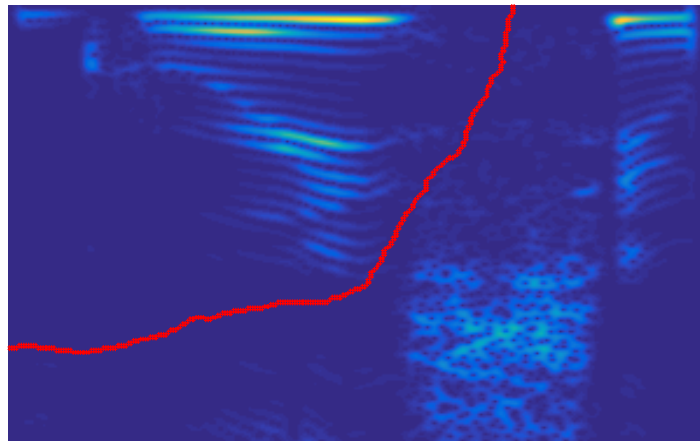


FIGURE 3. The partitioning of $|V_\varphi f|$ estimates the Cheeger constant $h_D(f) \approx 0.0019119$.



FIGURE 4. Partitioning of the two subdomains and estimating their respective Cheeger constants.

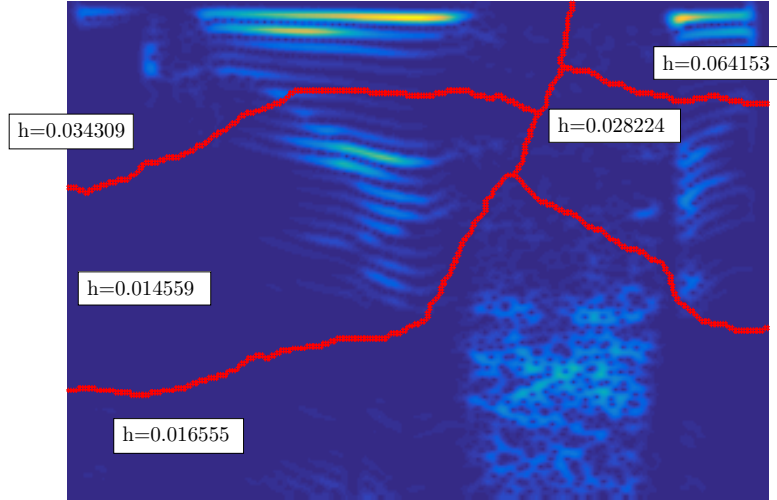


FIGURE 5. The algorithm terminates as soon as the (estimated) Cheeger constants of all subdomains are above a given threshold.

2.3. Architecture of the Proof. The proof of our main result is quite convoluted and draws on techniques from different mathematical fields such as complex analysis, functional analysis or spectral Riemannian geometry. For the benefit of the reader we provide a short sketch of our argumentation, before we go to the details in the later sections.

Let us start with the following observation: Given two functions $F_1, F_2 : D \rightarrow \mathbb{C}$, we have

$$(2.7) \quad \inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)}^p = \inf_{a \in \mathbb{C}, |a|=1} \int_D \left| \frac{F_2(z)}{F_1(z)} - a \right|^p w(z) dz,$$

where $w(z)dz$ is the Lebesgue measure with density $w(z) = |F_1(z)|^p$.

Now suppose that we could just disregard the constraint $|a| = 1$ in the above formula (2.7) (it turns out that one cannot do this but in Section 4 we develop tools which effectively amount to an equivalent result). Then, using the notation $L^p(D, w)$ for the L^p space with respect to the measure $w dz$, we would need to estimate a term of the form

$$(2.8) \quad \inf_{a \in \mathbb{C}} \left\| \frac{F_2}{F_1} - a \right\|_{L^p(D, w)}.$$

The Poincaré inequality tells us that (provided w and D are ‘nice’) there exists a constant $C_{poinc}(p, D, w) < \infty$, depending only on the domain and the weight, such that (2.7) can be bounded by

$$(2.9) \quad C_{poinc}(p, D, w) \cdot \left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(D, w)}.$$

Now spectral geometry enters the picture. Cheeger’s inequality [16] says that the Poincaré constant on a Riemannian manifold can be controlled by the reciprocal of the Cheeger constant. We would like to apply this result to the metric induced by the metric tensor $\left(w(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)_{z \in D}$ in order to get a bound on $C_{poinc}(p, D, w)$. However, since w in our case arises from Gabor measurements it generally has zeros and therefore does not qualify as Riemannian manifold. In Appendix B.1 we will show that for $F_1 = V_\varphi f$

$$C_{poinc}(p, D, w) \leq \frac{4p}{h_{p,D}(f)},$$

where $h_{p,D}(f)$ is defined as in (2.8) holds true, nevertheless.

Assuming that all heuristics up to this point were correct, we get a bound of the form

$$\inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)} \leq c \cdot h_{p,D}(f)^{-1} \cdot \left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(D,w)},$$

where here and in the following c denotes an unspecified constant.

We are faced with the problem of converting $\left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(D,w)}$ into a useful estimate in the difference $|F_1| - |F_2|$.

Now complex analysis enters the picture. If $F_1(z) = V_\varphi f(x, -y)$ and $F_2(z) = V_\varphi g(x, -y)$ it is known that the quotient $\frac{F_2}{F_1}$ is a meromorphic function (Theorem 2.4) which, almost everywhere, satisfies the Cauchy-Riemann equations. It is a simple exercise (Lemma 3.4) to verify that for any meromorphic function it holds that

$$\left| \nabla \frac{F_2}{F_1} \right| = \sqrt{2} \left| \nabla \frac{|F_2|}{|F_1|} \right|$$

almost everywhere. This is great, since we now can get a bound that only depends on the absolute values $|F_1|, |F_2|$!

To summarize, if all our heuristics were correct, we would get a bound of the form

$$\inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)} \leq c \cdot h_{p,D}(f)^{-1} \cdot \left\| \nabla \frac{|F_2|}{|F_1|} \right\|_{L^p(D,w)}.$$

If we now apply the quotient rule to the estimate above and utilize the fact that $w = |F_1|^p$ we would get a bound of the form

$$(2.10) \quad \inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)} \leq c \cdot h_{p,D}(f)^{-1} \cdot \left(\left\| \left(\frac{\nabla |F_1|}{|F_1|} \right) \cdot (|F_1| - |F_2|) \right\|_{L^p(D)} + \|\nabla |F_1| - \nabla |F_2|\|_{L^p(D)} \right).$$

This is precisely Proposition 3.3 and Theorem 5.3 (although the details of these results and their proofs are significantly more delicate than this informal discussion may suggest, see Section 4).

The estimate (2.10) is already close to what one would like to have, were it not for the term $\frac{\nabla |F_1|}{|F_1|}$ in the first summand of the right hand side of (2.10). Indeed, since F_1 will in general have zeroes, this term will not be bounded.

Here again complex analysis will come to our rescue: The function $F_1(z) = V_\varphi f(x, -y)$ is, after multiplication with a suitable function η , an entire function of order 2. Jensen's formula [20] provides bounds for the distribution of zeros of F_1 and this allows us to show that, for $1 \leq p < 2$ the norms $\left\| \frac{\nabla |F_1|}{|F_1|} \right\|_{L^p(B_R(0))}$ grow at most like a low-order polynomial in R which is, remarkably, independent of f ! These arguments are carried out in Section 5.2.

Finally we can put all our estimates together and arrive at our main stability theorems which are summarized in Section 5.3.

2.4. Outline. The outline of this article is as follows. In Section 3 we start by proving a general stability result, Proposition 3.3, for phase retrieval problems. This result, which depends on some at this point unspecified constants, namely an analytic Poincaré constant and a sampling constant, is inspired by and generalizes the main result of [1]. In Section 4 we gain control of the two unspecified constants of the main result in Section 3 and show that they can be controlled in terms of the global variation of the measurements as defined in Definition 4.5, see Proposition 4.7. In Section 5 we specialize to the case of Gabor phase retrieval. We first show that the global variation of Gabor measurements is independent of the signal to be analyzed, which will yield an estimate of the type (2.10), see Theorem 5.3. Finally, in Section 5.2 we remove the logarithmic derivative in the estimate of Theorem 5.3 at the expense of introducing weighted norms in the error estimate, see Proposition 5.7 whose proof requires deep function-theoretic properties of the Gabor transform. In Section 5.3 we formulate and prove our main stability result.

Finally, Appendix **A** is concerned with auxilliary properties of the Gabor phase retrieval problem and in Appendix **B** we state and prove several auxilliary facts related to Cheeger- and Poincaré constants. In Appendix **C** we provide some details on Spectral clustering algorithms that aim at estimating Cheeger constants of graphs.

2.5. Notation. We pause here to collect some notation that will be used throughout this article. Since some proofs will turn out to be quite technical we hope that this will prevent the reader to get lost in his or her reading.

- For $1 \leq p < \infty$, $D \subset \mathbb{R}^d$ and a weight function $w : D \rightarrow \mathbb{R}_+$ we write (somewhat informally)

$$L^p(D, w) := \{F : D \rightarrow \mathbb{C} : \|f\|_{L^p(D, w)} < \infty\},$$

where

$$\|F\|_{L^p(D, w)}^p := \int_D |F(u)|^p w(u) du.$$

If $w \equiv 1$ we simply write $L^p(D)$ instead of $L^p(D, 1)$.

- For $w : D \rightarrow \mathbb{R}_+$ we shall write

$$w(D) := \int_D w(u) du.$$

- For $F : D \rightarrow \mathbb{C}$ and $w : D \rightarrow \mathbb{R}_+$ we shall write

$$F_D^w := \frac{1}{w(D)} \int_D F(u) w(u) du.$$

- For $1 \leq p < \infty$, $D \subset \mathbb{R}^d$, $l \in \mathbb{N}$ and a weight function $w : D \rightarrow \mathbb{R}_+$ we write (somewhat informally)

$$W^{p, l}(D, w) := \{f : D \rightarrow \mathbb{C} : \|f\|_{W^{p, l}(D, w)} < \infty\},$$

where

$$\|f\|_{W^{p, l}(D, w)}^p := \sum_{k=0}^l \|\nabla^k f\|_{L^p(D, w)}^p.$$

If $w \equiv 1$ we simply write $W^{p, l}(D)$ instead of $W^{p, l}(D, 1)$.

- We shall often identify \mathbb{R}^2 with \mathbb{C} via the isomorphism $(x, y) \in \mathbb{R}^2 \leftrightarrow z := x + iy \in \mathbb{C}$. Using this identification we may also interpret a subset $D \subset \mathbb{R}^2$ as a subset of \mathbb{C} .
- For $z \in \mathbb{C}$ we shall write $B_r(z)$ for the ball of radius r around z .
- For a set $C \subset \mathbb{R}^2$ let $|C|$ denote the 2-dimensional Lebesgue measure of C . For a smooth curve $A \subset \mathbb{R}^2$ let $\ell(A)$ denote the Euclidean length of A .
- For $D \subset \mathbb{C}$ we denote $\mathcal{O}(D)$ the ring of holomorphic functions on D and $\mathcal{M}(D)$ the field of meromorphic functions on D .
- For $F \in \mathcal{M}(D)$ we may write (somewhat informally)

$$F(z) = u(x, y) + iv(x, y)$$

where u, v denote the real resp. imaginary part of F . We shall also write

$$F'(z) := \frac{\partial}{\partial x} u(x, y) + i \frac{\partial}{\partial x} v(x, y),$$

whenever defined.

- For $D \subset \mathbb{C}$ we denote χ_D the indicator function of D .

3. A FIRST STABILITY RESULT FOR PHASE RECONSTRUCTION FROM HOLOMORPHIC MEASUREMENTS

The starting point of our work will be a general stability result, Proposition **3.3** that we prove in the present section. The estimate will essentially depend on two quantities: an analytic Poincaré constant and a sampling constant. We will see later on how these two constants can be controlled, but for the time being we simply present their definitions.

Definition 3.1. Given a domain $D \subset \mathbb{C}$, $1 \leq p \leq \infty$, a number $\delta > 0$, a point $z_0 \in D$ such that $B_\delta(z_0) \subset D$ and a weight $w : D \rightarrow \mathbb{R}_+$, we define $C_{\text{poinc}}^a(p, D, z_0, \delta, w) > 0$ as the smallest constant such that

$$(3.1) \quad \|F - F(z_0)\|_{L^p(D, w)} \leq C_{\text{poinc}}^a(p, D, z_0, \delta, w) \|F'\|_{L^p(D, w)}$$

for all $F \in \mathcal{M}(D) \cap \mathcal{O}(B_\delta(z_0)) \cap W^{1,p}(D, w)$.

We will refer to $C_{\text{poinc}}^a(p, D, z_0, \delta, w)$ as an ‘analytic Poincaré constant’. We will see later on, in Section 4.1, how one can control this quantity.

Next we define what we call a ‘sampling constant’.

Definition 3.2. Let D be a domain, $w : D \rightarrow \mathbb{R}_+$ and $G \in L^p(D, w)$. Then we define, for $z_0 \in D$ $1 \leq p \leq \infty$ the Sampling constant

$$C_{\text{samp}}(p, D, z_0, G, w) := \frac{\|G(z_0)\|_{L^p(D, w)}}{\|G\|_{L^p(D, w)}}.$$

Later on, in Section 4.2 we will see how to control this quantity.

Having defined the notion of analytic Poincaré constant and sampling constant we can now state and prove the following general stability result.

Theorem 3.3. Let $D \subset \mathbb{C}$ and $1 \leq p < \infty$. Suppose that $F_1, F_2 \in L^p(D)$ are smooth function such that there exists a continuous, nowhere vanishing function $\eta : D \rightarrow \mathbb{C}$ for which both functions $\eta \cdot F_1, \eta \cdot F_2 \in \mathcal{O}(D)$.

Suppose that $z_0 \in D$ and $\delta > 0$ with $B_\delta(z_0) \subset D$ and

$$|F_1(z)| > 0 \quad \text{for all } z \in B_\delta(z_0).$$

Then the following estimate holds:

$$(3.2) \quad \inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)} \leq C_{\text{samp}}(p, D, z_0, |F_2/F_1| - 1, |F_1|^p) \| |F_2| - |F_1| \|_{L^p(D)} \\ + 2^{1/p-1/2} C_{\text{poinc}}^a(p, D, z_0, \delta, |F_1|^p) (\|\nabla|F_1| - \nabla|F_2|\|_{L^p(D)} + \|\nabla \log(F_1)(|F_1| - |F_2|)\|_{L^p(D)}).$$

We remark that this result draws its inspiration from, and generalizes, the main result of [1]. At its heart lies the following elementary lemma which is proved in [1] and which follows directly from the Cauchy-Riemann equations.

Lemma 3.4. Suppose that $F \in \mathcal{M}(D)$. Then for any $z = x + iy \in D$ which is not a pole of F we have the equality

$$|F'(z)| = |\nabla|F|(x, y)|.$$

Having Lemma 3.4 at hand we can now proceed to the proof of the main result of this section.

Proof of Theorem 3.3. We need to bound the quantity

$$(3.3) \quad \|F_2(z) - e^{i\alpha} F_1(z)\|_{L^p(D)}$$

for suitable $\alpha \in \mathbb{R}$

Step 1. As a first step we start by developing a basic estimate. Consider

$$F := F_2/F_1.$$

By assumption it holds that $F \in \mathcal{M}(D)$.

