THE MATHEMATICS OF PHASE RETRIEVAL

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ABSTRACT. The problem of phase retrieval, i.e., the problem of recovering a function from the magnitudes of its Fourier transform, naturally arises in various fields of physics, such as astronomy, radar, speech recognition, quantum mechanics and, perhaps most prominently, diffraction imaging. The mathematical study of phase retrieval problems possesses a long history with a number of beautiful and deep results drawing from different mathematical fields, such as harmonic analysis, complex analysis, or Riemannian geometry. The present paper aims to present a summary of some of these results with an emphasis on recent activities. In particular we aim to summarize our current understanding of uniqueness and stability properties of phase retrieval problems.

1. Introduction

The problem of phase retrieval, i.e., the problem of recovering a function from the magnitudes of its Fourier transform, naturally arises in various fields of physics, such as astronomy [19], radar [37], speech recognition [51] and quantum mechanics [49]. The most prominent example, however, is diffraction imaging, where in a basic experiment an object is placed in front of a laser which emits coherent electromagnetic radiation. The object interacts with the incident wave in a diffractive manner creating a new wave front, which is described by Kirchhoff’s diffraction equation. An adequate approximation of the resulting wave front in the far field is given by the Fraunhofer diffraction equation, which essentially states that the wave front in a plane in a sufficiently large distance from the object is given by the Fourier transform (with appropriate spatial scaling) of the function representing the object, cf. [27] for an introduction to diffraction theory.

The aim in diffractive imaging is to determine the object from measurements of the diffracted wave. This objective is seriously impeded by the fact that measurement devices usually are only capable of capturing the intensities and a loss of phase information takes place. Reconstructing the object from the far field diffraction intensities, the so-called diffraction pattern, therefore requires to solve the Fourier phase retrieval problem

Given $|\hat{f}|$, find $f$ (up to trivial ambiguities).

The name “phase retrieval” accounts for the fact that recovery of the phase of $\hat{f}$ is equivalent to recovering $f$ itself.

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In microscopy a lens is employed to essentially invert the Fourier transformation and create the image of the object. While this is possible in case of visible light, which has a wavelength of approximately $10^{-7} \text{m}$, lenses which perform this task are not available for waves of much shorter wavelength (e.g. for x-rays with a wavelength in the range between $10^{-8} \text{m}$ and $10^{-11} \text{m}$). Since the spatial resolution of the optical system is proportional to the wavelength of light the direct approach using lenses can only achieve a certain level of resolution. In order to obtain high resolution it is necessary to compute the image from the diffraction pattern.

Determining objects from diffraction patterns—and therefore the question of phase retrieval—for the first time became relevant when Max von Laue discovered in 1912 that x-rays are diffracted when interacting with crystals, an insight for which he would be awarded the Nobel prize in physics only two years later. The discovery of this phenomenon launched the field of x-ray crystallography. Crystallography seeks to determine the atomic and molecular structure of a crystal, i.e., a material whose atoms are arranged in a periodic fashion. In the diffraction pattern the periodicity of the crystalline sample manifests itself in form of strong peaks (Bragg peaks) lying on the so-called reciprocal lattice, cf. [46]. From the position and the intensities of these peaks crystallographers can deduce the electron density of the crystal. Over the course of the past century the methods of x-ray crystallography have developed into the most powerful tool for analyzing the atomic structure of various materials and have enabled scientists to achieve breakthrough results in different fields such as chemistry, medicine, biology, physics and material sciences. This is highlighted by the fact that more than a dozen Nobel prizes have been awarded for work involving x-ray crystallography, the discovery of the double helix structure of DNA [59] being just one example. For an exhaustive introduction to x-ray crystallography the interested reader may have a look at [33, 42].

In 1980 it was proposed by David Sayre [53] to extend the approach of x-ray crystallography to non-crystalline specimens. Almost twenty years later, facilitated by the development of new powerful x-ray sources, Sayre et al.[45] for the first time successfully reconstructed the image of a sample with resolution at nanometer scale from its x-ray diffraction pattern, see Figure 1. This approach is nowadays known under name of Coherent Diffraction Imaging (CDI). The process consists of two principal steps. Firstly, the acquisition of one or multiple diffraction patterns and secondly, processing the diffraction patterns in order to obtain the image of the sample, which is usually done by applying iterative phase retrieval algorithms. Plenty of CDI methods have been developed over recent years and have been employed to great success in physics, biology and chemistry. See [44, 54] for very recent overviews on CDI methods, their limitations and their achievements in various applications and for algorithmic phase retrieval methods in diffraction imaging.
Phase retrieval in the most general formulation is concerned with reconstructing a function $f$ in a space $\mathcal{X}$ from the phaseless information of some transform of $f$. The operator describing the transform, which will be denoted by $T$, is mapping elements of $\mathcal{X}$ into another space $\mathcal{Y}$ of either real- or complex valued functions and is usually linear, i.e.,

$$T : \mathcal{X} \rightarrow \mathcal{Y}.$$ 

Furthermore, $T$ is usually nicely invertible, which means that $T : \mathcal{X} \rightarrow \text{ran}T$ has a bounded inverse.

In order to have a concrete example in mind one may think of $\mathcal{X} = \mathcal{Y} = L^2(\mathbb{R}^d)$ and $T = \mathcal{F}$, the Fourier transform operator. In this case it is well-known that $T$ is a unitary map.
Under the above assumptions the linear measurement process does not introduce
a loss of information. However, the situation changes significantly if the phase in-
formation of the transform is absent. The problem arises of studying the obviously nonlinear mapping
\[ \mathcal{A} : f \mapsto |Tf|, \quad f \in \mathcal{X} \]
and its invertibility properties. Well-posedness in the sense of Hadamard of an inverse problem associated with \( f \mapsto \mathcal{A}f \) requires
(1) \textit{existence} of a solution, i.e., \( \mathcal{A} \) to be surjective,
(2) \textit{uniqueness}, i.e., \( \mathcal{A} \) to be injective and
(3) \textit{stability}, meaning that the solution continuously depends on the data, i.e., \( \mathcal{A}^{-1} \)
to be continuous.

For the problem of phase retrieval, condition (1) amounts to identifying the image of the operator \( \mathcal{A} \). The question is often of minor importance compared to (2) and (3) as it is simply assumed that the input data arise from the measurement process described by \( \mathcal{A} \).

Provided that \( \mathcal{X} \) is a vector space—excluding trivial cases—\( \mathcal{A} \) is not injective due to the simple observation that
\[ \mathcal{A}f = \mathcal{A}(cf), \quad f \in \mathcal{X}, \quad |c| = 1. \]

Further ambiguities may occur, such as translations in the Fourier example but also less trivial ones. The first key question in the mathematical analysis of a phase retrieval problem is to identify all ambiguities. Depending on the context a particular source of ambiguities is either classified as trivial or as severe. If there exist severe ambiguities the phase retrieval problem is hopeless as there exist different objects yielding identical measurements. If on the other hand all occurring ambiguities are considered trivial, \( f \) and \( g \) may be identified \( (f \sim g) \) whenever \( \mathcal{A}f = \mathcal{A}g \). Let \( \tilde{\mathcal{X}} = \mathcal{X}/\sim \) denote the quotient set, then—by definition—\( \mathcal{A} \) is injective as mapping acting on \( \tilde{\mathcal{X}} \) and uniqueness is in this new sense is ensured.

In order to study stability \( \tilde{\mathcal{X}} \) has to be endowed with a reasonable topology first. In case \( \mathcal{X} \) is a normed space and the only ambiguities occurring are of the type as in (1) usually the quotient metric
\[ d([(f)_{\sim}], [(g)_{\sim}]) := \inf_{|c|=1} \|f - cg\| \]
is used. If there are other ambiguities a suitable choice may be less obvious.

Beyond determining whether the mapping \( \mathcal{A} \) on \( \tilde{\mathcal{X}} \) is continuously invertible further continuity properties of the inverse are often studied such as (local) Lipschitz continuity.

If there are non-trivial ambiguities, i.e., if injectivity is not attained after identifying all trivial ambiguities or if the inverse is not continuous, one or both of the following measures may be taken in order to render the phase retrieval problem well-posed:

(A) Restriction of \( \mathcal{A} \): The restriction \( \mathcal{A} : \tilde{\mathcal{X}}' \to \mathcal{A}(\tilde{\mathcal{X}}') \), where \( \tilde{\mathcal{X}}' \subset \tilde{\mathcal{X}} \) is eventually injective (has a continuous inverse) if \( \tilde{\mathcal{X}}' \) is chosen sufficiently small, \( \tilde{\mathcal{X}}' \)
consisting of a single element being the extremal, trivial example. Restriction of $\mathcal{A}$ to a smaller domain can be understood as imposing additional a priori constraints on the function $f$ to be determined. In applications of the phase retrieval problem from Fourier measurements, for instance, it is typically sensible to demand that $f$ is non-negative, as other functions do not hold a physical meaning.

**B** Modification of $T$: The idea is to suitably modify $T$ in order to soften the setback which is suffered by the subsequent removal of the phase information. In case of the Fourier phase retrieval problem this can be achieved by applying several different manipulations of $f$ before computation of the Fourier transform, e.g. using

\[
T'f := (\hat{f}g_1, \ldots, \hat{f}g_m),
\]

for known functions $g_1, \ldots, g_m$ instead of $Tf = \hat{f}$. In the context of diffraction imaging this approach is common practice as a physical system which produces measurements $|T'f| = (|\hat{f}g_1|, \ldots, |\hat{f}g_m|)$ can often be implemented.

In ptychography—a concept proposed by Walter Hoppe in the sixties [36]—different sections of an object are illuminated one after another and the object is to be reconstructed from several diffraction patterns. For suitable, localized window functions $g_1, \ldots, g_m$ equation (2) serves as a reasonable mathematical model.

As a second example let us mention holography, invented by Dennis Gabor in 1947 [26]. In holography the diffracted waves interfere with the wave field of a known object. This idea amounts to an additive distortion of the wave field $T'f := \hat{f} + g$, where $g$ is a known reference wave.

When studying a concrete phase retrieval problem with an application in the background it is useful to keep in mind that often there is a certain degree of freedom in the way how the measurements are acquired. For instance, in diffraction imaging there is the fundamental observation that the wave in the object plane and the wave in the far field are connected in terms of the Fourier transform. However there are many different options in how to generate one or several diffraction patterns. Instead of viewing a phase retrieval problem as the analysis of a fixed operator $\mathcal{A}$ one may as well include the question of how to design the measurement process in order to get a well-posed problem.

Beyond the question of well-posedness it is desirable to provide a method that recovers a function $f$ (at least the equivalence class $[f]$) from the observed measurements $\mathcal{A}f$. Such a method could be an explicit expression of the inverse of $\mathcal{A}$. Mostly the aim of coming up with an explicit expression is rather hopeless. In practice iterative algorithms are employed, which serve as approximate inverses of the measurement mapping $\mathcal{A}$. 
Phase retrieval problems have been studied in a rich variety of shapes. It can be distinguished between finite and infinite dimensional as well as between discrete and continuous phase retrieval problems. Furthermore phase retrieval problems differ in what kind of measurements are considered, i.e., the choice of the operator $T$. The most common choices are that either $T$ involves some sort of Fourier transform or that $T$ is assembled in a random fashion. Moreover there is a huge body of research in the more abstract setting of frames, where it is assumed that $T$ is induced by a frame. Phase retrieval problems where the quantity of interest is assumed to satisfy certain differential equations have also been studied.

2. Abstract Phase Retrieval

From an abstract point of view, Fourier phase retrieval lends itself to the following interpretation: Of a function $f$, we are given the absolute values of measurements given by bounded linear functionals. In the case of Fourier phase retrieval, the family of linear functionals are just the pointwise evaluation of the Fourier transform $\{f \mapsto \hat{f}(x) : x \in \mathbb{R}^d\}$.

With this interpretation in mind, we can consider phase retrieval more general. Throughout this section let $\mathcal{B}$ denote a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{B}'$ its topological dual space. Furthermore, let $\Lambda$ be a not necessarily countable index set. For a family of bounded linear functionals $\Phi := \{\varphi_{\lambda} : \lambda \in \Lambda\} \subseteq \mathcal{B}'$, we define the operator of phaseless measurements by

$$A_{\Phi} f := (|\langle f, \varphi_{\lambda} \rangle|)_{\lambda \in \Lambda},$$

where $\langle \ldots, \ldots \rangle$ denotes the dual pairing. Due to the linearity, it is clear that $A_{\Phi}(cf) = A_{\Phi} f$ for phase factors $|c| = 1$. We therefore introduce the equivalence relation $cf \sim f$ and say $\Phi$ does phase retrieval if the mapping $A_{\Phi} : \mathcal{B}/\sim \to \mathbb{R}_{+}^{\Lambda}$ is injective.

2.1. Injectivity. Suppose $\Phi := \{\varphi_{\lambda} : \lambda \in \Lambda\} \subseteq \mathcal{B}'$ is a family of bounded linear functionals and $S \subseteq \Lambda$, we then write $\Phi_S := \{\varphi_{\lambda} : \lambda \in S\} \subseteq \Phi$. For a linear subspace $V$ of $\mathcal{B}'$, let $V_{\perp} := \{f \in \mathcal{B} : \langle f, v \rangle = 0 \ \forall v \in V\}$ denote the annihilator of $V$ in $\mathcal{B}$.