Pick α such that

$$(3.4) \quad |F(z_0) - e^{i\alpha}| = ||F(z_0)| - 1|.$$

Now consider for $z \in D$ arbitrary

$$(3.5) \quad |F_2(z) - e^{i\alpha} F_1(z)| = |F_1(z)| |F(z) - e^{i\alpha}| \\ \leq |F_1(z)| (|F(z) - F(z_0)| + |F(z_0) - e^{i\alpha}|) \\ = |F_1(z)| \cdot |F(z) - F(z_0)| + |F_1(z)| \cdot ||F(z_0)| - 1|$$

It follows that

$$\|F_2(z) - e^{i\alpha} F_1(z)\|_{L^p(D)} \leq \|F(z) - F(z_0)\|_{L^p(D, |F_1|^p)} + \| |F(z_0)| - 1 \|_{L^p(D, |F_1|^p)} =: (I) + (II).$$

Step 2 (Estimating (II)). By Definition 3.2 with $w = |F_1|^p$ we see that

$$(II) = C_{\text{samp}}(p, D, z_0, |F| - 1, w) \| |F_1| - |F_2| \|_{L^p(D)}.$$

Step 3 (Estimating (I)). By Definition 3.1 with $w = |F_1|^p$ and $F \in \mathcal{O}(B_\delta(z_0))$ (which follows from the fact that F_1 is nonzero on $B_\delta(z_0)$) we get that

$$(3.6) \quad (I) \leq C_{\text{poinc}}^a(p, D, z_0, \delta, w) \cdot \|F'\|_{L^p(D, w)}.$$

We now need to get a bound on $\|F'\|_{L^p(D, w)}$ in terms of $\| |F_1| - |F_2| \|_{W^{1,p}(D)}$ to finish the proof. This is where our key lemma, Lemma 3.4 comes into play, stating that

$$\|F'\|_{L^p(D, w)} = \|\nabla |F|\|_{L^p(D, w)}.$$

It thus remains to achieve a bound for $\|\nabla |F|\|_{L^p(D, w)}$. To this end we consider

$$\begin{aligned} \frac{\partial}{\partial x} |F| &= \frac{|F_1| \frac{\partial}{\partial x} |F_2| - |F_2| \frac{\partial}{\partial x} |F_1|}{|F_1|^2} \\ &= \frac{\frac{\partial}{\partial x} |F_1| (|F_1| - |F_2|) + |F_1| (\frac{\partial}{\partial x} |F_2| - \frac{\partial}{\partial x} |F_1|)}{|F_1|^2} \end{aligned}$$

which holds at least for all points where neither F_1 nor F_2 vanishes. Since this set of points is discrete by our assumptions we see that

$$\left\| \frac{\partial}{\partial x} |F| \right\|_{L^p(D, |F_1|^p)}^p \leq \int_D \frac{|F_1|_x^p}{|F_1|^p} \cdot \| |F_1| - |F_2| \|^p + \int_D \left| \frac{\partial}{\partial x} |F_2| - \frac{\partial}{\partial x} |F_1| \right|^p.$$

Since it always holds that $\left| \frac{\partial}{\partial x} |f(x)| \right| \leq \left| \frac{\partial}{\partial x} f(x) \right|$ (by the inverse triangle inequality) we get that

$$\left\| \frac{\partial}{\partial x} |F| \right\|_{L^p(D, |F_1|^p)}^p \leq \int_D |\log(F_1)'|^p \cdot \| |F_1| - |F_2| \|^p + \int_D \left| \frac{\partial}{\partial x} |F_2| - \frac{\partial}{\partial x} |F_1| \right|^p,$$

where we have used that $\log(F_1)' = F_1'/F_1$, the logarithmic derivative.

The norm of the derivative of $|F|$ w.r.t. y can be estimated analogously. Therefore we get

$$\begin{aligned} \|\nabla |F|\|_{L^p(D, w)}^p &= \int_D \left(|F|_x^2 + |F|_y^2 \right)^{p/2} \cdot |F_1|^p \\ &\leq \max\{2^{p/2-1}, 1\} \cdot \int_D \left(|F|_x^p + |F|_y^p \right) \cdot |F_1|^p \\ &\leq \max\{2^{p/2-1}, 1\} \cdot \left[\int_D (|\log(F_1)_x|^p + |\log(F_1)_y|^p) \cdot \| |F_1| - |F_2| \|^p \right. \\ &\quad \left. + \int_D \left(\left| |F_2|_x - |F_1|_x \right|^p + \left| |F_2|_y - |F_1|_y \right|^p \right) \right] \\ &\leq \max\{2^{p/2-1}, 1\} \cdot \max\{2^{1-p/2}, 1\} \left[\int_D |\nabla \log(F_1)|^p \| |F_1| - |F_2| \|^p \right. \\ &\quad \left. + \int_D \left| \nabla |F_2| - \nabla |F_1| \right|^p \right] \\ &\leq 2^{1-p/2} \cdot \left[\|\nabla \log(F_1)\|_{L^p(D)} \cdot \| |F_1| - |F_2| \|_{L^p(D)} + \|\nabla |F_2| - \nabla |F_1|\|_{L^p(D)} \right]^p. \end{aligned}$$

□

As it stands, Proposition 3.3 is not yet satisfactory for at least two reasons. First, it is not yet clear how the analytic Poincaré constant and the sampling constant can be (simultaneously) controlled. Second, the term $\|\nabla \log(F_1)\|_{L^p(D)} \| |F_1| - |F_2| \|_{L^p(D)}$ in the estimate (3.2) is difficult to interpret since the logarithmic derivative $\nabla \log(F_1)$ will in general be unbounded. The purpose of

the remainder of this article is to show that all these dependencies can be absorbed into a natural quantity which describes the degree of disconnectedness of the measurements.

4. BALANCING THE CONSTANTS

Having the technical result in Proposition 3.3 main at hand, the next task is to get a grip on the error term on the right hand side of (3.2). Indeed, we will show that both the analytic Poincaré constant, as well as the sampling constant can be simultaneously controlled.

4.1. Weighted Analytic Poincaré Inequalities. While the concept of analytic Poincaré constant seems not to be very widely studied, the classical Poincaré constant as defined next is certainly much more well-known.

Definition 4.1. For $1 \leq p < \infty$ denote by $C_{poinc}(p, D, w)$ the Poincaré constant of the domain D w.r.t. the weight w , i.e. the optimal constant C such that for all $F \in W^{1,p}(D, w) \cap \mathcal{M}(D)$ we have

$$\|F - F_D^w\|_{L^p(D, w)} \leq C \|\nabla F\|_{L^p(D, w)},$$

where we put $F_D^w := \frac{1}{w(D)} \int_D F(z)w(z)dz$, and $w(D) := \int_D w(z)dz$.

There exists a huge body of work devoted to the study of weighted Poincaré inequalities as just described. In Appendix B we present a collection of results which are especially relevant for the present paper.

Remark 4.2. Observe that in Definition 4.1, the defining inequality only needs to be satisfied for meromorphic functions. This is certainly non-standard but sufficient for our purposes, where F will always be the quotient of two (up to normalization) holomorphic functions. The reason for this somewhat odd definition is that we will ultimately estimate the Poincaré constant in terms of the Cheeger constant related to the measurements. The proof of this estimate is carried out in Appendix B but it does not necessarily apply to all functions $F \in W^{1,p}(D, w)$, the reason being the famous Lavrentiev phenomenon which states that smooth functions need not necessarily be dense in $W^{1,p}(D, w)$ [49].

The next result shows that analytic Poincaré constants as defined in Definition 3.1 can be, to some extent, controlled by the usual Poincaré constant as defined in Definition 4.1.

Lemma 4.3. With the notation of Definition 3.1 we have the estimate

$$(4.1) \quad C_{poinc}^a(p, D, z_0, \delta, w) \leq C_{poinc}(p, D, w) \cdot \left(1 + w(D)^{1/p} \cdot \inf_{0 < a \leq \delta} \frac{w(B_a(z_0))^{1-1/p} \|w^{-1}\|_{L^\infty(B_a(z_0))}}{|B_a(z_0)|} \right).$$

Proof. The analytic Poincaré inequality as defined in Definition 3.1 applies to functions F which are holomorphic in $B_\delta(z_0)$, so, for any disc $B := B_a(z_0) \subset D$ with $0 < a < \delta$, it holds that $F(z_0) = F_B := \frac{1}{|B|} \int_B F(z)dz$, so we need to estimate

$$(4.2) \quad \|F - F(z_0)\|_{L^p(D, w)} = \|F - F_B\|_{L^p(D, w)} \leq \|F - F_D^w\|_{L^p(D, w)} + \|F_B - F_D^w\|_{L^p(D, w)}.$$

The first summand above is bounded by $C_{poinc}(p, D, w) \|\nabla F\|_{L^p(D, w)}$, by the definition of the Poincaré constant.

For the second summand we estimate

$$\|F_B - F_D^w\|_{L^p(D, w)} \leq \frac{1}{|B|} \int_B |F(z) - F_D^w| dz \leq \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|} \int_B |F(z) - F_D^w| w(z) dz,$$

which, by Hölder's inequality, can be bounded by

$$\frac{\|w^{-1}\|_{L^\infty(B)}}{|B|} \|F - F_D^w\|_{L^p(D, w)} w(B)^{1-1/p}.$$

Thus, it holds that

$$\|F_B - F_D^w\|_{L^p(D, w)} \leq \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|} \|F - F_D^w\|_{L^p(D, w)} w(B)^{1-1/p} w(D)^{1/p}.$$

Applying the Poincaré inequality again yields that the second summand in (4.2) can be bounded by

$$C_{\text{poinc}}(p, D, w) \cdot \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|} w(B)^{1-1/p} w(D)^{1/p} \cdot \|\nabla F\|_{L^p(D, w)}$$

Since the expression above continuously depends on $a > 0$ we can also admit $a = \delta$. This proves the claim. \square

Taking a close look at the statement of Lemma 4.3 we see that the analytic Poincaré constant at z_0 can be controlled by the classical Poincaré constant whenever there exists a not too small neighbourhood around z_0 such that the weight function w is lower bounded on this neighbourhood. Since we will later on apply this result to very specific weight functions we will see that such z_0 can always be found.

4.2. Weighted Stable Point Evaluations. Having obtained an estimate for the analytic Poincaré constant in the previous subsection, we go on to develop bounds for the sampling constant which occurs in the right hand side of (3.2). We start with the following Lemma which shows that there exist 'many' points with a given sampling constant.

Lemma 4.4. *Suppose that $D \subset \mathbb{C}$ is a domain, $w : D \rightarrow \mathbb{R}_+$ a weight function and let $G \in L^p(D, w)$ for $1 \leq p < \infty$. For $C > 0$ we denote*

$$D_C(G) := \left\{ z \in D : \|G(z)\|_{L^p(D, w)} \leq C \|G\|_{L^p(D, w)} \right\}.$$

Then

$$w(D_C(G)) \geq w(D) \cdot \left(1 - \frac{1}{C^p}\right).$$

Proof. We compute

$$\int_{D \setminus D_C(G)} |G(x)|^p w(x) dx + \int_{D_C(G)} |G(x)|^p w(x) dx = \|G\|_{L^p(D, w)}^p$$

By the definition of $D_C(G)$ we have that

$$|G(x)|^p > \frac{C^p}{w(D)} \|G\|_{L^p(D, w)}^p \text{ for all } x \in D \setminus D_C(G),$$

and this implies that

$$w(D \setminus D_C(G)) \frac{C^p}{w(D)} \|G\|_{L^p(D, w)}^p + \int_{D_C(G)} |G(x)|^p w(x) dx \leq \|G\|_{L^p(D, w)}^p.$$

Consequently,

$$(w(D) - w(D_C(G))) \frac{C^p}{w(D)} \leq 1$$

and this yields the statement. \square

4.3. Simultaneously Balancing Poincaré- and Sampling Constants. Since Theorem 5.3 requires simultaneous control of the Poincaré and the Sampling constant we now show how the results of the previous two subsections may be combined to achieve this. We consider, for simplicity the case that $D \subset \mathbb{C}$ is convex such that the boundary of D has bounded curvature – the more general case would be more technical and is therefore omitted (see however Remark 4.8).

Definition 4.5. *Let $D \subset \mathbb{C}$ and $F_1 : \bar{D} \rightarrow \mathbb{C}$ be differentiable. We define the global variation of F_1 as*

$$(4.3) \quad \delta_D(F_1) := \min \left\{ \frac{1}{2} \cdot \frac{\|F_1\|_{L^\infty(D)}}{\|\nabla F_1\|_{L^\infty(D)}}, 1 \right\}.$$

The following elementary result will be used later on.

Lemma 4.6. *Let $D \subset \mathbb{C}$ be convex and $F_1 : \overline{D} \rightarrow \mathbb{C}$ be a differentiable function. Suppose that z_0 is a maximum of $|F_1|$ in D , e.g., $|F_1(z_0)| = \|F_1\|_{L^\infty(D)}$. Then it holds that*

$$(4.4) \quad \inf_{z \in B_{\delta_D(F_1)}(z_0) \cap D} |F_1(z)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

Proof. This is a simple consequence of the fact that for all $z \in D$

$$(4.5) \quad \left| |F_1|(z_0) - |F_1|(z) \right| \leq |z - z_0| \cdot \|\nabla |F_1|\|_{L^\infty(D)}$$

and

$$(4.6) \quad |F_1(z_0)| = \|F_1\|_{L^\infty(D)} :$$

Suppose that $z \in B_{\delta_D(F_1)}(z_0) \cap D$. Then

$$|z - z_0| \cdot \|\nabla |F_1|\|_{L^\infty(D)} \leq \frac{1}{2} |F_1(z_0)|.$$

By (4.5) and (4.6) it follows that

$$|F_1(z)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

□

The following proposition shows that the analytic Poincaré and Sampling constants can always be balanced, provided that the quantity $\delta(F_1)$ is not too small.

Proposition 4.7. *Let $1 \leq p < \infty$. Suppose that $D \subset \mathbb{C}$ is convex and that the curvature of the boundary ∂D is everywhere bounded by 1. Suppose $F_1 : \overline{D} \rightarrow \mathbb{C}$ is differentiable, $\delta_D(F_1)$ as defined in (4.5) is positive and $G \in L^p(D, |F_1|^p)$. Then there exists $z \in D$ with*

$$(4.7) \quad C_{poinc}^a(p, D, z, \delta_D(F_1)/4, |F_1|^p) \leq C_{poinc}(p, D, |F_1|^p) \cdot \left(1 + \frac{2^p}{\pi \delta_D(F_1)^2 / 16} \cdot \frac{\|F_1\|_{L^p(D)}^p}{\|F_1\|_{L^\infty(D)}^p} \right),$$

and

$$(4.8) \quad C_{samp}(p, D, z, G, |F_1|^p) \leq \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}} \cdot (\delta_D(F_1)^2 \pi)^{-1/p} \cdot 2 \cdot 16^{-1/p}.$$

Furthermore it holds that

$$(4.9) \quad \inf_{u \in B_{\delta_D(F_1)/4}(z)} |F_1(u)| > 0.$$

Proof. Suppose that $z_0 \in \overline{D}$ is a maximum of $|F_1|$ and put $\delta := \delta_D(F_1) \leq 1$ as defined in (4.3). First we note that by our assumptions on D and by the definition (4.3) it holds that the set $B_\delta(z_0) \cap D$ contains a ball of radius $\delta/2$, i.e., there exists \tilde{z}_0 such that

$$B_{\delta/2}(\tilde{z}_0) \subset D \quad \text{and} \quad \inf_{z \in B_{\delta/2}(\tilde{z}_0)} |F_1(z)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

But this implies that for all $z \in B_{\delta/4}(\tilde{z}_0)$ it holds that

$$B_{\delta/4}(z) \subset D \quad \text{and} \quad \inf_{z \in B_{\delta/4}(z)} |F_1(z)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

Using this fact, we start by estimating the analytic Poincaré constant for such a z , with the estimate from Lemma 4.3. More precisely we will use the estimate (4.1) with $a = \delta/4$ and $B_a := B_{\delta/4}(z)$, which yields that

$$\begin{aligned} C_{poinc}^a(p, D, z, \delta/4, |F_1|^p) &\leq C_{poinc}(p, D, |F_1|^p) \cdot \left(1 + \|F_1\|_{L^p(D)} \cdot \frac{\|F_1\|_{L^p(B_{\delta/4})}^{p-1} 2^p \|F_1\|_{L^\infty(D)}^{-p}}{\pi \delta^2 / 16} \right) \\ &\leq C_{poinc}(p, D, |F_1|^p) \cdot \left(1 + \frac{2^p}{\pi \delta^2 / 16} \cdot \frac{\|F_1\|_{L^p(D)}^p}{\|F_1\|_{L^\infty(D)}^p} \right). \end{aligned}$$

Recall that the above estimate holds for any $z \in B_{\delta/4}(\tilde{z}_0)$.

We now abbreviate $B_{\delta/4} := B_{\delta/4}(\tilde{z}_0)$ and show that there exists such a $z \in B_{\delta/4}$ which also generates good sampling constants. Put $w = |F_1|^p$. By (4.4), we have that

$$(4.10) \quad w(B_{\delta/4}) = \int_{B_{\delta/4}} |F_1(z)|^p dz \geq \|F_1\|_{L^\infty(D)}^p \frac{\pi \delta^2}{2^p \cdot 16}.$$

The measure of ‘good’ sampling points

$$D_C(G) := \{z \in D : C_{\text{samp}}(p, D, z, G, |F_1|^p) \leq C\},$$

by Lemma 4.4, satisfies

$$w(D_C(G)) \geq w(D) \cdot \left(1 - \frac{1}{C^p}\right).$$

Therefore, if

$$C > \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}} \cdot (\delta^2 \pi)^{-1/p} \cdot 2 \cdot 16^{1/p},$$

by (4.10) it holds that

$$w(D_C(G)) > w(D) - w(B_{\delta/4})$$

which implies that

$$D_C(G) \cap B_{\delta/4} \neq \emptyset.$$

Any z in this intersection will satisfy the desired estimates. \square

The result of Proposition 4.7 may still seem very technical. However, we have succeeded in providing bounds for both the analytic Poincaré constant as well as the sampling constant which appear in the right hand side of (3.2).

Indeed, from Proposition 4.7 we can infer that these constants essentially depend only on the Poincaré constant of the measurements $|F_1|^p$ and the quantity $\delta_D(F_1)$.