Definition 2.1. The family $\Phi \subseteq \mathcal{B}'$ satisfies the complement property in $\mathcal{B}$ if we have $(\text{span } \Phi_S)_{\perp} = \{0\}$ or $(\text{span } \Phi_{\Lambda \setminus S})_{\perp} = \{0\}$ for every $S \subseteq \Lambda$.

Then the complement property is necessary for $A_{\Phi}$ to be injective. In the real case, it is even sufficient.

Theorem 2.2. Let $\mathcal{B}$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi \subseteq \mathcal{B}'$ a family of bounded linear functionals. Then the following holds:

(i) If $A_{\Phi}$ is injective, then $\Phi$ satisfies the complement property.

(ii) If $\mathbb{K} = \mathbb{R}$ and $\Phi$ satisfies the complement property, then $A_{\Phi}$ is injective.
Theorem 2.2 has quite the history. It was first stated for finite dimensions in Balan et al. [8]. The arguments for the complex case should have been given more care. Bandeira et al. [10] spotted this oversight and gave an alternative proof for the complex case in finite dimensions. In doing so, they produced a series of characterizations for injectivity in finite dimensions. Moreover, they had the crucial insight for stability of phase retrieval by introducing a “numerical version” of the complement property (see Section 2.2).

Ultimately, only a minor correction was necessary to repair Balan et al.’s proof and the same arguments also work in infinite dimensions. This is the proof we present below, which can also be found in [5, 16, 20].

**Proof.** (i) Let $A_\Phi$ be injective for $\Phi = \{\varphi_\lambda : \lambda \in \Lambda\}$ and $S \subseteq \Lambda$ arbitrary. We need to show that $(\text{span } \Phi_S)_\perp = \{0\}$ or $(\text{span } \Phi_{\Lambda\setminus S})_\perp = \{0\}$. Suppose $(\text{span } \Phi_S)_\perp \neq \{0\}$, then there exists a non-zero $f \in (\text{span } \Phi_S)_\perp$.

For any $h \in (\text{span } \Phi_{\Lambda \setminus S})_\perp$, we have

$$|\langle f \pm h, \varphi_\lambda \rangle|^2 = |\langle f, \varphi_\lambda \rangle|^2 \pm 2 \text{Re}(\langle f, \varphi_\lambda \rangle \langle h, \varphi_\lambda \rangle) + |\langle h, \varphi_\lambda \rangle|^2 = 0 \quad \forall \lambda \in \Lambda.$$ 

Hence $A_\Phi(f + h) = A_\Phi(f - h)$ for all $h \in (\text{span } \Phi_{\Lambda \setminus S})_\perp$.

Since $A_\Phi$ is injective by assumption, there exists a phase factor $|c| = 1$ such that $f + h = c(f - h)$, or equivalently $(c - 1)f = (1 + c)h$. Note that we can exclude $c = -1$, as this would imply $f = 0$. Therefore, we obtain

$$h = \frac{c - 1}{1 + c} f \in (\text{span } \Phi_S)_\perp \cap (\text{span } \Phi_{\Lambda \setminus S})_\perp,$$

which implies that $A_\Phi h = 0$. By injectivity of $A_\Phi$, we have $h = 0$ and since $h \in (\text{span } \Phi_{\Lambda \setminus S})_\perp$ was arbitrary, we conclude that $(\text{span } \Phi_{\Lambda \setminus S})_\perp = \{0\}$.

(ii) Suppose $A_\Phi$ is not injective, this means that there exist $f, h \in \mathcal{B}$ such that $A_\Phi f = A_\Phi h$. Since $\Phi = \{\varphi_\lambda : \lambda \in \Lambda\}$ consists of real-valued linear functionals, the signed measurements of $f$ and $h$ with respect to $\Phi$ can only differ by a factor of $c = -1$. We therefore consider the following partition of the index set $\Lambda$: Let $S := \{\lambda \in \Lambda : \langle f, \varphi_\lambda \rangle = \langle h, \varphi_\lambda \rangle\}$, then $\Lambda \setminus S = \{\lambda \in \Lambda : \langle f, \varphi_\lambda \rangle = -\langle h, \varphi_\lambda \rangle\}$.

Consequently, $f - h \in (\text{span } \Phi_S)_\perp$ and $f + h \in (\text{span } \Phi_{\Lambda \setminus S})_\perp$. But by assumption, at least one of those annihilators consist only of 0. Hence $f = h$ or $f = -h$ and therefore $A_\Phi$ is injective. 

For the Paley-Wiener space $PW^{p,b}_R := \{f \in L^p(\mathbb{R}, \mathbb{R}) : \text{supp } \hat{f} \subseteq [-b/2, b/2]\}$ ($1 < p < \infty$) of real-valued bandlimited functions, one can show that the complement property holds for families of point-evaluations $\Phi = \{\delta_\lambda : \lambda \in \Lambda\}$ if the sampling rate exceeds twice the critical density [5]. Since $PW^{p,b}_R$ is a real-valued Banach space, this implies that phase retrieval is possible.

For complex Banach spaces, the complement property is not sufficient. Hence other methods need to be employed to study injectivity. For Fourier-type measurements, these tools often come from complex analysis (see Section 3).
We now turn to the finite dimensional case. The complement property implies that $\Phi \subseteq \mathbb{K}^d$ needs to span the whole space and must be overcomplete for phase retrieval to be possible. Or in other words, $\Phi$ must be a frame.

In the remainder of this section, we state necessary and sufficient conditions on the number of frame elements of $\Phi$ to do phase retrieval. The first result is an easy consequence of the complement property.

**Corollary 2.3.** If $N < 2d - 1$, then $A_\Phi$ cannot be injective for any family $\Phi \subseteq \mathbb{K}^d$ with $N$ elements.

**Proof.** We partition $\Phi$ into two sets $\Phi_S, \Phi_{A\setminus S}$ with at most $d - 1$ elements. This yields $\text{span} \Phi_S \neq \mathbb{K}^d$ and $\text{span} \Phi_{A\setminus S} \neq \mathbb{K}^d$, clearly violating the complement property. □

For $\mathbb{K} = \mathbb{R}$, the converse statement also holds for “almost all” frames. To make this more precise, we need some terminology of algebraic geometry.

An *algebraic variety* in $\mathbb{K}^d$ is the common zero set of finitely many polynomials in $\mathbb{K}[x_1, \ldots, x_d]$. By defining algebraic varieties in $\mathbb{K}^d$ as closed, we obtain the *Zariski topology*. Note that this topology is coarser than the Euclidean topology on $\mathbb{K}^d$, meaning that every Zariski-open set is also open with respect to the Euclidean topology. Furthermore, non-empty Zariski-open sets are dense with respect to the Euclidean topology and have full Lebesgue-measure in $\mathbb{K}^d$ [10, 22].

We say a *generic point* in $\mathbb{K}^d$ satisfies a certain property, if there exists a non-empty Zariski-open set with this property. By the above, this means that if a certain property holds for a generic point, it holds for almost all points in $\mathbb{K}^d$.

Now we identify a frame $\Phi \subseteq \mathbb{K}^d$ of $N$ elements with a $d \times N$ matrix of full rank. Hence the set of frames with $N$ elements in $\mathbb{K}^d$, i.e., the set of matrices of full rank in $\mathbb{K}^{d \times N}$, is a non-empty Zariski-open set and it makes sense to study generic points within the set of frames. We call those generic points *generic frames*.

The following theorem is due to Balan et al. [8]. Together with Corollary 2.3, it (almost) characterizes the injectivity of phase retrieval in $\mathbb{R}^d$.

**Theorem 2.4 ([8]).** If $N \geq 2d - 1$, then $A_\Phi$ is injective for a generic frame $\Phi \subseteq \mathbb{K}^d$ with $N$ elements.

For phase retrieval in $\mathbb{C}^d$, Bandeira et al. [10] conjectured an analogous characterization with $4d - 4$ being the critical number of frame elements. They also gave a proof in dimensions $d = 2, 3$. Conca et al. [22], see also [39], proved the following theorem, confirming the sufficient part of the $(4d - 4)$-Conjecture.

**Theorem 2.5 ([22]).** Let $d \geq 2$. If $N \geq 4d - 4$, then $A_\Phi$ is injective for a generic frame $\Phi \subseteq \mathbb{C}^d$ with $N$ elements.

Conversely, a frame in $\mathbb{C}^d$ with $N < 4d - 4$ elements does not allow phase retrieval in dimensions $d = 2^k + 1$ [22]. But the $(4d - 4)$-Conjecture does not hold in general: Vinzant [56] gave an example of a frame with $11 = 4d - 5$ elements in $\mathbb{C}^4$ which does phase retrieval. For necessary lower bounds in general dimension, we refer the interested reader to Wang and Xu [58]. A more indepth account of the history...
of necessary and sufficient bounds for phase retrieval in $\mathbb{C}^d$ can be found in [16]. Furthermore, Bodmann and Hammen [13, 14] developed concrete algorithms and error bounds for phase retrieval with low-redundancy frames.

2.2. Stability. Once the question of injectivity is answered positively, the question of stability arises. Stability refers to the continuity of the operator $A_{\Phi}^{-1} : \text{ran } A_{\Phi} \to B/\sim$. To this end, we need to introduce a topology on $B/\sim$ and find a suitable Banach space $\mathfrak{B}$ with $\text{ran } A_{\Phi} \subseteq \mathfrak{B} \subseteq \mathbb{K}^\Lambda$. The natural choice for $B/\sim$ is the quotient metric

$$d(f, h) := \inf_{|c|=1} \|f - ch\|_B.$$ 

The analysis space for frames in separable Hilbert spaces is the sequence space $\ell^2(\Lambda)$. We will consider the stability of phase retrieval for continuous Banach frames in this section. There, the appropriate generalization of $\ell^2(\Lambda)$ is an “admissible” Banach space $\mathfrak{B}$ such that the range of the coefficient operator $C_{\Phi} f := (\langle f, \varphi_\lambda \rangle)_{\lambda \in \Lambda}$ is contained in $\mathfrak{B}$.

**Definition 2.6.** Let $\Lambda$ be a $\sigma$-compact topological space. A Banach space $\mathfrak{B} \subseteq \mathbb{K}^\Lambda$ is called admissible if it satisfies the following properties:

(i) The indicator function $\chi_K$ of every compact set $K \subseteq \Lambda$ satisfies $\|\chi_K\|_\mathfrak{B} < \infty$.

(ii) The Banach space $\mathfrak{B}$ is solid, this means that $\|w\|_\mathfrak{B} \leq \|z\|_\mathfrak{B}$ whenever $|w(\lambda)| \leq |z(\lambda)|$ for all $\lambda \in \Lambda$.

(iii) The elements of $\mathfrak{B}$ with compact support are dense in $\mathfrak{B}$.

These properties are quite reasonable. Indeed, all $L^p$-spaces for $1 \leq p < \infty$ are admissible Banach spaces and $L^\infty$ violates only the last point unless $\Lambda$ is already compact.

Now we are in the position to define stability of phase retrieval precisely.

**Definition 2.7.** Let $\Phi \subseteq B'$ be a family of bounded linear functionals and $\mathfrak{B}$ and admissible Banach space such that $C_{\Phi} : B \to \mathfrak{B}$. We say that the phase retrieval of $\Phi$ is stable (with respect to $\mathfrak{B}$) if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha d(f, h) \leq \|A_{\Phi}(f) - A_{\Phi}(h)\|_\mathfrak{B} \leq \beta d(f, h) \quad \forall f, h \in B$$

Moreover, let $\alpha_{\text{opt}}(\Phi), \beta_{\text{opt}}(\Phi)$ denote the optimal lower and upper Lipschitz bound respectively.

**Definition 2.8.** Suppose that $\Phi := \{\varphi_\lambda : \lambda \in \Lambda\} \subseteq B'$ is a family of bounded linear functionals such that $\lambda \mapsto \varphi_\lambda$ is continuous. We call $\Phi$ a continuous Banach frame if there exists an admissible Banach space such that the following is satisfied:

(i) There exist positive constants $0 < A \leq B < \infty$ such that

$$A\|f\|_\mathfrak{B} \leq \|C_{\Phi} f\|_\mathfrak{B} \leq B\|f\|_\mathfrak{B} \quad \forall f \in B.$$ 

Moreover, let $A_{\text{opt}}(\Phi), B_{\text{opt}}(\Phi)$ denote the optimal constants satisfying (4).
(ii) There exists a continuous operator $R : \mathcal{B} \to \mathcal{B}$, the so-called reconstruction operator, satisfying

$$RC_\Phi f = f \quad \forall f \in \mathcal{B}.$$ 

The requirement for $\Phi$ to be a frame is a natural one. In fact, if $C_\Phi$ maps into an admissible Banach space, the solidity implies $\|A_\Phi f\|_\mathcal{B} = \|C_\Phi f\|_\mathcal{B}$. Hence, stability in the sense of (3) implies the frame inequality (4) by taking $h = 0$. For the upper inequalities, we even have equivalence:

**Proposition 2.9.** If $\Phi \subseteq \mathcal{B}'$ is a family of continuous linear functionals such that $C_\Phi$ maps into an admissible Banach space, then $\beta_{\text{opt}} = B_{\text{opt}}$.

Again the solidity of the admissible Banach space plays an integral role in the proof. As the rest follows from straightforward estimates, we omit the proof and refer the interested reader to [5, 20].

The remainder of the section deals with the lower inequality in (3). We start by mentioning an interesting result about the continuity of the inverse operator $A_\Phi^{-1}$, which can be regarded as a weaker form of stability.

**Theorem 2.10.** Let $\Phi \subseteq \mathcal{B}'$ be a continuous Banach frame and $A_\Phi$ injective. Then $A_\Phi^{-1}$ is continuous on the range of $A_\Phi$.

**Proof idea.** One needs to show that the convergence of the image sequence $A_\Phi f_n \to A_\Phi f$ in $\mathcal{B}$ implies the convergence of $f_n \to f$ in $\mathcal{B}$. The idea is to link the convergence of $A_\Phi f_n$ to the convergence of the signed measurements $C_\Phi f_n$. This is the technical and lengthy part of the proof and we refer the interested reader to [5] for the details. Once this relation is established, one can use use the continuous reconstruction operator $R$ to obtain $f_n \to f$. □

As an easy consequence of Theorem 2.10, we obtain stability of phase retrieval in finite-dimensional Banach spaces.

**Theorem 2.11.** Let $\mathcal{B}$ be a finite-dimensional Banach space. If $\Phi$ is a frame that does phase retrieval, then $A_\Phi$ has a lower Lipschitz bound $\alpha_{\text{opt}} > 0$.

**Proof.** Note that the existence of a positive lower Lipschitz bound $\alpha_{\text{opt}} > 0$ in (3) is equivalent to $A_\Phi^{-1} : \text{ran } A_\Phi \to \mathcal{B}/\sim$ being Lipschitz continuous with constant $L = \alpha_{\text{opt}}^{-1}$.

By Theorem 2.10, the inverse $A_\Phi^{-1}$ is continuous on $\text{ran } A_\Phi$. Since $\mathcal{B}$ is finite-dimensional, the closed unit ball $B(0, 1)$ is compact and therefore $A_\Phi^{-1}$ is uniformly continuous on $\text{ran } A_\Phi \cap B(0, 1)$. By using the scaling invariance of $A_\Phi^{-1}$ and playing everything back into the unit ball $B(0, 1)$, the Lipschitz continuity follows in a series of straightforward estimates. □

The result of Theorem 2.11 was proved first for the real case in [9, 10]. Cahill et al. [20] gave a proof for the complex case. The proof above is from [5].

For their proof of stability in finite dimensions, Bandeira et al. [10] introduced the following “numerical” version of the complement property, which relates to stability as the complement property relates to injectivity.
Definition 2.12. The family $\Phi \subseteq B'$ satisfies the $\sigma$-strong complement property in $B$ if there exists a $\sigma > 0$ such that
\[
\max\{A_{opt}(\Phi_S), A_{opt}(\Phi_{\Lambda \setminus S})\} \geq \sigma \quad \forall S \subseteq \Lambda.
\]
Moreover, let $\sigma_{opt}(\Phi)$ denote the supremum over all $\sigma > 0$ satisfying (5).

Theorem 2.13. Let $B$ be a Banach space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi \subseteq B'$ a continuous Banach frame. Then there exists a constant $C > 0$ such that
\[
\alpha_{opt} \leq C\sigma_{opt}.
\]
In the real case, the constant is $C = 2$. For the complex case, the constant can be chosen $C = 2B_{opt}/A_{opt}$.

Remark 2.14. For the real case, one can also show that $\sigma_{opt} \leq C\alpha_{opt}$ for some $C > 0$. This implies that the $\sigma$-strong complement property is not only necessary, but also sufficient for stability in real Banach spaces. In this sense, it mirrors the behavior of the complement property.

Unfortunately, the sufficiency cannot be exploited for (global) stability: On one hand, phase retrieval is always stable in finite dimensions by Theorem 2.11 and on the other hand, we will see that the $\sigma$-strong complement property can never hold in infinite dimensions.

Proof. Let $\sigma > \sigma_{opt}$. Then there exist a subset $S \subseteq \Lambda$ and $f, h \in B$ with $\|f\|_B = \|h\|_B = 1$ such that
\[
\|C_{\Phi_S}f\|_B < \sigma \quad \text{and} \quad \|C_{\Phi_{\Lambda \setminus S}}h\|_B < \sigma.
\]
Now set $x := f + h$ and $y := f - h$. Due to the solidity of $B$, we obtain
\[
\|A_{\Phi}(x) - A_{\Phi}(y)\|_B \leq \|(|\langle x, \varphi_{\lambda} \rangle| - |\langle y, \varphi_{\lambda} \rangle|)_{\lambda \in S}\|_B + \|(|\langle x, \varphi_{\lambda} \rangle| - |\langle y, \varphi_{\lambda} \rangle|)_{\lambda \in \Lambda \setminus S}\|_B
\leq 2\|C_{\Phi_S}f\|_B + 2\|C_{\Phi_{\Lambda \setminus S}}h\|_B
\leq 4\sigma,
\]
where we used the reverse triangle inequality in the second line.

By definition of $\alpha_{opt}$, we conclude
\[
\alpha_{opt}d(x, y) \leq \|A_{\Phi}(x) - A_{\Phi}(y)\|_B \leq 4\sigma.
\]

In the real case, we are done since $d(x, y) = \min\{\|x + y\|_B, \|x - y\|_B\} = 2\min\{\|f\|_B, \|h\|_B\} = 2$.

The complex case proves to be more difficult. A series of elementary estimates are necessary to bound $d(x, y)$ away from zero. We refer the interested reader to the original article [5].

Remark 2.15. The computations in the proof of Theorem 2.13 also yield an estimate on local stability constants. More precisely, suppose a fixed $x \in B$ can be decomposed according to $x = f + h$ such that $\|f\|_B \approx 1$, $\|h\|_B \approx 1$ and that (6) holds for $\sigma \ll 1$. Then there exists $y \in B$ such that
\[
\|A_{\Phi}(x) - A_{\Phi}(y)\|_B \lesssim \sigma \quad \text{and} \quad d(x, y) \gtrsim 1.
\]
Thus, $x$ and $y$ yield similar measurements even though they are very different from each other.

Theorem 2.13 implies that the $\sigma$-strong complement property is necessary for stability. Bandeira et al. [10] gave a proof of this for the real case and conjectured the complex case, which was proved in [5].

For finite dimensions, phase retrieval is always stable by Theorem 2.11. In particular, the $\sigma$-strong complement property is satisfied. In infinite dimensions, we will see that continuous Banach frames cannot satisfy the $\sigma$-strong complement property, hence phase retrieval is always unstable in this case. To show this, we follow [5] and prove an intermediate result, which is interesting in its own right. It states that there cannot exist continuous Banach frames in infinite dimensions with compact index set $\Lambda$.

**Proposition 2.16.** Suppose $\mathcal{B}$ is an infinite-dimensional Banach space and $\Lambda$ a compact index set. Then any family $\Phi := \{\varphi_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{B}'$ with continuous mapping $\lambda \mapsto \varphi_\lambda$ fails to satisfy the lower frame inequality. This means that for every $\varepsilon > 0$ there exists an $f \in \mathcal{B}$ such that

$$\|C_\Phi f\|_{\mathcal{B}} < \varepsilon \|f\|_{\mathcal{B}}.$$

**Proof.** Let $\varepsilon > 0$. By continuity of the mapping $\lambda \mapsto \varphi_\lambda$, there exists for every $\lambda \in \Lambda$ an open neighborhood $U_\lambda$ such that

$$\|\varphi_\omega - \varphi_\lambda\|_{\mathcal{B}'} < \frac{\varepsilon}{\|\chi_\lambda\|_{\mathcal{B}}} \quad \forall \omega \in U_\lambda.$$

Since $\Lambda$ is compact, the open covering $\{U_\lambda : \lambda \in \Lambda\}$ admits a finite subcover $\{U_{\lambda_1}, \ldots, U_{\lambda_N}\}$. Now set $U_1 := U_{\lambda_1}$ and $U_j := U_{\lambda_j} \setminus \bigcup_{k=1}^{j-1} U_k$ for $j = 2, \ldots, N$ to obtain a partition of $\Lambda$ which satisfies for all $j = 1, \ldots, N$

$$\|\varphi_\lambda - \varphi_{\lambda_j}\|_{\mathcal{B}'} < \frac{\varepsilon}{\|\chi_\lambda\|_{\mathcal{B}}} \quad \forall \lambda \in U_j.$$

Clearly, we have

$$|\langle f, \varphi_\lambda \rangle| \leq |\langle f, \varphi_{\lambda_j} \rangle| + |\langle f, \varphi_\lambda - \varphi_{\lambda_j} \rangle|$$

for all $j = 1, \ldots, N$. After multiplication with the characteristic function $\chi_{U_j}$ and summing over $j$, we obtain

$$\mathcal{A}_\Phi f(\lambda) = \sum_{j=1}^{N} |\langle f, \varphi_\lambda \rangle| \chi_{U_j}(\lambda)$$

$$\leq \sum_{j=1}^{N} |\langle f, \varphi_{\lambda_j} \rangle| \chi_{U_j}(\lambda) + \sum_{j=1}^{N} |\langle f, \varphi_\lambda - \varphi_{\lambda_j} \rangle| \chi_{U_j}(\lambda)$$

$$< \sum_{j=1}^{N} |\langle f, \varphi_{\lambda_j} \rangle| \chi_{U_j}(\lambda) + \varepsilon \|f\|_{\mathcal{B}} \chi_{\lambda}(\lambda) \cdot \|\chi_\lambda\|_{\mathcal{B}} \chi_{U_j}(\lambda).$$
Now the solidity of $\mathfrak{B}$ implies
\[
\|C_\Phi f\|_{\mathfrak{B}} = \|A_\Phi f\|_{\mathfrak{B}} < \sum_{j=1}^{N} |\langle f, \varphi_{\lambda_j} \rangle| \|\chi_{U_j}\|_{\mathfrak{B}} + \varepsilon \|f\|_{\mathfrak{B}}
\]
for all $f \in \mathcal{B} \setminus \{0\}$. Since $\mathcal{B}$ is infinite-dimensional, there exists a non-zero $f_0 \in \mathcal{B}$ such that $\langle f_0, \varphi_{\lambda_j} \rangle = 0$ for all $j = 1, \ldots, N$. Consequently, the sum on the right-hand side vanishes for $f_0$ and we obtain the claim. \hfill \Box

**Theorem 2.17.** Let $\mathcal{B}$ be an infinite-dimensional Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi \subseteq \mathcal{B}'$ a continuous Banach frame. Then $\Phi$ does not satisfy the $\sigma$-strong complement property.

**Proof.** We need to show that the $\sigma$-strong complement property is not satisfied. This means that for every $\varepsilon > 0$ we can find a subset $S \subseteq \Lambda$ and $f, h \in \mathcal{B}$ such that
\[
\|C_{\Phi_S} f\|_{\mathfrak{B}} < \varepsilon \|f\|_{\mathcal{B}} \quad \text{and} \quad \|C_{\Phi_{\Lambda \setminus S}} h\|_{\mathfrak{B}} < \varepsilon \|h\|_{\mathcal{B}}.
\]

We start with an arbitrary $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} = 1$. Since $\mathfrak{B}$ is an admissible Banach space where compact elements are dense, there exists a nested sequence of compact subsets $K_n \subseteq K_{n+1}$ with $\bigcup_{n \in \mathbb{N}} K_n = \Lambda$ such that
\[
\|C_\Phi f - C_\Phi f \cdot \chi_{K_n}\|_{\mathfrak{B}} \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence, there exists a $K_N$ such that
\[
\|C_\Phi f - C_\Phi f \cdot \chi_{K_N}\|_{\mathfrak{B}} < \varepsilon.
\]
Setting $S := \Lambda \setminus K_N$, we obtain $\|C_{\Phi_S} f\|_{\mathfrak{B}} < \varepsilon \|f\|_{\mathfrak{B}}$.

On the other hand, we can use Theorem 2.16 for the compact set $\Lambda \setminus S = K_N$ to find an $h \in \mathcal{B}$ such that
\[
\|C_{\Phi_{\Lambda \setminus S}} h\|_{\mathfrak{B}} < \varepsilon \|h\|_{\mathcal{B}}.
\]

\hfill \Box

**Corollary 2.18.** Let $\mathcal{B}$ be an infinite-dimensional Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi \subseteq \mathcal{B}'$ a continuous Banach frame. Then $\Phi$ cannot do stable phase retrieval. This means that for every $\varepsilon > 0$, there exist $f, h \in \mathcal{B}$ with $\|A_\Phi(f) - A_\Phi(h)\|_{\mathfrak{B}} < \varepsilon$ but $d(f, h) \geq 1$.