Remark 4.8. *Alternatively to the global variation as defined in (4.5) we may look at the quantity*

$$(4.11) \quad \tilde{\delta}_D(F_1) := \min\{\sup\{r > 0 : B_r(z) \subset D, \inf_{\zeta \in B_r(z)} |F_1(\zeta)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}\}, 1\}$$

Replicating the proof of Proposition 4.7 reveals that for any $G \in L^p(D, |F_1|^p)$ there is a $z \in D$ such that

$$(4.12) \quad C_{\text{poinc}}^a(p, D, z, \tilde{\delta}_D(F_1)/2, |F_1|^p) \leq C_{\text{poinc}}(p, D, |F_1|^p) \cdot \left(1 + \frac{2^p}{\pi \tilde{\delta}_D(F_1)^2/4} \cdot \kappa_D^p\right),$$

and

$$(4.13) \quad C_{\text{samp}}(p, D, z, G, |F_1|^p) \leq \kappa_D \cdot (\tilde{\delta}_D(F_1)^2 \pi)^{-1/p} \cdot 2 \cdot 4^{-1/p},$$

where we denote $\kappa_D := \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}}$. Additionally it holds that

$$(4.14) \quad \inf_{u \in B_{\tilde{\delta}_D(F_1)/2}(z)} |F_1(u)| > 0.$$

Note that in contrast to Proposition 4.7 we do not need the domain D to be convex and its boundary does not have to meet any curvature assumptions.

In the next section we shall see that the quantity $\delta_D(F_1)$ can always be uniformly bounded if F_1 arises as the Gabor transform of any $f \in \mathcal{S}'(\mathbb{R})$, e.g., $F_1(z) = V_\varphi f(x, -y)$.

5. GABOR PHASE RETRIEVAL

Up to now all results have applied to general functions F_1, F_2 which map from a domain $D \subset \mathbb{C}$ to \mathbb{C} and which are holomorphic after multiplication with a function η . Indeed, by combining Proposition 3.3 with Proposition 4.7 we obtain a stability result which essentially depends only on the Poincaré constant $C_{poinc}(p, D, |F_1|^p)$ and the quantity $\delta_D(F_1)$.

We will, from now on, specialize to the case that F_1 is – up to a reflection – the Gabor transform of a function $f \in \mathcal{S}'(\mathbb{R})$, e.g.,

$$F_1(z) = V_\varphi f(x, -y),$$

where $\varphi(t) = e^{-\pi t^2}$ and $V_\varphi f$ is defined as in Definition 2.1. The Gabor transform enjoys a lot of structure which allows us to obtain major improvements in the general stability bound (3.2). In order to estimate

$$\inf_{\alpha \in \mathbb{R}} \|V_\varphi f - e^{i\alpha} V_\varphi g\|_{L^p(D)}$$

we will apply the results of Chapters 3 and 4 on $F_1, F_2(z) = V_\varphi g(x, -y)$ and the reflected domain $\{\bar{z} : z \in D\}$.

First, in Section 5.1 we shall see that the quantity $\delta_D(V_\varphi f)$ can essentially be bounded independently of f which will finally give us complete control over the implicit constants which appear in the estimate (3.2). Then, in Section 5.2 we will show that, in the case of Gabor measurements, the term involving a logarithmic derivative in (3.2) can be absorbed into an error term with respect to a norm $\mathcal{D}_{p,q}^{r,s}$ for suitable parameters. This latter result will exploit deep function theoretic properties of the Gabor transform. Finally, in Section 5.3 we will put all these results together and present our final stability estimates for Gabor phase retrieval.

5.1. Balancing the Constants. The goal of the present section is to establish the following result.

Proposition 5.1. *Let $D \subset \mathbb{C}$ and suppose $f \in M^{\infty, \infty}(\mathbb{R})$. Then there exists $\delta > 0$, only depending on $\|V_\varphi f\|_{L^\infty(D)} / \|V_{\varphi'} f\|_{L^\infty(D)}$ such that*

$$(5.1) \quad \delta_D(V_\varphi f) \geq \delta.$$

For $D = \mathbb{C}$ we get a stronger statement: There exists a universal constant $\delta > 0$ with

$$(5.2) \quad \inf_{f \in M^{\infty, \infty}(\mathbb{R})} \delta_{\mathbb{C}}(V_\varphi f) \geq \delta.$$

Proof. The proof proceeds by showing that the L^∞ norm of the gradient of $|V_\varphi f|$ cannot be much larger than the L^∞ norm of $|V_\varphi f|$. Indeed, a simple calculation (or a look at Equations (3,5) in [7]) reveals that

$$|\nabla |V_\varphi f|| = |V_{\varphi'} f|$$

which directly implies

$$\|\nabla |V_\varphi f|\|_{L^\infty(D)} = \|V_{\varphi'} f\|_{L^\infty(D)}$$

which, looking at (4.3) implies (5.1).

For the case $D = \mathbb{C}$ we can use the norm equivalence

$$\|V_\phi \cdot\|_{L^p(\mathbb{C})} \sim \|V_{\phi'} \cdot\|_{L^p(\mathbb{C})}$$

on $M^{p,p}(\mathbb{R})$ for any $p \in [1, \infty]$ (see [31, Proposition 11.3.2(c)]). \square

As a corollary we get the following result for $D = \mathbb{C}$.

Corollary 5.2. *Let $1 \leq p < \infty$. Suppose that $f \in M^{\infty, \infty}(\mathbb{R})$ and let $F_1(z) = V_\varphi f(x, -y)$. Further suppose that $G \in L^p(\mathbb{C}, |F_1|^p)$. Then there exist constants $c, \delta > 0$ (independent of f and G !) such that there exists $z \in \mathbb{C}$ with*

$$(5.3) \quad C_{poinc}^a(p, \mathbb{C}, z, \delta, |F_1|^p) \leq c \cdot C_{poinc}(p, \mathbb{C}, |F_1|^p) \cdot \left(1 + \frac{\|F_1\|_{L^p(\mathbb{C})}^p}{\|F_1\|_{L^\infty(\mathbb{C})}^p}\right),$$

$$(5.4) \quad C_{samp}(p, \mathbb{C}, z, G, |F_1|^p) \leq c \cdot \frac{\|F_1\|_{L^p(\mathbb{C})}}{\|F_1\|_{L^\infty(\mathbb{C})}},$$

and

$$(5.5) \quad \inf_{u \in B_\delta(z)} |F_1(u)| > 0.$$

Proof. This is a direct consequence of Propositions 4.7 and 5.1. \square

Observe that for $f, g \in M^{p,p}(\mathbb{R})$ the function $G := \left| \frac{F_2}{F_1} \right| - 1$ is an element of $L^p(\mathbb{C}, |F_1|^p)$, where we put $F_1(z) = V_\varphi f(x, -y)$ and $F_2(z) = V_\varphi g(x, -y)$. Applying Corollary 5.2, together with the well-known fact that $\eta \cdot V_\varphi f$ is holomorphic for suitable η (Theorem 2.4) to Proposition 3.3 we get the following stability result.

Theorem 5.3. *Suppose that $f \in M^{\infty,\infty}(\mathbb{R}) \cap M^{p,p}(\mathbb{R})$ and $g \in M^{p,p}(\mathbb{R})$. Then there exists a constant $c > 0$ only depending on $\|V_\varphi f\|_{L^p(\mathbb{C})}/\|V_\varphi f\|_{L^\infty(\mathbb{C})}$ such that*

$$(5.6) \quad \inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha} g\|_{M^{p,p}(\mathbb{R})} \leq c \cdot (1 + C_{\text{poinc}}(p, \mathbb{C}, |F_1|^p)) \cdot (\| |F_1| - |F_2| \|_{W^{1,p}(\mathbb{C})} + \|\nabla \log(F_1)(|F_1| - |F_2|)\|_{L^p(\mathbb{C})}),$$

where $F_1 := V_\varphi f$ and $F_2 := V_\varphi g$.

For general $D \subset \mathbb{C}$ we get the following result.

Corollary 5.4. *Suppose that $f \in M^{\infty,\infty}(\mathbb{R})$ and let $F_1(z) = V_\varphi f(x, -y)$. Suppose that $D \subset \mathbb{C}$ satisfies the assumptions of Proposition 4.7 and that $G \in L^p(D, |F_1|^p)$. Then there exists a constant c which only depends monotonically increasingly on $\|V_{\varphi'} f\|_{L^\infty(D)}/\|V_\varphi f\|_{L^\infty(D)}$ (and which is otherwise independent of f and $D!$), a constant $\delta > 0$ which depends monotonically decreasingly on $\|V_\varphi f\|_{L^\infty(D)}/\|V_{\varphi'} f\|_{L^\infty(D)}$ (and which is otherwise independent of f and $D!$), and $z \in D$ with $B_\delta(z) \subset D$, such that*

$$(5.7) \quad C_{\text{poinc}}^a(p, D, z, \delta, |F_1|^p) \leq c \cdot C_{\text{poinc}}(p, D, |F_1|^p) \cdot \left(1 + \frac{\|F_1\|_{L^p(D)}^p}{\|F_1\|_{L^\infty(D)}^p} \right),$$

$$(5.8) \quad C_{\text{samp}}(p, D, z, G, |F_1|^p) \leq c \cdot \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}}.$$

and

$$(5.9) \quad \inf_{u \in B_\delta(z)} |F_1(u)| > 0.$$

Proof. This is a direct consequence of Theorem 2.4, Proposition 4.7 and Proposition 5.1. \square

As before, Corollary 5.4 directly leads to a stability result for Gabor phase retrieval.

Theorem 5.5. *Suppose that $f, g \in M^{p,p}(\mathbb{R})$ and let $D \subset \mathbb{C}$ satisfy the assumptions of Proposition 4.7. Then there exists a constant $c > 0$ only depending on*

$$\max \left\{ \frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_{\varphi'} f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}} \right\}$$

such that

$$(5.10) \quad \inf_{\alpha \in \mathbb{R}} \|F_1 - e^{i\alpha} F_2\|_{L^p(D)} \leq c \cdot (1 + C_{\text{poinc}}(p, D, |F_1|^p)) \cdot (\| |F_1| - |F_2| \|_{W^{1,p}(D)} + \|\nabla \log(F_1)(|F_1| - |F_2|)\|_{L^p(D)}).$$

5.2. Controlling the Logarithmic Derivative. Compared to Theorem 2.9, Theorems 5.3 and 5.5 are now independent of any choice of z_0 which is very nice. However, the term involving the logarithmic derivative of F_1 in (5.6) and (5.10) is still bothersome, in particular because, in general $|\nabla \log(F_1)|$ will certainly be unbounded. It turns out that in the case of Gabor measurements this quantity can be absorbed into an error term with respect to a norm as defined in Definition 2.5. The proof of this fact is however quite difficult and involves deep function-theoretic properties of the Gabor transform.

We begin by estimating the norms of the logarithmic derivative of a Gabor transform $V_\varphi f$ on discs with growing radii. It turns out that these can be estimated independently of the original signal $f \in M^{\infty, \infty}(\mathbb{R})$. The reason for this perhaps surprising fact is that the holomorphic function $\eta \cdot V_\varphi f$, with suitable η , satisfies certain restricted growth properties. Jensen's formula relates the distribution of zeros of a holomorphic functions with its growth rate which allows us to bound the number of zeros of $V_\varphi f$ in a given disc. This in turn will yield a bound on the norm of the logarithmic derivative of $V_\varphi f$ as follows.

Proposition 5.6. *Let $1 \leq r < 2$. There exists a polynomial ρ of degree at most 5 such that for all $f \in M^{\infty, \infty}(\mathbb{R})$ and all $R > 0$ we have an estimate*

$$\|\nabla \log(V_\varphi f)\|_{L^r(B_R(z_0))} \leq \rho(R),$$

where z_0 is a maximum of $|V_\varphi f|$.

Proof. We assume w.l.o.g. that $z_0 = 0$ (otherwise translate and modulate). We will only estimate the norm of $\frac{\partial}{\partial x} \log(V_\varphi f)$; the derivative in direction of the second variable can be bounded in the same way.

Let $F_1(z) = V_\varphi f(x, -y)$ and $\eta(z) := e^{\pi(\frac{|z|^2}{2} - ixy)}$. Then, by Theorem 2.4, the function $G = \eta F_1$ is an entire function of order 2 and type $\frac{\pi}{2}$ [20, Chapter XI].

Let the zeros of G - and therefore of F_1 - be denoted by $(\zeta_i)_{i \in \mathbb{N}}$ (counted by multiplicity). By the Hadamard factorization theorem [20, Chapter XI, §3] we obtain

$$(5.11) \quad G(z) = e^{az^2 + bz + c} \cdot \prod_{i=1}^{\infty} E\left(\frac{z}{\zeta_i}\right),$$

where $E(u) := (1 - u)e^{u + \frac{u^2}{2}}$ and $|a| \leq \frac{\pi}{2}$. Next compute the logarithmic derivative of G :

$$(5.12) \quad \frac{G'}{G}(z) = \sum_i \frac{z^2}{(z - \zeta_i)\zeta_i^2} + 2az + b.$$

Since $\nabla |\eta|(0) = 0$ (by calculation) and since $\nabla |F_1|(0) = 0$ (by the assumption that 0 is a maximum of F_1) the product rule gives $\nabla |G|(0) = 0$. By Lemma 3.4 we have $G'(0) = 0$. Comparison with equation (5.12) implies $b = 0$.

Computing the logarithmic derivative of G in a second way yields

$$(5.13) \quad \frac{G'}{G} = \frac{\eta' F_1 + \eta F_1'}{\eta F_1} = \frac{\eta'}{\eta} + \frac{F_1'}{F_1}.$$

A direct calculation shows that $\eta'/\eta = \pi \bar{z}$ and we get

$$(5.14) \quad \log(V_\varphi f)'(z) = \frac{F_1'}{F_1}(z) = \sum_i \frac{z^2}{(z - \zeta_i)\zeta_i^2} + 2az - \pi \bar{z}.$$

We shall now develop an estimate for the expression on the right hand side of (5.14).

Let's get to the hardest part: Estimating the norm of the (infinite) sum $\sum_i \frac{z^2}{(z - \zeta_i)\zeta_i^2}$. The first step is to bound the number of zeros of G in balls of radius $R > 0$. We do this by using Jensen's formula (see [20, Chapter XI, 1.2]), which states that for any entire function G

$$\log |G(0)| - \sum_{|\zeta_i| < R} \log \left| \frac{\zeta_i}{R} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |G(Re^{i\theta})| d\theta.$$

Therefore, for any $R > 0$ it holds that

$$\begin{aligned}
|\{i : |\zeta_i| < R\}| &\leq \frac{1}{\log(2)} \sum_{\{i : |\zeta_i| < 2R\}} \log\left(\frac{2R}{|\zeta_i|}\right) \\
&= \frac{1}{\log(2)} \left(\frac{1}{2\pi} \int_0^{2\pi} \log |F_1(2Re^{i\theta})| + \log |\eta(2Re^{i\theta})| d\theta - \log |F_1(0)| - \log |\eta(0)| \right) \\
(5.15) \quad &\leq \frac{1}{\log(2)} (2\pi R^2 + \log \|F_1\|_\infty - \log |F_1(0)|) = \frac{2\pi}{\log(2)} R^2.
\end{aligned}$$

Let us assume that $R = 2^l$ for some $l \in \mathbb{N}$. We define subsets of \mathbb{N} by

$$I := \{i : |\zeta_i| < 2^{l+1}\} \quad \text{and} \quad I_j := \{i : 2^j \leq |\zeta_i| < 2^{j+1}\}, \quad j \geq l+1.$$

The above estimate (5.15) guarantees that $|I| \leq 8\pi 2^{2l+2}$ and $|I_j| \leq 8\pi 2^{2j+2}$.

Using this information we now split the sum (5.14) over $i \in \mathbb{N}$ into a sequence of sums over I and I_j , $j \geq l+1$ and estimate the $L^r(B_R(0))$ norm on each of these parts.

First we take care of the term $\sum_{i \in I} \frac{z^2}{(z-\zeta_i)\zeta_i^2}$: Since there exists $\delta > 0$ independent from f such that $|\zeta_i| \geq \delta$ (this follows from the assumption that 0 is a maximum of $V_\varphi f$, together with Proposition 5.1 and Equation (4.3)) we can estimate

$$(5.16) \quad \left\| \sum_{i \in I} \frac{z^2}{(z-\zeta_i)\zeta_i^2} \right\|_{L^r(B_R(0))} \leq \frac{R^2}{\delta^2} \cdot \sum_{i \in I} \left\| \frac{1}{z-\zeta_i} \right\|_{L^r(B_R(0))}.$$

The summands can be uniformly bounded:

$$(5.17) \quad \left\| \frac{1}{z-\zeta_i} \right\|_{L^r(B_R(0))}^r \leq \left\| \frac{1}{\cdot} \right\|_{L^r(B_R(0))}^r = 2\pi \int_0^R s^{1-r} ds = 2\pi \cdot \frac{R^{2-r}}{2-r} = \tau_{2-r}^r,$$

where $\tau_q = \tau_q(R) := \left(2\pi \cdot \frac{R^q}{q}\right)^{1/r}$.