**Proof.** This is an immediate consequence of the fact that the $\sigma$-strong complement property is necessary for stability by Theorem 2.13, but continuous Banach frames in infinite dimensions cannot satisfy it by Theorem 2.17. \hfill \Box

**Remark 2.19.** Phase retrieval in infinite dimensions cannot be stable for continuous Banach frames by Corollary 2.18. On the other hand, Theorem 2.11 states that it is always stable in finite dimensions. The natural question that arises is the following: Suppose $V_n \subseteq \mathcal{B}$ is a sequence of finite-dimensional subspaces and let $\alpha(V_n)$ denote the stability constant for the subspace $V_n$ in (3). How fast does the stability constant $\alpha(V_n)$ degenerate as the dimension increases?

It turns out, this can be rather rapidly: Cahill et al. [20] considered subspaces of increasing dimension in the Paley-Wiener space and showed that the stability
constant degrades exponentially fast in the dimension. Even worse degeneration can be observed for the Gabor transform on $L^2(\mathbb{R})$: Alaifari and one of the authors [6] constructed a sequence of subspaces whose stability constant degrades quadratically exponential in the dimension.

3. Phase Retrieval from Fourier-type Measurements

This chapter is devoted to phase retrieval problems where the underlying operator is the Fourier transform or a variant thereof. Such problems are typically studied within the scope of complex analysis. As it is widely known analytic functions of several complex variables behave very different from univariate holomorphic functions. As we shall see a qualitative gap between the one- and the multidimensional case is also encountered for the problem of Fourier phase retrieval.

3.1. The classical Fourier Phase Retrieval Problem. In this section we will discuss the problem of recovering a signal from its phaseless Fourier transform in the discrete as well as in the continuous setting.

We consider multidimensional discrete signals. This means that for $n \in \mathbb{N}^d$, a discrete signal is a complex-valued function on $J_n := \{0, \ldots, n-1\} \times \cdots \times \{0, \ldots, n_{d-1}\}$.

**Definition 3.1.** The Discrete-Time-Fourier-Transform (DTFT) $\hat{x}$ of a discrete signal $x = (x_j)_{j \in J_n} \in \mathbb{C}^{J_n}$ is defined by

$$\hat{x}(\omega) := \sum_{j \in J_n} x_j e^{-2\pi i j \cdot \omega/n} \quad \forall \omega \in \mathbb{R}^d,$$

where the normalization $\omega/n := (\omega_1/n_1, \ldots, \omega_d/n_d)$ is understood componentwise and $j \cdot \omega := \sum_{k=1}^d j_k \omega_k$ denotes the inner product on $\mathbb{R}^d$.

For signals of continuous variables we will use the following normalization of the Fourier transform.

**Definition 3.2.** Let $f \in L^1(\mathbb{R}^d)$. The Fourier transform $\hat{f}$ of $f$ is defined by

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx \quad \forall \xi \in \mathbb{R}^d.$$

For $f \in L^2(\mathbb{R}^d)$, the Fourier transform is to be understood as the usual extension.

The problem of Fourier phase retrieval can now be stated as follows.

**Problem 1** (Fourier Phase Retrieval). Suppose $x \in \mathbb{C}^{J_n}$ or $f \in L^2(\mathbb{R}^d)$ and compactly supported, respectively. Recover $x$ from $|\hat{x}|$ or $f$ from $|\hat{f}|$, respectively.

**Remark 3.3.** For $x \in \mathbb{C}^{J_n}$ the squared modulus of its DTFT $|\hat{x}|^2$ is a trigonometric polynomial and is uniquely defined by its values on a suitable, finite sampling set $\Omega \subseteq \mathbb{R}^d$. The problem of recovering $x$ from the full Fourier magnitude $|\hat{x}(\omega)|$, $\omega \in \mathbb{R}^d$ is therefore equivalent to the problem of recovering $x$ from finitely many samples of the Fourier magnitude $|\hat{x}(\omega)|$, $\omega \in \Omega$. 
In the following two sections we will collect uniqueness results for the discrete and continuous version of Problem 1, see [52] for a thorough discussion.

3.1.1. The discrete Fourier Phase Retrieval Problem. Before we identify ambiguities of the Fourier phase retrieval problem, we have to explain what it means to reflect and translate a signal \( x \in \mathbb{C}^{J_n} \). We define the reflection operator \( R \) on \( \mathbb{C}^{J_n} \) by
\[
(Rx)_j = x_{-j} \pmod{n} \quad \forall j \in J_n
\]
and the translation operator \( T_\tau \) for \( \tau \in \mathbb{Z}^d \) by
\[
(T_\tau x)_j = x_{j-\tau} \pmod{n} \quad \forall j \in J_n,
\]
where the modulo operation is to be understood componentwise. Similarly the conjugation operation will be understood componentwise. For \( z \in \mathbb{C}^d \) and \( j \in \mathbb{Z}^d \) we will write \( z^{-1} := (z_1^{-1}, \ldots, z_d^{-1}) \) and \( z^j := z_1^{j_1} \cdots z_d^{j_d} \) for short.

**Proposition 3.4.** Let \( x \in \mathbb{C}^{J_n} \). Then each of the following choices of \( y \) yields the same Fourier magnitudes as \( x \), i.e., \(|\hat{y}| = |\hat{x}|\):

(i) \( y = cx \), for \(|c| = 1\),
(ii) \( y = T_\tau x \), for \( \tau \in \mathbb{Z}^d \),
(iii) \( y = Rx \).

**Proof.** The statement follows from (i) linearity of the Fourier transform, (ii) translation amounts to modulation in the Fourier domain and (iii) reflection and conjugation amounts to conjugation in the Fourier domain. \(\Box\)

The ambiguities described in Proposition 3.4, as well as combinations thereof, are considered trivial. By identifying trivial ambiguities an equivalence relation \( \sim \) is introduced on \( \mathbb{C}^{J_n} \), i.e.,
\[
x \sim y \Leftrightarrow y = cT_\tau Rx \text{ or } y = cT_\tau x, \quad \text{for some } \tau \in \mathbb{Z}^d, \ |c| = 1.
\]
To determine all ambiguities we will study the so called Z-transform.

**Definition 3.5.** For \( x \in \mathbb{C}^{J_n} \) the Z-transform is defined by
\[
X(z) := (Zx)(z) := \sum_{j \in J_n} x_j z^j \quad \forall z \in \mathbb{C}^d.
\]

The question of uniqueness of Problem 1 is closely connected to whether the Z-transform has a non-trivial factorization, as we shall see.

**Definition 3.6.** A polynomial \( p \) of one or several variables is called reducible if there exist non-constant polynomials \( q \) and \( r \) such that \( p = q \cdot r \). Otherwise \( p \) is called irreducible.

In the subsequent, let \( p(z) = \sum_j c_j z^j \) denote a multivariate polynomial. Its degree \( \deg(p) \in \mathbb{N}_0^d \) is defined with respect to each coordinate, i.e.,
\[
\deg(p)_k = \max\{j_k : c_j \neq 0\} \quad k = 1, \ldots, d.
\]
Later we will need to consider the mapping $z \mapsto \overline{p(\bar{z}^{-1})}$. Clearly, its singularities can be removed by multiplication with a suitable monomial. Indeed,

$$q(z) := \overline{p(\bar{z}^{-1})} \cdot z^{\deg(p)}$$

is again a polynomial. Finally, let $\nu(p) \in \mathbb{N}_0^d$ denote the largest exponent (componentwise) such that $z^{\nu(p)}$ is a divisor of $p$. Thus there exists a unique polynomial $p_0$ such that

$$p(z) = z^{\nu(p)}p_0(z).$$

**Lemma 3.7.** Let $p \neq 0$ be a polynomial and let $q$ and $p_0$ be defined as in (7) and (8) respectively. Then

(i) $\nu(q) = 0$ and

(ii) $p_0$ is irreducible if and only if $q$ is irreducible.

**Proof.** (i) We show $\nu(q) = 0$ by contradiction. Assume that there exists $0 \neq \nu \in \mathbb{N}_0^d$ such that $z^\nu$ is a divisor of $q$. This means that

$$q(z) = z^{\deg(p)} \cdot \sum_j c_j z^{-j} \cdot z^{\deg(p) - \nu}$$

is a polynomial, where $p(z) = \sum_j c_j z^j$. Let $1 \leq i \leq d$ such that $\nu_i \geq 1$. By the definition of the degree there exists an index $l \in J_n$ such that $c_l \neq 0$ and $l_i = \deg(p)_i$. Then the sum on the right hand side of (9) contains the summand $\overline{c_l} z^{-l + \deg(p) - \nu}$, whose exponent is negative in the $i$-th coordinate. This is a contradiction to $z^\nu$ being a divisor of $q$.

(ii) First suppose that $p_0$ is reducible, i.e., that there exist non-trivial polynomials $p_1, p_2$ such that $p_0 = p_1 \cdot p_2$. Let $q_i(z) := p_i(\bar{z}^{-1})z^{\deg(p_i)}$ for $i = 1, 2$, then we have

$$q(z) = z^{\deg(p)} \cdot \overline{p(\bar{z}^{-1})} = z^{\deg(p) - \nu(p)} \cdot \overline{p_1(\bar{z}^{-1})} \cdot \overline{p_2(\bar{z}^{-1})} = q_1(z) \cdot q_2(z)$$

since $\deg(p) = \nu(p) + \deg(p_1) + \deg(p_2)$. By the maximality of $\nu(p)$, we have that $q_1$ and $q_2$ are non-constant polynomials. Thus $q$ is reducible.

Suppose now that $q = q_1 \cdot q_2$ for non-trivial polynomials $q_1, q_2$. Then, by making similar manipulations as in the first step, it follows that

$$p_0(z) = z^{\deg(p) - \nu(p)} \cdot \overline{q_1(\bar{z}^{-1})} \cdot \overline{q_2(\bar{z}^{-1})}.$$

This implies that $\deg(q_1) + \deg(q_2) = \deg(q) \leq \deg(p) - \nu(p)$, otherwise the right hand side would not be a polynomial. Note that the factors

$$z^{\deg(q_i)} \cdot \overline{q_i(\bar{z}^{-1})}, \quad i = 1, 2$$

are non-constant polynomials, otherwise $q$ would have a monomial divisor which is a contradiction to $\nu(q) = 0$. Consequently $p_0$ has non-trivial divisors. \qed

The following theorem characterizes all ambiguities of the discrete Fourier phase retrieval problem.
Theorem 3.8. Let $x, y \in \mathbb{C}^n$ and let $X, Y$ denote their respective $Z$-transforms. Then $|\hat{x}| = |\hat{y}|$ if and only if there exists a factorization $Y = Y_1 \cdot Y_2$, a constant $\gamma$ with $|\gamma| = 1$, and $\tau \in \mathbb{Z}^d$ such that

\begin{equation}
X(z) = \gamma z^{\tau} \cdot Y_1(z) \cdot Y_2(\bar{z}^{-1}).
\end{equation}

Proof. First we show the necessity of the statement. Suppose $y$ is an ambiguous solution with respect to $x$. By definition $X(z) = \sum_{j \in J_n} x_j z^j$ and thus, using the notation

\[ e^{-2\pi i \omega/n} = \left(e^{-2\pi i \omega_1/n_1}, \ldots, e^{-2\pi i \omega_d/n_d}\right) \quad \forall \omega \in \mathbb{R}^d, \]

we observe that $X(e^{-2\pi i \omega/n}) = \hat{x}(\omega)$. For the squared magnitude of the Fourier transform it therefore holds that

\[ |\hat{x}(\omega)|^2 = X(e^{-2\pi i \omega/n}) \cdot \overline{X(e^{-2\pi i \omega/n})} = X(e^{-2\pi i \omega}) \cdot \overline{X(e^{-2\pi i \omega^{-1}})}, \]

where conjugation and reciprocal operation are to be understood componentwise. By the assumption that $|\hat{x}| = |\hat{y}|$ and by analytic continuation we obtain

\begin{equation}
X(z) \cdot \overline{X(\bar{z}^{-1})} = Y(z) \cdot \overline{Y(\bar{z}^{-1})} \quad \forall z \in \mathbb{C}^d \setminus \{0\}.
\end{equation}

Now factorize $X$ and $Y$ into irreducible polynomials $p_1, \ldots, p_L$ and $p'_1, \ldots, p'_{L'}$, i.e.,

\[ X(z) = z^{\nu(X)} \prod_{i=1}^L p_i(z) \quad \text{and} \quad Y(z) = z^{\nu(Y)} \prod_{i=1}^{L'} p'_i(z). \]

After multiplying both sides of (11) with $z^n$, we obtain the following equality

\[ z^{n-L} \sum_{i=1}^L \deg(p_i) \prod_{i=1}^L p_i(z) \prod_{i=1}^L z^{-\deg(p_i)} \overline{p_i(\bar{z}^{-1})} = z^{n-L'} \sum_{i=1}^{L'} \deg(p'_i) \prod_{i=1}^{L'} p'_i(z) \prod_{i=1}^{L'} z^{\deg(p'_i)} \overline{p'_i(\bar{z}^{-1})}. \]

By Lemma 3.7 we have that $q_i(z) := z^{-\deg(p_i)} \overline{p_i(\bar{z}^{-1})}$ and $q'_i(z) := z^{\deg(p'_i)} \overline{p'_i(\bar{z}^{-1})}$ are irreducible, and furthermore that $\nu(q_i) = \nu(q'_i) = 0$ for all $i$. By uniqueness of the factorization it follows that

\begin{equation}
\prod_{i=1}^L p_i(z) \cdot \prod_{i=1}^L q_i(z) = \prod_{i=1}^{L'} p'_i(z) \cdot \prod_{i=1}^{L'} q'_i(z)
\end{equation}

and that $L = L'$.  