Next we estimate the norm of the expression $\sum_{i \in I_j} \frac{z^2}{(z-\zeta_i)\zeta_i^2}$ for fixed $j \geq l+1$: Since $|\zeta_i| \geq 2^j$ for $i \in I_j$ we obtain for $|z| \leq R = 2^l$

$$|z - \zeta_i| \geq ||z| - |\zeta_i|| \geq 2^j - 2^l = 2^l(2^{j-l} - 1) \geq 2^l 2^{j-l-1} = 2^{j-1}.$$

Together with the computation

$$\|z^2\|_{L^r(B_R(0))}^r = 2\pi \int_0^R s^{2r+1} ds = \tau_{2r+2}^r$$

this yields

$$(5.18) \quad \left\| \sum_{i \in I_j} \frac{z^2}{(z-\zeta_i)\zeta_i^2} \right\|_{L^r(B_R(0))} \leq |I_j| \cdot 2^{-2j} \cdot 2^{-j+1} \cdot \tau_{2r+2}$$

A straight forward computation shows us

$$(5.19) \quad \|2az\|_{L^r(B_R(0))} = 2|a| \tau_{r+2} \quad \text{and} \quad \|\pi\bar{z}\|_{L^r(B_R(0))} = \pi \tau_{r+2}.$$

The only thing that is left is to put all the pieces together:

$$\begin{aligned}
 \left\| \frac{\partial}{\partial x} \log(V_\varphi f) \right\|_{L^r(B_R(0))} &= \left\| \frac{F_1'}{F_1} \right\|_{L^r(B_R(0))} \\
 &\leq \left\| \sum_{i \in I} \frac{\cdot^2}{(\cdot - \zeta_i) \zeta_i^2} \right\|_{L^r(B_R(0))} + \sum_{j \geq l+1} \left\| \sum_{i \in I_j} \frac{\cdot^2}{(\cdot - \zeta_i) \zeta_i^2} \right\|_{L^r(B_R(0))} \\
 &\quad + \|2a\|_{L^r(B_R(0))} + \|\pi^-\|_{L^r(B_R(0))} \\
 &\leq \frac{R^2}{\delta^2} \cdot |I| \cdot \tau_{2-r} + \sum_{j \geq l+1} 2 |I_j| 2^{-3j} \cdot \tau_{2r+2} + (2|a| + \pi) \cdot \tau_{r+2} \\
 &\leq \frac{R^2}{\delta^2} \cdot 32\pi \cdot R^2 \cdot \tau_{2-r} + 64\pi \cdot \tau_{2r+2} \underbrace{\sum_{j \geq l+1} 2^{-j}}_{=2^{-l}-R^{-1}} + 2\pi \cdot \tau_{r+2}
 \end{aligned}$$

Having a look on the definition of τ_q we observe that the term on the right hand side of the estimate can be bounded by a polynomial of maximal order of 5; i.e. there exists a constant c that only depends on r such that

$$(5.20) \quad \left\| \frac{\partial}{\partial x} \log(V_\varphi f) \right\|_{L^r(B_R(0))} \leq c \cdot (R^5 + 1).$$

□

Proposition 5.6 yields important information on how fast the L^r norm of the logarithmic derivative of a Gabor transform can possibly grow as the size of the integration domain increases. It is remarkable that this quantity can be bounded independent of the original signal.

Moreover, with Proposition 5.6 in hand we can go on to control the bothersome logarithm term in the estimates of Theorems 5.3 and 5.5:

Proposition 5.7. *Let $p \in [1, 2)$ and $q \in (\frac{2p}{2-p}, \infty]$. Then there exists a polynomial σ of maximal order 6 such that for any $f \in \mathcal{S}'(\mathbb{R})$ with $V_\varphi f$ centered (see Definition 2.7), all $g \in \mathcal{S}'(\mathbb{R})$ and all domains $D \subset \mathbb{C}$ with $0 \in \mathbb{C}$ ($D = \mathbb{C}$ is allowed!) it holds that*

$$\|\nabla \log(V_\varphi f)(|V_\varphi f| - |V_\varphi g|)\|_{L^p(D)} \leq \|(|V_\varphi f| - |V_\varphi g|) \sigma(\cdot)\|_{L^q(D)}$$

Proof. We write $F_1(z) = V_\varphi f(x, -y)$ and $F_2(z) = V_\varphi g(x, -y)$. The statement is only proven for $D = \mathbb{C}$ since the general case can be proven in the same way.

We will only estimate the norm of $\frac{\partial}{\partial x} \log(F_1)$; the derivative w.r.t. the second variable y can be handled analogously.

Let $D_0 := B_1(0)$ and $D_j := B_{2^j}(0) \setminus B_{2^{j-1}}(0)$ for $j \geq 1$, then

$$(5.21) \quad \int_{\mathbb{C}} |\log(F_1)'|^p \cdot \|F_1\| - \|F_2\|^p = \sum_{j \geq 0} \int_{D_j} |\log(F_1)'|^p \cdot \|F_1\| - \|F_2\|^p$$

The numbers $s = \frac{q}{q-p}$ and $s' = \frac{q}{p}$ are Hölder conjugated. Denoting $r := ps = \frac{pq}{q-p}$ we have

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} > \frac{1}{p} - \frac{2-p}{2p} = \frac{1}{2},$$

and therefore $r < 2$. Applying Hölder's inequality and Proposition 5.6 we obtain

$$\begin{aligned} \int_{D_j} |\log(F_1)'|^p \cdot \|F_1 - F_2\|^p &\leq \|\log(F_1)'\|_{L^r(D_j)}^p \cdot \|F_1 - F_2\|_{L^q(D_j)}^p \\ &\leq (c \cdot (2^{5j} + 1))^p \cdot \|F_1 - F_2\|_{L^q(D_j)}^p \\ &\leq c' \cdot 2^{5pj} \cdot \left(\int_{D_j} \|F_1 - F_2\|^q \right)^{1/s'} \end{aligned}$$

where c' depends only on r and p . Using Hölder's inequality for sums yields

$$\begin{aligned} \int_{\mathbb{C}} |\log(F_1)'|^p \cdot \|F_1 - F_2\|^p &\leq c' \cdot \sum_{j \geq 0} 2^{-\frac{j}{s}} \cdot 2^{5pj + \frac{j}{s}} \cdot \left(\int_{D_j} \|F_1 - F_2\|^q \right)^{1/s'} \\ &\leq c' \cdot \left(\sum_{j \geq 0} 2^{-j} \right)^{1/s} \cdot \left(\sum_{j \geq 0} 2^{(5ps' + \frac{s'}{s})j} \int_{D_j} \|F_1 - F_2\|^q \right)^{1/s'} \\ &= 2^{1/s} \cdot c' \cdot \left(\int_{\mathbb{C}} \|F_1 - F_2\|^q \cdot \sum_{j \geq 0} 2^{(5ps' + \frac{s'}{s})j} \chi_{D_j} \right)^{1/s'} \end{aligned}$$

For $z \in D_j$

$$2^{(5ps' + \frac{s'}{s})j} \leq 2 \cdot \left(|z|^{5ps' + \frac{s'}{s}} + 1 \right) =: 2 \cdot \sigma^q(z)$$

holds. Therefore we conclude

$$(5.22) \quad \int_{\mathbb{C}} |\log(F_1)'|^p \|F_1 - F_2\|^p \leq \underbrace{2^{1/s+1/s'}}_{=2} c' \|(|F_1| - |F_2|)\sigma(\cdot)\|_{L^q(\mathbb{C})}^p.$$

Since $\frac{s'}{s} = s' - 1 = \frac{q}{p} - 1$ we have

$$(5.23) \quad \sigma(z) \approx |z|^{5 + \frac{1}{p} - \frac{1}{q}} + 1 \lesssim |z|^6 + 1$$

and the stated result holds. \square

5.3. Putting Everything Together. We can now apply Proposition 5.7 to control the logarithmic derivative in Theorem 5.3 and immediately get the following result.

Theorem 5.8. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$ only depending on p, q and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that*

$$d_{M^{p,p}(\mathbb{R})}(f, g) \leq c \cdot (1 + C_{\text{poinc}}(p, \mathbb{C}, |V_\varphi f|^p)) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}}.$$

We can also establish the following local version which follows by combining Proposition 5.7 and Theorem 5.5.

Theorem 5.9. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $D \subset \mathbb{C}$ satisfies the assumptions of Proposition 4.7. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$ only depending on p, q and*

$$\max \left\{ \frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_\varphi f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}} \right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that

$$\inf_{\alpha \in \mathbb{R}} \|V_\varphi f - e^{i\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot (1 + C_{\text{poinc}}(p, D, |V_\varphi f|^p)) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}(D)}.$$

It remains to interpret the weighted Poincaré constants $C_{poinc}(p, D, |V_\varphi f|^p)$ and $C_{poinc}(p, \mathbb{C}, |V_\varphi f|^p)$. In Appendix B we prove the following result.

Theorem 5.10. *For every connected domain $D \subset \mathbb{R}^2$ ($D = \mathbb{C}$ is allowed!) and every $f \in \mathcal{S}'(\mathbb{R})$ it holds that*

$$C_{poinc}(p, D, |V_\varphi f|^p) \leq \frac{4p}{h_{p,D}(f)},$$

where $h_{p,D}(f)$ is defined as in Definition 2.8.

Combining Theorem 5.10 with Theorem 5.8 we obtain the following fundamental stability result.

Theorem 5.11. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$ only depending on p, q and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that*

$$d_{M^{p,p}(\mathbb{R})}(f, g) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}}.$$

Combining Theorem 5.10 with Theorem 5.9 we obtain the following fundamental local stability result.

Theorem 5.12. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $D \subset \mathbb{C}$ satisfies the assumptions of Proposition 4.7. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$ only depending on p, q and*

$$\max \left\{ \frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_\varphi f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}} \right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that

$$\inf_{\alpha \in \mathbb{R}} \|V_\varphi f - e^{i\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot (1 + h_{p,D}(f)^{-1}) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}(D)}.$$

Remark 4.8 together with Proposition 5.7 and Theorem 5.10 gives us a slightly different version of the stability result in Theorem 5.12, where we can drop the assumptions on the domain D altogether.

Theorem 5.13. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$. Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f) and let $\delta := \tilde{\delta}_D(V_\varphi f)$ as defined in (4.11) and $\kappa := \frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}$. Then there exists a constant $c > 0$ only depending on p, q such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that*

$$\inf_{\alpha \in \mathbb{R}} \|V_\varphi f - e^{i\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot (1 + h_{p,D}(f)^{-1}) \cdot (1 + \frac{\kappa^p}{\delta^2}) \cdot \| |V_\varphi f| - |V_\varphi g| \|_{\mathcal{D}_{p,q}^{1,6}(D)}.$$

As a consequence we get the following multicomponent-type stability result.

Corollary 5.14. *Let $p \in [1, 2)$ and $q \in (2p/(2-p), \infty]$ and let D be partitioned in subdomains D_1, \dots, D_s , i.e.,*

$$D_i \subset D \text{ open, } D_i \cap D_j = \emptyset \text{ for } i \neq j \text{ and } \bigcup_{i=1}^s \overline{D_i} = \overline{D}.$$

Suppose that $f \in M^{p,p}(\mathbb{R}) \cap M^{\infty,\infty}(\mathbb{R})$ be such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Let

$$B := \max_{i=1, \dots, s} (1 + h_{p,D_i}(f)^{-1}) \cdot (1 + \frac{\kappa_i^p}{\delta_i^2})$$

where we set $\delta_i := \tilde{\delta}_{D_i}(V_\varphi f)$ as defined in (4.11) and

$$(5.24) \quad \kappa_i := \frac{\|V_\varphi f\|_{L^p(D_i)}}{\|V_\varphi f\|_{L^\infty(D_i)}}.$$

Then there exists a constant $c > 0$ only depending on p, q such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that

$$\sum_{i=1}^s \inf_{\alpha_i \in \mathbb{R}} \|V_{\varphi} f - e^{i\alpha_i} V_{\varphi} g\|_{L^p(D)} \leq c \cdot B \cdot \| |V_{\varphi} f| - |V_{\varphi} g| \|_{\mathcal{D}_{p,q}^{1,0}(D)}.$$

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APPENDIX A. THE STFT DOES PHASE RETRIEVAL

In this section we present two remarkable properties of the Gabor transform. First we show that by multiplication with a function η (which is independent from the signal f) the Gabor transform $V_\varphi f$ becomes an entire function (compare to [31, Proposition 3.4.1]).

Theorem A.1. *Let $z := x + iy \in \mathbb{C}$ and let $\eta(z) := e^{\pi\left(\frac{|z|^2}{2} - ix y\right)}$. Then for every $f \in \mathcal{S}'(\mathbb{R})$ the function $z \mapsto \eta(z) \cdot V_\varphi f(x, -y)$ is an entire function.*

Proof. For fixed f define $F(z) := \eta(z) \cdot V_\varphi f(x, -y)$. Since any tempered distribution is a derivative of finite order of a continuous function of polynomial growth ([28, Theorem 8.3.1.]) we can find a function h with these properties such that

$$\begin{aligned} F(z) &= \eta(z) \cdot \left(\frac{d^k}{dt^k} h(\cdot), e^{-\pi(\cdot-x)^2} e^{2\pi iy \cdot} \right)_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} \\ &= (-1)^k \eta(z) \int_{\mathbb{R}} h(t) \frac{d^k}{dt^k} \left(e^{-\pi(t-x)^2} e^{2\pi iy t} \right) dt \end{aligned}$$

for some $k \in \mathbb{N}$. With $g(t, z) := e^{-\pi(t-x)^2} e^{2\pi iy t}$ we get

$$\frac{\partial}{\partial t} g(t, z) = -2\pi(t-x)g(t, z)$$

A simple induction argument yields that any higher derivative of g w.r.t. t is of the form $p(t-x) \cdot g(t, z)$ where p is a polynomial. Since for any t the function $z \mapsto \eta(z)g(t, z)$ is holomorphic so

is the integrand of

$$F(z) = \int_{\mathbb{R}} (-1)^k h(t) \eta(z) p(t-z) g(t, z) dt.$$

To conclude that F is an entire function it suffices to show that for any bounded disc $D \subset \mathbb{C}$ centered at the origin there is an integrable function u_D such that the integrand is bounded by u_D uniformly for all $z \in D$ (see [23, IV Theorem 5.8]).

Let r be the radius of such a disc D then for any $z = x + iy \in D$ the estimate

$$e^{-\pi(t-x)^2} \leq g_D(t) := \begin{cases} e^{-\frac{\pi}{4}t^2}, & |t| \geq 2r \\ 1 & \text{otherwise} \end{cases}$$

holds. Further there is a polynomial \tilde{p} such that $p(t-z) \leq \tilde{p}(t)$ for all $z \in D$. Therefore

$$|h(t)\eta(z)p(t-z)g(t, z)| \leq \sup_{z \in D} |\eta(z)| h(t) \tilde{p}(t) g_D(t) =: u_D(t)$$

Since g_D decays exponentially and h and \tilde{p} each have polynomial growth we get the desired result. \square

The following theorem states that the Fourier transform of the spectrogram turns out to be the product of the ambiguity functions of the window g and the signal f (see [19], [18]). This result allows us to write down a reconstruction formula for our problem.

We will present a proof of the statement for the case where f is a tempered distribution. To that end we first of all have to give a meaningful definition of $\mathcal{A}f$ for f a tempered distribution. For any $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ we define two linear transforms by

$$(A.1) \quad TF(x, y) := F(x, x-y) \quad \text{and} \quad SF(x, y) := F(y, x).$$

Clearly $T^{-1} = T$ and $S^{-1} = S$ hold. For $F \in \mathcal{S}'(\mathbb{R}^2)$ let $TF \in \mathcal{S}'(\mathbb{R}^2)$ be defined by

$$(TF, \Theta)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)} := (F, T\Theta)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)}$$

and $SF \in \mathcal{S}(\mathbb{R}^2)$ analogously.

Note that this notation makes sense: If F is a regular tempered distribution we have

$$(TF, \Theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(x, y) T\Theta(x, y) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} TF(x, y) \Theta(x, y) dx dy$$

since $T^{-1} = T$ and T describes a linear coordinate transform with jacobian determinant -1 .

For $f \in \mathcal{S}'(\mathbb{R})$ we can define a tempered distribution by $\mathcal{A}f := S \circ \mathcal{F}_1 \circ T(f \otimes \bar{f})$, where \mathcal{F}_1 denotes the Fourier transform w.r.t. the first variable of a bivariate tempered distribution, i.e.

$$(\mathcal{F}_1 F, \Theta)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)} = \left(F, (x, y) \mapsto \int_{\mathbb{R}} \Theta(t, y) e^{2\pi i x t} dt \right)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)}$$

We call $\mathcal{A}f$ the ambiguity function of f . For $f \in \mathcal{S}(\mathbb{R})$ the calculation

$$(A.2) \quad \mathcal{A}f(x, y) = [\mathcal{F}_1 \circ T(f \otimes \bar{f})](y, x) = \int_{\mathbb{R}} f(t) \overline{f(t-x)} e^{-2\pi i y t} dt$$

shows that $\mathcal{A}f$ is indeed an extension of the definition of the ambiguity function (see Theorem 2.3).

In the following \mathcal{F} will denote the Fourier transform of bivariate functions. By duality \mathcal{F} can be defined on tempered distributions:

$$(\mathcal{F}F, \Theta)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)} = \left(F, (x, y) \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(s, t) e^{2\pi i (xs+yt)} ds dt \right)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)}$$

Theorem A.2. Let $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ then $\mathcal{F}|V_g f|^2 = \mathcal{S}A f \cdot \mathcal{S}A g$, i.e.

$$(A.3) \quad \left(\mathcal{F}|V_g f|^2, \Theta \right)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)} = (\mathcal{S}A f, \mathcal{S}A g \cdot \Theta)_{\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)}$$

holds for all $\Theta \in \mathcal{S}(\mathbb{R}^2)$.

Proof. First note that $V_g f$ has at most polynomial growth (see [31, Theorem 11.2.3.]). So $|V_g f|^2$ also has polynomial growth, therefore is in $\mathcal{S}'(\mathbb{R}^2)$ and its Fourier transform is well defined.

To simplify notation we will use duality brackets without explicitly stating in which spaces we take duality. From the context it will be clear if we mean duality either in $\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ or in $\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$.

Since compactly supported functions are dense in $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{F} : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$ is unitary it suffices to show

$$\left(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta \right) = (\mathcal{S}A f, \mathcal{S}A g \cdot \mathcal{F}\Theta)$$

for all $\Theta \in C_c^\infty(\mathbb{R}^2)$, where we denote by $C_c^\infty(\mathbb{R}^2)$ the space of infinitely often differentiable functions on \mathbb{R}^2 with compact support.

For the moment let us assume that also g is compactly supported. The spectrogram can be written as

$$\begin{aligned} |V_g f(x, y)|^2 &= (f, e^{-2\pi i y} \bar{g}(\cdot - x)) \cdot \overline{(f, e^{-2\pi i y} \bar{g}(\cdot - x))} \\ &= (f \otimes \bar{f}, (s, t) \mapsto e^{-2\pi i y s} \bar{g}(s - x) e^{2\pi i y t} g(t - x)) \end{aligned}$$

We obtain

$$(A.4) \quad \begin{aligned} \left(\mathcal{F}(|V_g f|^2), \mathcal{F}\Theta \right) &= \int_{\mathbb{R}} \int_{\mathbb{R}} |V_g f|^2 \Theta(x, y) dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x, y) (f \otimes \bar{f}, (s, t) \mapsto e^{-2\pi i y s} \bar{g}(s - x) e^{2\pi i y t} g(t - x)) dx dy. \end{aligned}$$

What we want to do next is to interchange integration and evaluation by the distribution $f \otimes \bar{f}$ in the equation above. To this end we approximate the integral by a sequence of Riemann sums and use the linearity of $f \otimes \bar{f}$.

Let functions J_n for $n \in \mathbb{N}$ and J be defined by

$$\begin{aligned} J_n(s, t) &:= n^{-2} \sum_{k, l \in \mathbb{Z}} \Theta\left(\frac{k}{n}, \frac{l}{n}\right) e^{-2\pi i \frac{k}{n} s} \bar{g}\left(s - \frac{k}{n}\right) e^{2\pi i \frac{l}{n} t} g\left(t - \frac{k}{n}\right), \\ J(s, t) &:= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x, y) e^{-2\pi i y s} \bar{g}(s - x) e^{2\pi i y t} g(t - x) dx dy. \end{aligned}$$

For $M > 0$ such that $\text{supp } \Theta \subset [-M/2, M/2]^2$ and $\text{supp } g \subset [-M/2, M/2]$ clearly both $\text{supp } J_n$ and $\text{supp } J$ are subsets of $[-M, M]^2$. Furthermore the indices k, l in the definition of J_n will in fact only run over the finite set $\mathbb{Z} \cap [-Mn, Mn]$.

Note that (A.4) is an integral of a continuous and compactly supported function and therefore

$$(A.5) \quad \left(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta \right) = \lim_{n \rightarrow \infty} (f \otimes \bar{f}, J_n)$$

holds. Clearly J_n converges to J pointwise. To interchange taking the limit and evaluation by $f \otimes \bar{f}$ we will show that J_n converges to J w.r.t. Schwartz space topology, i.e.

$$\sup_{s, t} \left| s^{\beta_1} t^{\beta_2} \frac{\partial^{\alpha_1 + \alpha_2}}{\partial s^{\alpha_1} \partial t^{\alpha_2}} (J_n(s, t) - J(s, t)) \right|$$

goes to zero for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N} \cup \{0\}$.

The polynomial factor can be omitted as there is a mutual compact support of $(J_n)_{n \in \mathbb{N}}$ and J .

Using $\mathfrak{D} := \frac{\partial^{\alpha_1 + \alpha_2}}{\partial s^{\alpha_1} \partial t^{\alpha_2}}$ let Ψ be defined by

$$\Psi(x, y, s, t) := \mathfrak{D} \left(e^{-2\pi i y s} \bar{g}(s - x) e^{2\pi i y t} g(t - x) \right).$$

Then obviously

$$\begin{aligned}\mathfrak{D}J(s, t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x, y) \Psi(x, y, s, t) dx dy, \\ \mathfrak{D}J_n(s, t) &= n^{-2} \sum_{k, l} \Theta\left(\frac{k}{n}, \frac{l}{n}\right) \Psi\left(\frac{k}{n}, \frac{l}{n}, s, t\right).\end{aligned}$$

Again for any fixed (s, t) the values $\mathfrak{D}J_n(s, t)$ can be interpreted as Riemann approximations for the integral $\mathfrak{D}J(s, t)$ and we can infer pointwise convergence. By showing that $(\mathfrak{D}J_n)_{n \in \mathbb{N}}$ is equicontinuous on the compact set $[-M, M]^2$ we can conclude that $\mathfrak{D}J_n \rightarrow \mathfrak{D}J$ uniformly. Since Ψ is a smooth function there exists a $c > 0$ such that

$$|\Psi(x, y, s, t) - \Psi(x, y, s', t')| \leq c |(s, t) - (s', t')|,$$

for all $(x, y, s, t), (x, y, s', t') \in [-\frac{M}{2}, \frac{M}{2}]^2 \times [-M, M]^2$. Equicontinuity holds by the estimate

$$\begin{aligned}|\mathfrak{D}J_n(s, t) - \mathfrak{D}J_n(s', t')| &\leq n^{-2} \sum_{k, l} \left| \Theta\left(\frac{k}{n}, \frac{l}{n}\right) \right| \left| \Psi\left(\frac{k}{n}, \frac{l}{n}, s, t\right) - \Psi\left(\frac{k}{n}, \frac{l}{n}, s', t'\right) \right| \\ &\leq n^{-2} \sum_{k, l} \|\Theta\|_{L^\infty(\mathbb{R}^2)} \cdot c \cdot |(s, t) - (s', t')| \\ &\leq \|\Theta\|_{L^\infty(\mathbb{R}^2)} M^2 c |(s, t) - (s', t')|\end{aligned}$$

where we used the fact that for every n the indices run over the finite set $|k|, |l| \leq \frac{Mn}{2}$.

Defining $h_\tau(\cdot) := g(\cdot)\bar{g}(\cdot - \tau)$ we obtain

$$J(s, t) = [\mathcal{F}_2 \Theta(\cdot, s - t) * h_{s-t}(\cdot)](s)$$

and therefore

$$TJ(s, t) := J(s, s - t) = [\mathcal{F}_2 \Theta(\cdot, t) * h_t(\cdot)](s).$$

Fourier transform in the first variable gives

$$\mathcal{F}_1 \circ TJ(s, t) = \mathcal{F} \Theta(s, t) \cdot \widehat{h}_t(s).$$

Now \widehat{h}_t turns out to be the ambiguity function of g :

$$\widehat{h}_t(s) = \mathcal{F}(g(\cdot)\bar{g}(\cdot - t))(s) = \mathcal{A}g(t, s)$$

Putting it all together we get

$$\begin{aligned}(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta) &= (f \otimes \bar{f}, J) = (T(f \otimes \bar{f}), TJ) = (\mathcal{F}_1 \circ T(f \otimes \bar{f}), \mathcal{F}_1 \circ TJ) \\ &= (\mathcal{F}_1 \circ T(f \otimes \bar{f}), SAg \cdot \mathcal{F}\Theta) = (SAf, SAg \cdot \mathcal{F}\Theta).\end{aligned}$$

It remains to proof that the result holds true for any Schwartz function g . We will do this by a density argument: For $g \in \mathcal{S}(\mathbb{R})$ one can find a sequence $(g_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ of compactly supported functions converging to g .

Since for any $f \in \mathcal{S}'(\mathbb{R})$ there exist $C > 0$ and $L > 0$ such that

$$|(f, h)| \leq C \sum_{\alpha, \beta \leq L} \left\| \frac{d^\alpha}{d \cdot^\alpha} (\cdot^\beta \cdot h(\cdot)) \right\|_{L^\infty(\mathbb{R})} \quad \text{for all } h \in \mathcal{S}(\mathbb{R})$$

(see [28, chapter 8.3]) we can estimate

$$\begin{aligned}
|V_{g-g_n}f(x, y)| &= \left| \left(f, e^{2\pi i y \cdot \overline{(g_n - g)}(\cdot - x)} \right) \right| \\
&\leq C \sum_{\alpha, \beta \leq L} \left\| \frac{d^\alpha}{d \cdot \alpha} \left(\cdot^\beta e^{2\pi i y \cdot \overline{(g_n - g)}(\cdot - x)} \right) \right\|_{L^\infty(\mathbb{R})} \\
&= C \sum_{\alpha, \beta \leq L} \left\| \frac{d^\alpha}{d \cdot \alpha} \left((\cdot + x)^\beta e^{2\pi i y \cdot \overline{(g_n - g)}(\cdot)} \right) \right\|_{L^\infty(\mathbb{R})} \\
&\leq p(x, y) \cdot \max_{\alpha, \beta \leq L} \left\| \frac{d^\alpha}{d \cdot \alpha} \cdot^\beta (g_n(\cdot) - g(\cdot)) \right\|_{L^\infty(\mathbb{R})}
\end{aligned}$$

for some polynomial p .

In particular for any compact $K \subset \mathbb{R}^2$ there is a constant C_K independent from n such that the function on the right hand side of the inequality above can be bounded by C_K for all $(x, y) \in K$.

The STFT is continuous therefore also the function

$$|V_{g_n}f(x, y)| \leq |V_{g_n-g}f(x, y)| + |V_gf(x, y)|$$

can be bounded by a constant independent from n on any compact K . Obviously $|V_{g_n}f|^2$ converges to $|V_gf|^2$ pointwise. By dominated convergence we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\mathcal{F} |V_{g_n}f|^2, \mathcal{F}\Theta \right) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} |V_{g_n}f(x, y)|^2 \Theta(x, y) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |V_gf(x, y)|^2 \Theta(x, y) dx dy = \left(\mathcal{F} |V_gf|^2, \mathcal{F}\Theta \right)
\end{aligned}$$

Since $g \mapsto g \otimes \bar{g}$ is continuous as mapping from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R}^2)$ and so are S, \mathcal{F}_1 and T on $\mathcal{S}(\mathbb{R}^2)$ so is their composition $\mathcal{A}g = S \circ \mathcal{F}_1 \circ T(f \otimes \bar{f})$ which implies convergence of $(\mathcal{A}g_n)_{n \in \mathbb{N}}$ to $\mathcal{A}g$. Multiplication by a fixed Schwartz function is again a continuous operator on $\mathcal{S}(\mathbb{R}^2)$ therefore

$$(A.6) \quad \lim_{n \rightarrow \infty} (S\mathcal{A}f, S\mathcal{A}g_n \cdot \mathcal{F}\Theta) = (S\mathcal{A}f, S\mathcal{A}g \cdot \mathcal{F}\Theta)$$

□

As a consequence of Theorem A.2 we obtain that a window function g whose ambiguity function has no zeros allows phase retrieval:

Theorem A.3. *Let $g \in \mathcal{S}(\mathbb{R})$ be such that its ambiguity function $\mathcal{A}g$ has no zeros. Then for any $f, h \in \mathcal{S}'(\mathbb{R})$ with $|V_gf| = |V_g h|$ there exists $\alpha \in \mathbb{R}$ such that $h = e^{i\alpha} f$. If $f \in \mathcal{S}(\mathbb{R})$ then*

$$(A.7) \quad f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1} \left(S\mathcal{F} |V_gf|^2 / \mathcal{A}g \right) (t, t), \quad t \in \mathbb{R}$$

holds true, where S is defined by (A.1) and \mathcal{F}_2^{-1} denotes the inverse Fourier transform w.r.t. the second variable.

Proof. For $\Theta \in C_c^\infty(\mathbb{R}^2)$ so is the function $(S\mathcal{A}g)^{-1} \cdot \Theta$. By Theorem A.2 the tempered distributions $\mathcal{A}f$ and $\mathcal{A}h$ coincide on the dense subspace $C_c^\infty(\mathbb{R}^2)$ and are therefore equal. For arbitrary $\phi, \psi \in \mathcal{S}(\mathbb{R})$ we get

$$(\mathcal{A}f, S \circ \mathcal{F}_1 \circ T(\phi \otimes \bar{\psi})) = (f \otimes \bar{f}, \phi \otimes \bar{\psi}) = (f, \phi) \cdot \overline{(f, \psi)}$$

and further

$$(f, \phi) \cdot \overline{(f, \psi)} = (h, \phi) \cdot \overline{(h, \psi)}.$$

The choice $\psi = \phi$ implies $|(f, \phi)| = |(h, \phi)|$.

Let ψ be such that $(h, \psi) \neq 0$ then we obtain the equation

$$(h, \phi) = \frac{\overline{(f, \psi)}}{(h, \psi)} (f, \phi)$$

Since the fraction has modulus one the statement holds.

For $f \in \mathcal{S}(\mathbb{R})$ equation (A.3) implies

$$S\mathcal{F}|V_g f|^2 = \mathcal{A}f \cdot \mathcal{A}g$$

pointwise. Looking at (A.2) shows $f(t) \cdot \overline{f(t-x)} = \mathcal{F}_2^{-1} \mathcal{A}f(x, t)$. Combining these observations yields

$$f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1} \mathcal{A}f(t, t) = \mathcal{F}_2^{-1} \left(S\mathcal{F}|V_g f|^2 / \mathcal{A}g \right) (t, t).$$

□

Remark A.4. *If we restrict the signals f and h to be in $L^2(\mathbb{R})$ the ambiguity function $\mathcal{A}g$ can in fact vanish on a set of measure zero and the statement of Theorem A.3 still holds.*

By calculating the ambiguity function for the Gaussian we can conclude that the Gabor transform does phase retrieval:

Lemma A.5. *Let $\varphi(\cdot) = e^{-\pi \cdot^2}$ be the Gaussian. Then we have*

$$\mathcal{A}\phi(x, y) = c \cdot e^{-\pi i x y} \cdot e^{-\pi/2(x^2 + y^2)}$$

for some positive constant c .