Now let $I$ be a maximal subset of $\{1, \ldots, L\}$ such that $\prod_{i \in I} p_i$ divides $\prod_{i=1}^{L'} p'_i$ and let $J := \{1, \ldots, L\} \setminus I$. Then there exists $I' \subseteq \{1, \ldots, L\}$, with $I' = J I$ and a constant $c$ such that

\[ \prod_{i \in I} p_i = c \prod_{i \in I'} p'_i \]

and therefore also

\[ \prod_{i \in I} q_i = d \prod_{i \in I'} q'_i \]
for a suitable constant $d$. From (12) it follows that
\[
  cd \cdot \prod_{i \in J} p_i \cdot \prod_{i \in J'} q_i = \prod_{i \in J} p'_i \cdot \prod_{i \in J'} q'_i
\]
where $J' := \{1, \ldots, n\} \setminus I'$. Since $I$ is chosen to be maximal and since $\sharp J = \sharp J'$ it follows that
\[
  \prod_{i \in J} p_i = a \prod_{i \in J'} q'_i
\]
for a suitable constant $a$. Thus, we obtain
\[
  X(z) = z^{\nu(X)} \cdot \prod_{i=1}^L p_i(z) = z^{\nu(X)} \cdot \prod_{i \in I} p_i(z) \cdot \prod_{i \in J} p_i(z) = z^{\nu(X)} \cdot \prod_{i \in J'} p'_i \cdot a \prod_{i \in J'} q'_i.
\]
Note that $|ac| = 1$, since
\[
  |X(1)| = |ac| \prod_{i \in I'} |p'_i(1)| \cdot \prod_{i \in J'} |q'_i(1)| = |ac| \prod_{i \in I'} |p'_i(1)| \cdot \prod_{i \in J'} |q'_i(1)| = |ac| |Y(1)|.
\]
Consequently, we obtain for suitable $m \in \mathbb{Z}^d$ and $\gamma := ac$ the factorization
\[
  X(z) = \gamma z^m \cdot Y_1(z) \cdot \overline{Y_2(\bar{z}^{-1})},
\]
with $Y_1 := \prod_{i \in I'} p'_i$ and $Y_2 := \prod_{i \in J'} p'_i$.

For the sufficiency let $X$ be a polynomial of the form (10). Then
\[
  |\hat{x}(\omega)|^2 = X(e^{-2\pi i \omega / n}) \cdot \overline{X(e^{-2\pi i \omega / n})}
  = Y_1(e^{-2\pi i \omega / n}) \cdot \overline{Y_1(e^{-2\pi i \omega / n})} \cdot Y_2(e^{-2\pi i \omega / n}) \cdot \overline{Y_2(e^{-2\pi i \omega / n})} = |\hat{y}(\omega)|^2.
\]
\[\square\]

The factor $\gamma$ in formula (10) corresponds to multiplication with a unimodular constant. Multiplication by $z^\tau$ corresponds to a translation by $\tau$ in the signal domain. Flipping the $Z$-transform, i.e., choosing $Y(z) = Y_2(z) = \overline{X(\bar{z}^{-1})}$, amounts to reflection and conjugation in the signal domain.

For $x$ to have non-trivial ambiguities it is therefore necessary that its $Z$-transform $X$ is reducible. Note that this is not sufficient in general, as the factors of $X$ may possess symmetry properties such that a flipping does not introduce non-trivial ambiguities. Nevertheless, this observation yields an upper bound on the number of ambiguous solutions for $x \in \mathbb{C}^{J_n}$ denoted by
\[
  \mathcal{N}(x) := \# \{[y] \sim \in \mathbb{C}^{J_n} / \sim : |\hat{y}| = |\hat{x}| \}. 
\]

**Corollary 3.9.** Let $x \in \mathbb{C}^{J_n}$ and let $X$ denote its $Z$-transform. Then $\mathcal{N}(x) \leq 2^{L-1}$, where $L$ denotes the number of non-trivial factors of $X$.

In the one-dimensional case $d = 1$ the $Z$-transform $X$ is a polynomial of one variable of order $k \leq n$. By the fundamental theorem of algebra $X$ has $k$ roots and can be expressed as product of $k$ linear factors. The situation in the higher dimensional case is radically different, as shown by Hayes and McClellan [34]:
Theorem 3.10 ([34]). Let $\mathcal{P}^{d,k}$ denote the set of complex polynomial of $d > 1$ variables with order $k$ and let $m$ denote the degrees of freedom of $\mathcal{P}^{d,k}$. We identify $\mathcal{P}^{d,k}$ with $\mathbb{C}^m \simeq \mathbb{R}^{2m}$. Then the set of reducible polynomials in $\mathcal{P}^{d,k}$ is a set of measure zero (as subset of $\mathbb{C}^m$).

Corollary 3.9 together with Theorem 3.10 yields the following result.

Corollary 3.11. If $d = 1$, then for any fixed $n \in \mathbb{N}$ the set \( \{ x \in \mathbb{C}^n : N(x) < 2^{n-1} \} \) is of measure zero.

If $d > 1$, then for any fixed $n \in \mathbb{N}^d$ the set \( \{ x \in \mathbb{C}^d : N(x) > 1 \} \) is of measure zero.

The gap between the one- and the multidimensional situation has also been encountered in computational experiments, such as those carried out in [25], where it was observed that

"Experimental results suggest that the uniqueness problem is severe for one-dimensional objects but may not be severe for complicated two-dimensional objects."

3.1.2. The continuous Fourier Phase Retrieval problem. We now consider the continuous case and start with identifying the trivial ambiguities. As in the discrete case, let $T_\tau$ denote the translation operator $T_\tau f(x) = f(x - \tau)$ for $\tau \in \mathbb{R}^d$ and $R$ the reflection operator $Rf(x) = f(-x)$.

Proposition 3.12. Let $f \in L^2(\mathbb{R}^d)$. Then each of the following choices of $g$ yields the same Fourier magnitudes as $f$, i.e., $|\hat{g}| = |\hat{f}|$:

(i) $g = cf$, for $|c| = 1$,
(ii) $g = T_\tau f$, for $\tau \in \mathbb{R}^d$,
(iii) $g = \overline{Rf}$.

The proof is straightforward. Again the ambiguities of Proposition 3.12 and their combinations are considered trivial ambiguities.

A standard assumption is to only consider compactly supported functions. In the context of imaging applications, this restriction is rather mild as it requires the object of interest to be of finite extent. The great advantage of this assumption is that the Fourier transform of compactly supported functions extends analytically to all of $\mathbb{C}^d$ and one can draw upon complex analysis and the theory of entire functions in particular. By the well-known Paley-Wiener theorem [48] for functions of one variable also the converse holds true. The extension to higher dimensions is due to Plancherel and Pólya [50].

Theorem 3.13 (Paley-Wiener). Let $f \in L^2(\mathbb{R}^d)$ be compactly supported. Then its Fourier-Laplace transform

$$F(z) := \int_{\mathbb{R}^d} f(x)e^{-2\pi iz \cdot x} \, dx \quad \forall z \in \mathbb{C}^d$$
is an entire function of exponential type, i.e., there exist $C_1, C_2 > 0$ such that

$$|F(z)| \leq C_1 e^{C_2|z|} \quad \forall z \in \mathbb{C}^d.$$  

Conversely, suppose $F : \mathbb{C}^d \to \mathbb{C}$ is an entire function of exponential type and its restriction to the real plane $F|_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{C}$ is square integrable. Then $F$ is the Fourier-Laplace transform of a compactly supported function $f \in L^2(\mathbb{R}^d)$.

**Definition 3.14.** An entire function $F$ of one or several variables is called *reducible* if there exist entire functions $G, H \neq 0$ both having a non-empty zero set such that $F = G \cdot H$. Otherwise $F$ is called *irreducible*.

The decomposition of an entire function of exponential type into irreducible factors is unique up to non-vanishing factors. For functions of one variable this is due to Weierstrass factorization theorem [41], for functions of several variables due to Osgood [47]. A similar result as in the discrete case, cf. Theorem 3.8, can be established.

**Theorem 3.15.** Let $f, g \in L^2(\mathbb{R}^d)$ be compactly supported and let $F, G$ denote the Fourier-Laplace transform of $f$ and $g$ respectively. Then $|\hat{f}| = |\hat{g}|$ if and only if there exists a factorization $G = G_1 \cdot G_2$, a constant $\gamma$ with $|\gamma| = 1$, and $\tau \in \mathbb{R}^d$ such that

$$F(z) = \gamma e^{2\pi i \tau \cdot z} \cdot G_1(z) \cdot \overline{G_2(z)}. \tag{13}$$

**Proof.** The proof is quite similar to the proof of Theorem 3.8. Therefore we only give a sketch. Firstly, from the assumption that $|\hat{f}| = |\hat{g}|$ it follows by analytic extension that

$$F(z) \cdot \overline{F(z)} = G(z) \cdot \overline{G(z)} \quad \forall z \in \mathbb{C}^d. \tag{14}$$

Both $F$ and $G$ can be represented as (infinite) products of irreducible functions, where the representations are essentially unique. By plugging the product expansions into (14) one can finally deduce in a similar way as in the proof of Theorem 3.8 that equation (13) holds true.

Sufficiency follows from the observation that the function defined by the right hand side of (13) has the same modulus as $G$ for arguments in $\mathbb{R}^d$.  

The constant $\gamma$ and the modulation $e^{2\pi i \tau \cdot z}$ in formula (13) correspond to multiplication by a unimodular constant and translation in the signal domain respectively. Flipping the whole Fourier-Laplace transform, i.e., choosing $G(z) = G_2(z) = \overline{F(z)}$, amounts to reflection and conjugation of the underlying function.

By making use of the Paley-Wiener theorem, we can characterize all ambiguous solutions of a given function $f$:

**Corollary 3.16.** Let $f \in L^2(\mathbb{R}^d)$ be compactly supported and let $F$ denote its Fourier-Laplace transform. Furthermore, suppose that $F = F_1 \cdot F_2$ be such that the entire function $G(z) := F_1(z) \cdot F_2(\overline{z})$ is of exponential type. Then for any constant $\gamma$ with $|\gamma| = 1$ and $\tau \in \mathbb{R}^d$ the function

$$g := \gamma \cdot T_\tau F^{-1}(G|_{\mathbb{R}^d})$$
is ambiguous with respect to $f$, i.e., $|\hat{g}| = |\hat{f}|$. Here $G|_{\mathbb{R}^d}$ denotes the restriction of $G$ to real-valued inputs and $\mathcal{F}$ is the usual Fourier transform on $\mathbb{R}^d$.

For functions of one variable the question of uniqueness has been studied in the late 50s by Akutowicz [2, 3] and a few years later independently by Walther [57] and Hofstetter [35]. Their results reveal that all ambiguous solutions of the phase retrieval problem are obtained by flipping a set of zeros of the holomorphic extension of the Fourier transform across the real axis.

**Theorem 3.17** (Akutowicz-Walther-Hofstetter). Let $f, g \in L^2(\mathbb{R})$ be compactly supported and let $F, G$ denote their respective Fourier-Laplace transforms. Furthermore, let $m \in \mathbb{N}_0$ denote the order of zeros at the origin and let $z_1, z_2, \ldots \in \mathbb{C} \setminus \{0\}$ denote the remaining zeros including multiplicity. Then $|\hat{f}| = |\hat{g}|$ if and only if there exist $a, b \in \mathbb{R}$ and $J \subseteq \mathbb{N}$ such that

$$G(z) = e^{i(a+bx)}z^m \cdot \prod_{j \in J} \left(1 - \frac{z}{z_j}\right)e^{z/z_j} \cdot \prod_{j \in \mathbb{N} \setminus J} \left(1 - \frac{z}{z_j}\right)e^{z/z_j}.$$ 

Theorem 3.17 can be deduced from Theorem 3.15 by applying Hadamard’s factorization theorem (see for example [1]), which states that an entire function of one complex variable is essentially determined by its zeros. More precisely, suppose $F$ is an entire function of exponential type with a zero of order $m$ at the origin and remaining zeros at $z_1, z_2, \ldots \in \mathbb{C} \setminus \{0\}$ including multiplicity. Then there exist $a, b \in \mathbb{C}$ such that

$$F(z) = e^{az+b}z^m \cdot \prod_{j \in \mathbb{N}} \left(1 - \frac{z}{z_j}\right)e^{z/z_j}.$$ 

While for functions of one variable expecting uniqueness is in general hopeless, it is commonly asserted that—similar to the finite dimensional case—the situation changes drastically when switching to multivariate functions, see [11] where it is referred to Theorem 3.10 and stated that

“Irreducibility extends to general functions of two variables with infinite sets of zeros, so that exact alternative solutions are most unlikely in 2-D phase retrieval.”