Proof. Using the substitution $\tau = t - x/2$ gives

$$\begin{aligned} \mathcal{A}\phi(x, y) &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-\pi(t-x)^2} e^{-2\pi i y t} dt \\ &= \int_{\mathbb{R}} e^{-\pi(\tau+x/2)^2} e^{-\pi(\tau-x/2)^2} e^{-2\pi i y(\tau+x/2)^2} d\tau \\ &= e^{-\pi i y x} e^{-\pi/2 \cdot x^2} \int_{\mathbb{R}} e^{-2\pi \tau^2} e^{-2\pi i y \tau} d\tau \end{aligned}$$

It is well known that the Fourier transform of a Gaussian is again a Gaussian. We will still do the calculation to get the constants: Let $g(\cdot) := e^{-2\pi \cdot^2}$. Then

$$\widehat{g}'(y) = - \int_{\mathbb{R}} 2\pi i t \cdot e^{-2\pi t^2} e^{-2\pi i y t} dt = \frac{1}{2} \cdot \widehat{g}(y) = -\pi y \cdot \widehat{g}(y).$$

Therefore with $c = \int_{\mathbb{R}} g > 0$ we have $\widehat{g}(y) = c \cdot e^{-\pi/2 \cdot y^2}$. □

Applying Lemma A.5 with the value $\beta = \pi$ and the using the fact that $\mathcal{A}\varphi = V_\varphi \varphi$ implies that the ambiguity function of the Gaussian is again a (two-dimensional) Gaussian. Therefore by Theorem A.3 the Gabor transform does Phase retrieval:

Corollary A.6. *Let $\varphi(\cdot) := e^{-\cdot^2}$ be the Gauss window. Let $f, h \in \mathcal{S}'(\mathbb{R})$ be such that $|V_\varphi f| = |V_\varphi h|$ then there exists $\alpha \in \mathbb{R}$ such that $h = e^{i\alpha} f$.*

APPENDIX B. POINCARÉ AND CHEEGER CONSTANTS

In this section we relate the Poincaré constant to a geometric quantity, the so called Cheeger constant. This concept goes back to Jeff Cheeger [16]. We will further show that on a bounded domain which is equipped with a weight arising from a Gabor measurement there always holds a Poincaré inequality. Moreover we find that when the weight w is chosen to be a Gaussian there exists a finite constant $C > 0$ independent from $R > 0$ such that

$$C_{\text{poinc}}(p, B_R(0), w) \leq C.$$

B.1. Cheeger constant. The goal is to estimate the Poincaré constant from above in terms of the Cheeger constant. First recall the definition of Lipschitz and locally Lipschitz functions:

Definition B.1. Let $A \subset \mathbb{R}^d$ and $f : A \rightarrow \mathbb{R}$. Then f is called

(i) Lipschitz (on A) if there exists a $C > 0$ such that

$$|f(x) - f(y)| \leq C |x - y|$$

for all $x, y \in A$.

(ii) locally Lipschitz (on A) if f is Lipschitz on any compact subset of A .

Let \mathcal{H}^{d-1} denote the $(d - 1)$ -dimensional Hausdorff measure. A definition and some basic properties about Hausdorff measures as well as a proof of the following formula can be found in [24, Chapter 3.4.3.].

Theorem B.2 (Coarea formula or Change of Variables formula). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be Lipschitz and $g : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function. Then the restriction $g|_{f^{-1}\{t\}}$ is integrable w.r.t. \mathcal{H}^{d-1} for almost all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}^d} g(x) |\nabla f(x)| \, dx = \int_{\mathbb{R}} \int_{f^{-1}\{t\}} g(s) \, d\mathcal{H}^{d-1}(s) \, dt.$$

We will need the Coarea formula to hold under slightly different assumptions:

Lemma B.3. Let $D \subset \mathbb{R}^d$ be an open set and w a nonnegative, integrable function on D . Let u be a measurable and realvalued function and assume that there exists a set $E \subset D$ such that

- (i) u restricted to $D \setminus E$ is locally Lipschitz,
- (ii) there exists a sequence $D_1 \subset D_2 \subset \dots \subset D \setminus E$ of bounded open sets such that $\bigcup_n D_n = D \setminus E$ and $\overline{D_n} \subset D \setminus E$ for all $n \in \mathbb{N}$,
- (iii) w vanishes for \mathcal{H}^{d-1} almost all $x \in E$.

Then

$$\int_D w(x) |\nabla u(x)| \, dx = \int_{\mathbb{R}} \int_{u^{-1}\{t\}} w(s) \, d\mathcal{H}^{d-1}(s) \, dt.$$

Proof. By extending w by 0 outside of D we consider w as a function defined on the whole space \mathbb{R}^d . By assumptions (i) and (ii) the restriction of u on the compact set $\overline{D_n}$ is Lipschitz and thus also u restricted to D_n is Lipschitz and therefore has an extension u_n which is Lipschitz on \mathbb{R}^d (compare [24, Chapter 3.1.]). With $w_n := w \cdot \chi_{D_n}$ we can apply Theorem B.2 on $f = u_n$ and $g = w_n$ for any n and obtain

$$(B.1) \quad \int_{\mathbb{R}^d} w_n(x) |\nabla u_n(x)| \, dx = \int_{\mathbb{R}} \int_{u_n^{-1}\{t\}} w_n(s) \, d\mathcal{H}^{d-1}(s) \, dt.$$

Since $u_n(x) = u(x)$ for $x \in D_n$ and D_n is open Rademacher's theorem implies $\nabla u_n(x) = \nabla u(x)$ makes sense for almost all $x \in D_n$.

By monotone convergence we can take the limit $n \rightarrow \infty$ in the left hand side of equation (B.1):

$$(B.2) \quad \lim_n \int_{\mathbb{R}^d} w_n(x) |\nabla u_n(x)| \, dx = \lim_n \int_{D_n} w(x) |\nabla u(x)| \, dx = \int_{D \setminus E} w(x) |\nabla u(x)| \, dx.$$

Assumption (iii) in particular implies that w vanishes for almost every $x \in E$ (w.r.t. d -dimensional Lebesgue measure). Therefore we can conclude

$$\lim_n \int_{\mathbb{R}^d} w_n(x) |\nabla u_n(x)| \, dx = \int_D w(x) |\nabla u(x)| \, dx.$$

Let us have a closer look at the right hand side of equation (B.1): Since u_n and u coincide on D_n we have

$$\begin{aligned} \lim_n \int_{\mathbb{R}} \int_{u_n^{-1}\{t\}} w_n(s) d\mathcal{H}^{d-1}(s) dt &= \lim_n \int_{\mathbb{R}} \int_{u_n^{-1}\{t\} \cap D_n} w(s) d\mathcal{H}^{d-1}(s) dt \\ &= \lim_n \int_{\mathbb{R}} \int_{u^{-1}\{t\} \cap D_n} w(s) d\mathcal{H}^{d-1}(s) dt \\ &= \int_{\mathbb{R}} \int_{u^{-1}\{t\} \cap (D \setminus E)} w(s) d\mathcal{H}^{d-1}(s) dt, \end{aligned}$$

where we used again monotone convergence. Assumption (iii) tells us that E is a zero set w.r.t. $w(\cdot) d\mathcal{H}^{d-1}(\cdot)$ and so we finally obtain the claimed equality. \square

We consider now a domain $D \subset \mathbb{R}^2$ which can be bounded or unbounded, together with a nonnegative and integrable weight w . We define a measure μ on D by

$$(B.3) \quad \mu(C) := \int_C w(x) dx, \quad C \subset D \text{ measurable w.r.t. Lebesgue measure.}$$

For $A \subset D$ a 1-dimensional manifold we use the notation

$$(B.4) \quad \nu(A) := \int_A w(s) d\sigma(s),$$

where σ denotes the surface measure on A . Further let us define a system of subsets of D by

$$(B.5) \quad \mathcal{C} = \mathcal{C}(D, w) := \left\{ \emptyset \neq C \subset D \text{ open} : \partial C \cap D \text{ is a 1-dim. manifold and } \mu(C) \leq \frac{1}{2} \mu(D) \right\}.$$

The Cheeger constant h w.r.t. D and w is defined by

$$(B.6) \quad h = h(D, w) := \inf_{C \in \mathcal{C}} \frac{\nu(\partial C \cap D)}{\mu(C)}.$$

Remark B.4. *If D is not connected, there is a component C of D such that $C \in \mathcal{C}$. Since $\partial C \cap D = \emptyset$ we clearly have $h = 0$ in that case.*

For a measurable, realvalued function u on D we denote the sub-, super- and levelsets of u by

$$\mathcal{S}_t := \{x \in D : u(x) < t\}, \quad \mathcal{U}_t := \{x \in D : u(x) > t\} \quad \text{and} \quad \mathcal{A}_t := \{x \in D : u(x) = t\}.$$

In Proposition B.5 we will now establish a first connection between the Cheeger constant and a Poincare-type inequality:

Proposition B.5. *Let $D \subset \mathbb{R}^2$ be a domain and w a weight on D . Let h denote the Cheeger constant of D and w and let μ be defined as in Equation (B.3). Let u be a nonnegative function on D and assume there exists $E \subset D$ such that the Conditions (i)-(iii) of Lemma B.3 hold. Let \mathcal{U}_t and \mathcal{A}_t denote the super- and the levelsets w.r.t. u . Further suppose that*

- (1) $\mu(\mathcal{U}_0) \leq \frac{1}{2} \mu(D)$,
- (2) \mathcal{A}_t is a 1-dimensional manifold for almost all $t > 0$,
- (3) $\mathcal{A}_t = \partial \mathcal{U}_t \cap D$ for almost all $t > 0$,
- (4) \mathcal{U}_t is open for almost all $t > 0$.

Then the inequality

$$(B.7) \quad h \int_D u(x) d\mu(x) \leq \int_D |\nabla u(x)| d\mu(x)$$

holds true.

Proof. Let $\Gamma \subset (0, \infty)$ be such that

$$\mathcal{A}_t \text{ is a 1-dim. manifold} \quad \text{and} \quad \mathcal{A}_t = \partial \mathcal{U}_t \cap D \quad \text{for all } t \in \Gamma$$

and that $(0, \infty) \setminus \Gamma$ is of Lebesgue measure zero.

Applying Lemma B.3 and using the fact that \mathcal{H}^1 coincides with the surface measure on 1-dimensional manifolds gives us

$$\begin{aligned} \int_D |\nabla u(x)| w(x) dx &= \int_0^\infty \int_{\mathcal{A}_t} w(s) d\mathcal{H}^1(s) dt = \int_\Gamma \int_{\mathcal{A}_t} w(s) d\mathcal{H}^1(s) dt \\ &= \int_\Gamma \nu(\mathcal{A}_t) dt = \int_\Gamma \nu(\partial\mathcal{U}_t \cap D) dt, \end{aligned}$$

where ν is defined as in (B.4). Now we can estimate

$$\begin{aligned} \int_D |\nabla u(x)| d\mu(x) &\geq \int_{\Gamma \cap \{t: \mu(\mathcal{U}_t) > 0\}} \frac{\nu(\partial\mathcal{U}_t \cap D)}{\mu(\mathcal{U}_t)} \cdot \mu(\mathcal{U}_t) dt \\ &\geq h \int_{\Gamma \cap \{t: \mu(\mathcal{U}_t) > 0\}} \mu(\mathcal{U}_t) dt = h \int_{(0, \infty)} \mu(\mathcal{U}_t) dt = h \int_D u(x) d\mu(x) \end{aligned}$$

□

Before we prove Theorem B.7 we need one more lemma:

Lemma B.6. *Let $D \subset \mathbb{R}^2$ be a domain, w a weight on D and $p \in [1, \infty)$. Then for any $F = u + iv \in L^p(D, w) \cap L^1(D, w)$ and any $a, b \in \mathbb{R}$ we have*

$$\|F - F_D^w\|_{L^p(D, w)}^p \leq \max\{2^{3p/2-1}, 2^p\} \left(\|u - a\|_{L^p(D, w)}^p + \|v - b\|_{L^p(D, w)}^p \right).$$

Proof. Step 1. Assume $f \in L^p(D, w) \cap L^1(D, w)$ is a realvalued function with $f_D^w = 0$. We first show that

$$(B.8) \quad \|f\|_{L^p(D, w)} \leq \|f + a\|_{L^p(D, w)}$$

for arbitrary $a \in \mathbb{R}$. This statement can be found in [22] but we still give a proof.

W.l.o.g. we may assume that a is positive. Let μ be defined as in (B.3). Then we have

$$\int_{\{x \in D: f(x) > 0\}} |f(x)|^p d\mu(x) \leq \int_{\{x \in D: f(x) > 0\}} |f(x) + a|^p d\mu(x)$$

and

$$\int_{\{x \in D: f(x) < -2a\}} |f(x) + a|^p d\mu(x) \geq 2^{-p} \int_{\{x \in D: f(x) < -2a\}} |f(x)|^p d\mu(x).$$

Furthermore, using $\int_{\{x \in D: f(x) < 0\}} |f(x)| d\mu(x) = \int_{\{x \in D: f(x) \geq 0\}} |f(x)| d\mu(x)$ we obtain

$$\begin{aligned} \int_{\{x \in D: -2a \leq f(x) \leq 0\}} |f(x)|^p d\mu(x) &\leq (2a)^{p-1} \int_{\{x \in D: -2a \leq f(x) \leq 0\}} |f(x)| d\mu(x) \\ &\leq (2a)^{p-1} \int_{\{x \in D: f(x) > 0\}} |f(x)| d\mu(x) \\ &\leq 2^{p-1} \int_{\{x \in D: f(x) > 0\}} |f(x) + a|^p d\mu(x). \end{aligned}$$

Combining these estimates and since $2^{p-1} + 1 \leq 2^p$ we see that the inequality (B.8) holds.

Step 2. For any nonnegative numbers α, β we have

$$(\alpha^2 + \beta^2)^{p/2} \leq \max\{2^{p/2-1}, 1\} (\alpha^p + \beta^p).$$

Using this inequality and applying (B.8) on u and v respectively we obtain

$$\begin{aligned} \|F - F_D^w\|_{L^p(D, w)}^p &= \int_D ((u(x) - u_D^w)^2 + (v(x) - v_D^w)^2)^{p/2} d\mu(x) \\ &\leq \max\{2^{p/2-1}, 1\} \cdot 2^p \int_D |u(x) - u_D^w|^p + |v(x) - v_D^w|^p d\mu(x) \\ &\leq \max\{2^{p/2-1}, 1\} \cdot 2^p \int_D |u(x) - a|^p + |v(x) - b|^p d\mu(x). \end{aligned}$$

□

Finally we establish a weighted Poincaré inequality for certain meromorphic functions. Looking at (B.7) it is not surprising that the corresponding Poincaré constant can be controlled by the reciprocal of the Cheeger constant:

Theorem B.7. *Let $D \subset \mathbb{R}^2$ be a domain, w a weight on D and $p \in [1, \infty)$. Let h denote the Cheeger constant of D and w . Assume that h is positive. Then a weighted Poincaré inequality holds and $C_{\text{poinc}}(D, w, p) \leq \frac{4p}{h}$, i.e.*

$$(B.9) \quad \|F - F_D^w\|_{L^p(D, w)} \leq \frac{4p}{h} \|\nabla F\|_{L^p(D, w)}$$

for all $F \in W^{1,p}(D, w) \cap L^1(D, w) \cap \mathcal{M}(D)$.

Proof. Let u and v denote real and imaginary part of F and let μ be defined as in (B.3). Let m_u be a median of u , i.e.:

$$\mu(\mathcal{U}_{m_u}^u) \leq \frac{1}{2}\mu(D) \quad \text{and} \quad \mu(\mathcal{S}_{m_u}^u) \leq \frac{1}{2}\mu(D),$$

where \mathcal{U}_t^u and \mathcal{S}_t^u denote super- and sublevelsets of u .

To see why such a number exists observe that the mapping $t \mapsto \mu(\mathcal{U}_t^u)$ is continuous and takes its values in the interval $[0, \mu(D)]$. Therefore there has to be a number m_u such that $\mu(\mathcal{U}_{m_u}^u) = \frac{1}{2}\mu(D)$ and since

$$\mu(\mathcal{S}_{m_u}^u) \leq \mu(D) - \mu(\mathcal{U}_{m_u}^u) = \frac{1}{2}\mu(D)$$

m_u is a median of u .

Let m_v be a median of v , defined analogously. By Lemma B.6 we get

$$\|F - F_D^w\|_{L^p(D, w)}^p \leq \max\{2^{3p/2-1}, 2^p\} \left(\|u - m_u\|_{L^p(D, w)}^p + \|v - m_v\|_{L^p(D, w)}^p \right).$$

We will only estimate the first term in the right hand side of the inequality above. The second one can be dealt with in the exact same way.

Let u_+ and u_- denote the positive and the negative part of the function $u - m_u$. Then we have

$$\psi := (u - m_u) \cdot |u - m_u|^{p-1} = u_+^p - u_-^p$$

and

$$\|\psi\|_{L^p(D, w)}^p = \|u - m_u\|_{L^p(D, w)}^p = \|u_+^p\|_{L^1(D, w)} + \|u_-^p\|_{L^1(D, w)}.$$

We want to apply Proposition B.5 on both u_+^p and u_-^p .

First we check that the Assumptions (i)-(iii) of Lemma B.3 are satisfied:

ad (i). Let E denote the set of points $x \in D$ such that x is a pole of F . The restriction of ψ on $D \setminus E$ is a smooth function and therefore locally Lipschitz. Since for $x, y \in D \setminus E$ we have

$$|u_{\pm}^p(x) - u_{\pm}^p(y)| \leq |\psi(x) - \psi(y)|$$

also u_{\pm}^p and u_{\pm}^p are locally Lipschitz on $D \setminus E$.

ad (ii). Obviously, setting

$$D_n := (D \cap B_n(0)) \setminus \bigcup_{x \in E} \overline{B_{1/n}(x)}$$

for $n \in \mathbb{N}$ is a valid choice.

ad (iii). Since E is a discrete set this property holds for any weight w .

Next we verify that also the Assumptions (1)-(4) of Proposition B.5 hold for u_+^p and u_-^p : We write \mathcal{U}_t^+ , \mathcal{U}_t^- and \mathcal{U}_t^ψ for the superlevelsets of u_+^p , u_-^p and ψ and accordingly for the sub- and levelsets of the same functions.

ad (1). This property follows directly from the definition of ψ .

ad (4). Since ψ is continuous on $D \setminus E$ its super- and sublevelsets are open in $D \setminus E$. The set of poles E is discrete, therefore any open set in $D \setminus E$ also is open in D . The property then follows by the observation that for $t > 0$ the following two equalities hold:

$$\mathcal{U}_t^+ = \mathcal{U}_t^\psi \quad \text{and} \quad \mathcal{U}_t^- = \mathcal{S}_{-t}^\psi.$$

ad (3). Let $x \in \partial\mathcal{U}_t^\psi \cap D$, then for any $\varepsilon > 0$ the ball $B_\varepsilon(x)$ contains points y, y' such that $\psi(y) > t$ and $\psi(y') \leq t$. By continuity of ψ we infer $\psi(x) = t$, i.e. $x \in \mathcal{A}_t^\psi$. Thus the inclusion $\partial\mathcal{U}_t^\psi \cap D \subset \mathcal{A}_t$ holds for all t .

Assume next that the set $J := \{t \in \mathbb{R} : \mathcal{A}_t^\psi \supsetneq \partial\mathcal{U}_t^\psi \cap D\}$ is of positive measure. For $t \in J$ there exists an $x \in \mathcal{A}_t^\psi$ such that $x \notin \partial\mathcal{U}_t^\psi \cap D$. Therefore for sufficiently small $\varepsilon > 0$ we have $B_\varepsilon(x) \cap \mathcal{U}_t^\psi = \emptyset$, so ψ has a local maximum in x , i.e. $\nabla\psi(x) = 0$. Sard's theorem tells us that $\psi(\{x : \nabla\psi(x) = 0\}) \supseteq J$ is a zero set, which contradicts our assumption. This proves that $\mathcal{A}_t^\psi = \partial\mathcal{U}_t^\psi \cap D$ for almost all $t \in \mathbb{R}$. A similar argument shows that also $\mathcal{A}_t^\psi = \partial\mathcal{S}_t^\psi \cap D$ holds true for almost all t .

The observation

$$\partial\mathcal{U}_t^+ \cap D = \partial\mathcal{U}_t^\psi \cap D = \mathcal{A}_t^\psi = \mathcal{A}_t^+ \quad \text{and} \quad \partial\mathcal{U}_t^- \cap D = \partial\mathcal{S}_{-t}^\psi \cap D = \mathcal{A}_{-t}^\psi = \mathcal{A}_t^-$$

concludes the argument.

ad (2). Clearly it suffices to show that \mathcal{A}_t^ψ is a 1-dimensional manifold for almost all $t \in \mathbb{R}$.

Let $t \notin J$ and $x \in \mathcal{A}_t^\psi$ then by the implicit function theorem \mathcal{A}_t^ψ is locally the graph of a smooth function which implies that both \mathcal{A}_t^+ and \mathcal{A}_t^- are 1-dimensional manifolds for almost all $t > 0$.

Proposition B.5 can now be applied on u_+^p and u_-^p :

$$\begin{aligned} h \|u - m_u\|_{L^p(D,w)}^p &= h \int_D u_+^p(x) + u_-^p(x) \, d\mu(x) \\ &\leq \int_D |\nabla u_+^p(x)| + |\nabla u_-^p(x)| \, d\mu(x) = \int_D |\nabla\psi(x)| \, d\mu(x). \end{aligned}$$

We want to show that $\|u - m_u\|_{L^p(D,w)} \leq p/h \cdot \|\nabla u\|_{L^p(D,w)}$:

For $p = 1$ we are done. For any $p > 1$ we have

$$|\nabla\psi(x)| \leq p |u(x) - m_u|^{p-1} |\nabla u(x)|;$$

using Hölder's inequality we obtain

$$\begin{aligned} \int_D |\nabla\psi(x)| \, d\mu(x) &\leq p \int_D |u(x) - m_u|^{p-1} \cdot |\nabla u(x)| \, d\mu(x) \\ &\leq p \left(\int_D |u(x) - m_u|^{(p-1)p'} \, d\mu(x) \right)^{1/p'} \cdot \left(\int_D |\nabla u(x)|^p \, d\mu(x) \right)^{1/p} \\ &= p \|u - m_u\|_{L^p(D,w)}^{p-1} \|\nabla u\|_{L^p(D,w)}. \end{aligned}$$

Note that by Cauchy Riemann equations we have for any $x \in D \setminus E$ that

$$|\nabla F(x)| = \sqrt{2} |\nabla u(x)| = \sqrt{2} |\nabla v(x)|.$$

Combining our estimates we finally obtain

$$\begin{aligned} \|F - F_D^w\|_{L^p(D,w)}^p &\leq \max\{2^{3p/2-1}, 2^p\} \left(\frac{p}{h}\right)^p \int_D |\nabla u(x)|^p + |\nabla v(x)|^p \, d\mu(x) \\ &= \max\{2^{3p/2-1}, 2^p\} \left(\frac{p}{h}\right)^p 2^{1-p/2} \|\nabla F\|_{L^p(D,w)}^p. \end{aligned}$$

Since $\max\{2^p, 2^{p/2+1}\} < 4$ Inequality (B.9) holds true. \square

B.2. Positivity of the Cheeger Constant for Finite Domains. The goal of this section is to prove the following statement.

Theorem B.8. *Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Further, let $f \in \mathcal{S}'(\mathbb{R})$ such that $V_\varphi f$ has no zeros on ∂D and $1 \leq p < \infty$. Then*

$$h_{p,D}(f) > 0.$$

To this end we will need the fact that in the definition of the Cheeger constant it suffices to consider connected sets C .

Lemma B.9. *Let w be a nonnegative weight on a domain $D \subset \mathbb{R}^2$ and let h denote the Cheeger constant (see Equation (B.6)). Let \mathcal{C} be defined as in Equation (B.5). Then*

$$(B.10) \quad h = \inf_{\substack{C \in \mathcal{C} \\ C \text{ connected}}} \frac{\int_{\partial C} w \, d\sigma}{\int_C w}.$$

Proof. The inequality " \leq " in equation (B.10) is trivial.

It is an easy exercise to see that for positive numbers $(a_l)_{l \in \mathbb{N}}, (b_l)_{l \in \mathbb{N}}$ the following inequality holds:

$$(B.11) \quad \frac{\sum_l a_l}{\sum_l b_l} \geq \inf_l \frac{a_l}{b_l}.$$

For arbitrary $C \in \mathcal{C}$ – since C is open – we can write C as a disjoint union of at most countably many connected, open sets C_l , $l \in \mathbb{N}$. Applying Inequality (B.11) on $a_l = \int_{\partial C_l \cap D} w \, d\sigma$ and $b_l = \int_{C_l} w$ gives

$$\frac{\int_{\partial C \cap D} w \, d\sigma}{\int_C w} = \frac{\sum_l \int_{\partial C_l \cap D} w \, d\sigma}{\sum_l \int_{C_l} w} \geq \inf_l \frac{\int_{\partial C_l \cap D} w \, d\sigma}{\int_{C_l} w} \geq \inf_{\substack{C \in \mathcal{C} \\ C \text{ connected}}} \frac{\int_{\partial C \cap D} w \, d\sigma}{\int_C w}.$$

Taking the infimum over all $C \in \mathcal{C}$ yields the desired result. \square

First we will show that Theorem B.8 holds in the case $w \equiv 1$. Recall that for $D \subset \mathbb{R}^d$ open a function $u \in L^1(D)$ is of bounded variation ($u \in BV(D)$) if

$$\sup_{\substack{\phi \in C_c^1(D, \mathbb{R}^d) \\ |\phi| \leq 1}} \int_D u \operatorname{div} \phi \leq \infty,$$

where $C_c^1(D, \mathbb{R}^d)$ denotes the set of continuously differentiable functions from D to \mathbb{R}^d whose support is a compact subset of D . We will make use of the following properties of functions of bounded variation (see [24, Chapter 5] for details): For any $u \in BV(D)$ there exists a Radon measure μ on D and a μ -measurable function $\sigma : D \rightarrow \mathbb{R}^d$ such that $|\sigma| = 1$ μ -a.e. and

$$\int_D u \operatorname{div} \phi = - \int_D \phi \cdot \sigma \, d\mu, \quad \text{for all } \phi \in C_c^1(D, \mathbb{R}^d).$$

We will use the notation $|\nabla u| = \mu$ in the following. Equipped with the norm

$$\|\cdot\|_{BV(D)} := \|\cdot\|_{L^1(D)} + |\nabla \cdot|(D)$$

$BV(D)$ becomes a Banach space.

To show that Theorem B.8 holds in the case $w \equiv 1$ let us consider the functional

$$(B.12) \quad \mathcal{F} : u \mapsto |\nabla u|(D), \quad u \in BV(D)$$

and the minimization problem

$$(B.13) \quad \text{minimize } \mathcal{F} \text{ in } V := \{u \in BV(D) : \|u\|_{L^1(D)} = 1, |\operatorname{supp} u| \leq \frac{1}{2}|D|\}.$$

Proving that $h(D, 1) > 0$ by showing that (B.13) has a solution is inspired by [14, 33] where they considered a slightly different problem, namely proving positivity of the quantity

$$\inf_{C \subset D \text{ open}, \partial C \text{ smooth}} \frac{\ell(\partial C)}{|C|}.$$

Note that this situation corresponds to estimating Poincaré constants for functions which satisfy Dirichlet conditions:

$$\|u\|_{L^p(D)} \leq C \|\nabla u\|_{L^p(D)}, \quad \text{for all } u \text{ vanishing on } \partial D.$$

Proposition B.10. *Let $D \subset \mathbb{R}^2$ be a bounded and connected Lipschitz domain and let $\mathcal{C} := \{C \subset D : \partial C \cap D \text{ is smooth}, |C| \leq \frac{1}{2}|D|\}$. Let \mathcal{F} and V be defined as in (B.12), (B.13). Then*

(i) There exists $u^* \in V$ such that

$$\mathcal{F}(u^*) = \inf_{u \in V} \mathcal{F}(u) > 0.$$

(ii) For any $C \in \mathcal{C}$ we have $\mathcal{F}(\chi_C) = \ell(\partial C \cap D)$.

Proof. ad (i). Choose a minimizing sequence $(u_n)_{n \in \mathbb{N}} \subset V$, i.e.

$$\lim_n \mathcal{F}(u_n) = \inf_{u \in V} \mathcal{F}(u).$$

Then $(u_n)_{n \in \mathbb{N}}$ is bounded in $BV(D)$ and therefore by [24, Chapter 5.2.3, Theorem 4] there is a subsequence which we still call $(u_n)_{n \in \mathbb{N}}$ and a $u^* \in BV(D)$ such that $u_n \rightarrow u^*$ in $L^1(D)$. Obviously $\|u^*\|_{L^1(D)} = 1$.

To show that $u^* \in V$ it remains to verify that $|\text{supp } u^*| \leq \frac{1}{2}|D|$.

Assume that $|\text{supp } u^*| > \frac{1}{2}|D|$. Then for $\varepsilon > 0$ sufficiently small we have $|D_\varepsilon| > \frac{1}{2}|D|$ where we set $D_\varepsilon := \{x : |u^*(x)| > \varepsilon\}$. Since L^1 -convergence implies almost uniformly convergence there must be a set $E_\varepsilon \subset D$ such that

$$|E_\varepsilon| < \frac{1}{2}|D| - |D_\varepsilon| \quad \text{and} \quad u_n \rightarrow u^* \text{ uniformly on } D \setminus E_\varepsilon.$$

In particular convergence is uniformly on $D_\varepsilon \setminus E_\varepsilon$. Therefore there exists $N \in \mathbb{N}$ such that $|u_n(x) - u^*(x)| < \varepsilon/2$ for all $x \in D_\varepsilon \setminus E_\varepsilon$. Applying the inverse triangle inequality yields $|u_n(x)| > \varepsilon/2$ for these n and x . By construction we have $|D_\varepsilon \setminus E_\varepsilon| > \frac{1}{2}|D|$ which contradicts the assumption that $(u_n)_{n \in \mathbb{N}} \subset V$.

By [24, Chapter 5.2.1, Theorem 1] the functional \mathcal{F} is lower semicontinuous w.r.t. L^1 -norm. Thus we obtain

$$\inf_{u \in V} \mathcal{F}(u) \leq \mathcal{F}(u^*) \leq \liminf_n \mathcal{F}(u_n) = \lim_n \mathcal{F}(u_n) = \inf_{u \in V} \mathcal{F}(u).$$

What remains is to show that $\mathcal{F}(u^*)$ is strictly positive:

Assume $\mathcal{F}(u^*) = 0$. Then $|\nabla u^*|$ is the zero measure which by [4, Proposition 3.2(a)] amounts to u^* being constant on the connected set D . Since there is no constant function u^* such that

$$|\text{supp } u^*| \leq \frac{1}{2}|D| \quad \text{and} \quad \|u^*\|_{L^1(D)} = 1,$$

we have a contradiction. **ad (ii).** Any $C \in \mathcal{C}$ can be extended to a set $C' \supseteq C$ such that

$$C' \cap D = C \quad \text{and} \quad \partial C' \text{ is smooth.}$$

Note that if $\partial C \subset D$ we can choose $C' = C$. Since C' has smooth boundary the length of $\partial C' \cap D$ can be measured by the total variation of the gradient of $\chi_{C'}$, i.e.

$$|\nabla \chi_{C'}|(D) = \mathcal{H}^{n-1}(\partial C' \cap D),$$

see [24, Chapter 5.1, Example 2]. Therefore we obtain

$$|\nabla \chi_C|(D) = |\nabla \chi_{C'}|(D) = \mathcal{H}^{n-1}(\partial C' \cap D) = \mathcal{H}^{n-1}(\partial C \cap D) = \ell(\partial C \cap D).$$

□

As a direct consequence we obtain the following theorem.

Theorem B.11. *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then*

$$h(D, 1) > 0,$$

where $h(D, 1)$ is defined as in (B.6).

Proof. We use the notation of Proposition B.10. Since $\mathcal{F}(\chi_C) = \ell(\partial C \cap D)$ for any $C \in \mathcal{C}$ we obtain

$$\frac{\ell(\partial C \cap D)}{|C|} = \frac{\mathcal{F}(\chi_C)}{\|\chi_C\|_{L^1(D)}} = \mathcal{F}\left(\|\chi_C\|_{L^1(D)}^{-1} \cdot \chi_C\right) \geq \mathcal{F}(u^*) > 0.$$

□

Let us get to the general case where w emerges from an Gabor measurement, i.e. $w = |V_\varphi f|^p$. On a bounded domain D one can construct a rather simple, equivalent weight $w_r \sim w$ which we will analyze:

Proof of Theorem B.8. The idea of the proof is to construct a weight that is equivalent to $|V_\varphi f|^p$ which locally is either constant or looks like $z \mapsto |z|^q$ for some positive number q . We then seek to exploit results on Cheeger constants for the case $w \equiv 1$ as well as isoperimetric inequalities w.r.t weights of the form $|z|^p$.

Up to multiplication with a nonzero function η and a reflection in the plane the function $V_\varphi f$ is an entire function (see Theorem 2.4), therefore can only have a finite number of zeros $(\zeta_i)_{i=1}^N$ in D . We have for every $z = x + iy \in D$ that

$$(B.14) \quad V_\varphi f(z) = \prod_{i=1}^N (z - \zeta_i)^{m_i} \cdot g(z),$$

where $m_i \in \mathbb{N}$ denotes the multiplicity of the zero ζ_i and g is a continuous function without zeros on \overline{D} . Due to the compactness of \overline{D} the function $|g|$ assumes a nonzero minimum and a maximum on \overline{D} .

For $r > 0$ let $D_0^r := \bigcup_{i=1}^N B_r(\zeta_i)$ and define

$$w_r(z) := \begin{cases} 1 & z \in D \setminus D_0^r \\ |z - \zeta_i|^{m_i \cdot p} & z \in B_r(\zeta_i) \text{ for some } i \in \{1, \dots, N\} \end{cases}$$

This definition is ambiguous if $z \in B_r(\zeta_i) \cap B_r(\zeta_j)$ for $i \neq j$. But clearly there is a $r_0 > 0$ such that all these intersections will be empty for $r < r_0$.

For $\delta > 0$ let us define the set

$$D_\delta := \{x \in D : \text{dist}(x, \partial D) < \delta\}.$$

Obviously for $\delta \rightarrow 0$ we have that $|D_\delta| \rightarrow 0$. Since $V_\varphi f$ has no zeros on the boundary ∂D we can choose δ such that

$$|D_\delta| \leq \frac{1}{2} |D| \quad \text{and} \quad \bigcup_{i=1}^N B_\delta(\zeta_i) \cap D_\delta = \emptyset.$$

From now on we consider the weight w_r for fixed $0 < r < \min\{r_0, \delta, 1\}$. Since $w_r \sim |V_\varphi f|^p$ in D and by Lemma B.9 it suffices to show that the quantity

$$\frac{\int_{\partial C} w_r \, d\sigma}{\int_C w_r}$$

can be uniformly bounded from below by a positive constant for all connected, open sets $C \subset D$ with smooth boundary such that $\int_C |V_\varphi f|^p \leq \frac{1}{2} \int_C |V_\varphi f|^p$.