However, we are not aware of a rigorous argument of this claim.

### 3.1.3. Restriction and Modification of the 1D Fourier Phase Retrieval Problem.

Common restriction approaches to achieve uniqueness include (1) to demand the function to be real valued or even positive, (2) to satisfy certain symmetry properties, (3) to be monotonic or (4) its support to satisfy stronger restrictions than only compactness. We will only state an incomplete, deliberate selection of results into this direction. Before that we mention that requiring positivity as the only a priori assumption (apart from compact support) does not suffice for $|\hat{f}|$ to uniquely determine $f$ up to trivial ambiguities, as it has been shown in [23].
Theorem 3.18. Suppose that \( f \in L^2(\mathbb{R}) \) is compactly supported and that there exists \( t_0 \in \mathbb{R} \) such that
\[
\hat{f}(t_0 - t) = \hat{f}(t_0 + t) \quad \forall t \in \mathbb{R}.
\]
Then \( f \) is uniquely (up to trivial ambiguities) determined by \( |\hat{f}| \).

Proof. As translations are trivial ambiguities, we may assume w.l.o.g. that \( t_0 = 0 \).
Due to the symmetry of \( f \), its Fourier-Laplace transform \( F \) satisfies
\[
(15) \quad F(\bar{z}) = \int_{\mathbb{R}} f(t)e^{2\pi izt} dt = \int_{\mathbb{R}} f(-t)e^{2\pi izt} dt = F(z) \quad \forall z \in \mathbb{C}.
\]
Particularly, the zeros of \( F \) appear symmetrically with respect to the real axis. In fact \( z_0 \) is a zero of multiplicity \( m \) if and only if \( \bar{z}_0 \) is a zero of multiplicity \( m \).
Furthermore, the zeros \( z_1, z_2, \ldots \) of the product \( I(z) = F(z)F(\bar{z}) \) occur with even multiplicity \( m_1, m_2, \ldots \) due to (15).
Since \(|\hat{f}|\) uniquely determines \( I(z) = F(z)F(\bar{z}) \), and therefore its zeros including multiplicity, the zeros of \( F \) appear with multiplicity \( m_1/2, m_2/2, \ldots \) and are also uniquely determined by \(|\hat{f}|\). Furthermore, \( F \) is uniquely given by its zeros up to a factor \( \gamma e^{2\pi i\tau z} \) with \(|\gamma| = 1 \) and \( \tau \in \mathbb{R}^d \). Consequently, \( f \) is uniquely determined by \(|\hat{f}|\) up to trivial ambiguities. \( \square \)

We have seen in the previous theorem that by requiring \( f \) to be symmetric, the zeros of its Fourier-Laplace transform appear in a symmetric way, which ensures uniqueness. By requiring that \( f \) is monotonically non-decreasing, it can be shown that all the zeros of the Fourier-Laplace transform are located in the lower half plane, which gives the following result.

Theorem 3.19 ([40]). Suppose that \( f \in L^2(\mathbb{R}) \) is compactly supported, positive, and monotonically non-decreasing. Then \( f \) is uniquely (up to trivial ambiguities) determined by \(|\hat{f}|\).

A further method to enforce uniqueness is to require the function to be supported on two intervals which are sufficiently far apart from each other.

Theorem 3.20 ([28, 24]). Suppose that \( f = f_1 + f_2 \in L^2(\mathbb{R}) \) where the support of \( f_1 \) and \( f_2 \) is contained in finite, disjoint intervals \( I_1 \) and \( I_2 \) respectively. Suppose further that the distance between the intervals \( I_1 \) and \( I_2 \) is greater than the sum of their lengths and that the Fourier-Laplace-transforms of \( f_1 \) and \( f_2 \) have no common zeros. Then \( f \) is uniquely (up to trivial ambiguities) determined by \(|\hat{f}|\).

The use of a second measurement obtained by additive distortion by a known signal has also been considered:

Theorem 3.21 ([40]). Suppose \( g \in L^2(\mathbb{R}) \) is compactly supported and its Fourier transform is real valued and suppose \( f \in L^2(\mathbb{R}) \) with compact support in \([0, \infty)\). Then \( f \) is uniquely determined by \(|\hat{f}| \) and \(|\hat{f} + \hat{g}| \).

If the additive distortion \( g \) is chosen to be a suitable multiple of the delta distribution the magnitude information of \( \hat{f} \) is dispensable, i.e. if \( c \) is sufficiently large
compared to $f$, $f$ can be recovered from $|\hat{f} + c|$. The interference of $f$ with such a $g$ pushes all the zeros of the analytic extension of the Fourier transform to the upper half plane. In this case the relation between phase and magnitude is described by the Hilbert transform [17, 18] and remarkably, phase retrieval is rendered not only unique but also stable.

**Theorem 3.22.** For $a, b > 0$ let $B_{a,b} := \{ f \in L^2(\mathbb{R}) : \|f\|_{L^\infty(\mathbb{R})} < a \text{ and } \text{supp}(f) \subseteq [0,b] \} \subseteq L^2(\mathbb{R})$ and for $c \in \mathbb{R}$ let $L^2_c(\mathbb{R}) := \{ f + c : f \in L^2(\mathbb{R}) \}$ endowed with the metric

$$d_{L^2_c(\mathbb{R})}(f, g) := \|f - g\|_{L^2(\mathbb{R})}.$$  

Suppose $c > ab$. Then $A : f \mapsto |\hat{f} + c|$ is an injective mapping from $B_{a,b}$ to $L^2_c(\mathbb{R})$ and $A^{-1} : A(B_{a,b}) \subseteq L^2_c(\mathbb{R}) \rightarrow B_{a,b}$ is uniformly continuous, i.e. there exists a constant $C > 0$ such that

$$\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq C \cdot d_{L^2_c(\mathbb{R})}(|\hat{f}_1 + c|, |\hat{f}_2 + c|) \quad \forall f_1, f_2 \in B_{a,b}. $$

**Proof.** In order to show that $A$ maps from $B_{a,b}$ to $L^2_c(\mathbb{R})$ let $f \in B_{a,b} \subseteq L^2(\mathbb{R})$ be arbitrary. We have to verify that $A f - c \in L^2(\mathbb{R})$. By the reverse triangle inequality we have that

$$|A f - c| = ||\hat{f} + c| - c| \leq |\hat{f} + c| = |\hat{f}|.$$  

Since $\hat{f} \in L^2(\mathbb{R})$ also $A f - c \in L^2(\mathbb{R})$.

Let us denote by $g$ the analytic extension of $\hat{f} + c$, i.e.,

$$g(z) = \int_\mathbb{R} f(t) e^{-2\pi i z t} \, dt + c \quad \forall z \in \mathbb{C}.$$  

Then—provided that $g$ has all its zeros in the upper (or lower) half-plane—phase and magnitude of $g$ are related via the Hilbert transform [18], i.e.,

$$|\alpha(x) := H(\ln|g|)(x) := -\frac{1}{\pi} P.V. \int_\mathbb{R} \frac{\ln|g(t)|}{t - x} \, dt \quad \forall x \in \mathbb{R},$$

satisfies $g = |g| e^{i\alpha}$.  

In order to make use of this identity, we check that $g$ has no zeros in the lower half-plane: For $\text{Im} \, z \leq 0$ it holds that

$$\left| \int_\mathbb{R} f(t) e^{-2\pi i z t} \, dt \right| \leq \|f\|_{L^1(\mathbb{R})} \leq ab$$  

and we have $|g(z)| \geq ||\hat{f}(z)| - |c|| \geq c - ab > 0$ in the lower half-plane since $c > ab$.

For $f_1, f_2 \in B_{a,b}$ let $g_k := \hat{f}_k + c$ and let $\alpha_k := H(\ln|g_k|)$. Then we have for $k = 1, 2$ that

$$|g_k(x)| = |\hat{f}_k(x) + c| \geq c - |\hat{f}_k(x)| \geq c - ab > 0 \quad \forall x \in \mathbb{R}$$
and similarly that $|g_k(x)| \leq c + ab$. It follows that there exists a constant $C_1 > 0$ (depending on $a, b, c$) such that
\begin{equation}
|\ln|g_1(x)| - \ln|g_2(x)|| \leq C_1 \cdot ||g_1(x)| - |g_2(x)|| \quad \forall x \in \mathbb{R},
\end{equation}
which implies that the difference $\ln|g_1| - \ln|g_2|$ is an element of $L^2(\mathbb{R})$. According to (16) the phase difference $\Delta := \alpha_1 - \alpha_2$ can be computed by $\Delta = H(\ln|g_1| - \ln|g_2|)$. By using the well-known fact that the Hilbert transform is an isometry on $L^2(\mathbb{R})$ [55] and (17) it follows that there exists a constant $C_2$ (depending on $a, b, c$) such that
\[ ||\Delta||_{L^2(\mathbb{R})} \leq C_2 \cdot |||g_1| - |g_2|||_{L^2(\mathbb{R})}. \]

Thus we obtain by using the elementary estimate $|1 - e^{it}| \leq |t|$, $t \in \mathbb{R}$ that
\begin{align*}
||f_1 - f_2||_{L^2(\mathbb{R})} &= ||\hat{f}_1 - \hat{f}_2||_{L^2(\mathbb{R})} = ||g_1 - g_2||_{L^2(\mathbb{R})} \\
&= ||g_1 e^{i\alpha_1} - g_2 e^{i\alpha_2}||_{L^2(\mathbb{R})} \\
&\leq ||g_1 |(1 - e^{-i\Delta})||_{L^2(\mathbb{R})} + |||g_1| - |g_2|||_{L^2(\mathbb{R})} \\
&\leq ||g_1||_{L^\infty(\mathbb{R})} \cdot ||\Delta||_{L^2(\mathbb{R})} + |||g_1| - |g_2|||_{L^2(\mathbb{R})} \\
&\leq C_3 |||g_1| - |g_2|||_{L^2(\mathbb{R})},
\end{align*}
for suitable constant $C_3 > 0$. □

**Remark 3.23.** Note that the assumption $\text{supp } f \subseteq [0, b]$ implies not only that $\hat{f}$ is bandlimited but also $|\hat{f}|^2$ and $\text{Re } \hat{f}$. Therefore the function
\[ |\hat{f} + c|^2 - c^2 = |\hat{f}|^2 + 2c \text{ Re } \hat{f} \]
is also bandlimited and $|\hat{f} + c|$ can be uniquely and stably determined from samples. Together with Theorem 3.22, this implies that any $f \in B_{a,b}$ can be recovered stably from the samples of $|\hat{f} + c|$ on a suitable discrete set.

3.2. **The Short-Time Fourier Phase Retrieval Problem.** As we have seen in the previous section, Fourier phase retrieval has numerous ambiguities even after accounting for the trivial ones. One workaround is to restrict the set of functions we aim to recover. In Subsection 3.1.3 the assumption of symmetry or stricter conditions on the support of $f$ ensured injectivity of the phase retrieval problem.

Another approach is to slightly change the experiment to obtain more measurements. Introducing a window $\tilde{g}$ in front of the object $f$ yields the (signed) measurements
\[ V_g f(x, \xi) := (f \cdot T_x \tilde{g})^\wedge (\xi) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \xi} \, dx \quad \forall x, \xi \in \mathbb{R}^d, \]
depending on the position $x \in \mathbb{R}^d$ of the “sliding window” $\tilde{g}$. This modified Fourier transform is called the short-time Fourier transform (STFT) and is well studied in time-frequency analysis.

For $g, f \in L^2(\mathbb{R}^d)$ this transform is well-defined. By duality one can extend this to tempered distributions with windows in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. To mimic the $L^2$-inner product, it is customary to use the complex conjugate $\tilde{g}$ and still refer to
This change in the set-up introduces additional information, which makes phase retrieval possible for a suitable choice of window. Central to the reconstruction of a function from its phaseless short-time Fourier transform measurements is the following fundamental formula of time-frequency analysis

\[(V_g f \cdot \overline{V_u h})(x, \xi) = (V_h f \cdot \overline{V_u g})(-\xi, x).\]

By taking the Fourier transform of the squared phaseless measurements, one can therefore recover \(V_f f\) as long as \(V_g g\) does not vanish anywhere. Applying the inverse Fourier transform to \(V_f f(x, \xi) = (f \cdot T_x \bar{f})^\sim(\xi)\) yields the original function \(f\) up to a global sign.

The fundamental formula (18) lies at the heart of much of the structure in time-frequency analysis. Its proof is elementary and the consequence of two things: The resulting phase factors that occur when interchanging translation with modulation and a version of Plancherel’s theorem for the short-time Fourier transform.