Let us fix a C with these properties. We can now look at the following two cases separately:

Case A: $\ell(\partial C \cap D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)) \geq r/2$,

Case B: $\ell(\partial C \cap D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)) < r/2$.

Within Case B we further distinguish

Case B.1: $\partial C \cap \partial D \neq \emptyset$,

Case B.2: $\partial C \cap \partial D = \emptyset$ and $C \cap B_{r/2}(\zeta_i) \neq \emptyset$ for some i ,

Case B.3: $\partial C \cap \partial D = \emptyset$ and $C \cap \bigcup_{i=1}^N B_{r/2}(\zeta_i) = \emptyset$.

ad A. Let $m := \max_i \{m_i\}$. Since $w_r(z) \geq (r/2)^{mp}$ for $z \in D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)$ we get the estimate

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq \frac{(r/2)^{mp+1}}{\int_D w_r}.$$

ad B.1. By construction we have $C \subset D_\delta$. Since $w_r \equiv 1$ in D_δ and $|C| \leq \frac{1}{2} |D|$ we obtain

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq h(D, 1).$$

By Theorem B.11 $h(D, 1)$ is positive.

ad B.2. We can infer that $C \subset B_r(\zeta_i)$ for suitable i and let $q = m_i p$. Let us assume for simplicity that $\zeta_i = 0$. Let μ and ν be defined as in (B.3) and (B.4) w.r.t. the weight $|\cdot|^{m_i p}$. Since

$$\mu(B_{2r}(2r)) \geq \mu(B_r(0)) \geq \mu(C)$$

by continuity of $s \mapsto \mu(B_s(s))$ there exists $s \in (0, 2r]$ such that $\mu(B_s(s)) = \mu(C)$. We can now appeal to a result about weighted isoperimetric problems ([21, see Theorem 3.16]) which guarantees $\nu(\partial B_s(s)) \leq \nu(\partial C)$. Since

$$\nu(B_s(s)) \geq s\pi(\sqrt{2}s)^q \quad \text{and} \quad \mu(B_s(s)) \leq 2s^2\pi(2s)^q$$

we obtain

$$\frac{\nu(\partial C)}{\mu(C)} \geq \frac{\nu(\partial B_s(s))}{\mu(B_s(s))} \gtrsim s^{-1}$$

The function s^{-1} is bounded from below by a positive constant on the interval $(0, 2r]$ and we are done in this case.

ad B.3. We estimate

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq \frac{(r/2)^{mp} \ell(\partial C)}{|C|} \geq \frac{(r/2)^{mp} \ell(\partial C)}{|D|^{1/2} |C|^{1/2}}$$

By the isoperimetric inequality the fraction $\ell(\partial C)/|C|^{1/2}$ has a positive lower bound independent from C . \square

B.3. Cheeger Constant of a Gaussian. In this subsection we will study the Cheeger constant of the Gaussian $\varphi = e^{-\pi \cdot^2}$ on disks centered at 0.

Theorem B.12. *For $p \in [1, \infty)$ there exists a constant $\delta > 0$, depending on p , but independent of $R > 0$ such that*

$$h_{p, B_R(0)}(\varphi) \geq \delta.$$

Proof. By Lemma A.5 there exists a positive constant r such that

$$|V_\varphi \varphi(x, y)|^p = r e^{-p\pi/2(x^2+y^2)}.$$

For $q \geq \pi/2$ let $w_q(x, y) := e^{-q(x^2+y^2)}$. Proving the statement amounts to showing that $h(B_R(0), w_q)$ is uniformly bounded away from zero for any fixed $q \geq \pi/2$. The restriction is however not necessary and it suffices to assume $q > 0$.

Let $\beta > 0$ be such that $\int_{\mathbb{R}^2} \beta w_q = 1$. For $C \subset \mathbb{R}^2$ and A a 1-dimensional manifold we will use the notations

$$\mu(C) := \beta \int_C w_q \quad \text{and} \quad \nu(A) := \beta \int_A w_q \, d\sigma,$$

where σ denotes the surface measure on A . For $R > 0$ let us define

$$\begin{aligned} \mathcal{C}_R &:= \{C \subset B_R(0) \text{ open, connected} : \partial C \cap B_R(0) \text{ is smooth and } \mu(C) \leq \frac{1}{2} \mu(B_R(0))\}, \\ \mathcal{C}_R^i &:= \{C \in \mathcal{C}_R : \partial C \cap \partial B_R(0) = \emptyset\}, \\ \mathcal{C}_R^b &:= \{C \in \mathcal{C}_R : \partial C \cap \partial B_R(0) \neq \emptyset\}. \end{aligned}$$

Clearly $\mathcal{C}_R = \mathcal{C}_R^i \cup \mathcal{C}_R^b$.

Our proof will heavily rely on the fact that on probability spaces with log-concave measures a isoperimetric inequality holds true ([9]), i.e. there exists $c > 0$ such that

$$(B.15) \quad \nu(\partial C) \geq cI(\mu(C)) \quad \text{for all } C \subset \mathbb{R}^2 \text{ with smooth boundary,}$$

where $I := \gamma \circ \Gamma^{-1}$ with

$$\gamma(t) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \Gamma(t) := \int_{-\infty}^t \gamma(s) \, ds.$$

The function $I : [0, 1] \rightarrow [0, \frac{1}{\sqrt{2\pi}}]$ is strictly positive on $(0, 1)$ and satisfies $I(0) = I(1) = 0$. An elementary calculation yields $I''(t) = -1/\gamma(\Gamma^{-1}(t)) \leq 0$, therefore I is concave. Since $I(\frac{1}{2}) = 1/\sqrt{2\pi}$ we have for any C such that $\mu(C) \leq \frac{1}{2}$ that

$$(B.16) \quad \nu(\partial C) \geq c \cdot I(\mu(C)) \geq c \cdot \sqrt{\frac{2}{\pi}} \mu(C).$$

Note that since \mathbb{R}^2 has no boundary Equation (B.16) tells us that $h(\mathbb{R}^2, w_q) \geq c\sqrt{\frac{2}{\pi}}$.

First let $C \in \mathcal{C}_R^i$. Since $\mu(C) \leq \frac{1}{2}\mu(B_R(0)) \leq \frac{1}{2}$ we have

$$\frac{\nu(\partial C \cap B_R(0))}{\mu(C)} = \frac{\nu(\partial C)}{\mu(C)} \geq c \cdot \sqrt{\frac{2}{\pi}}.$$

Thus it remains to look at sets $C \in \mathcal{C}_R^b$:

For $R > 0$ let $\rho = \rho(R) > 0$ be such that $\mu(B_\rho(0)) = \frac{3}{4}\mu(B_R(0))$. Since C is connected there is exactly one connected component A_0 of ∂C such that $A_0 \cap \partial B_R(0) \neq \emptyset$.

We will now have a look at the ratio $(R - \rho(R))/R$: Let $R < 1/\sqrt{q}$ and let $\alpha := \sqrt{\frac{3}{4e}}$, then

$$\mu(B_{\alpha R}(0)) = \beta \int_{B_{\alpha R}(0)} e^{-q|z|^2} dz \leq \beta \alpha^2 R^2 \pi = e \alpha^2 \cdot \beta e^{-1} R^2 \pi \leq e \alpha^2 \mu(B_R(0)) = \frac{3}{4} \mu(B_R(0))$$

and thus we obtain for $R \in (0, 1/\sqrt{q})$

$$\frac{R - \rho(R)}{R} \geq 1 - \alpha > 0.$$

Note that $(R - \rho(R))/R$ is a nonnegative and continuous function of $R > 0$. Since $\rho(R)$ converges to a finite limit for $R \rightarrow \infty$ there exists a $\kappa > 0$ that only depends on q such that

$$(B.17) \quad R - \rho(R) \geq \kappa R \quad \text{for all } R > 0.$$

We will now distinguish three cases:

Case A: $A_0 \cap B_\rho \neq \emptyset$,

Case B: $A_0 \cap B_\rho = \emptyset$ and $C_0 \cap B_\rho \neq \emptyset$,

Case C: $A_0 \cap B_\rho = \emptyset$ and $C_0 \cap B_\rho = \emptyset$.

In the first two cases we will show that there exists a positive λ that does not depend on R and C such that

$$\ell(A_0 \cap \partial B_R(0)) \leq \lambda \cdot \ell(A_0 \cap B_R(0)).$$

This implies that $\nu(A_0 \cap \partial B_R(0)) \leq \lambda \nu(A_0 \cap B_R(0))$ and therefore

$$\nu(A_0) = \nu(A_0 \cap B_R(0)) + \nu(A_0 \cap \partial B_R(0)) \leq (1 + \lambda) \nu(A_0 \cap B_R(0)).$$

Now we can estimate

$$\begin{aligned} \frac{\nu(\partial C \cap B_R(0))}{\mu(C)} &= \frac{\nu(A_0 \cap B_R(0)) + \nu(\partial C \setminus A_0)}{\mu(C)} \\ &\geq (1 + \lambda)^{-1} \frac{\nu(\partial C)}{\mu(C)} \geq (1 + \lambda)^{-1} c \sqrt{\frac{2}{\pi}}, \end{aligned}$$

where we used (B.16).

ad A. By (B.17) we have

$$\ell(A_0 \cap B_R(0)) \geq 2|R - \rho(R)| \geq 2\kappa R.$$

Since $\ell(A_0 \cap \partial B_R(0)) \leq 2R\pi$ we can choose $\lambda = \pi/\kappa$.

ad B. If C is such that $A_0 \cap \partial B_R(0)$ is not contained in any open halfplane H such that $0 \in \partial H$ then there has to be a connected component of $A_0 \cap B_R(0)$ with Euclidean length at least R .

If $A_0 \cap \partial B_R(0)$ however is contained in some halfplane H there is an arch Λ of $B_R(0)$ in H whose endpoints are contained in $A_0 \cap \partial B_R(0)$ and $\Lambda \supseteq A_0 \cap \partial B_R(0)$. Then $\ell(A_0 \cap B_R(0)) \geq l$, where l denotes the distance between the endpoints of Λ .

From basic geometry we know that $\ell(\Lambda)/2R = \arcsin(l/2R)$. Since $\arcsin(x) \leq \frac{\pi}{2}x$ for $x \in [0, 1]$ we obtain

$$\frac{\ell(A_0 \cap \partial B_R(0))}{\ell(A_0 \cap B_R(0))} \leq \frac{\ell(\Lambda)}{l} = \frac{\arcsin(l/2R)}{l/2R} \leq \frac{\pi}{2}.$$

ad C. Let C' denote the open and bounded set with boundary A_0 and set $E := C' \setminus \bar{C}$. Since $B_\rho(0)$ is contained in C' we have $\mu(C') \geq \frac{3}{4}\mu(B_R(0))$ and

$$\mu(E) = \mu(C' \setminus C) = \mu(C') - \mu(C) \geq \frac{3}{4}\mu(B_R(0)) - \frac{1}{2}\mu(B_R(0)) = \frac{1}{4}\mu(B_R(0)).$$

In case $\mu(E) \leq \frac{1}{2}$ we estimate using (B.16)

$$\nu(\partial C \cap B_R(0)) \geq \nu(\partial E) \geq c\sqrt{\frac{2}{\pi}}\mu(E) \geq c\sqrt{\frac{2}{\pi}}\frac{1}{4}\mu(B_R(0)) \geq c\sqrt{\frac{2}{\pi}}\frac{1}{2}\mu(C)$$

If $\mu(E) > \frac{1}{2}$ we can apply (B.16) on the unbounded set $E' := \mathbb{R}^2 \setminus \bar{E}$. Since $B_R(0) \supseteq E$ it follows that $\mu(B_R(0)) > \frac{1}{2}$ and we obtain

$$\begin{aligned} \nu(\partial C \cap B_R(0)) &\geq \nu(\partial E) = \nu(\partial E') \geq c\sqrt{\frac{2}{\pi}}\mu(E') \\ &= c\sqrt{\frac{2}{\pi}}(1 - \mu(E)) \geq c\sqrt{\frac{2}{\pi}}(2\mu(B_R(0)) - \mu(B_R(0))) \geq c\sqrt{\frac{2}{\pi}}\mu(C) \end{aligned}$$

□

APPENDIX C. SPECTRAL CLUSTERING ALGORITHM

In this section we provide some details on the partitioning algorithm used for our experiment in Section 2.2.3. Spectral clustering methods are based on relating optimal partitioning of a graph to the eigenvector corresponding to the second eigenvalue of the so called graph Laplacian.

Suppose we are given a set of finitely many points $V := \{v_1, \dots, v_l\} \subset \mathbb{R}^d$ and a similarity measure w on V , i.e.,

$$w : V \times V \rightarrow [0, \infty), \quad w \text{ is symmetric and } w(v_i, v_i) = 0 \text{ for all } i \in \{1, \dots, l\}.$$

A weighted, undirected graph G is associated to the pair (V, w) in a very natural way: The vertices of G are exactly the points v_1, \dots, v_l . Two vertices v_i, v_j are connected if and only if $w(v_i, v_j) > 0$ and in this case their connecting edge has weight $w(v_i, v_j)$. The matrix $W := (w(v_i, v_j))_{i,j=1}^l$ is called the *weight matrix* of G . For any $i \in \{1, \dots, l\}$ and $C \subset V$ we will use the notations

$$\begin{aligned} d_i &:= \sum_{j=1}^l W_{i,j}, \quad \text{vol}(C) := \sum_{j:v_j \in C} d_j, \\ \text{cut}_G(C) &:= \sum_{i,j:v_i \in C, v_j \in V \setminus C} W_{i,j} \end{aligned}$$

for the *degree* of the i -th vertex, the *volume* of C and the *cut* of C in V . The *Cheeger ratio* of a set $C \subset V$ is defined by

$$h_G(C) := \frac{\text{cut}_G(C)}{\min\{\text{vol}(C), \text{vol}(V \setminus C)\}}$$

and the *Cheeger constant* of G by $h_G := \min_{C \subset V} h_G(C)$.

Let us from now on assume the graph is connected, i.e. $d_i > 0$ for all i . To compute a partition such that the corresponding Cheeger ratio is quasioptimal we will draw onto the results in [12].

Let I denote the $l \times l$ identity matrix and D the diagonal matrix with entries d_1, \dots, d_l then the *normalized graph Laplacian* is given by the matrix

$$L := I - D^{-\frac{1}{2}} W D^{-\frac{1}{2}}.$$

The vector $(1, \dots, 1)^T$ is an eigenvector of L with corresponding eigenvalue 0. By the assumption that G is connected all other eigenvalues of L will be positive. The partition is computed by thresholding an eigenvector corresponding to the smallest positive eigenvalue of L .

Let u be an element of the eigenspace of the smallest positive eigenvalue of L and let

$$C^* \text{ be a minimizer of } h_G(\cdot) : \{C_t : t \in \mathbb{R}\} \rightarrow [0, \infty),$$

where $C_t := \{v_i : u_i > t\}$. Then for $h_G^* := h_G(C^*)$ it holds that

$$h_G \leq h_G^* \leq 2 \cdot \sqrt{h_G}.$$

In our case we have samples of $|V_\varphi f(x, y)|$ available for $(x, y) \in Z \subset \Delta \cdot \mathbb{Z}^2 + d$, where $d \in \mathbb{R}^2$ and $\Delta > 0$. On the set Z we define a similarity measure w by

$$w(z, z') := \begin{cases} \frac{1}{2}(|V_\varphi f|^p(z) + |V_\varphi f|^p(z')), & \text{if } |z - z'| = \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C \subset Z$. Since $|V_\varphi f|$ is smooth we obtain (for small Δ) that

$$\begin{aligned} \text{cut}_G(C) &= \sum_{\substack{z \in C \\ z' \in Z \setminus C}} w(z, z') = \sum_{\substack{z \in C, z' \in Z \setminus C \\ |z - z'| = \Delta}} \frac{1}{2}(|V_\varphi f|^p(z) + |V_\varphi f|^p(z')) \\ &\approx \sum_{\substack{z \in C, z' \in Z \setminus C \\ |z - z'| = \Delta}} |V_\varphi f|^p\left(\frac{z + z'}{2}\right). \end{aligned}$$

Therefore – up to the factor Δ – $\text{cut}_G(D)$ can be interpreted as a discrete version of the boundary integral in the nominator in Equation (2.1). Similarly $\text{vol}(C)$ can be interpreted as an approximation of $\int_C |V_\varphi f|$ that occurs in the denominator.

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