### 3.2.1. The discrete STFT Phase Retrieval Problem

In this section, we consider finite signals \(x\) in the complex Hilbert space \(\mathbb{C}^N\) with inner product

\[\langle x, y \rangle := \sum_{n=0}^{N-1} x_n \bar{y}_n.\]

The discrete Fourier transform maps finite signals to finite signals and is defined as

\[\hat{x}(j) := \sum_{n=0}^{N-1} x_n e^{-2\pi inj/N} \quad \forall j \in \mathbb{Z}_N.\]

Its inverse is given by

\[\hat{x}(j) := \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{2\pi inj/N} \quad \forall j \in \mathbb{Z}_N,\]

and with the normalization above, Plancherel’s Theorem is of the form

\[\langle \hat{x}, \hat{y} \rangle = N \langle x, y \rangle.\]

We define the (circular) translation and modulation operators by

\[(T_k x)_j := x_{j-k} \quad (\text{mod } N) \quad \text{and} \quad (M_l x)_j := e^{2\pi ij l/N} x_j\]

for \(k, l \in \mathbb{Z}_N\). In the following, we identify the finite signal \(x \in \mathbb{C}^N\) with its periodic extension and just write \((T_k x)_j = x_{j-k}\) for the circular translated signal.

Since a modulation in time corresponds to a shift in frequency, operators of the form \(\pi(\lambda) = \pi(k, l) := M_l T_k\) are called time-frequency shifts for \(\lambda = (k, l)\). Note that time-frequency shifts do not commute, but satisfy the following commutation relation.
Lemma 3.24. Let \( \lambda = (k, l), \mu = (p, q) \in \mathbb{Z}_N^2 \). Then
\[
\pi(\lambda)\pi(\mu) = e^{2\pi i(k-q+l-p)/N} \pi(\mu)\pi(\lambda) = e^{2\pi i\mu \cdot \mathcal{I} \lambda/N} \pi(\mu)\pi(\lambda)
\]
where \( \mathcal{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) denotes the standard symplectic matrix.

We omit the proof, as it is a straight forward verification.

The discrete short-time Fourier transform of \( x \in \mathbb{C}^N \) with respect to the window \( g \in \mathbb{C}^N \) is defined by
\[
V_g x(\lambda) := \langle x, \pi(\lambda)g \rangle = (x \cdot T_\lambda \bar{g})^\sim(l) = \sum_{n=0}^{N-1} x_n \bar{g}_{n-k} e^{-2\pi i n l/N}
\]
for \( \lambda = (k, l) \in \mathbb{Z}_N^2 \).

For fixed window \( g \), the short-time Fourier transform \( V_g \) is a linear operator that maps finite signals in \( \mathbb{C}^N \) to finite signals in \( \mathbb{C}^{N^2} \). Due to the linearity, we again have the trivial ambiguity \( |V_g(cx)| = |V_g x| \) for phase factors \( |c| = 1 \). The question now is whether this are the only ambiguities and how to recover the original signal.

**Problem 2** (Discrete STFT Phase Retrieval). Suppose \( x \in \mathbb{C}^N \). Recover \( x \) from \( |V_g x| \) up to a global phase factor when \( g \in \mathbb{C}^N \) is known.

Whether Problem 2 has a solution depends on the choice of the window \( g \). A sufficient condition is that the short-time Fourier transform \( V_g \) does not vanish anywhere on \( \mathbb{Z}_N^2 \). The remainder of this subsection is devoted to proving this fact.

The main insight for short-time Fourier transform phase retrieval comes from formula (18). Its discrete analogue will be proved in the subsequent and reads
\[
(V_g x \cdot \overline{V_h y})^\sim(k, l) = N(V_g x \cdot \overline{V_h g})(-l, k) .
\]
Consequently, we can recover \( x \) by taking the Fourier transform of the squared phaseless measurements \( |V_g x|^2 \). Since \( V_x x(k, l) = (x \cdot T_\lambda \bar{x})^\sim(l) \), applying the inverse Fourier transform with respect to \( l \in \mathbb{Z}_N \) recovers the signal up to a global phase.

The proof of formula (20) is elementary and requires only two things: The covariance property, which is an easy consequence of the commutation relations (19), and a version of Plancherel’s Theorem for the short-time Fourier transform.

Lemma 3.25 (Covariance Property). Let \( \lambda, \mu \in \mathbb{Z}_N^2 \). Then
\[
V_{\pi(\lambda)g}(\pi(\lambda)x)(\mu) = e^{2\pi i\mu \cdot \mathcal{I} \lambda/N} V_g x(\mu) .
\]

**Proof.** Note that time-frequency shifts are unitary operators on \( \mathbb{C}^N \). Hence
\[
V_{\pi(\lambda)g}(\pi(\lambda)x)(\mu) = (\pi(\lambda)x, \pi(\mu)\pi(\lambda)g)
= e^{2\pi i\mu \cdot \mathcal{I} \lambda/N} (\pi(\lambda)x, \pi(\lambda)\pi(\mu)g)
= e^{2\pi i\mu \cdot \mathcal{I} \lambda/N} (x, \pi(\mu)g)
= e^{2\pi i\mu \cdot \mathcal{I} \lambda/N} V_g x(\mu) ,
\]
where we used the commutation relation (19) in the second line.
Proposition 3.26 (Orthogonality Relations). Let \( g, h, x, y \in \mathbb{C}^N \). Then

\[
\langle V_g x, V_h y \rangle = N \langle x, y \rangle \langle h, g \rangle .
\]

Proof. We write the short-time Fourier transform as \( V_g x(k, l) = (x \cdot T_k \tilde{g})^\ast(l) \) and use Plancherel’s Theorem in the sum over \( l \in \mathbb{Z}_N^2 \):

\[
\langle V_g x, V_h y \rangle = \sum_{k,l=0}^{N-1} V_g x(k, l) V_h y(k, l) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \left(x \cdot T_k \tilde{g}\right)^\ast(l) \left(y \cdot T_l \tilde{h}\right)^\ast(l)
\]

\[
\sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x_n \tilde{g}_{n-k} \tilde{h}_{n-k} = N \langle x, y \rangle \langle h, g \rangle .
\]

\( \square \)

Proposition 3.27. Let \( x, y, g, h \in \mathbb{C}^N \). Then

\[
(V_g x \cdot \overline{V_h y})(\lambda) = N (V_g x \cdot \overline{V_h g})(-\mathcal{I} \lambda) \quad \forall \lambda \in \mathbb{Z}_N^2 ,
\]

where \( \mathcal{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) denotes the standard symplectic matrix.

Proof. First note that \( \mathcal{I}^2 = -I \), where \( I \) denotes the identity matrix. Consequently,

\[
e^{-2\pi i \mu \cdot \mathcal{I}^x / N} V_g x(\mu) = e^{2\pi i \mu \cdot \mathcal{I}^y / N} V_g x(\mu) = V_{\pi(\mathcal{I})} g(\pi(\mathcal{I}) x)(\mu)
\]

by Lemma 3.25. Hence, we obtain

\[
(V_g x \cdot \overline{V_h y})(\lambda) = \sum_{\mu \in \mathbb{Z}_N^2} V_g x(\mu) \overline{V_h y(\mu)} e^{-2\pi i \mu \cdot \mathcal{I}^x / N} = \sum_{\mu \in \mathbb{Z}_N^2} V_{\pi(\mathcal{I})} g(\pi(\mathcal{I}) x)(\mu) \overline{V_h y(\mu)}
\]

\[
= \langle V_{\pi(\mathcal{I})} g(\pi(\mathcal{I}) x), V_h y \rangle = N \langle \pi(\mathcal{I}) x, y \rangle \langle h, \pi(\mathcal{I}) y \rangle ,
\]

where we used the orthogonality relations (21) in the last step.

Note that \( \pi(\lambda)^* = c \pi(-\lambda) \) for a suitable phase factor \( |c| = 1 \). But these phase factors cancel when we bring both time-frequency shifts to the other side, hence

\[
(V_g x \cdot \overline{V_h y})(\lambda) = N \langle x, \pi(-\mathcal{I}) y \rangle \langle g, \pi(-\mathcal{I}) h \rangle = N (V_g x \cdot \overline{V_h g})(-\mathcal{I} \lambda) .
\]

\( \square \)

We can now prove a sufficient condition on the window to allow phase retrieval.

Theorem 3.28. Let \( g \in \mathbb{C}^N \) be a window with \( V_g g(\lambda) \neq 0 \) for all \( \lambda \in \mathbb{Z}_N^2 \). Then any \( x \in \mathbb{C}^N \) can be recovered from \( |V_g x| \) up to a global phase factor.

Proof. By Proposition 3.27 we have

\[
|V_g x|^2(k, l) = NV_x x(-l, k) \cdot \overline{V_g g(-l, k)} \quad \forall k, l \in \mathbb{Z}_N .
\]

If \( V_g g \) has no zeros, we can recover \( V_x x \). Now we apply the inverse discrete Fourier transform to \( V_x x(k, l) = (x \cdot T_k \tilde{g})^\ast(l) \) and obtain

\[
x_j \cdot \overline{x}_{j-k} = \frac{1}{N} \sum_{l=0}^{N-1} V_x x(k, l) e^{2\pi i l j / N} .
\]
Setting \( k = j \) yields
\[
x_j \cdot \bar{x}_0 = \frac{1}{N} \sum_{l=0}^{N-1} V_x x(j, l) e^{2\pi i j/N}
\]
and we recover the signal \( x \) up to a global phase factor after dividing by \( |x_0| \). \( \square \)

Theorem 3.28 also appears in [15], where it is proved with the methods introduced in [10]. Moreover, the authors also give examples and counter-examples of window functions \( g \) satisfying \( V_g g(\lambda) \neq 0 \) for all \( \lambda \in \mathbb{Z}_N^2 \).

### 3.2.2. The Continuous STFT Phase Retrieval Problem

We return to the continuous case. Recall that the short-time Fourier transform of \( f \in L^2(\mathbb{R}^d) \) with respect to the window \( g \in L^2(\mathbb{R}^d) \) is defined by
\[
V_g f(x, \xi) := (f \cdot T_x g)^\wedge(\xi) = \int_{\mathbb{R}^d} f(t) g(t - x) e^{-2\pi i t \cdot \xi} \, dx \quad \forall x, \xi \in \mathbb{R}^d.
\]

If we fix the window \( g \), the short-time Fourier transform \( V_g \) is a linear operator from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^{2d}) \). Consequently, a multiplication of \( f \) with a unimodular constant produces the same phaseless short-time Fourier transform measurements and is therefore considered a trivial ambiguity. The problem of phase retrieval now reads as follows.

**Problem 3** (STFT Phase Retrieval). Suppose \( f \in L^2(\mathbb{R}^d) \). Recover \( f \) from \( |V_g f| \) up to a global phase factor when \( g \in L^2(\mathbb{R}^d) \) is known.

Whether Problem 3 is well-posed depends on the choice of the window \( g \). Again a sufficient condition for uniqueness is given in terms of the zero set of its short-time Fourier transform \( V_g g \). The proof of this result is analogous to the discrete case with the following fundamental formula at its core.

**Proposition 3.29.** Let \( f, h, g, u \in L^2(\mathbb{R}^d) \). Then
\[
(V_g f \cdot \overline{V_u h})^\wedge(x, \xi) = (V_h f \cdot \overline{V_u g})(-\xi, x) \quad \forall x, \xi \in \mathbb{R}^d.
\]

Proposition 3.29 is obtained as in the discrete setting by combining the covariance property with the orthogonality relations. The relevant properties of the short-time Fourier transform and their detailed proof can be found in [29, 30].

We can now prove the following theorem.

**Theorem 3.30.** Let \( g \in L^2(\mathbb{R}^d) \) with \( V_g g(x, \xi) \neq 0 \) for almost all \( x, \xi \in \mathbb{R}^d \). Then for any \( f, h \in L^2(\mathbb{R}^d) \) with \( |V_g f| = |V_g h| \) there exists \( \alpha \in \mathbb{R} \) such that \( h = e^{i\alpha} f \).

Moreover, any \( f \in S(\mathbb{R}^d) \) can be recovered from \( |V_g f| \) up to a global phase factor.

**Proof.** By Proposition 3.29 we obtain that
\[
(|V_g f|^2)^\wedge(x, \xi) = V_f f(-\xi, x) \cdot \overline{V_g g(-\xi, x)} \quad \forall x, \xi \in \mathbb{R}^d
\]
and recover \( V_f f \) almost everywhere.
If \( f \in \mathcal{S}(\mathbb{R}^d) \), then \( f \cdot T_x \widehat{f} \in \mathcal{S}(\mathbb{R}^d) \) for every \( x \in \mathbb{R}^d \) and we can apply the inverse Fourier transform to \( V_f f(x, \xi) = (f \cdot T_x \widehat{f})^\wedge(\xi) \). This yields

\[
f(t) \cdot \widehat{f}(t-x) = \int_{\mathbb{R}^d} V_f f(x, \xi) e^{2\pi i \xi \cdot t} \, d\xi \quad \forall x, t \in \mathbb{R}^d.
\]

Since this holds pointwise, we can chose \( t = x \) and obtain

\[
f(x) \cdot \widehat{f}(0) = \int_{\mathbb{R}^d} V_f f(x, \xi) e^{2\pi i \xi \cdot x} \, d\xi \quad \forall x \in \mathbb{R}^d.
\]

Again, we recover \( f \) up to a global sign after dividing by \( |f(0)| \).

If \( f \in L^2(\mathbb{R}^d) \), the inversion formula need not hold pointwise anymore and we cannot argue as above. But suppose that \( |V_g f| = |V_h h| \) for an \( h \in L^2(\mathbb{R}^d) \), then \( V_f f = V_h h \) holds almost everywhere by (22).

Now we use Proposition 3.29 in the reverse direction with arbitrary \( u \in L^2(\mathbb{R}^d) \) and obtain

\[
(V_u f \cdot \overline{V_g f})(-\xi, x) = (V_f f \cdot \overline{V_g u})^\wedge(x, \xi) = (V_h h \cdot \overline{V_g h})^\wedge(-\xi, x)
\]

for all \( x, \xi \in \mathbb{R}^d \). Setting \((x, \xi) = (0, 0)\) yields

\[
\int_{\mathbb{R}^d} f(t) u(t) \, dt = V_u f(0, 0) = \left( \frac{V_h h}{V_g f} \right)(0, 0) \cdot V_u h(0, 0)
\]

\[
= \left( \frac{V_h h}{V_g f} \right)(0, 0) \int_{\mathbb{R}^d} h(t) u(t) \, dt.
\]

Since \( u \in L^2(\mathbb{R}^d) \) was arbitrary, we conclude that \( f = c h \) with the unimodular constant \( c = (V_h h / V_g f)(0, 0) \). \( \square \)

**Remark 3.31.** The condition on the zero set of the ambiguity function in Theorem 3.30 is at least almost necessary in the following sense: Suppose the ambiguity function vanishes on an open set. Then one can construct functions \( f_1, f_2 \neq 0 \) such that \( f_\pm := f_1 \pm f_2 \) have the same short-time Fourier transform magnitudes, i.e., \( |V_g f_\pm| = |V_g f_-| \). By construction \( f_+ \) and \( f_- \) cannot be trivial associates, i.e., there is no \( \alpha \in \mathbb{R} \) such that \( f_- = e^{i\alpha} f_+ \).

Let us mention some examples for window functions that allow phase retrieval because their short-time Fourier transform does not vanish. The obvious candidate is the Gaussian \( \varphi(x) = e^{-\pi |x|^2} \), whose short-time Fourier transform \( V_{\varphi} \varphi \) is again a (generalized) Gaussian. A lesser known example is the one-sided exponential \( g(x) = e^{-\alpha x} \chi_{[0, \infty)} \) for parameter \( \alpha > 0 \). Already Janssen [38] computed its short-time Fourier transform \( V_{\varphi} g = e^{-|x| (\alpha + \pi i \xi)} / (2\alpha + 2\pi i \xi) \), which clearly does not vanish. More examples can be found in the recent paper by Gröchenig et al. [31].

The choice of the one-dimensional Gaussian \( \varphi(x) = e^{-\pi x^2} \) is special in one crucial point: it is the only window for which \( V_{\varphi} f \) yields a holomorphic function after a slight modification [7]. Hence the full toolbox of complex analysis becomes available when working with a Gaussian window. The short-time Fourier transform with Gaussian window is also called the *Gabor transform* and will be denoted by
In the remainder, we present a result of two of the authors [32], which gives a characterization of instabilities of the Gabor phase retrieval problem. The work in [32] builds upon results by one of the authors and his collaborators [4], where for phaseless measurements arising from holomorphic functions it is shown that the phase can be stably recovered on so-called atolls.

By an instability we mean, roughly speaking, a signal $f$ for which there exists a signal $g$ which is very different from $f$, but at the same time produces very similar phaseless measurements. This intuition is formalized by the local Lipschitz constant of the solution operator $|Gf| \mapsto f \sim e^{i\alpha} f$.

**Definition 3.32.** Let $\mathcal{A}$ be a mapping from $\mathcal{X}$ to $\mathcal{Y}$, where $(\mathcal{X}, d_X)$ and $(\mathcal{Y}, d_Y)$ are metric spaces. Then the local stability constant $C_\mathcal{A}(f)$ of $\mathcal{A}$ at $f \in \mathcal{X}$ is defined as the smallest positive number $C$ such that

$$d_X(f, g) \leq C \cdot d_Y(\mathcal{A}f, \mathcal{A}g), \quad \forall g \in \mathcal{X}.$$

Instabilities are routinely constructed by fixing a well localized function $f_0$; then for large $\tau$ the functions

$$f^\tau_\pm := f_0(\cdot - \tau) \pm f_0(\cdot + \tau)$$

yield approximately the same phaseless Gabor measurements. Even more so the stability constant degenerates exponentially in $\tau$, i.e., $C_{|G|}(f^\tau_+) \gtrsim e^{c\tau^2}$ for suitable metrics [6].

As we shall see, the stability constant for Gabor phase retrieval can be controlled in terms of a concept which has been introduced by Cheeger in the field of Riemannian geometry [21].

**Definition 3.33.** Let $\Omega \subseteq \mathbb{R}^d$ be open. For a continuous, non-negative, integrable function $w$ on $\Omega$ the Cheeger constant is defined as

$$(23) \quad h(w, \Omega) := \inf_{C \subseteq \Omega, \partial C \text{ smooth}} \frac{\int_{\partial C \cap \Omega} w}{\min\{\int_C w, \int_{\Omega \setminus C} w\}}.$$

A small Cheeger constant indicates that the domain can be partitioned into two subdomains such that the weight is rather small on the separating boundary of the two subdomains and that, at the same time both subdomains carry approximately the same amount of $L^1$-energy. In that sense the Cheeger constant captures the disconnectedness of the weight, cf. Figure 2.

Before we state the stability result, both the signal space and the measurement space have to be endowed with suitable metrics. To this end we define Feichtinger’s algebra and a family of weighted Sobolev norms.

**Definition 3.34.** Feichtinger’s algebra is defined as

$$\mathcal{M}^1 := \{ f \in L^2(\mathbb{R}) : |\mathcal{G}f| \in L^1(\mathbb{R}^2) \},$$
Figure 2. If the weight has its mass concentrated on two or more disjoint subdomains a partition can be found such that both components of the partition carry approximately the same amount of energy and at the same time the weight is small along the separating boundary (left figure), i.e., the Cheeger constant is small in that case. If on the other hand the mass is concentrated on a single connected domain a partition which satisfies both requirements cannot be found: Aiming for small values of the weight along the separating boundary will not distribute the mass well between the two components (center), whereas a fair distribution of the mass entails that the weight is substantially large on parts of the boundary.

with induced norm $\|f\|_{M^1} := \|Gf\|_{L^1(\mathbb{R}^2)}$.

**Definition 3.35.** For $1 \leq p, q < \infty$, $s > 0$ and $F : \mathbb{R}^2 \to \mathbb{C}$ sufficiently smooth we define

$$\|F\|_{D_{p,q}} := \|F\|_{L^p(\mathbb{R}^2)} + \|\nabla F\|_{L^p(\mathbb{R}^2)} + \|\|x| + |y|\|^s F(x,y)\|_{L^q(\mathbb{R}^2)}.$$  

The main stability result in [32] now reads as follows.

**Theorem 3.36.** Let $q > 2$. Let $\mathcal{X} := M^1 / \sim^1$ be endowed with the metric

$$d_\mathcal{X}([f]_\sim, [g]_\sim) := \inf_{\alpha \in \mathbb{R}} \|f - e^{i\alpha}g\|_{M^1},$$

and let $\mathcal{Y} := |G|(M^1)$ be endowed with the metric induced by the norm $\|\cdot\|_{D_{4,1}^1}$. Suppose that $f \in M^1$ is such that $|Gf|$ has a global maximum at the origin. Then there exists a constant $c$ that only depends on $q$ and the quotient $\|f\|_{M^1} / \|Gf\|_{L^\infty(\mathbb{R}^2)}$ such that

$$C|G|((f)_\sim) \leq c(1 + h(|Gf|, \mathbb{R}^2))^{-1}).$$

Disregarding the weak dependence of $c$ on $f$ the estimate (24) can be informally summarized by

“The only instabilities for Gabor phase retrieval are of disconnected type.”

Before we give a sketch of the proof we set the stability result in relation to the general results in the abstract setting in Section 2.2, where the concept of the $\sigma$-strong

\[1f \sim g \text{ if and only if } g = e^{i\alpha}f \text{ for some } \alpha \in \mathbb{R}.\]
complement property was introduced. In the context of Gabor phase retrieval Remark 2.15 can be qualitatively understood in the following way. A function $x$ is rather instable if it can be written as $x = f + h$ with $\|f\|_{L^2(\mathbb{R})}, \|h\|_{L^2(\mathbb{R})} \approx 1$ such that their respective Gabor measurements are essentially supported on two disjoint domains. In other words the time-frequency plane can be split up into $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \setminus S$ such that both $\|Gf\|_{L^2(S)}$ and $\|Gh\|_{L^2(\mathbb{R}^2 \setminus S)}$ are small. If the metrics on the signal and measurement space are both induced by the respective $L^2$-norm it holds that

$$ C_{|g|}(x) \gtrsim \sup_{f,h:x=f+h, S \subseteq \mathbb{R}^2} \max\{\|Gf\|_{L^2(S)}, \|Gh\|_{L^2(\mathbb{R}^2 \setminus S)}\} \div \min\{\|f\|_{L^2(\mathbb{R})}, \|h\|_{L^2(\mathbb{R})}\}. $$

Theorem 3.36 nicely complements this result in the sense that the disconnectedness as quantified by the Cheeger constant—which to some extent resembles the lower bound in equation (25)—also gives an upper bound on the local stability constant.

**Architecture of the proof.** Let us start with the observation that for any $f, g \in \mathcal{M}$ it holds that

$$ d_X([f]_\sim, [g]_\sim) = \inf_{|c|=1} \| Gg - Gf \|_{L^1(\mathbb{R}^2, w)}, $$

where $w = |Gf|$.

Now suppose that we could just disregard the constraint $|c| = 1$ in (26) (this can be justified with considerable effort). The Poincaré inequality tells us that there exists a constant $C_{\text{poinc}}(w)$ such that (26) can be bounded by

$$ C_{\text{poinc}}(w) \cdot \left\| \nabla \frac{Gg}{Gf} \right\|_{L^1(\mathbb{R}^2, w)}. $$

Now spectral geometry enters the picture. Cheeger’s inequality [21] says that the Poincaré constant on a Riemannian manifold can be controlled by the reciprocal of the Cheeger constant. We would like to apply this result to the metric induced by the metric tensor $\left( w(x, y) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ in order to get a bound on $C_{\text{poinc}}(w)$. However, since $w$ in our case arises from Gabor measurements it generally has zeros and therefore does not qualify as Riemannian manifold. Nevertheless a version of Cheeger’s inequality can be established, i.e.

$$ C_{\text{poinc}}(w) \lesssim h(w, \mathbb{R}^2)^{-1}, $$

where $h(w, \mathbb{R}^2)$ is defined as in (23).

Next we will make use of the fact that for any $h \in L^2(\mathbb{R})$

$$ z = x + iy \mapsto \mathcal{G}h(x, y) \cdot e^{\pi(|z|^2/2 - ixy)} $$

is an entire function (up to reflection). Thus $\mathcal{G}g/\mathcal{G}f$ is meromorphic (again up to reflection) and by applying the Cauchy-Riemann equations one elementarily
computes that
\begin{equation}
\left| \nabla \frac{Gg}{Gf} \right| = \sqrt{2} \cdot \left| \nabla \frac{Gg}{Gf} \right|
\end{equation}
almost everywhere. Combining (26), (27), (28) and (30) yields that
\begin{equation}
d_X([f]_\sim, [g]_\sim) \lesssim h(w, \mathbb{R}^2)^{-1} \cdot \left\| \nabla \frac{Gg}{Gf} \right\|_{L^1(\mathbb{R}^2, w)}.
\end{equation}
This means that we already succeeded in bounding the distance between the signals in terms of their phaseless Gabor measurements. The aim, however, is to get a bound in terms of the difference of the Gabor transform magnitudes. In order to obtain this, we estimate
\begin{equation}
\left\| \nabla \frac{|Gf|}{|Gf|} \right\|_{L^1(\mathbb{R}^2, w)} \leq \left( \left\| \frac{|Gf|}{|Gf|} \right\|_{L^1(\mathbb{R}^2)} \right) \left( \left\| \frac{|Gf| - |Gg|}{|Gf|} \right\|_{L^1(\mathbb{R}^2)} + \left\| \nabla |Gf| - \nabla |Gg| \right\|_{L^1(\mathbb{R}^2)} \right).
\end{equation}
The final ingredient of the proof lies in the treatment of the logarithmic derivative \( \nabla \frac{|Gf|}{|Gf|} \). The norm of the logarithmic derivative on balls centered at the origin can essentially be controlled by the product of the volume of the ball and the number of its singularities in a ball of twice the radius, which are precisely the zeros of \( Gf \). Jensen’s formula relates the number of zeros of the function in (29), and therefore of \( Gf \), to its growth. Since the growth of the entire functions in (29) can be uniformly bounded for functions \( f \in \mathcal{M}^1 \) this argument allows to absorb the logarithmic derivative in a lower order polynomial, which is independent from \( f \). \( \square \)

References


